Logik für Informatiker
Logic for computer scientists

The logic of quantifiers

Till Mossakowski

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First-order equivalence

Two well-formed formulas P and Q (possibly containing free variables) are logically equivalent, if in all circumstances, they are satisfied by the same objects. This is written as

$$P \Leftrightarrow Q$$

Substitution principle

If $P \Leftrightarrow Q$, then $S(P) \Leftrightarrow S(Q)$.

Here, $S(\_)$ is a sentence with a “hole”.

DeMorgan laws for quantifiers

$$\neg\forall x P(x) \quad\Leftrightarrow\quad \exists x\neg P(x)$$
$$\forall x P(x) \quad\Leftrightarrow\quad \neg\exists x\neg P(x)$$

$$\neg\exists x P(x) \quad\Leftrightarrow\quad \forall x\neg P(x)$$
$$\exists x P(x) \quad\Leftrightarrow\quad \neg\forall x\neg P(x)$$

More quantifier equivalences

$$\forall x (P(x) \land Q(x)) \quad \Leftrightarrow \quad \forall x P(x) \land \forall x Q(x)$$
$$\forall x (P(x) \lor Q(x)) \quad \not\Leftrightarrow \quad \forall x P(x) \lor \forall x Q(x)$$

$$\exists x (P(x) \lor Q(x)) \quad \Leftrightarrow \quad \exists x P(x) \lor \exists x Q(x)$$
$$\exists x (P(x) \land Q(x)) \quad \not\Leftrightarrow \quad \exists x P(x) \land \exists x Q(x)$$

TW consequence $\neq$ FO consequence

We have encountered arguments that are valid in Tarski's World but not FO valid.

$$
\begin{array}{|l}
\forall x(\mathsf{Cube}(x) \leftrightarrow \mathsf{SameShape}(x, c)) \\
\hline
\mathsf{Cube}(c)
\end{array}
$$

The replacement method yields an invalid argument:

$$
\begin{array}{|l}
\forall x(P(x) \leftrightarrow Q(x, c)) \\
\hline
P(c)
\end{array}
$$

The axiomatic method

Axiomatic method: bridge the gap between Tarski's World validity and FO validity by systematically expressing facts about the meanings of the predicates, and introduce them as *axioms*. Axioms restrict the possible interpretation of predicates.

Axioms may be used as premises within arguments/proofs.

The argument revisited

$\forall x(\mathsf{Cube}(x) \leftrightarrow \mathsf{SameShape}(x, c))$
$\forall x \mathsf{SameShape}(x, x)$

$\mathsf{Cube}(c)$

The replacement method yields a valid argument:

$\forall x(P(x) \leftrightarrow Q(x, c))$
$\forall x Q(x, x)$

$P(c)$

The basic shape axioms

1. $\neg\exists x(Cube(x) \land Tet(x))$
2. $\neg\exists x(Tet(x) \land Dodec(x))$
3. $\neg\exists x(Dodec(x) \land Cube(x))$
4. $\forall x(Tet(x) \lor Dodec(x) \lor Cube(x))$

An argument using the shape axioms

$$\neg\exists x(\mathrm{Dodec}(x) \land \mathrm{Cube}(x))$$
$$\forall x(\mathrm{Tet}(x) \lor \mathrm{Dodec}(x) \lor \mathrm{Cube}(x))$$
$$\neg\exists x\ \mathrm{Tet}(x)$$
$$\vdash$$
$$\forall x(\mathrm{Cube}(x) \leftrightarrow \neg\mathrm{Dodec}(x))$$

$$\neg\exists x(P(x) \land Q(x))$$
$$\forall x(R(x) \lor P(x) \lor Q(x))$$
$$\neg\exists x\ R(x)$$
$$\vdash$$
$$\forall x(Q(x) \leftrightarrow \neg P(x))$$

SameShape introduction and elimination axioms

1. $\forall x \forall y((Cube(x) \land Cube(y)) \rightarrow SameShape(x, y))$
2. $\forall x \forall y((Dodec(x) \land Dodec(y)) \rightarrow SameShape(x, y))$
3. $\forall x \forall y((Tet(x) \land Tet(y)) \rightarrow SameShape(x, y))$

4. $\forall x \forall y((SameShape(x, y) \land Cube(x)) \rightarrow Cube(y))$
5. $\forall x \forall y((SameShape(x, y) \land Dodec(x)) \rightarrow Dodec(y))$
6. $\forall x \forall y((SameShape(x, y) \land Tet(x)) \rightarrow Tet(y))$

Euclid's axiomatization of geometry

1. Any two points can be joined by a straight line.
2. Any straight line segment can be extended indefinitely in a straight line.
3. Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.
4. All right angles are congruent.
5. Parallel postulate. If two lines intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough.

