Logik für Informatiker
Logic for computer scientists

Quantifiers

Till Mossakowski

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Quantifiers: Motivating examples

$$\frac{\begin{array}{l}\forall x\ \mathit{Cube}(x)\ \text{(“All objects are cubes.”)}\\ \forall x\ (\mathit{Cube}(x) \rightarrow \mathit{Large}(x))\ \text{(“All cubes are large.”)}\end{array}}{\forall x\ \mathit{Large}(x)\ \text{(“All objects are large.”)}}$$

$$\exists x\ \mathit{Cube}(x)$$

“There exists a cube.”

$$\exists x\ (\mathit{Cube}(x) \wedge \mathit{Large}(x))$$

“There exists a large cube.”

The four Aristotelian forms

All P's are Q's.	$\forall x(P(x) \rightarrow Q(x))$
Some P's are Q's.	$\exists x(P(x) \wedge Q(x))$
No P's are Q's.	$\forall x(P(x) \rightarrow \neg Q(x))$
Some P's are not Q's.	$\exists x(P(x) \wedge \neg Q(x))$

Note:
$\forall x(P(x) \rightarrow Q(x))$ does not imply that there are some $P's$.
$\exists x(P(x) \wedge Q(x))$ does not imply that not all $P's$ are $Q's$.

First-order signatures

A first-order signature consists of

- a set of predicate symbols with arities, like $Smaller^{(2)}, Dodec^{(1)}, Between^{(3)}, \leq^{(2)}$, including propositional symbols (nullary predicate symbols), like $A^{(0)}, B^{(0)}, C^{(0)}$, (written uppercase)
- its names or constants for individuals, like a, b, c, (written lowercase)
- its function symbols with arities, like $f^{(1)}, +^{(2)}, \times^{(2)}$.

Usually, arities are omitted.
In the book, the terminology "language" is used. "Signature" is more precise, since it exactly describes the ingredients that are needed to generate a (first-order) language.

Terms

$t ::= a$	constant
$t ::= x$	variable
$\mid f^{(n)}(t_1, \ldots, t_n)$	application of function symbols to terms

Usually, arities are omitted.
Variables are: t, u, v, w, x, y, z, possibly with subscripts.

Well-formed formulas

$F ::= p^{(n)}(t_1, \ldots, t_n)$	application of predicate symbols
$\mid \bot$	contradiction
$\mid \neg F$	negation
$\mid (F_1 \wedge \ldots \wedge F_n)$	conjunction
$\mid (F_1 \wedge \ldots \vee F_n)$	disjunction
$\mid (F_1 \rightarrow F_2)$	implication
$\mid (F_1 \leftrightarrow F_2)$	equivalence
$\mid \forall \nu F$	universal quantification
$\mid \exists \nu F$	existential quantification

The variable ν is said to be bound in $\forall \nu F$ and $\exists \nu F$.

Parentheses

The outermost parenthese of a well-formed formula can be omitted:

$$Cube(x) \wedge Small(x)$$

In general, parentheses are important to determine the scope of a quantifier (see next slide).

Free and bound variables

An occurrence of a variable in a formula that is not bound is said to be free.

$\exists y\ LeftOf(x, y)$	x is free, y is bound
$(Cube(x) \wedge Small(x)) \rightarrow \exists y\ LeftOf(x, y)$	x is free, y is bound
$\exists x\ (Cube(x) \wedge Small(x))$	Both occurrences of x are bound
$\exists x\ Cube(x) \wedge Small(x)$	The first occurrence of x is bound, the second one is free

Sentences

A sentence is a well-formed formula without free variables.

$$\bot \qquad\qquad A \wedge B$$

$$Cube(a) \vee Tet(b)$$

$$\forall x\ (Cube(x) \rightarrow Large(x))$$

$$\forall x\ ((Cube(x) \wedge Small(x)) \rightarrow \exists y\ LeftOf(x, y))$$

Semantics of quantification

- We need to fix some domain of discourse.
- $\forall x\ S(x)$ is true iff for every object in the domain of discourse with name n, $S(n)$ is true.
- $\exists x\ S(x)$ is true iff for some object in the domain of discourse with name n, $S(n)$ is true.
- Not all objects need to have names — hence we assume that for objects, names $n_1, n_2, \ldots$ can be invented "on the fly".

The game rules

Form	Your commitment	Player to move	Goal
$P \lor Q$	TRUE FALSE	you Tarski's World	Choose one of P, Q that is true.
$P \land Q$	TRUE FALSE	Tarski's World you	Choose one of P, Q that is false.
$\exists x\, P(x)$	TRUE FALSE	you Tarski's World	Choose some ***b*** that satisfies the wff P(x).
$\forall x\, P(x)$	TRUE FALSE	Tarski's World you	Choose some ***b*** that does not satisfy P(x).

Logical consequence for quantifiers

$$\begin{array}{|l} \forall x(\mathrm{Cube}(x) \rightarrow \mathrm{Small}(x)) \\ \forall x\ \mathrm{Cube}(x) \\ \hline \forall x\ \mathrm{Small}(x) \end{array}$$

$$\begin{array}{|l} \forall x\ \mathrm{Cube}(x) \\ \forall x\ \mathrm{Small}(x) \\ \hline \forall x(\mathrm{Cube}(x) \wedge \mathrm{Small}(x)) \end{array}$$

However: ignoring quantifiers does not work!

$$\begin{array}{|l} \exists x(\mathrm{Cube}(x) \rightarrow \mathrm{Small}(x)) \\ \exists x\ \mathrm{Cube}(x) \\ \hline \exists x\ \mathrm{Small}(x) \end{array}$$

$$\begin{array}{|l} \exists x\ \mathrm{Cube}(x) \\ \exists x\ \mathrm{Small}(x) \\ \hline \exists x(\mathrm{Cube}(x) \wedge \mathrm{Small}(x)) \end{array}$$

Tautologies do not distribute over quantifiers

$$\exists x\ Cube(x) \lor \exists x\ \neg Cube(x)$$

is a logical truth, but

$$\forall x\ Cube(x) \lor \forall x\ \neg Cube(x)$$

is not. By contrast,

$$\forall x\ Cube(x) \lor \neg\forall x\ Cube(x)$$

is a tautology.