Interval Computation

Regivan Hugo Nunes Santiago

Departamento de Informática e Matemática Aplicada
Universidade Federal do Rio Grande do Norte
Natal - Rio Grande do Norte - Brasil

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Parte I

What is it?

Rump's example:

$$y = 333.75b^6 + a^2(11a^2b^2 - b^6 - 121b^4 - 2) + 5.5b^8 + a/(2b) \quad (1)$$

for $a = 77617.0$ and $b = 33096.0$.
Rump computed this function in an IBM S/370 main frame, he obtained the following results:

1. single precision: $y = 1.172603\ldots;$
2. double precision: $y = 1.1726039400531\ldots;$
3. extended precision: $y = 1.172603940053178\ldots;$

All results lead any user to conclude that IBM S/370 returned the correct result. However this result is WRONG and the correct result lies in the interval

$$-0.827396059946821 35 \pm 5 \times 10^{-17} \quad (2)$$

*** Even the sign is wrong.**

- This example was revised in [3] using IEEE 754 arithmetic in the Forte Developer 6 update 2 Fortran 95 compiler from Sun Microsystem Inc. and the results are even more incorrect.
- It means that floating-point computations can be even dangerous, if we imagine that, for example, human lives depend on a computer application which implements (or in some step of a computation evaluates to) Rump's function.
- Therefore, correctness is the fundamental condition in numerical computations.
- Aiming to provide methods for performing machine *correct* numerical computations, Moore[4, 5] proposed interval analysis which is, the development of an arithmetic, a topology, relations, etc. for closed intervals.

Interval Algebras

- A **closed interval** or just **Moore Interval** is a continuum of real numbers, defined by: $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$. Intervals whose endpoints are equals are called **degenerate intervals**.
- The **set of all closed intervals** is the set $\mathbb{IR} = \{[a, b] : a, b \in \mathbb{R} \wedge a \leq b\}$
- An n-ary **interval operation** is a function $F : \mathbb{IR}^n \rightarrow \mathbb{IR}$.
- An **interval algebra** is an structure $\langle \mathbb{IR}, \{F_i\}_{i \in I} \rangle$, where F_i is an n-ary interval operation

Interval Arithmetic — Moore Arithmetic

Given $X = [\underline{x}, \overline{x}]$ and $Y = [\underline{y}, \overline{y}]$, **Moore Interval Arithmetic** can be defined as follows:

$$X + Y = [\underline{x} + \underline{y}, \overline{x} + \overline{y}] \tag{3}$$

$$X - Y = [\underline{x} - \overline{y}, \overline{x} - \underline{y}] \tag{4}$$

$$X \times Y = [\min\ W, \max\ W] \tag{5}$$

where $W = \{\underline{x} \times \underline{y}, \underline{x} \times \overline{y}, \overline{x} \times \underline{y}, \overline{x} \times \overline{y}\}$

$$1/Y = [1/\overline{y}, 1/\underline{y}], \quad \text{for } 0 \notin Y. \tag{6}$$

Correctness

- Moore arithmetic guarantees correctness, since all the operations have the following property:

$$X \odot Y = \{x \odot y : x \in X \wedge y \in Y\} \tag{7}$$

for $\odot \in \{+, -, /, \times\}$.

Therefore, if we want to "calculate" some $x \odot y$, where $x \in X$ and $y \in Y$, then $x \odot y \in X \odot Y$

Correctness

- Interval correctness can be formalized in the following way:

  An interval function $F(X_1, \ldots, X_n)$ *is correct with respect a real function* $f(x_1, \ldots, x_n)$*, whenever*

$$x_1 \in X_1, \ldots, x_n \in X_n \Rightarrow f(x_1, \ldots, x_n) \in F(X_1, \ldots, X_n). \tag{8}$$

Correctness, order and precision

- The **width** of an interval $[a, b]$, $w([a, b])$, is $b - a$.
- It is possible to order intervals as sets, using set inclusion order; that is:

$$[a, b] \subseteq [c, d], \text{ if } c \leq a \leq b \leq d \tag{9}$$

- In this case $w([a, b]) \geq w([c, d])$
- A function F is inclusion-monotonic if $F([a, b]) \subseteq F([c, d])$, whenever $[a, b] \subseteq [c, d]$.
- In this case, inclusion-monotonic functions are correct.
- **Fundamental theorem of interval arithmetic:**

  "if F is a inclusion monotonic extension of a real function f, then $f(X_1, \ldots, X_n) \subseteq F(X_1, \ldots, X_n)$.

