The rule system of Fitch (natural deduction)

Logik für Informatiker
Proofs in propositional logic

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WiSe 2009/10
The rule system of Fitch (natural deduction)

Logical consequence

- **Q** is a **logical consequence** of \( P_1, \ldots, P_n \), if all worlds that make \( P_1, \ldots, P_n \) true also make \( Q \) true.

- **Q** is a **tautological consequence** of \( P_1, \ldots, P_n \), if all valuations of atomic formulas with truth values that make \( P_1, \ldots, P_n \) true also make \( Q \) true.

- **Q** is a **TW-logical consequence** of \( P_1, \ldots, P_n \), if all worlds from Tarski’s world that make \( P_1, \ldots, P_n \) true also make \( Q \) true.
Proofs

- With proofs, we try to show (tauto)logical consequence
- Truth-table method can lead to very large tables, proofs are often shorter
- Proofs are also available for consequence in full first-order logic, not only for tautological consequence
Limits of the truth-table method

1. truth-table method leads to *exponentially growing* tables
   - 20 atomic sentences $\Rightarrow$ more than 1,000,000 rows

2. truth-table method cannot be extended to *first-order logic*
   - *model checking* can overcome the first limitation (up to 1,000,000 atomic sentences)
   - *proofs* can overcome both limitations
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Proofs

- A proof consists of a sequence of *proof steps*
- Each proof step is known to be valid and should
  - be significant but easily understood, in *informal* proofs,
  - follow some *proof rule*, in *formal* proofs.
- Some valid patterns of inference that generally go unmentioned in informal (but not in formal) proofs:
  - From $P \land Q$, infer $P$.
  - From $P$ and $Q$, infer $P \land Q$.
  - From $P$, infer $P \lor Q$. 
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To prove $S$ from $P_1 \lor \ldots \lor P_n$, prove $S$ from each of $P_1, \ldots, P_n$.

Claim: there are irrational numbers $b$ and $c$ such that $b^c$ is rational.

Proof: $\sqrt{2}^{\sqrt{2}}$ is either rational or irrational.

Case 1: If $\sqrt{2}^{\sqrt{2}}$ is rational: take $b = c = \sqrt{2}$.

Case 2: If $\sqrt{2}^{\sqrt{2}}$ is irrational: take $b = \sqrt{2}^{\sqrt{2}}$ and $c = \sqrt{2}$.

Then $b^c = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{\sqrt{2} \cdot \sqrt{2}} = \sqrt{2}^2 = 2$. 
Proof by cases (disjunction elimination)

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To prove $\neg S$, assume $S$ and prove a contradiction $\bot$.
($\bot$ may be inferred from $P$ and $\neg P$.)
Assume $Cube(c) \lor Dodec(c)$ and $Tet(b)$.
Claim: $\neg (b = c)$.
Proof: Let us assume $b = c$.
Case 1: If $Cube(c)$, then by $b = c$, also $Cube(b)$, which contradicts $Tet(b)$.
Case 2: $Dodec(c)$ similarly contradicts $Tet(b)$.
In both case, we arrive at a contradiction. Hence, our assumption $b = c$ cannot be true, thus $\neg (b = c)$. 
Proof by contradiction

To prove \( \neg S \), assume \( S \) and prove a contradiction \( \bot \).
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A proof of a contradiction $\bot$ from premises $P_1, \ldots, P_n$ (without additional assumptions) shows that the premises are inconsistent. An argument with inconsistent premises is always valid, but more importantly, always unsound.

$$
\begin{align*}
\text{Home(max)} \lor \text{Home(claire)} \\
\neg \text{Home(max)} \\
\neg \text{Home(claire)} \\
\text{Home(max)} \land \text{Happy(carl)}
\end{align*}
$$
A proof without any premises shows that its conclusion is a *logical truth*.
Example: $\neg(P \land \neg P)$. 
Well-defined set of *formal proof rules*

- Formal proofs in Fitch can be *mechanically checked*.
- For each connective, there is
  - an *introduction rule*, e.g. “from $P$, infer $P \lor Q$”.
  - an *elimination rule*, e.g. “from $P \land Q$, infer $P$”.
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Conjunction Elimination
($\land$ Elim)

\[
\begin{array}{c}
P_1 \land \ldots \land P_i \land \ldots \land P_n \\

\vdots \\

\Delta \quad P_i
\end{array}
\]
Conjunction Introduction
(∧ Intro)

| \( P_1 \)
| \( \Downarrow \)
| \( P_n \)
| \( \vdots \)
| \( \triangleright \)
| \( P_1 \land \ldots \land P_n \)
Disjunction Introduction
($\lor$ Intro)

\[
\begin{array}{c}
P_i \\
\vdots \\
\hline
\hline
P_1 \lor \ldots \lor P_i \lor \ldots \lor P_n
\end{array}
\]
Disjunction Elimination
(∨ Elim)

\[ P_1 \lor \ldots \lor P_n \]
\[ \vdots \]
\[ P_1 \]
\[ \vdots \]
\[ S \]
\[ \Downarrow \]
\[ S \]
\[ \vdots \]
\[ S \]
\[ \Downarrow \]
\[ S \]
In the following two exercises, determine whether the sentences are consistent. If they are, use Tarski's World to build a world where the sentences are both true. If they are inconsistent, use Fitch to give a proof that they are inconsistent (that is, derive $\bot$ from them). You may use Ana Con in your proof, but only applied to literals (that is, atomic sentences or negations of atomic sentences).

6.15 $\neg(Larger(a, b) \land Larger(b, a))$ $\neg$ SameSize(a, b)

6.16 $\neg$ Smaller(a, b) $\lor$ Smaller(b, a) $\&$ SameSize(a, b)

Subproofs are the characteristic feature of Fitch-style deductive systems. It is important that you understand how to use them properly, since if you are not careful, you may "prove" things that don't follow from your premises. For example, the following formal proof looks like it is constructed according to our rules, but it purports to prove that $A \land B$ follows from $(B \land A) \lor (A \land C)$, which is clearly not right.

1. $(B \land A) \lor (A \land C)$
2. $B \land A$
3. $B$
4. $A$ $\land$ Elim: 2
5. $A \land C$ $\land$ Elim: 5
6. $A$ $\land$ Elim: 2
7. $A$ $\lor$ Elim: 1, 2–4, 5–6
8. $A \land B$ $\land$ Intro: 7, 3
In justifying a step of a subproof, you may cite any *earlier step* contained in the main proof, or in any subproof whose assumption is *still in force*. You may *never* cite individual steps inside a subproof that has *already ended*.

Fitch enforces this automatically by not permitting the citation of individual steps inside subproofs that have ended.
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⊥ Introduction
(⊥ Intro)

\[ \begin{array}{c}
\vdots \\
\neg P \\
\vdots \\
\bot
\end{array} \]
Negation Introduction
(¬ Intro)

\[ \begin{array}{c}
\vdash P \\
\vdash \bot \\
\vdash \neg P
\end{array} \]
Negation Elimination
(¬ Elim)

\[ \neg \neg P \]
\[ \vdots \]
\[ \neg P \]
\[ \vdots \]
\[ P \]
\[ \bot \text{ Elimination} \]
\[ (\bot \text{ Elim}) \]
\[ \begin{array}{c}
\bot \\
\vdots \\
\triangleright \\
P
\end{array} \]