Logik für Informatiker
Logic for computer scientists

Boolean connectives

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Negation — Truth table

P	¬P
TRUE	FALSE
FALSE	TRUE

The Henkin-Hintikka game

"Checkmate!"

The Henkin-Hintikka game

Is a sentence true in a given world?

- Players: you and the computer (Tarski's world)
- You claim that a sentence is true (or false), Tarski's world will claim the opposite
- In each round, the sentence is reduced to a simpler one
- When an atomic sentence is reached, its truth can be directly inspected in the given world

You have a winning strategy exactly in those cases where your claim is correct.

Negation — Game rule

Form	Your commitment	Player to move	Goal
$\neg P$	either	—	Replace $\neg P$ by P and switch commitment

Conjunction — Truth table

P	Q	P ∧ Q
TRUE	TRUE	TRUE
TRUE	FALSE	FALSE
FALSE	TRUE	FALSE
FALSE	FALSE	FALSE

Conjunction — Game rule

Form	Your commitment	Player to move	Goal
$P \wedge Q$	TRUE	Tarski's World	
	FALSE	you	Choose one of P, Q that is false.

Disjunction — Truth table

P	Q	$P \lor Q$
TRUE	TRUE	TRUE
TRUE	FALSE	TRUE
FALSE	TRUE	TRUE
FALSE	FALSE	FALSE

Disjunction — Game rule

Form	Your commitment	Player to move	Goal
$P \vee Q$	TRUE	you	Choose one of P, Q that is true.
	FALSE	Tarski's World	

Formalisation

- Sometimes, natural language double negation means logical single negation
- The English expression and sometimes suggests a temporal ordering; the FOL expression $\wedge$ never does.
- The English expressions but, however, yet, nonetheless, and moreover are all stylistic variants of and.
- Natural language disjunction can mean invlusive-or ($\vee$) or exclusive-or: $A\ xor\ B \Leftrightarrow (A \vee B) \wedge (\neg A \vee \neg B)$

Logical necessity

A sentence is

- logically necessary, or logically valid, if it is true in all circumstances (worlds),
- logically possible, if it is true in some circumstances (worlds),
- logically impossible, if it is true in no circumstances (worlds).

