Definition 3.1 Linear-time temporal logic (LTL) has the following syntax given in Backus Naur form:

$$\begin{aligned}\phi ::= &\top \mid \bot \mid p \mid (\neg\phi) \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \rightarrow \phi)\\ &\mid (\mathrm{X}\,\phi) \mid (\mathrm{F}\,\phi) \mid (\mathrm{G}\,\phi) \mid (\phi\ \mathrm{U}\ \phi) \mid (\phi\ \mathrm{W}\ \phi) \mid (\phi\ \mathrm{R}\ \phi)\end{aligned} \tag{3.1}$$

where p is any propositional atom from some set `Atoms`.

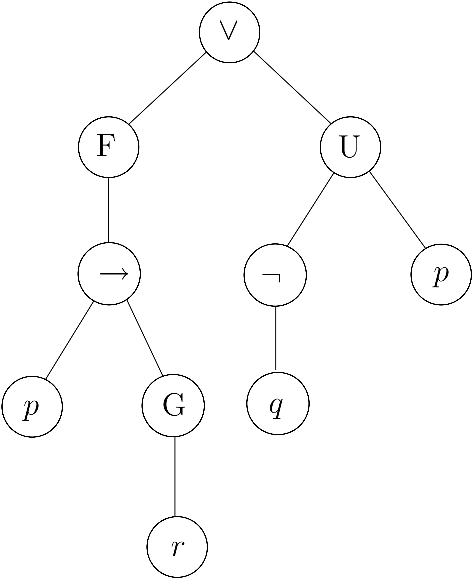

Figure 3.1. The parse tree of $(\mathrm{F}\,(p \rightarrow \mathrm{G}\,r) \vee ((\neg q)\ \mathrm{U}\ p))$.

3.2.2 Semantics of LTL

The kinds of systems we are interested in verifying using LTL may be modelled as transition systems. A transition system models a system by means of *states* (static structure) and *transitions* (dynamic structure). More formally:

Definition 3.4 A transition system $\mathcal{M} = (S, \rightarrow, L)$ is a set of states S endowed with a transition relation $\rightarrow$ (a binary relation on S), such that every $s \in S$ has some $s' \in S$ with $s \rightarrow s'$, and a labelling function $L\colon S \rightarrow \mathcal{P}(\mathtt{Atoms})$.

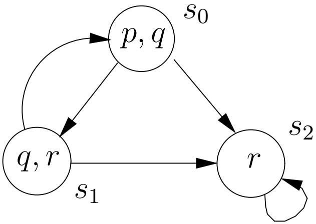

Figure 3.3. A concise representation of a transition system $\mathcal{M} = (S, \rightarrow, L)$ as a directed graph. We label state s with l iff $l \in L(s)$.

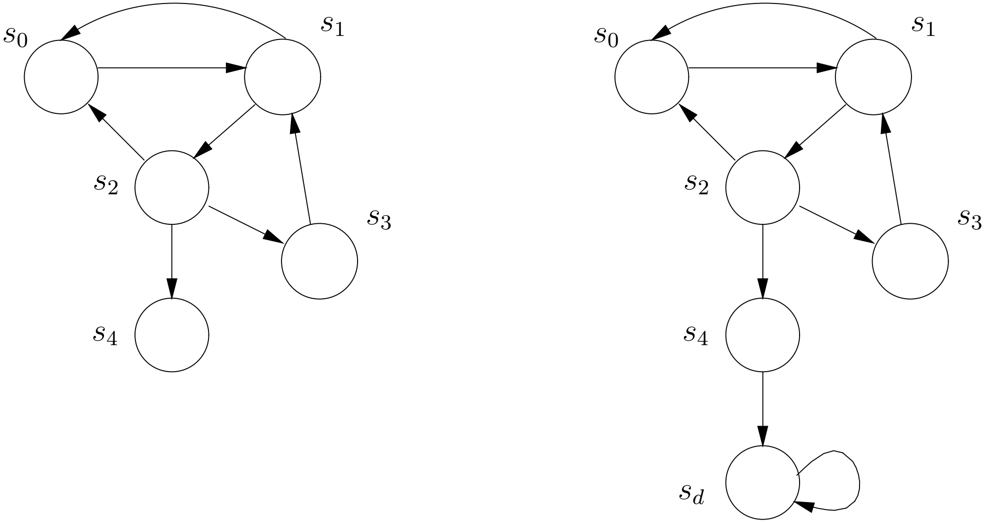

Figure 3.4. On the left, we have a system with a state s_4 that does not have any further transitions. On the right, we expand that system with a 'deadlock' state s_d such that no state can deadlock; of course, it is then our understanding that reaching the 'deadlock' state s_d corresponds to deadlock in the original system.

