Logik für Informatiker
Logic for computer scientists

Soundness an completeness

Till Mossakowski

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Object and meta theory

Object theory = reasoning within a formal proof system (e.g. Fitch)

Meta theory = reasoning about a formal proof system

Tautological consequence

A sentence S is a *tautological consequence* of a set of sentences $\mathcal{T}$, written

$$\mathcal{T} \models_T S,$$

if all valuations of atomic formulas with truth values that make all sentences in $\mathcal{T}$ true also make S true.

$\mathcal{T}$ is called *tt-satisfiable*, if there is a valuation making all sentences in $\mathcal{T}$ true. (Note: $\mathcal{T}$ may be infinite.)

Propositional proofs

S is $\mathcal{F}_T$-*provable* from $\mathcal{T}$, written

$$\mathcal{T} \vdash_T S,$$

if there is a formal proof of S with premises drawn from $\mathcal{T}$ using the elimination and introduction rules for $\vee, \wedge, \neg, \rightarrow, \leftrightarrow$ and $\bot$.

Again note: $\mathcal{T}$ may be infinite.

Consistency

A set of sentences $\mathcal{T}$ is called *formally inconsistent*, if

$$\mathcal{T} \vdash_T \bot.$$

Example: $\{A \vee B, \neg A, \neg B\}$.

Otherwise, $\mathcal{T}$ is called *formally consistent*.

Example: $\{A \vee B, A, \neg B\}$

Soundness

Theorem 1. The proof calculus $\mathcal{F}_T$ is sound, i.e. if

$$\mathcal{T} \vdash_T S,$$

then

$$\mathcal{T} \models_T S.$$

Proof: Book: by contradiction, using the first invalid step.
Here: by induction on the length of the proof.

Theorem 2 (Bernays, Post). The proof calculus $\mathcal{F}_T$ is complete, i.e. if

$$\mathcal{T} \models_T S,$$

then

$$\mathcal{T} \vdash_T S.$$

Theorem 2 follows from:

Theorem 3. Every formally consistent set of sentences is tt-satisfiable.

Lemma 4. $\mathcal{T} \cup \{\neg S\} \vdash_T \bot$ if and only if $\mathcal{T} \vdash_T S$.

Proof of Theorem 3

A set $\mathcal{T}$ is *formally complete*, if for any sentence S, either $\mathcal{T} \vdash_T S$ or $\mathcal{T} \vdash_T \neg S$.

Proposition 5. Every formally complete and formally consistent set of sentences is tt-satisfiable.

Proposition 6. Every formally consistent set of sentences can be expanded to a formally complete and formally consistent set of sentences.

Lemma 7. Let $\mathcal{T}$ be formally complete and formally consistent. Then

1. $\mathcal{T} \vdash_T (R \wedge S)$ iff $\mathcal{T} \vdash_T R$ and $\mathcal{T} \vdash_T S$
2. $\mathcal{T} \vdash_T (R \vee S)$ iff $\mathcal{T} \vdash_T R$ or $\mathcal{T} \vdash_T S$
3. $\mathcal{T} \vdash_T (\neg S)$ iff $\mathcal{T} \nvdash_T S$
4. $\mathcal{T} \vdash_T (R \rightarrow S)$ iff $\mathcal{T} \nvdash_T R$ or $\mathcal{T} \vdash_T S$
5. $\mathcal{T} \vdash_T (R \leftrightarrow S)$ iff ($\mathcal{T} \vdash_T R$ iff $\mathcal{T} \vdash_T S$)

Proof of Proposition 6

Lemma 8. A set of sentences $\mathcal{T}$ is formally complete if and only if for any *atomic* sentence A,

$$\text{either } \mathcal{T} \vdash_T A \text{ or } \mathcal{T} \vdash_T \neg A.$$

Compactness Theorem

Theorem 9. Let $\mathcal{T}$ be any set of sentences. If every finite subset of $\mathcal{T}$ is tt-satisfiable, then $\mathcal{T}$ itself is satisfiable.