Proof of a fact about the factorial function in P-logic

Till Mossakowski

We put

$$Geq\ P = \{x \mid \exists y\ .\ y \geq x \wedge y ::: P\}$$

From the rule about sections, we get

$$\vdash 1 ::: \$!(== 1)$$

By disjunction introduction,

$$\vdash 1 ::: \$!(== 1) \vee Geq\ \xi$$

From this and $n ::: \$!(== 0) \vdash n == 0 ::: \$True$, we get by the if-then-else rule

$$n ::: \$!(== 0) \vdash if\ \ n == 0\ then 1\ else\ n * fact(n-1) ::: \$!(== 1) \vee Geq\ \xi$$

By the abstraction rule, we get

$$\vdash \backslash n \to if\ \ n == 0\ then 1\ else\ n * fact(n-1) ::: \$(\$!(== 0) \to \$!(== 1) \vee Geq\ \xi) \tag{1}$$

By arithmetic reasoning (which either is coded in extra rules, or is based on reasoning about algebraic datatypes), we get

$$n ::: \$!(> 0) \vdash n - 1 ::: \$(\geq 0)$$

and

$$fact ::: \$!(> 0) \to \xi,\ n ::: \$!(> 0) \vdash fact(n-1) ::: Geq\ \xi$$

From these two, again by arithmetic reasoning, we get

$$fact ::: \$!(> 0) \to \xi,\ n ::: \$!(> 0) \vdash n * fact(n-1) ::: Geq\ \xi$$

By disjunction introduction,

$$fact ::: \$!(> 0) \to \xi,\ n ::: \$!(> 0) \vdash n * fact(n-1) ::: \$!(== 1) \vee Geq\ \xi$$

From this and $n ::: \$!(> 0) \vdash n == 0 ::: \$False$, we get by the if-then-else rule

$$fact ::: \$!(> 0) \to \xi,\ n ::: \$!(> 0) \vdash if\ \ n == 0\ then 1\ else\ n * fact(n-1) ::: \$!(== 1) \vee Geq\ \xi$$

By the abstraction rule, we get

$$fact ::: \$!(> 0) \to \xi \vdash \backslash n \to if\ \ n == 0\ then 1\ else\ n * fact(n-1) ::: \$!(> 0) \to \$!(== 1) \vee Geq\ \xi$$

From this (the inductive step) and (1), the inductive base, we get by the Lfp rule:

$$fact = \backslash n \to if\ \ n == 0\ then 1\ else\ n * fact(n-1) \vdash fact ::: \$(\$(\geq 0) \to Lfp\ \xi\ .\ \$!(== 1) \vee Geq\ \xi)$$