Logik für Informatiker
Logic for computer scientists

Identity, induction and XMas

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Identity Elimination (= Elim)

$$
\begin{array}{l|l}
 & P(n) \\
 & \vdots \\
 & n = m \\
 & \vdots \\
\triangleright & P(m)
\end{array}
$$

Example for =-Elim

$$\begin{array}{|l} \mathsf{SameCol}(a, b) \\ b = c \\ \hline \mathsf{SameCol}(a, c) \end{array}$$

Identity Introduction (= Intro)

$$\triangleright \;\Big|\; \mathsf{n} = \mathsf{n}$$

Reflexivity, symmetry and transitivity

$\vdash \forall x\ x = x$

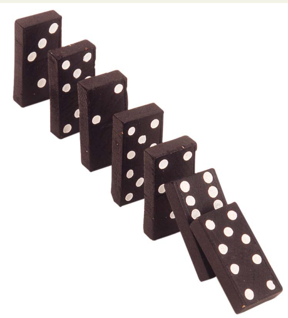
$\vdash \forall x\ \forall y\ x = y \rightarrow y = x$

$\vdash \forall x\ \forall y\ \forall z\ (x = y \wedge y = z) \rightarrow x = z$

Induction

Induction is like a chain of dominoes. You need

- the dominoes must be close enough together $\Rightarrow$ one falling dominoe knocks down the next (inductive step)
- you need to knock down the first dominoe (inductive basis)



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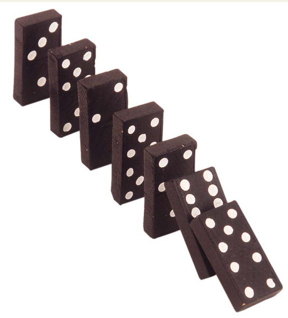

Note: in the inductive step, branching is possible.

Inductive definition: Natural numbers

1. 0 is a natural number.
2. If n is natural number, then $suc(n)$ is a natural number.
3. There is no natural number whose successor is 0.
4. Two different natural numbers have different successors.
5. Nothing is a natural number unless generated by repeated applications of (1) and (2).

Recursive definition of functions

$$\forall y(0 + y = y)$$
$$\forall x \forall y(suc(x) + y = suc(x + y))$$

$$\forall y(0 * y = 0)$$
$$\forall x \forall y(suc(x) * y = (x * y) + y)$$

Formalization of Peano's axioms

1. a constant 0
2. a unary function symbol suc
3. $\forall n \neg suc(n) = 0$
4. $\forall m \forall n \; suc(m) = suc(n) \rightarrow m = n$
5. $(\Phi(x/0) \wedge \forall n(\Phi(x/n) \rightarrow \Phi(x/suc(n)))) \rightarrow \forall n \; \Phi(x/n)$
   if Φ is a formula with a free variable x, and
   $\Phi(x/t)$ denotes the replacement of x with t within Φ

Inductive proofs

Take $\Phi(x) := \forall y \forall z(x + (y + z) = (x + y) + z)$. Then

$$(\Phi(x/0) \land \forall n(\Phi(x/n) \to \Phi(x/suc(n)))) \to \forall n\ \Phi(x/n)$$

is just

$$\begin{array}{l}(\forall y \forall z(0 + (y + z) = (0 + y) + z) \\ \quad \land \forall n \forall y \forall z\ (n + (y + z) = (n + y) + z \\ \qquad \to suc(n) + (y + z) = (suc(n) + y) + z)) \\ \quad \to \forall n \forall y \forall z\ (n + (y + z) = (n + y) + z)\end{array}$$

With this, we can prove $\forall n \forall y \forall z\ (n + (y + z) = (n + y) + z)$

Inductive datatypes: Lists of natural numbers

1. The empty list $[\,]$ is a list.
2. If l is a list and n is natural number, then $cons(n, l)$ is a list.
3. Nothing is a list unless generated by repeated applications of (1) and (2).

Note: This needs many-sorted first-order logic.
We have two sorts of objects: natural numbers and lists.

Recursive definition of functions over lists

$$length([\,]) = 0$$
$$\forall n : Nat\ \forall l : List\ (length(cons(n, l)) = suc(length(l)))$$

$$\forall l : List\ ([\,] ++ l = l)$$
$$\forall n : Nat\ \forall l_1 : List\ \forall l_2 : List$$
$$(cons(n, l_1) ++ l_2 = cons(n, l_1 ++ l_2))$$

Inductive proofs over lists

$$\forall l_1 : List\ \forall l_2 : List\ \forall l_3 : List$$
$$(\ l_1\ ++\ (l_2\ ++\ l_3)\ =\ (l_1\ ++\ l_2)\ ++\ l_3\ )$$

$$\forall l_1 : List\ \forall l_2 : List$$
$$(\ length(l_1\ ++\ l_2)\ =\ length(l_1)\ +\ length(l_2)\ )$$

XMas special

Existence of Santa Clause

Theorem. Santa Clause exists.

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Proof.

Assume to the contrary, that Santa Clause does not exist.

By $\exists$-Intro, there exists something that does not exist.

This is a contradiction. Hence, the assumption that Santa Clause does not exist must be wrong.

Thus, Santa Clause exists. $\square$

All reindeers have the same color

Theorem. Any number of reindeers have the same color.

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Inductive step: suppose that any collection of n reindeers has the same color. We need to show that $n+1$ reindeers have the same color, too. By induction hypothesis, the first n reindeers have the same color. Take out the last reindeer of these and replace it with the $n+1$st. Again by induction hypothesis, these have the same color. Hence, all $n+1$ reindeers have the same color. $\square$

Why the date of XMas cannot be surprising

Son: It is boring that XMas always is on the 24th.

Father: OK. This year, we will celebrate XMas on a day in the week from 23th to 29th. You will not know the date beforehand.

Son: Good! Then it cannot be the 29th — if we hadn't celebrated it until the 28th, I would know beforehand that it must be the 29th, since this is the last day of the week! Moreover, it cannot be the 28th — if we hadn't celebrated it until the 27th, I would know beforehand that it must be the

28th (the 29th already has been excluded above).

Son (cont'd): Similarly, it can be neither the 27th, nor the 26th, nor the 25th, nor the 24th, nor the 23th.

Hence, you cannot fulfill you promise that I won't know the date beforehand.

Father: You will see, you won't know the date beforehand.

Why the date of XMas can be surprising

After all, XMas was celebrated on the 27th.

The son was quite surprised.

A scheduling problem

A camel must travel 1000 miles across a desert to the nearest city. She has 3000 bananas but can only carry 1000 at a time. For every mile she walks, she needs to eat a banana. What is the maximum number of bananas she can transport to the city?