Logik für Informatiker
Logic for computer scientists

Logical consequence

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A circumstance is

- in propositional logic: a valuation of the atomic formulas in the set { true, false }
- in Tarski's world: a block world

Logical consequence — examples

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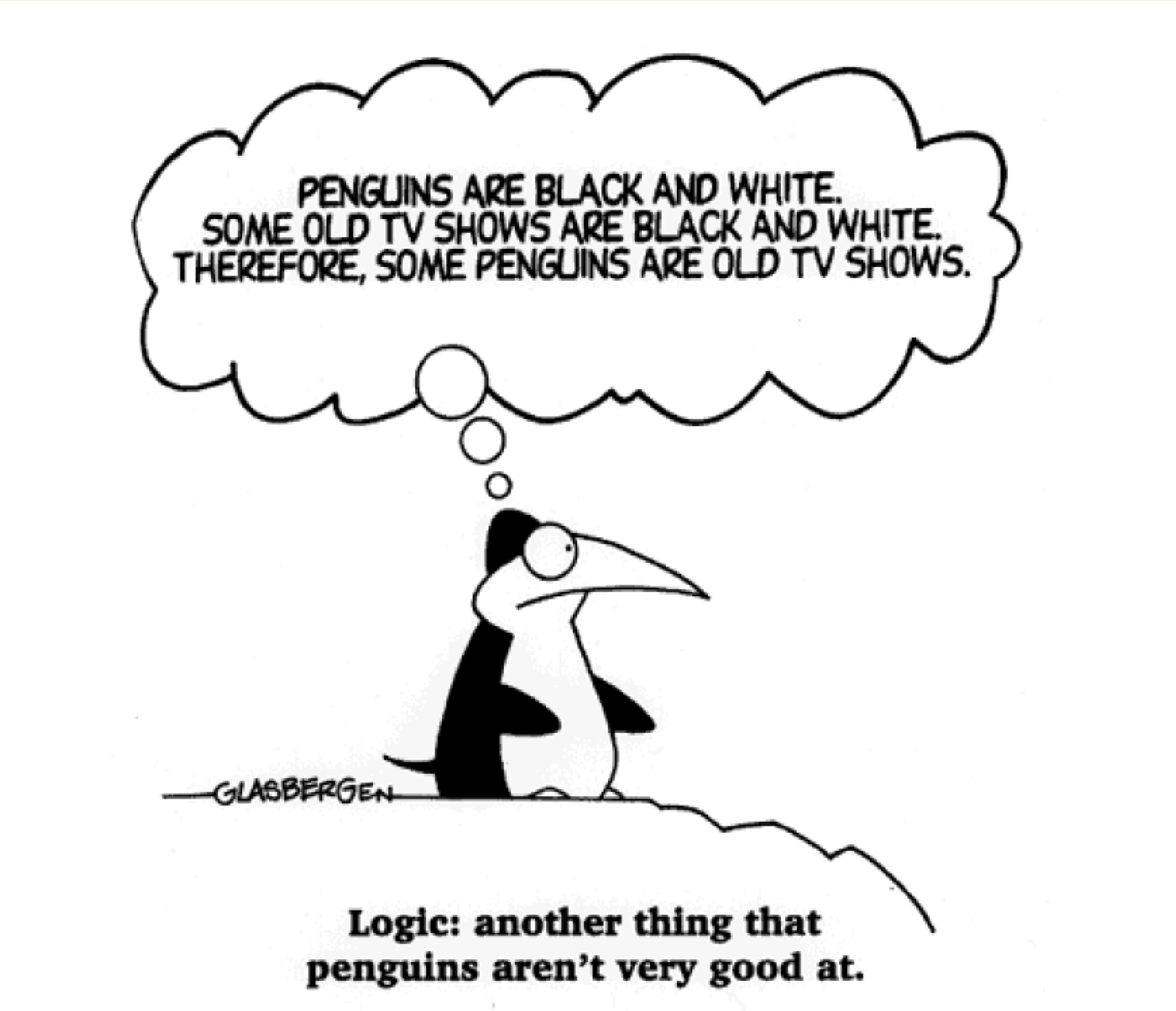
PENGUINS ARE BLACK AND WHITE.
SOME OLD TV SHOWS ARE BLACK AND WHITE.
THEREFORE, SOME PENGUINS ARE OLD TV SHOWS.
GLASBERGEN
Logic: another thing that
penguins aren't very good at.

Fitch notation

All men are mortal
Socrates is a man

So, Socrates is mortal

A_1	Premise$_1$
$\ldots$	$\ldots$
A_n	Premise$_n$
—	—
B	Conclusion

Methods for showing (in)validity of arguments

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Invalidity An argument can shown to be invalid by finding a counterexample (model), i.e. a circumstance where the premises are true, but the conclusion is false.

Informal and formal proofs

- informal reasoning is used in everyday life
- semi-formal reasoning is used in mathematics and theoretical computer science
- balance between readability and precision
- formal proofs: follow some specific rule system,
- and are entirely rigorous
- and can be checked by a computer

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- But all mortals will eventually die, since that is what it means to be mortal.
- So Socrates will eventually die.
- But we are given that everyone who will eventually die sometimes worries about it.
- Hence Socrates sometimes worries about dying.