Logically possible

Logically and physically possible

Logically impossible

$P \wedge \neg P$ $\qquad a \neq a$

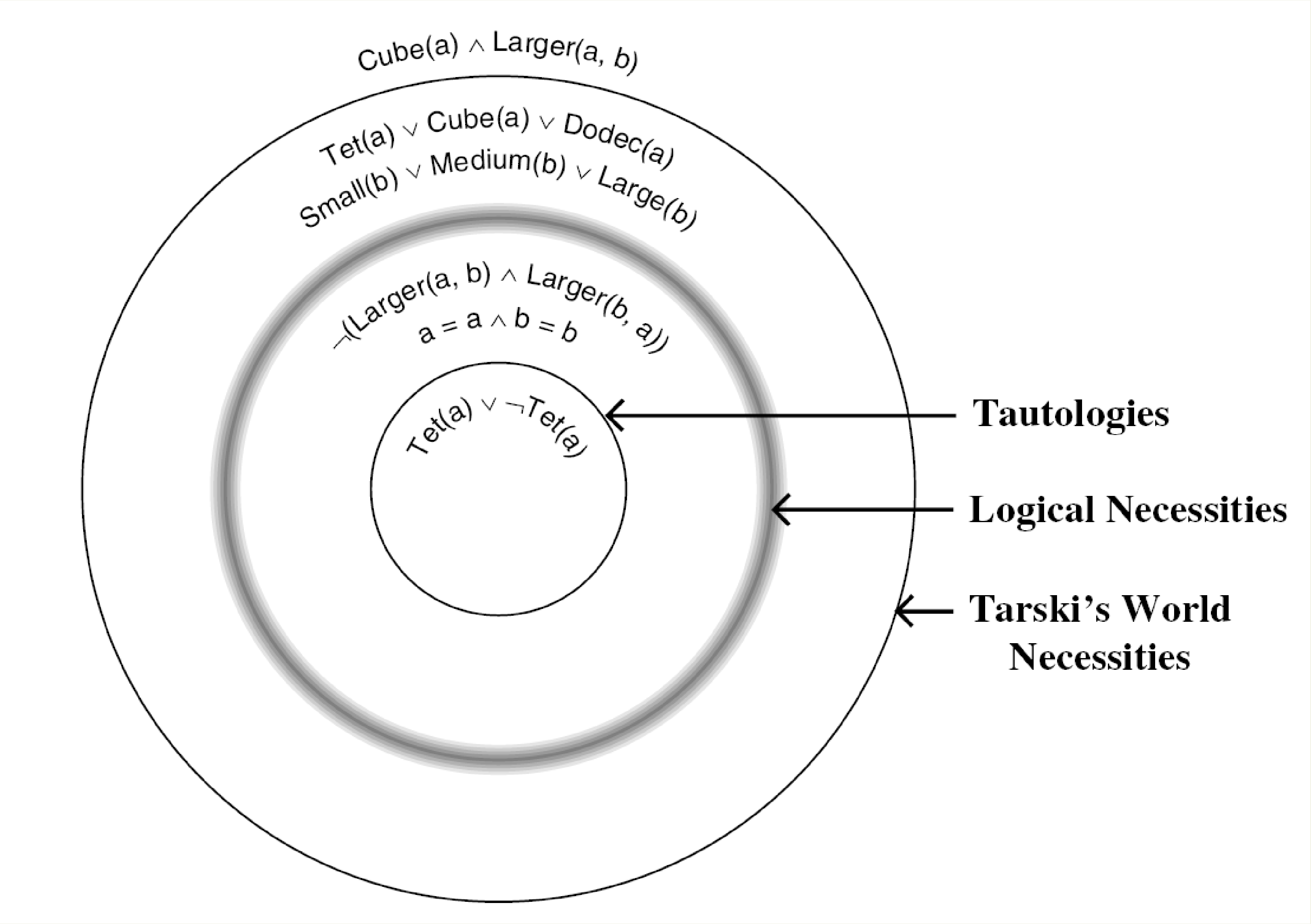
Logically necessary

$P \vee \neg P$ $\qquad a = a$

Logic, Boolean logic and Tarski's world

A sentence is

- logically necessary, or logically valid, if it is true in all circumstances (worlds),
- TW-necessary, if it is true in all worlds of Tarski's world,
- a tautology, if it is true in all valuations of the atomic sentences with $\{\mathrm{TRUE}, \mathrm{FALSE}\}$.


Cube(a) ∧ Larger(a, b)
Tet(a) ∨ Cube(a) ∨ Dodec(a)
Small(b) ∨ Medium(b) ∨ Large(b)
¬(Larger(a, b) ∧ Larger(b, a))
a = a ∧ b = b
Tet(a) ∨ ¬Tet(a)
Tautologies
Logical Necessities
Tarski's World Necessities

The truth table method

- A sentence is a tautology if and only if it evaluates to TRUE in all rows of its complete truth table.
- Truth tables can be constructed with the program Boole.

Tautological equivalence and consequence

- Two sentences P and Q are tautologically equivalent, if they evaluate to the same truth value in all valuations (rows of the truth table).
- Q is a tautological consequence of $P_1, \ldots, P_n$ if and only if every row that assigns TRUE to each of $P1, \ldots P_n$ also assigns TRUE to Q.
- If Q is a tautological consequence of $P_1, \ldots P_n$, then Q is also a logical consequence of $P_1, \ldots, P_n$.
- Some logical consequences are not tautological ones.

The Con rules in Fitch

- **Taut Con** proves all tautological consequences.
- **FO Con** proves all first-order consequences (like $a = c$ follows from $a = b \wedge b = c$).
- **Ana Con** proves (almost) all Tarski's world consequences.

de Morgan's laws and double negation

$$\neg(P \wedge Q) \Leftrightarrow (\neg P \vee \neg Q)$$

$$\neg(P \vee Q) \Leftrightarrow (\neg P \wedge \neg Q)$$

$$\neg\neg P \Leftrightarrow P$$

Note: $\neg$ binds stronger than $\wedge$ and $\vee$. Bracktes are needed to override this.

Negation normal form

- Substitution of equivalents: If P and Q are logically equivalent: $P \Leftrightarrow Q$ then the results of substituting one for the other in the context of a larger sentence are also logically equivalent: $S(P) \Leftrightarrow S(Q)$
- A sentence is in negation normal form (NNF) if all occurrences of $\neg$ apply directly to atomic sentences.
- Any sentence built from atomic sentences using just $\wedge$, $\vee$, and $\neg$ can be put into negation normal form by repeated application of the de Morgan laws and double negation.

Distributive laws

For any sentences P, Q, and R:

- Distribution of $\wedge$ over $\vee$:

$$P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R).$$

- Distribution of $\vee$ over $\wedge$:

$$P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R).$$

Conjunctive and disjunctive normal form

- A sentence is in conjunctive normal form (CNF) if it is a conjunction of one or more disjunctions of one or more literals.
- Distribution of $\vee$ over $\wedge$ allows you to transform any sentence in negation normal form into conjunctive normal form.

Disjunctive normal form

- A sentence is in disjunctive normal form (DNF) if it is a disjunction of one or more conjunctions of one or more literals.
- Distribution of $\wedge$ over $\vee$ allows you to transform any sentence in negation normal form into disjunctive normal form.
- Some sentences are in both CNF and DNF.