Software specification in CASL - The Common Algebraic Specification Language

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January 2007

SFB/TR 8 SPATIAL COGNITION

Semantics of CASL basic specifications (recalled)

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- Satisfaction of sentences in models

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- an S^*-indexed set $(P_w)_{w \in S^*}$ of predicate symbols

Signature morphisms map these components in a compatible way

Example signatures

- $\Sigma^{Nat} = (\{Nat\}, \{0 : Nat, succ\colon Nat \longrightarrow Nat\},$
  $\{pre\colon Nat \longrightarrow ?Nat\}, \emptyset)$

- $(\{Elem\}, \emptyset, \emptyset, \{\_\_ < \_\_ : Elem * Elem\})$

- $(\{Elem, List\},$
  $\{Nil : Elem, Cons\colon Elem * List \longrightarrow List\}, \emptyset, \emptyset)$

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- a predicate $p^M \subseteq w^M$ for each predicate symbol $p \in P_w$.

Example Σ^{Nat}-models

- $Nat^M = I\!N$, 0^M=0, $suc^M(x) = x + 1$,
  $pre^M(x) = \begin{cases} x - 1, x > 0 \\ \text{undefined, otherwise} \end{cases}$

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Semantics of terms

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- conjunctions, disjunctions, implications, equivalences of formulae
- universal, existential, unique-existential quantifications

Satisfaction of atomic formulae

A formula φ is satisfied in a model M w.r.t. a valuation $\nu\colon X \longrightarrow M$ (short notation: $M, \nu \models \varphi$), if

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- a universal (existential) quantification is satisfied when all (some) of the changes of the valuation for the quantified variable lead to satisfcation in the model: $M, \nu \models \forall x : s \,.\, \phi$ iff $M, \xi \models \phi$ for all valuation ξ that differ from ν only on $x : s$

Satisfaction of closed formulae

A closed formula (sentences) is satisfied in a model iff it is satisfied w.r.t. the empty valuation:

$$M \models \varphi \text{ iff } M, \emptyset \models \varphi$$

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i.e. for each $s \in S'$, $a \in s^M$, there is some term t (with variables of sorts outside S') and some valuation ν with $\nu^{\#}(t) = a$.

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Semantics of CASL Structured Specifications

Institutions

- Basic idea: abstract away from the details of signature, model, sentence, satisfaction.
- The semantics of CASL structured specifications is defined for an arbitrary institution.
- first-order, higher-order, polymorphic, modal, temporal, process, behavioural, ASM- und Z-like and object-oriented logics have been shown to be institutions.
- Hence, you may replace the CASL institution with your favourite institution.

The CASL institution revisited

Given a signature morphism $\sigma\colon \Sigma \longrightarrow \Sigma'$, $\Sigma = (S, TF, PF, P)$ and a Σ'-model M', the reduct $M'|_\sigma$ is defined as follows

- $s^M := \sigma(s)^{M'}$ for $s \in S$,
- $f_{w,s}^M := \sigma(f_{w,s})^{M'}$ for $f \in TF_{w,s} \cup PF_{w,s}$,
- $p_w^M := \sigma(p_w)^{M'}$ for $p \in P_w$.

A Σ-formula φ is translated along σ by just replacing the symbols in φ according to σ.