A formal proof

1. Cube(c)

A formal proof

1. Cube(c)
2. $c = b$

A formal proof

1. Cube(c)
2. c = b

---

3. Cube(b)   = **Elim:** 1,2

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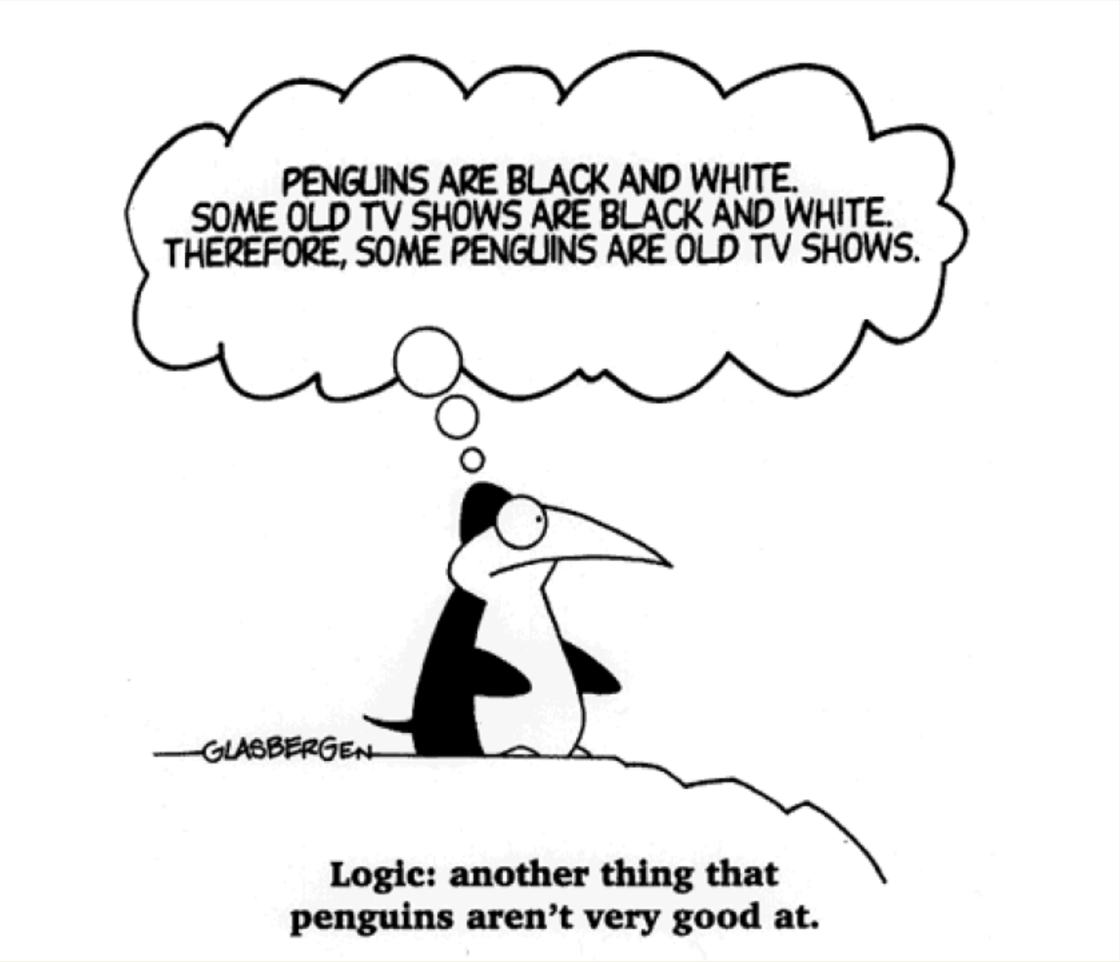
The latter two principles follow from the first two.

Transitivity . . .



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- Suppose that $a = b$.
- We know that $a = a$, by the reflexivity of identity.
- Now substitute the name b for the first use of the name a in $a = a$, using the indiscernibility of identicals.
- We come up with $b = a$, as desired.

Formal proofs

P	
Q	
R	
S_1	Justification 1
...	
...	
S_n	Justification n
S	Justification n+1

Formal proof of symmetry of identity

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2. $a = a$ $\quad$ $=$**Intro:**

Formal proof of symmetry of identity

1. $a = b$

2. $a = a$ $\quad$ **=Intro:**
3. $b = a$ $\quad$ **=Elim:** 2,1

Fitch rule: Identity introduction

Identity Introduction (= Intro):

$\triangleright \mid \mathsf{n} = \mathsf{n}$

Fitch rule: Identity elimination

Identity Elimination (= Elim):

$$
\begin{array}{l|l}
 & P(n) \\
 & \vdots \\
 & n = m \\
 & \vdots \\
\triangleright & P(m)
\end{array}
$$

Fitch rule: Reiteration

Reiteration (Reit):

	P
	⋮
▷	P

Example proof in fitch

Properties of predicates in Tarski's world

$$\begin{array}{|l} \mathrm{Larger}(a,b) \\ \mathrm{Larger}(b,c) \\ \hline \mathrm{Larger}(a,c) \end{array}$$

$$\begin{array}{|l} \mathrm{RightOf}(b,c) \\ \hline \mathrm{LeftOf}(c,b) \end{array}$$

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Such arguments can be proved in Fitch using the special rule **Ana Con**.

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Such arguments can be proved in Fitch using the special rule **Ana Con**.

This rule is only valid for reasoning about Tarski's world!

Showing invalidity using counterexamples

Al Gore is a politician
Hardly any politicians are honest

Al Gore is dishonest

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Imagine a situation where there are 10,000 politicians, and that Al Gore is the only honest one of the lot. In such circumstances both premises would be true but the conclusion would be false.

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Imagine a situation where there are 10,000 politicians, and that Al Gore is the only honest one of the lot. In such circumstances both premises would be true but the conclusion would be false.

This demonstrates that the argument is invalid.

Are the following arguments valid?

Small(a)
Larger(b, a)

Large(b)

Small(a)
Larger(a, b)

Large(b)