Peano's axiomatization of the naturals

1. 0 is a natural number.
2. For every natural number, its successcor is a natural number.
3. There is no natural number whose successor is 0.
4. Two different natural numbers have different successors.
5. If K is a set such that:
   - 0 is in K, and
   - for every natural number in K, its successor also is in K,

then K contains every natural number.

Formalization of Peano's axioms

1. a constant 0
2. a unary function symbol suc
3. $\forall n\ \neg suc(n) = 0$
4. $\forall m \forall n\ suc(m) = suc(n) \rightarrow m = n$
5. $(\Phi(x/0) \wedge \forall n(\Phi(x/n) \rightarrow \Phi(x/suc(n)))) \rightarrow \forall n\ \Phi(x/n)$
   if Φ is a formula with a free variable x, and
   $\Phi(x/n)$ denotes the replacement of x with t within Φ

Other famous axiom systems

- Zermelo-Fraenkel axiomatization of set theory
- axiomatizations in algebra: monoids, groups, rings, fields, vector spaces . . .
- Hoare's axiomatization of imperative programming with while-loops, if-then-else and assignment

Multiple quantifiers

$$\forall x \exists y \; Likes(x, y)$$

is very different from

$$\exists y \forall x \; Likes(x, y)$$

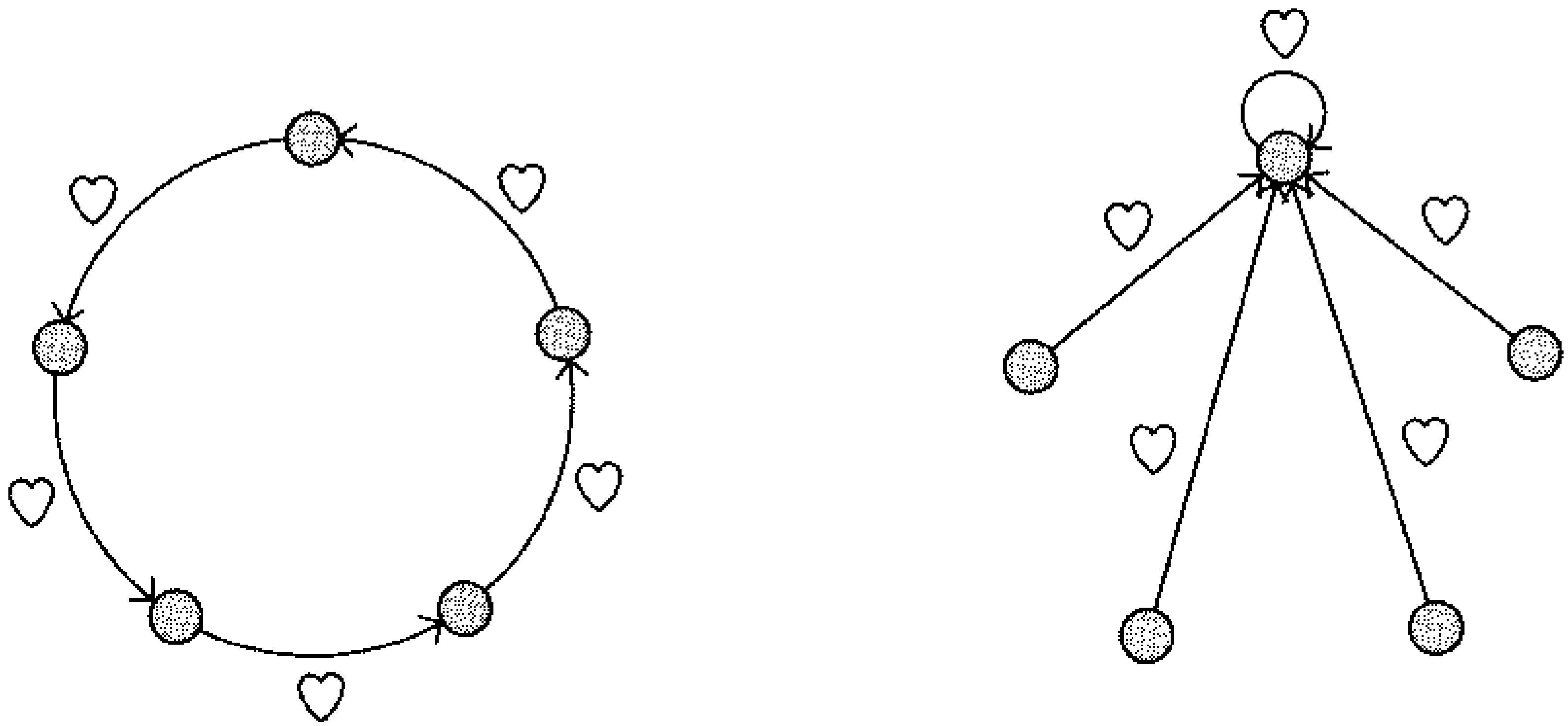


Figure 11.1: A circumstance in which $\forall x \, \exists y \, \mathsf{Likes}(x, y)$ holds versus one in which $\exists y \, \forall x \, \mathsf{Likes}(x, y)$ holds. It makes a big difference to someone!

Prenex Normal Form

Goal: shift all quantifiers to the top-level

Rules for conjunctions and disjunctions

$$(\forall x P) \wedge Q \rightsquigarrow \forall x (P \wedge Q) \qquad (\exists x P) \wedge Q \rightsquigarrow \exists x (P \wedge Q)$$

$$P \wedge (\forall x Q) \rightsquigarrow \forall x (P \wedge Q) \qquad P \wedge (\exists x Q) \rightsquigarrow \exists x (P \wedge Q)$$

$$(\forall x P) \vee Q \rightsquigarrow \forall x (P \vee Q) \qquad (\exists x P) \vee Q \rightsquigarrow \exists x (P \vee Q)$$

$$P \vee (\forall x Q) \rightsquigarrow \forall x (P \vee Q) \qquad P \vee (\exists x Q) \rightsquigarrow \exists x (P \vee Q)$$

Prenex Normal Form (cont'd)

Rules for negations, implications, equivalences