The Satisfaction Condition

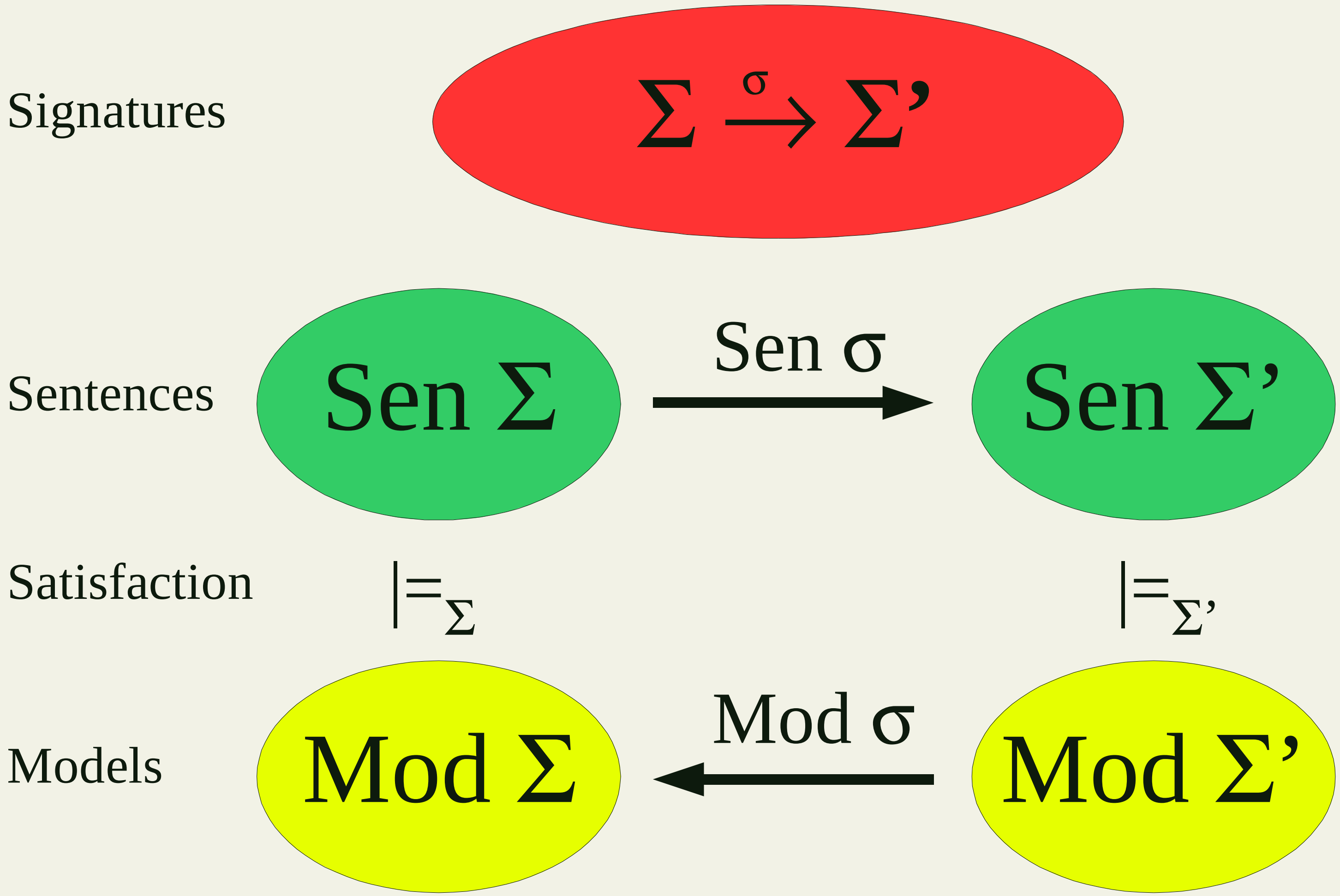
Theorem

$$M' \models \sigma(\varphi) \text{ iff } M'|_\sigma \models \varphi$$

That is:

Truth is invariant under change of notation and enlargement of context.

Institutions



Institutions, formally

- category $\mathbf{Sign}$ of signatures,
- a sentence functor $\mathbf{Sen}\colon \mathbf{Sign} \longrightarrow \mathbf{Set}$,
- a model functor $\mathbf{Mod}\colon \mathbf{Sign}^{op} \longrightarrow \mathcal{CAT}$,
- a satisfaction relation $\models_\Sigma \subseteq |\mathbf{Mod}(\Sigma)| \times \mathbf{Sen}(\Sigma)$,

such that the following satisfaction condition holds:

$$M' \models_{\Sigma'} \mathbf{Sen}(\sigma)(\varphi) \Leftrightarrow \mathbf{Mod}(\sigma)(M)' \models_\Sigma \varphi$$

or shortly

$$M' \models_{\Sigma'} \sigma(\varphi) \Leftrightarrow M'|_\sigma \models_\Sigma \varphi.$$

Benefits of institutions

- Institution independent semantics (and proof system) of structured specifications, architectural specifications, refinement, behavioural abstraction etc.
- ASMs over arbitrary institutions (Zucca 1999, TCS 216)
- Borrowing of parts of a logic from other logics
- Combination of logics
- Heterogeneous specification and tools
- Abstract model theory with deep results (Diaconescu)