Correctness, order and precision

> *"This theorem is an astounding result: with a single evaluation of a function over an interval, access to information about the function over a continuum is obtained" (In Walster [11] section 1.2).*

Yes, Walster is right! This is really a strong result since it states that from a finite evaluation it is possible to access information of an infinite function. However, as we show below, correctness is not restricted only to inclusion monotonic.

Correctness, order and precision

- The opposite order: $[a, b] \sqsubseteq [c, d]$ iff $[c, d] \subseteq [a, b]$ establishes that $[c, d]$ is a **better or an equally better representation** for all real numbers in it than $[a, b]$, since $w([a, b]) \geq w([c, d])$.
- In this sense, $[c, d]$ is more precise than or equally precise $[a, b]$.
- Any inclusion-monotonic function is also monotonic here; therefore this order captures aspects of precision and correctness.
- This order was introduced on interval by D. Scott [10], and in 1990 Acióly [1] discovered that it is possible to obtain a **continuous domain structure** (domain theory), in such a way that the set of total elements in the space is isomorphic with real numbers.
- This discover gives another interpretation for interval; as **information about real numbers**.

Interval Semantics

Therefore, an interval can be interpreted as:

- a set
- a number extension
- an information

The point now is: What are the relations between those different aspects?

Algebraic Properties

Equality		**Consistency**
$x + (y + z) = (x + y) + z$		$x + (y + z) \asymp (x + y) + z$
$x + [0, 0] = x$		$x + [0, 0] \asymp x$
$x + y = y + x$		$x + y \asymp y + x$
$x - x \supseteq [0, 0]$		$x - x \asymp [0, 0]$
$x \times (y \times z) = (x \times y) \times z$		$x \times (y \times z) \asymp (x \times y) \times z$
$x \times [1, 1] = x$		$x \times [1, 1] \asymp x$
$x \times y = y \times x$		$x \times y \asymp y \times x$
$x/x \supseteq [1, 1]$		$x/x \asymp [1, 1]$
$x \times (y + z) \subseteq (x \times y) + (x \times z)$		$x \times (y + z) \asymp (x \times y) + (x \times z)$

Parte II

Recent Theoretical Results

Interval Representation Theory [7, 8, 9]

- What kind of interval functions are suitable to “represent” real functions?
- What is the relation between classical topological aspects of real numbers and the topologies derived from those interpretations of intervals?

Topological aspects of Intervals

Moore Continuity for Intervals

- Moore in [5] proposed the following notion of distance for intervals:

$$di(X, Y) = \max(|\ \underline{x} - \underline{y}\ |, |\ \overline{x} - \overline{y}\ |) \tag{10}$$

- This distance is proved to be a metric and therefore provide a continuity notion for interval functions.
- The ϵ**-open ball of center** A is the set $B(A, \epsilon) = \{X \in \mathbb{I}(\mathbb{R}) : di(A, X) < \epsilon\}$.

Scott Continuity for Intervals

- The information order "$\sqsubseteq$" induces a topology called Scott topology;
- A function $F : \mathbb{IR} \rightarrow \mathbb{IR}$ is **ord-continuous** if for every directed set Δ, $F(\bigsqcup \Delta) = \bigsqcup F(\Delta)$;
- A function is ord-continuous iff it is continuous with respect to Scott topology — **Scott-continuous**.
- A Scott-continuous function is also monotonic; and therefore a correct function;
- From Scott topology, it is also possible to derive a quasi-metric space [2]:

$$qi(X, Y) = max\{\underline{y} - \underline{x}, \overline{x} - \overline{y}, 0\} \tag{11}$$

- The ϵ-**open ball of center** A is the set $B(A, \epsilon) = \{X \in \mathbb{I}(\mathbb{R}) : qi(A, X) < \epsilon\}$.
- $F : \mathbb{I}(\mathbb{R}) \rightarrow \mathbb{I}(\mathbb{R})$ is Scott-continuous iff F is qi-continuous.

Moore-continuity vs Scott-continuity

- Moore-continuity does not imply Scott-continuity
- The following function is Moore but not Scott-continuous; where $m([a, b]) = \frac{a+b}{2}$ is the **midpoint** of $[a, b]$:

$$F(X) = m(X) + \frac{1}{2}(X - m(X)) \tag{12}$$

Moore-continuity vs Scott-continuity

- Scott-continuity does not imply Moore-continuity
- The following function is Scott but not Moore-continuous:

$$F(X) = \begin{cases} [-1, 1] & \text{, if } 0 \in X, \text{ or} \\ [0, 0] & \text{, otherwise.} \end{cases} \tag{13}$$

Distances for Intervals and it's interpretations

- A **metric** is a function $d : A \times A \longrightarrow \mathbb{R}$, such that for all $a, b, c \in A$,
  - $d(a, b) = 0 \Leftrightarrow a = b$;
  - $d(a, c) \leq d(a, b) + d(b, c)$; and
  - $d(a, b) = d(b, a)$.

