Logik für Informatiker
Logic for computer scientists

Quantifiers

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Motivating examples

$\forall x\ Cube(x)$ ("All objects are cubes.")
$\forall x\ (Cube(x) \rightarrow Large(x))$ ("All cubes are large.")

$\forall x\ Large(x)$ ("All objects are large.")

$$\exists x \; Cube(x)$$

"There exists a cube."

$$\exists x \; (Cube(x) \wedge Large(x))$$

"There exists a large cube."

The four Aristotelian forms

All P's are Q's.	$\forall x(P(x) \rightarrow Q(x))$
Some P's are Q's.	$\exists x(P(x) \land Q(x))$
No P's are Q's.	$\forall x(P(x) \rightarrow \neg Q(x))$
Some P's are not Q's.	$\exists x(P(x) \land \neg Q(x))$

Note:

$\forall x(P(x) \rightarrow Q(x))$ does not imply that there are some $P's$.

$\exists x(P(x) \land Q(x))$ does not imply that not all $P's$ are $Q's$.

First-order signatures

A first-order signature consists of

- a set of predicate symbols with arities, like $Smaller^{(2)}, Dodec^{(1)}, Between^{(3)}, \leq^{(2)}$, including propositional symbols (nullary predicate symbols), like $A^{(0)}, B^{(0)}, C^{(0)}$, (written uppercase)
- its names or constants for individuals, like a, b, c, (written lowercase)
- its function symbols with arities, like $f^{(1)}, +^{(2)}, \times^{(2)}$.

Usually, arities are omitted.

In the book, the terminology “language” is used.

“Signature” is more precise, since it exactly describes the ingredients that are needed to generate a (first-order) language.

Terms

$t ::= a$	constant
$t ::= x$	variable
$\quad \mid f^{(n)}(t_1, \ldots, t_n)$	application of function symbols to terms

Usually, arities are omitted.

Variables are: t, u, v, w, x, y, z, possibly with subscripts.

Well-formed formulas

$F ::= p^{(n)}(t_1, \ldots, t_n)$	application of predicate symbols
$\mid \bot$	contradiction
$\mid \neg F$	negation
$\mid (F_1 \wedge \ldots \wedge F_n)$	conjunction
$\mid (F_1 \wedge \ldots \vee F_n)$	disjunction
$\mid (F_1 \rightarrow F_2)$	implication
$\mid (F_1 \leftrightarrow F_2)$	equivalence
$\mid \forall \nu F$	universal quantification
$\mid \exists \nu F$	existential quantification

The variable ν is said to be bound in $\forall \nu F$ and $\exists \nu F$.

Parentheses

The outermost parenthese of a well-formed formula can be omitted:

$$Cube(x) \wedge Small(x)$$

In general, parentheses are important to determine the scope of a quantifier (see next slide).

Free and bound variables

An occurrence of a variable in a formula that is not bound is said to be free.

$\exists y\ LeftOf(x, y)$	x is free, y is bound
$(Cube(x) \land Small(x)) \rightarrow \exists y\ LeftOf(x, y)$	x is free, y is bound
$\exists x\ (Cube(x) \land Small(x))$	Both occurrences of x are bound
$\exists x\ Cube(x) \land Small(x)$	The first occurrence of x is bound, the second one is free

Sentences

A sentence is a well-formed formula without free variables.

$$\bot \qquad\qquad A \wedge B$$

$$Cube(a) \vee Tet(b)$$

$$\forall x\ (Cube(x) \rightarrow Large(x))$$

$$\forall x\ ((Cube(x) \wedge Small(x)) \rightarrow \exists y\ LeftOf(x, y))$$

Semantics of quantification

We need to fix some domain of discourse.

$\forall x\ S(x)$ is true iff for every object in the domain of discourse with name n, $S(n)$ is true.

$\exists x\ S(x)$ is true iff for some object in the domain of discourse with name n, $S(n)$ is true.

Not all objects need to have names — hence we assume that for objects, names $n_1, n_2, \ldots$ can be invented "on the fly".