$$\neg\forall x P \rightsquigarrow \exists x(\neg P) \qquad \neg\exists x P \rightsquigarrow \forall x(\neg P)$$

$$(\forall x P) \rightarrow Q \rightsquigarrow \exists x(P \rightarrow Q) \qquad (\exists x P) \rightarrow Q \rightsquigarrow \forall x(P \rightarrow Q)$$

$$P \rightarrow (\forall x Q) \rightsquigarrow \forall x(P \rightarrow Q) \qquad P \rightarrow (\exists x Q) \rightsquigarrow \exists x(P \rightarrow Q)$$

$$P \leftrightarrow Q \rightsquigarrow (P \rightarrow Q) \wedge (Q \rightarrow P)$$

Prenex Normal Form: example

What is the prenex normal form of

$$\exists x Cube(x) \rightarrow \forall y Small(y)$$

Proof methods for quantifiers

Universal elimination

Universal statments can be instantiated to any object.

From $\forall x S(x)$, we may infer $S(c)$.

Existential introduction

If we have established a statement for an instance, we can also establish the corresponding existential statement.

From $S(c)$, we may infer $\exists x S(x)$.

Example

$$\forall x[\mathrm{Cube}(x) \rightarrow \mathrm{Large}(x)]$$
$$\forall x[\mathrm{Large}(x) \rightarrow \mathrm{LeftOf}(x, b)]$$
$$\mathrm{Cube}(d)$$

$$\exists x[\mathrm{Large}(x) \land \mathrm{LeftOf}(x, b)]$$

Existential instantiation (elimination)

From $\exists x S(x)$, we can infer $S(c)$, if c is a new name not used otherwise.

Example: Scotland Yard searched a serial killer. The did not know who he was, but for their reasoning, they called him "Jack the ripper".
This would have been an unfair procedure if there had been a real person named Jack the ripper.

Example

$$\forall x[\mathsf{Cube}(x) \longrightarrow \mathsf{Large}(x)]$$
$$\forall x[\mathsf{Large}(x) \longrightarrow \mathsf{LeftOf}(x, b)]$$
$$\exists x \mathsf{Cube}(x)$$
$$\exists x[\mathsf{Large}(x) \land \mathsf{LeftOf}(x, b)]$$

Universal generalization (introduction)

If we introduce a new name c that is not used elsewhere, and can prove $S(c)$, then we can also infer $\forall x S(x)$.

Example:

Theorem Every even number greater zero is the sum of two odd numers.

Proof Let $n > 0$ be even, i.e. $n = 2m$ with $m > 0$. If m is odd, then $m + m = n$ does the job. If m is even, consider $(m - 1) + (m + 1) = n$.