  The pair (A, d) is called **metric space**.
- A **quasi-metric** generalizes the notion of metric. It is a function $d : A \times A \longrightarrow \mathbb{R}$, such that
  - $d(a, a) = 0$;
  - $d(a, c) \leq d(a, b) + d(b, c)$; and
  - $d(a, b) = d(b, a) = 0 \Rightarrow a = b$.
- Observe the counter-positive for the last axiom, i.e. $a \neq b \Rightarrow [d(a, b) \neq d(b, a) \vee d(a, b) \neq 0 \vee d(b, a) \neq 0]$. So, it is possible to have $a \neq b$, $d(a, b) \neq 0$, and $d(b, a) = 0$.
- The concept of distance in a set A can be formalized by the notion of **quasi-metric**.

Distances for Intervals and it's interpretations

- For every quasi-metric q, it is always possible to define another quasi-metric called **conjugated quasi-metric** defined by $\overline{q}(a,b) = q(b,a)$ and a metric q^*, such that

$$q^*(a,b) = max\{q(a,b), \overline{q}(a,b)\}. \tag{14}$$

- Since a quasi-metric q generalizes the notion of distance, it induces the following open balls:
  - $B(a,\epsilon) = \{s \in A : q(a,s) < \epsilon\}$,
  - $\overline{B}(a,\epsilon) = \{s \in A : \overline{q}(a,s) < \epsilon\}$ and
  - $B^*(a,\epsilon) = \{s \in A : q^*(a,s) < \epsilon\}$.

  Those three kinds of balls define three topological spaces: $\mathcal{T}(q)$ and $\mathcal{T}(\overline{q})$, and the metric $\mathcal{T}(q^*)$.

Distances for Intervals and it's interpretations

- The quasi-metric for intervals which is associated with Scott-continuity was introduced by Acióly and Bedregal in [2]:

$$d_I([a,b],[c,d]) = \max\{a-c, d-b, 0\} \qquad (15)$$

- It's conjugated is:

$$\overline{d_I}([a,b],[c,d]) = \max\{c-a, b-d, 0\} \qquad (16)$$

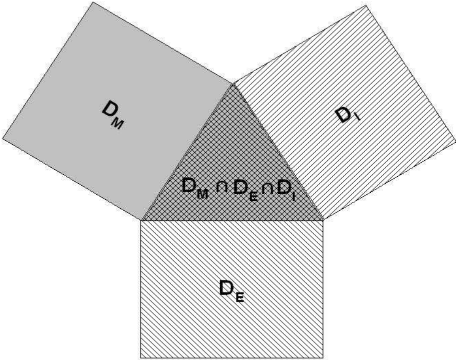
- **Proposition**: Moore metrics [6] coincides with d_I^*.
- **Obs:** Therefore, d_I is a distance to measure intervals as **information**, d_E as **sets** and d_I^* as **numbers**. From now on d_I^* will be replaced by d_M.

Characterizing D_I, D_E and D_M-continuities

- **Definition**: A function $F : \mathbb{I}(\mathbb{R}) \to \mathbb{I}(\mathbb{R})$ is bi-continuous if it is d_I and d_E-continuous.
- **Definition**: The sets D_M, D_I and D_E are the sets of d_M, d_I and d_E-continuous functions.
- **Proposition:**
  - $D_I - (D_E \cup D_M) \neq \emptyset$.
  - $D_E - (D_I \cup D_M) \neq \emptyset$.
  - $D_M - (D_I \cup D_E) \neq \emptyset$.
- **Proposition:** $F : \mathbb{I}(\mathbb{R}) \to \mathbb{I}(\mathbb{R})$ **is bi-continuous** iff F is **Moore and Scott** continuous.
- **Corolary:** $(D_E \cap D_I) - D_M = (D_M \cap D_I) - D_E = \emptyset$.
- **Proposition:** If $F : \mathbb{I}(\mathbb{R}) \to \mathbb{I}(\mathbb{R})$ is **Moore and $\mathbf{d_E}$-continuous** then F is **bi-continuous**.
- **Corolary:** $(D_M \cap D_E) - D_I = \emptyset$.
- **Corolary:** $D_M \cap D_E \cap D_I = D_E \cap D_I = D_E \cap D_M = D_M \cap D_I$.