The game rules

Form	Your commitment	Player to move	Goal
$P \vee Q$	TRUE FALSE	you Tarski's World	Choose one of P, Q that is true.
$P \wedge Q$	TRUE FALSE	Tarski's World you	Choose one of P, Q that is false.
$\exists x\, P(x)$	TRUE FALSE	you Tarski's World	Choose some ***b*** that satisfies the wff $P(x)$.
$\forall x\, P(x)$	TRUE FALSE	Tarski's World you	Choose some ***b*** that does not satisfy $P(x)$.

Logical consequence for quantifiers

$\forall x(\mathsf{Cube}(x) \longrightarrow \mathsf{Small}(x))$

$\forall x\ \mathsf{Cube}(x)$

$\forall x\ \mathsf{Small}(x)$

$\forall x\ \mathsf{Cube}(x)$

$\forall x\ \mathsf{Small}(x)$

$\forall x(\mathsf{Cube}(x) \land \mathsf{Small}(x))$

However: ignoring quantifiers does not work!

$$\exists x(\mathsf{Cube}(x) \rightarrow \mathsf{Small}(x))$$
$$\exists x\ \mathsf{Cube}(x)$$
$$\vdash$$
$$\exists x\ \mathsf{Small}(x)$$

$$\exists x\ \mathsf{Cube}(x)$$
$$\exists x\ \mathsf{Small}(x)$$
$$\vdash$$
$$\exists x(\mathsf{Cube}(x) \land \mathsf{Small}(x))$$

Tautologies do not distribute over quantifiers

$$\exists x\ Cube(x) \lor \exists x\ \neg Cube(x)$$

is a logical truth, but

$$\forall x\ Cube(x) \lor \forall x\ \neg Cube(x)$$

is not. By contrast,

$$\forall x\ Cube(x) \lor \neg \forall x\ Cube(x)$$

is a tautology.

Truth-functional form

Replace all top-level quantified sub-formulas (i.e. those not ocurring below another quantifier) by propositional letters.

Replace multiple occurrences of the same sub-formula by the same propositional letter.

A quantified sentence of FOL is said to be a tautology iff its truth-functional form is a tautology.

$$\forall x\ Cube(x) \lor \neg\forall x\ Cube(x)$$

becomes

$$A \lor \neg A$$

Truth functional form — examples

FO sentence	t.f. form
$\forall x\, Cube(x) \lor \neg\forall x\, Cube(x)$	$A \lor \neg A$
$(\exists y\, Tet(y) \land \forall z\, Small(z)) \rightarrow \forall z\, Small(z)$	$(A \land B) \rightarrow B$
$\forall x\, Cube(x) \lor \exists y\, Tet(y)$	$A \lor B$
$\forall x\, Cube(x) \rightarrow Cube(a)$	$A \rightarrow B$
$\forall x\, (Cube(x) \lor \neg Cube(x))$	A
$\forall x\, (Cube(x) \rightarrow Small(x)) \lor \exists x\, Dodec(x)$	$A \lor B$

Examples of →-Elim

$\exists x(\mathrm{Cube}(x) \rightarrow \mathrm{Small}(x))$	A	No!
$\exists x\ \mathrm{Cube}(x)$	B	
───	───	
$\exists x\ \mathrm{Small}(x)$	C	

$\exists x \mathrm{Cube}(x) \rightarrow \exists x\ \mathrm{Small}(x)$	$A \rightarrow B$	Yes!
$\exists x\ \mathrm{Cube}(x)$	A	
───	───	
$\exists x\ \mathrm{Small}(x)$	B	

Tautologies and logical truths

Every tautology is a logical truth, but not vice versa.