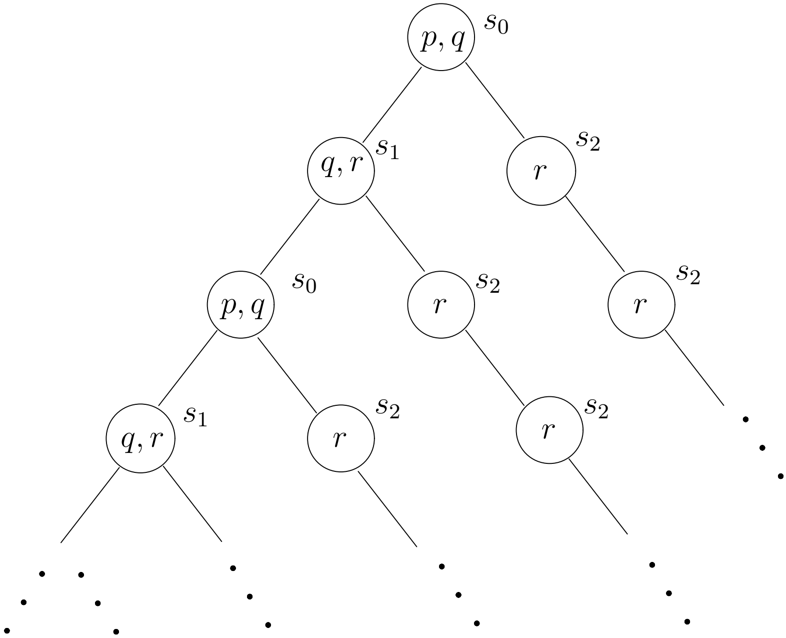

Figure 3.5. Unwinding the system of Figure 3.3 as an infinite tree of all computation paths beginning in a particular state.

Definition 3.6 Let $\mathcal{M} = (S, \rightarrow, L)$ be a model and $\pi = s_1 \rightarrow \ldots$ be a path in $\mathcal{M}$. Whether π satisfies an LTL formula is defined by the satisfaction relation $\vDash$ as follows:

1. $\pi \vDash \top$
2. $\pi \nvDash \bot$
3. $\pi \vDash p$ iff $p \in L(s_1)$
4. $\pi \vDash \neg\phi$ iff $\pi \nvDash \phi$
5. $\pi \vDash \phi_1 \wedge \phi_2$ iff $\pi \vDash \phi_1$ and $\pi \vDash \phi_2$
6. $\pi \vDash \phi_1 \vee \phi_2$ iff $\pi \vDash \phi_1$ or $\pi \vDash \phi_2$
7. $\pi \vDash \phi_1 \rightarrow \phi_2$ iff $\pi \vDash \phi_2$ whenever $\pi \vDash \phi_1$
8. $\pi \vDash \mathrm{X}\,\phi$ iff $\pi^2 \vDash \phi$
9. $\pi \vDash \mathrm{G}\,\phi$ iff, for all $i \geq 1$, $\pi^i \vDash \phi$

10. $\pi \models \mathrm{F}\,\phi$ iff there is some $i \geq 1$ such that $\pi^i \models \phi$
11. $\pi \models \phi \mathrel{\mathrm{U}} \psi$ iff there is some $i \geq 1$ such that $\pi^i \models \psi$ and for all $j = 1, \ldots, i-1$ we have $\pi^j \models \phi$
12. $\pi \models \phi \mathrel{\mathrm{W}} \psi$ iff either there is some $i \geq 1$ such that $\pi^i \models \psi$ and for all $j = 1, \ldots, i-1$ we have $\pi^j \models \phi$; or for all $k \geq 1$ we have $\pi^k \models \phi$
13. $\pi \models \phi \mathrel{\mathrm{R}} \psi$ iff either there is some $i \geq 1$ such that $\pi^i \models \phi$ and for all $j = 1, \ldots, i$ we have $\pi^j \models \psi$, or for all $k \geq 1$ we have $\pi^k \models \psi$.