Characterizing D_I, D_E and D_M-continuities

- According to those corollaries, the following figure shows the classification of interval continuous functions with respect to various viewpoints of interval analysis; namely the extensional, informational and metrical — i.e. D_E, D_I and D_M, respectively:



Correctness and Optimality

- Correctness or Representation: **An interval function F is correct with respect to a real function f** if it satisfies the following property:

$$x \in [a, b] \Rightarrow f(x) \in F([a, b]) \tag{17}$$

- Let $f : \mathbb{R} \to \mathbb{R}$ and $F : \mathbb{I}(\mathbb{R}) \to \mathbb{I}(\mathbb{R})$ be real and interval functions, respectively. F **represent** f, if for each $X \in \mathbb{I}(\mathbb{R})$ and $x \in X$, $f(x) \in F(X)$. The set of all real functions represented by F is $Rep(F) = \{f : F \text{ represents } f\}$.

Some results about Correctness

- Some interval function $F : \mathbb{I}(\mathbb{R}) \longrightarrow \mathbb{I}(\mathbb{R})$ does not represent any real function f. For example:

$$F([a,b]) = \begin{cases} [5,5] & \text{, if } a = b\text{, or} \\ [0,1] & \text{, otherwise} \end{cases} \tag{18}$$

- Not every real function f admits interval representations. For example:

$$f(x) = \begin{cases} \frac{1}{x} & \text{, if } x > 0\text{, or} \\ 1 & \text{, otherwise} \end{cases} \tag{19}$$

Correctness vs d_I, d_E and d_M-continuities

- Not every interval representation is Moore-continuous.
- Every monotonic function $F : \mathbb{I}(\mathbb{R}) \to \mathbb{I}(\mathbb{R})$ represents some real function f
- Every Scott-continuous function $F : \mathbb{I}(\mathbb{R}) \to \mathbb{I}(\mathbb{R})$ represents some real function f.
- Not every interval representation is a monotone function and hence a Scott-continuous function.
- **Corollary:** Correctness is weaker than monotonicity; i.e. **The Fundamental Theorem for Interval Arithmetics** is very strong to capture the notion of correctness.
- **Corollary:** So the set of interval monotonic functions is a proper subset of interval correct functions.

Canonical representations

- x^2 and $x \cdot x$ are the same functions;
- But the interval counterpart "$X \cdot X$", does not always return the optimum interval approximation.
- For example, $[-1, 2] \cdot [-1, 2] = [-2, 4]$;
- $X \cdot X$ is correct, since the image of x^2 under $[-1, 2]$ is a subset of $[-1, 2] \cdot [-1, 2]$.
- The function bellow is also a representation for x^2:

$$SQR([a, b]) = \begin{cases} [a^2, b^2] & \text{, if } a \geq 0, \text{ or} \\ [b^2, a^2] & \text{, if } b < 0, \text{ or} \\ [0, 0] & \text{, otherwise.} \end{cases} \tag{20}$$

- However this is the best one, since for every representation F, $F([a, b]) \sqsubseteq SQR([a, b])$.
- We called it the **canonical interval representation**.

Canonical representations and Euclidean Topology

- Let $f : \mathbb{R} \to \mathbb{R}$ be a real function. If f is a total non-asymptotic real function, then the interval function:

$$CIR(f)([a,b]) = [min\ f([a,b]), max\ f([a,b])]$$

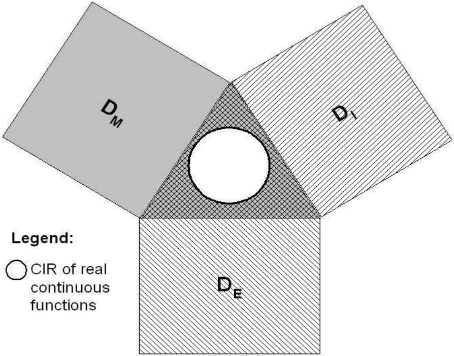
  is well defined and it is an interval representation called **canonical interval representation** for f.
- **Lemma:** If $f : \mathbb{R} \to \mathbb{R}$ is continuous then for every $[a,b] \in \mathbb{I}(\mathbb{R})$, $CIR(f)([a,b]) = f([a,b])$.
- **Proposition:** For every interval representation F of a real function f, $F \sqsubseteq CIR(f)$.
- **First Representation Theorem:** For every real function f, f is continuous if and only if $CIR(f)$ is Scott-continuous.
- **Second Representation Theorem:** For every real function f, f is continuous iff $CIR(f)$ is Moore-continuous.
- **Corollary: Representation for bi-continuity:** f is continuous if and only if $CIR(f)$ is bi-continuous..