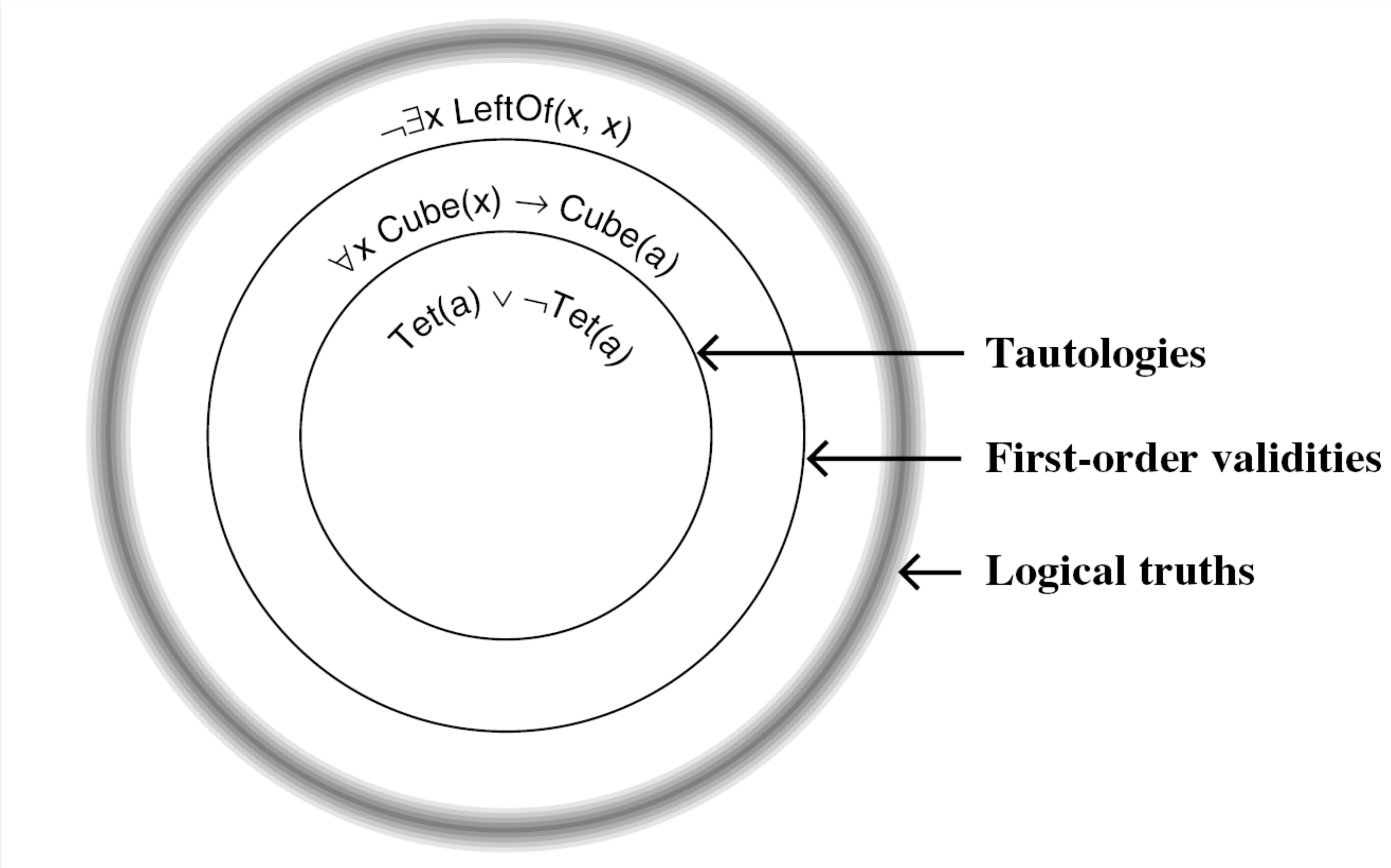
Example: $\exists x\ Cube(x) \lor \exists x\ \neg Cube(x)$

is a logical truth, but not a tautology.

Similarly, every tautologically valid argument is a logically valid argument, but not vice versa.

$$\begin{array}{|l} \forall x\ \mathrm{Cube}(x) \\ \hline \exists x\ \mathrm{Cube}(x) \end{array}$$

is a logically valid argument, but not tautologically valid.