Arguments involving multiple quantifiers

$$\exists y[\mathsf{Girl}(y) \land \forall x(\mathsf{Boy}(x) \to \mathsf{Likes}(x, y))]$$

$$\vdash$$

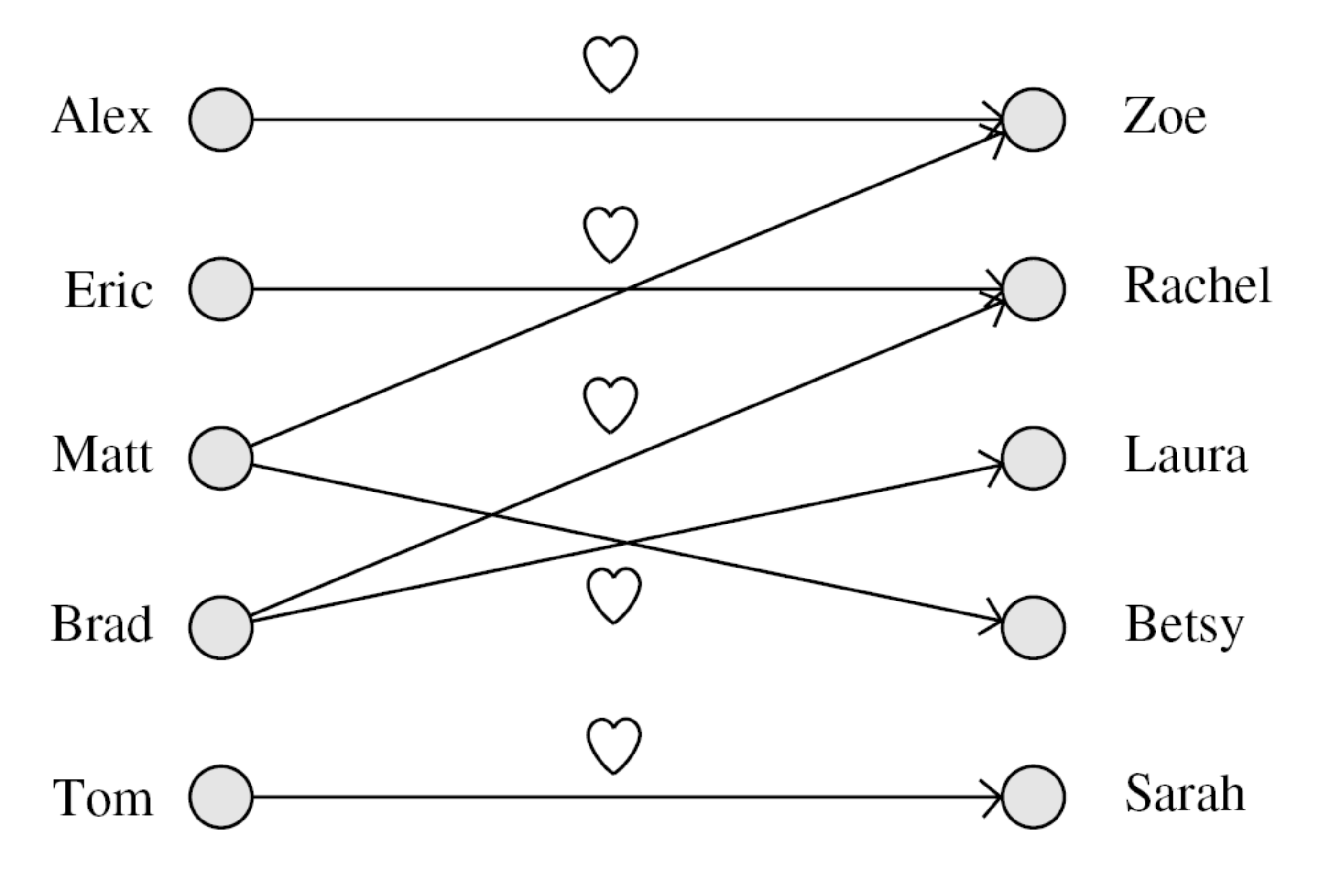
$$\forall x[\mathsf{Boy}(x) \to \exists y(\mathsf{Girl}(y) \land \mathsf{Likes}(x, y))]$$

$$\forall x[\mathsf{Boy}(x) \to \exists y(\mathsf{Girl}(y) \land \mathsf{Likes}(x, y))]$$

$$\vdash$$

$$\exists y[\mathsf{Girl}(y) \land \forall x(\mathsf{Boy}(x) \to \mathsf{Likes}(x, y))]$$

A (counter)example



Exercises

- chapter 10: 10.20 to 10.31
- chapters 11 and 12
- additional exercise (grade 1): write a complete axiomatization of Tarski's World in CASL.