Canonical representations and Euclidean Topology

- *CIR* is an operator which maps the set of real Euclidean continuous functions into the set of Scott-continuous functions and Moore-continuous functions.
- **Corollary:** If $CIR(f)$ is Scott-continuous or Moore-continuous then $CIR(f)([a,b]) = f([a,b])$.
- **Proposition:** There are discontinuous functions $f : \mathbb{R} \to \mathbb{R}$, such that $CIR(f)$ is monotonic.
- **Proposition:** $CIR(\lfloor\ \rfloor)$ is neither Moore-continuous nor Scott-continuous.
- **Corollary:** f is continuous iff $CIR(f)$ is d_I, d_E, and d_M-continuous.
- **Proposition:** Not every interval function Moore and Scott continuous is a canonical representation of some continuous function.
- **Corollary:** Not every interval bi-continuous function is the canonical interval representation of a real continuous function.

Canonical representations and Euclidean Topology

- **Corollary:** $(D_E \cap D_I \cap D_M) - \{CIR(f) : f \text{ is continuous}\} \neq \emptyset$



The basic idea of an interval system

Parallel resistors

- The equation to calculate the resistance of resistors connected in parallel is:

$$R_p = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}} \tag{21}$$

- Now suppose that a resistor $R_1 = 6.80$ ohms with 10% of tolerance (i.e. the resistance varies in the interval 6.80 ± 0.68, i.e. $[6.12, 7.48]$) is connected in parallel with another $R_2 = 4.7$ ohms with 5% of tolerance (i.e. the resistance varies in the interval 4.7 ± 0.23 i.e. $[4.47, 4.93]$).
- The values of the resulting resistance varies in the interval $[2.58, 2.97]$ with midpoint and tolerance equal to 2.77 ± 0.20.
- This is an interval application, since the result is also a variation, an uncertainty. Moreover, the interval function must be correct.
- The equation 21 is in fact an interval equation instead of a real equation, since tolerances are important data for the system.

Interval Fuzzy Logic

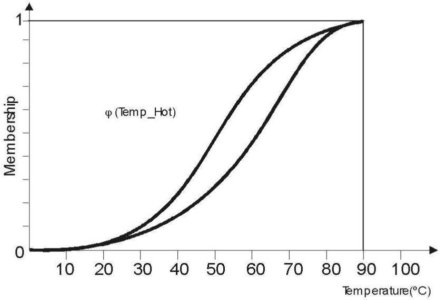
- Interval Valued Fuzzy Sets were introduced by Turksen in 1986;
- An interval Valued Fuzzy Set is a function:

$$A = \{(x, \varphi_A(x)) : x \in U \wedge \varphi_A(x) \in I[0,1]\} \tag{22}$$

where U is an universe and
$I[0,1] = \{[a,b] \in \mathbb{I}(\mathbb{R}) : 0 \leq a \leq b \leq 1\}$.

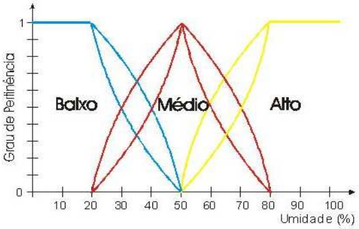
Interval Fuzzy Sets

- The following figure is a fuzzy interval set "Hot Temperature". The membership interval degrees for 40°C, 50°C and 60°C, are respectively $[0.16, 0.23]$, $[0.28, 0.39]$ and $[0.47, 0.71]$.



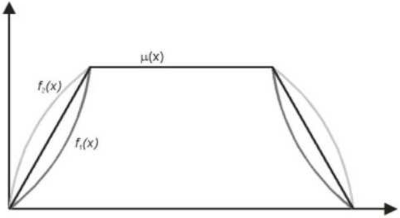
Interval Fuzzy Sets

- Notice that the usual fuzzy membership functions (linear, curve-S, bell curves, triangular, etc.) are continuous. Therefore, it's canonical interval representations, its fuzzy interval counterparts are continuous.
- **Example of interval set:**



Interval Fuzzy Sets

- **Example of interval set:**



Further Applications

- Signal Processing;
- Image Processing (Signal and Morphological approach);
- Computation with real numbers.

Programming Languages

- XSC-Languages
  - Pascal-XSC
  - C-XSC
  - Fortran-XSC
  - Java-XSC

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