Semantics of basic specifications

$$\frac{\Sigma \vdash \texttt{BASIC-SPEC} \rhd (\Sigma', \Psi)}{\Sigma \vdash \texttt{BASIC-SPEC}\ qua\ \texttt{SPEC} \rhd \Sigma'}$$

$$\frac{\Sigma \vdash \texttt{BASIC-SPEC} \rhd (\Sigma', \Psi) \qquad \mathcal{M}' = \{M \in \mathbf{Mod}(\Sigma') \mid M|_{\Sigma} \in \mathcal{M}, M \models \Psi\}}{\Sigma, \mathcal{M} \vdash \texttt{BASIC-SPEC}\ qua\ \texttt{SPEC} \Rightarrow \Sigma', \mathcal{M}'}$$

Semantics of translations

$$\frac{\Sigma \vdash \texttt{SPEC} \rhd \Sigma'}{\Sigma \vdash \texttt{SPEC}\ \textbf{with}\ \sigma\colon \Sigma' \longrightarrow \Sigma'' \rhd \Sigma''}$$

$$\frac{\begin{array}{c}\Sigma, \mathcal{M} \vdash \texttt{SPEC} \Rightarrow \Sigma', \mathcal{M}' \\ \mathcal{M}'' = \{M \in \mathbf{Mod}(\Sigma'') \mid M|_\sigma \in \mathcal{M}'\}\end{array}}{\Sigma, \mathcal{M} \vdash \texttt{SPEC}\ \textbf{with}\ \sigma\colon \Sigma' \longrightarrow \Sigma'' \Rightarrow \Sigma'', \mathcal{M}''}$$

Semantics of reductions

$$\frac{\Sigma \vdash \texttt{SPEC} \rhd \Sigma'}{\Sigma \vdash \texttt{SPEC}\ \textbf{hide}\ \sigma\colon \Sigma'' \longrightarrow \Sigma' \rhd \Sigma''}$$

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Semantics of extensions

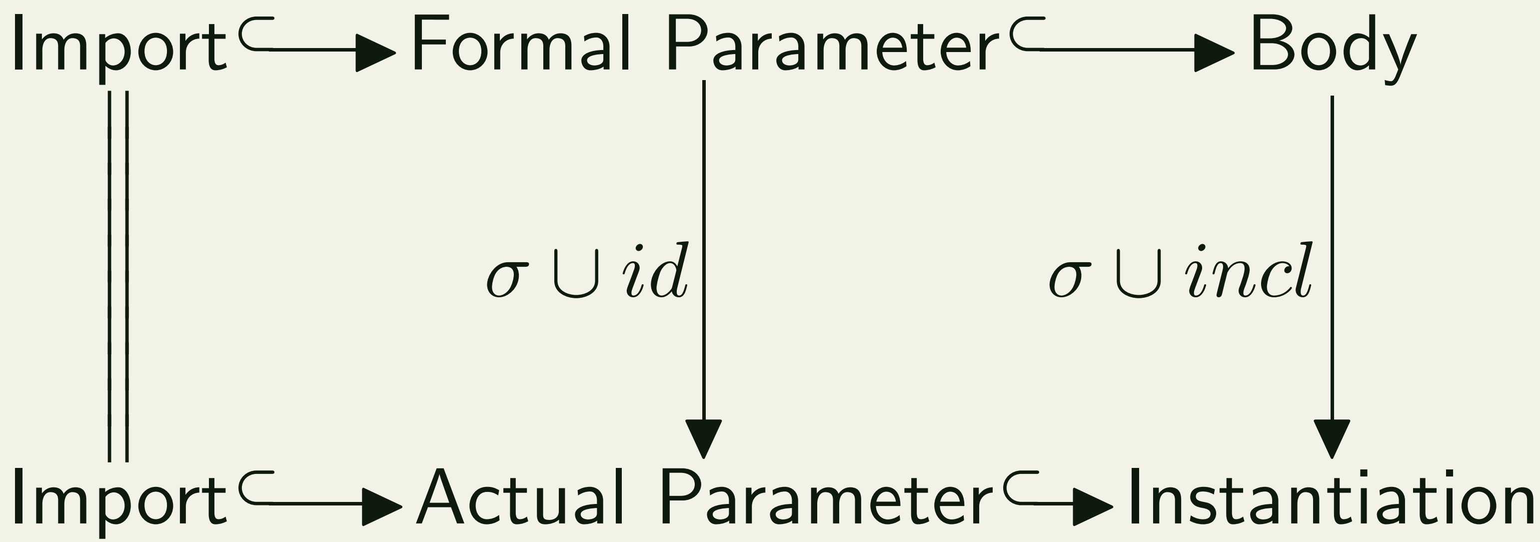
$$\frac{\Sigma \vdash \mathrm{SPEC}_1 \rhd \Sigma' \qquad \Sigma' \vdash \mathrm{SPEC}_2 \rhd \Sigma''}{\Sigma \vdash \mathrm{SPEC}_1 \textbf{ then } \mathrm{SPEC}_2 \rhd \Sigma''}$$