¬∃x LeftOf(x, x)
∀x Cube(x) → Cube(a)
Tet(a) ∨ ¬Tet(a)
Tautologies
First-order validities
Logical truths

Tautologies and logical truths, cont'd

Propositional logic	First-order logic	Tarski' World	General notion
Tautology	*FO validity*	*TW validity*	*Logical Truth*
Tautological consequence	*FO consequence*	*TW consequence*	*Logical consequence*
Tautological equivalence	*FO equivalence*	*TW equivalence*	*Logical equivalence*

Which ones are FO validities?

$$\forall x \; SameSize(x, x)$$
$$\forall x \; Cube(x) \rightarrow Cube(b)$$
$$(Cube(b) \land b = c) \rightarrow Cube(c)$$
$$(Small(b) \land SameSize(b, c)) \rightarrow Small(c)$$

Replacement method: Replace predicates by meaningless ones

$$\forall x \;\; Outgrabe(x, x)$$

$$\forall x \; Tove(x) \rightarrow Tove(b)$$

$$(Tove(b) \wedge b = c) \rightarrow Tove(c)$$

$$(Slithy(b) \wedge Outgrabe(b, c)) \rightarrow Slithy(c)$$

Is this a valid FO argument?

$$\forall x(\mathrm{Tet}(x) \rightarrow \mathrm{Large}(x))$$
$$\neg \mathrm{Large}(b)$$
$$\vdash$$
$$\neg \mathrm{Tet}(b)$$

Replacement with nonsense predicates:

$$\forall x(\mathrm{Borogove}(x) \rightarrow \mathrm{Mimsy}(x))$$
$$\neg \mathrm{Mimsy}(b)$$
$$\vdash$$
$$\neg \mathrm{Borogove}(b)$$

Is this a valid FO argument?

Replacement with a meaningless predicate:

$\neg\exists x\ \mathsf{Larger}(x, a)$	$\neg\exists x\ R(x, a)$
$\neg\exists x\ \mathsf{Larger}(b, x)$	$\neg\exists x\ R(b, x)$
$\mathsf{Larger}(c, d)$	$R(c, d)$
⊢	⊢
$\mathsf{Larger}(a, b)$	$R(a, b)$

The method of counterexamples

In order to show that the argument

$$\begin{array}{|l} P_1 \\ \ldots \\ P_n \\ \hline Q \end{array}$$

is not valid, it suffices to give a counterexample, i.e. a world that makes the premises $P_1, \ldots, P_n$ true, but the conclusion Q false.

(For now, "world" is understood informally. Later on, we will formalize "world" as "first-order structure".)

A counterexample

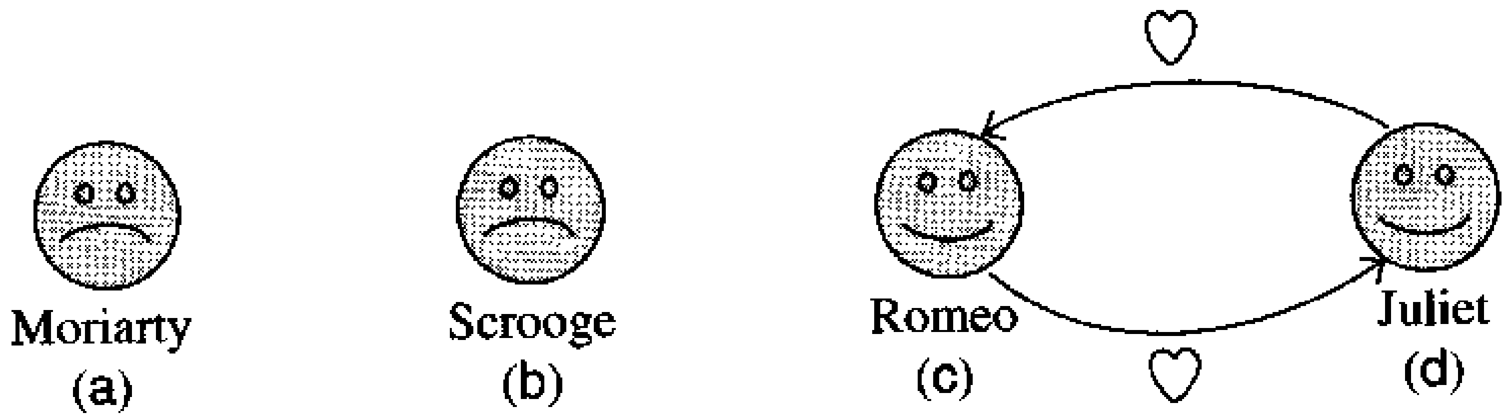


Figure 10.1: A first-order counterexample.

Exercises

- chapter 9
- chapter 10: 10.1 to 10.29