

Logik für Informatiker

Proofs in propositional logic

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Logical consequence

- Q is a **logical consequence** of P_1, \dots, P_n , if all worlds that make P_1, \dots, P_n true also make Q true.
- Q is a **tautological consequence** of P_1, \dots, P_n , if all valuations of atomic formulas with truth values that make P_1, \dots, P_n true also make Q true.
- Q is a **TW-logical consequence** of P_1, \dots, P_n , if all worlds from Tarski's world that make P_1, \dots, P_n true also make Q true.

Proofs

- With proofs, we try to show (tauto)logical consequence
- Truth-table method can lead to very large tables, proofs are often shorter
- Proofs are also available for consequence in full first-order logic, not only for tautological consequence

Limits of the truth-table method

- 1 truth-table method leads to *exponentially growing* tables
 - 20 atomic sentences \Rightarrow more than 1.000.000 rows
- 2 truth-table method cannot be extended to *first-order logic*
 - *model checking* can overcome the first limitation (up to 1.000.000 atomic sentences)
 - *proofs* can overcome both limitations

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Proofs

- A proof consists of a sequence of *proof steps*
- Each proof step is known to be valid and should
 - be significant but easily understood, in *informal* proofs,
 - follow some *proof rule*, in *formal* proofs.
- Some valid patterns of inference that generally go unmentioned in informal (but not in formal) proofs:
 - From $P \wedge Q$, infer P .
 - From P and Q , infer $P \wedge Q$.
 - From P , infer $P \vee Q$.

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Proof by cases (disjunction elimination)

To prove S from $P_1 \vee \dots \vee P_n$, prove S from each of P_1, \dots, P_n .

Claim: there are irrational numbers b and c such that b^c is rational.

Proof: $\sqrt{2}^{\sqrt{2}}$ is either rational or irrational.

Case 1: If $\sqrt{2}^{\sqrt{2}}$ is rational: take $b = c = \sqrt{2}$.

Case 2: If $\sqrt{2}^{\sqrt{2}}$ is irrational: take $b = \sqrt{2}^{\sqrt{2}}$ and $c = \sqrt{2}$.

$$\text{Then } b^c = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{(\sqrt{2} \cdot \sqrt{2})} = \sqrt{2}^2 = 2.$$

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Proof by contradiction

To prove $\neg S$, assume S and prove a contradiction \perp .
(\perp may be inferred from P and $\neg P$.)

Assume $Cube(c) \vee Dodec(c)$ and $Tet(b)$.

Claim: $\neg(b = c)$.

Proof: Let us assume $b = c$.

Case 1: If $Cube(c)$, then by $b = c$, also $Cube(b)$, which contradicts $Tet(b)$.

Case 2: $Dodec(c)$ similarly contradicts $Tet(b)$.

In both case, we arrive at a contradiction. Hence, our assumption $b = c$ cannot be true, thus $\neg(b = c)$.

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Arguments with inconsistent premises

A proof of a contradiction \perp from premises P_1, \dots, P_n (without additional assumptions) shows that the premises are *inconsistent*. An argument with inconsistent premises is always *valid*, but more importantly, always *unsound*.

Home(max) \vee Home(claire)

\neg Home(max)

\neg Home(claire)

Home(max) \wedge Happy(carl)

Arguments without premises

A proof without any premises shows that its conclusion is a *logical truth*.

Example: $\neg(P \wedge \neg P)$.

Formal proofs in Fitch

- Well-defined set of *formal proof rules*
- Formal proofs in Fitch can be *mechanically checked*
- For each connective, there is
 - an *introduction rule*, e.g. “from P , infer $P \vee Q$ ”.
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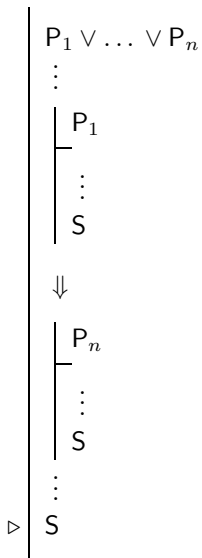
Conjunction Elimination (\wedge Elim)

$$\triangleright \left| \begin{array}{l} P_1 \wedge \dots \wedge P_i \wedge \dots \wedge P_n \\ \vdots \\ P_i \end{array} \right.$$

Disjunction Introduction (\vee Intro)

$$\triangleright \left| \begin{array}{l} P_i \\ \vdots \\ P_1 \vee \dots \vee P_i \vee \dots \vee P_n \end{array} \right.$$

Disjunction Elimination (\vee Elim)



The proper use of subproofs

1. $(B \wedge A) \vee (A \wedge C)$	
2. $B \wedge A$	
3. B	\wedge Elim : 2
4. A	\wedge Elim : 2
5. $A \wedge C$	
6. A	\wedge Elim : 5
7. A	\vee Elim : 1, 2–4, 5–6
8. $A \wedge B$	\wedge Intro : 7, 3

The proper use of subproofs (cont'd)

- In justifying a step of a subproof, you may cite any *earlier step* contained in the main proof, or in any subproof whose assumption is *still in force*. You may *never* cite individual steps inside a subproof that has *already ended*.
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\perp Introduction (\perp Intro)

	P
	⋮
	$\neg P$
	⋮
▷	\perp

Negation Elimination (\neg Elim)

$$\begin{array}{|l} \neg\neg P \\ \vdots \\ P \end{array}$$

\perp Elimination (\perp Elim)

$$\begin{array}{|l} \perp \\ \vdots \\ P \end{array}$$