$$\frac{\Sigma, \mathcal{M} \vdash \mathrm{SPEC}_1 \Rightarrow \Sigma', \mathcal{M}' \qquad \Sigma', \mathcal{M}' \vdash \mathrm{SPEC}_2 \Rightarrow \Sigma'', \mathcal{M}''}{\Sigma, \mathcal{M} \vdash \mathrm{SPEC}_1 \textbf{ then } \mathrm{SPEC}_2 \Rightarrow \Sigma'', \mathcal{M}''}$$

Semantics of views

$$\frac{\begin{array}{c}\emptyset, \mathcal{M}_\perp \vdash \mathtt{SPEC}_1 \Rightarrow \Sigma_1, \mathcal{M}_1 \\ \emptyset, \mathcal{M}_\perp \vdash \mathtt{SPEC}_2 \Rightarrow \Sigma_2, \mathcal{M}_2 \\ \text{for each } M \in \mathcal{M}_2,\ M|_\sigma \in \mathcal{M}_1\end{array}}{\vdash \textbf{view } \mathtt{SPEC}_1 \textbf{ to } \mathtt{SPEC}_2 = \sigma \Rightarrow \sigma, \mathcal{M}_1, \mathcal{M}_2}$$

Semantics of parameterization (simplified)



The right square is required to be a pushout, that is, all symbols shared between the body and the actual parameter must occur also in the formal parameter.

Models: those models of the instantiation whose reducts are models of the body and of the actual parameter.

Development graphs $\mathcal{S} = \langle \mathcal{N}, \mathcal{L} \rangle$

Nodes in $\mathcal{N}$: (Σ^N, Γ^N) with

- Σ^N signature,
- $\Gamma^N \subseteq \mathbf{Sen}(\Sigma^N)$ set of local axioms.

Links in $\mathcal{L}$:

- global $M \xrightarrow{\sigma} N$, where $\sigma : \Sigma^M \rightarrow \Sigma^N$,
- local $M \overset{\sigma}{\cdots\cdots\blacktriangleright} N$ where $\sigma : \Sigma^M \rightarrow \Sigma^N$, or
- hiding $M \xrightarrow[h]{\sigma} N$ where $\sigma : \Sigma^N \rightarrow \Sigma^M$ going against the direction of the link.

Semantics of development graphs

$\mathbf{Mod}_S(N)$ consists of those Σ^N-models n for which

1. n satisfies the local axioms Γ^N,
2. for each $K \xrightarrow{\sigma} N \in \mathcal{S}$, $n|_\sigma$ is a K-model,
3. for each $K \overset{\sigma}{\dashrightarrow} N \in \mathcal{S}$,
   $n|_\sigma$ satisfies the local axioms Γ^K,
4. for each $K \xrightarrow[h]{\sigma} N \in \mathcal{S}$,
   n has a σ-expansion k (i.e. $k|_\sigma = n$) that is a K-model.

Theorem links

Theorem links come in two versions:

- global theorem links $M \xrightarrow{\sigma} N$, where $\sigma\colon \Sigma^M \longrightarrow \Sigma^N$,
  - $\mathcal{S} \models M \xrightarrow{\sigma} N$ iff for all $n \in \mathbf{Mod}_{\mathcal{S}}(N)$, $n|_\sigma \in \mathbf{Mod}_{\mathcal{S}}(M)$.
- local theorem links $M \overset{\sigma}{\dashrightarrow} N$, where $\sigma\colon \Sigma^M \longrightarrow \Sigma^N$,
  - $\mathcal{S} \models M \overset{\sigma}{\dashrightarrow} N$ iff for all $n \in \mathbf{Mod}_{\mathcal{S}}(N)$, $n|_\sigma \models \Gamma^M$.
- the calculus reduces these to local proof obligations.