Multiple quantifiers

\(\forall x \exists y \ Likes(x, y)\)

is very different from

\(\exists y \forall x \ Likes(x, y)\)
Figure 11.1: A circumstance in which $\forall x \exists y \text{Likes}(x, y)$ holds versus one in which $\exists y \forall x \text{Likes}(x, y)$ holds. It makes a big difference to someone!
Prenex Normal Form

Goal: shift all quantifiers to the top-level

Rules for conjunctions and disjunctions

\((\forall x P) \land Q \leadsto \forall x (P \land Q)\)  \((\exists x P) \land Q \leadsto \exists x (P \land Q)\)

\(P \land (\forall x Q) \leadsto \forall x (P \land Q)\)  \(P \land (\exists x Q) \leadsto \exists x (P \land Q)\)

\((\forall x P) \lor Q \leadsto \forall x (P \lor Q)\)  \((\exists x P) \lor Q \leadsto \exists x (P \lor Q)\)

\(P \lor (\forall x Q) \leadsto \forall x (P \lor Q)\)  \(P \lor (\exists x Q) \leadsto \exists x (P \lor Q)\)
Rules for negations, implications, equivalences

\[ \neg \forall x P \leadsto \exists x (\neg P) \quad \neg \exists x P \leadsto \forall x (\neg P) \]

\[ (\forall x P) \rightarrow Q \leadsto \exists x (P \rightarrow Q) \quad (\exists x P) \rightarrow Q \leadsto \forall x (P \rightarrow Q) \]

\[ P \rightarrow (\forall x Q) \leadsto \forall x (P \rightarrow Q) \quad P \rightarrow (\exists x Q) \leadsto \exists x (P \rightarrow Q) \]

\[ P \leftrightarrow Q \leadsto (P \rightarrow Q) \land (Q \rightarrow P) \]
What is the prenex normal form of

$$\exists x \text{Cube}(x) \rightarrow \forall y \text{Small}(y)$$
Proof methods for quantifiers

**Universal elimination**
Universal statements can be instantiated to any object.
From $\forall x S(x)$, we may infer $S(c)$.

**Existential introduction**
If we have established a statement for an instance, we can also establish the corresponding existential statement.
From $S(c)$, we may infer $\exists x S(x)$.
∀x[Cube(x) \rightarrow Large(x)]
∀x[Large(x) \rightarrow LeftOf(x, b)]

Cube(d)

\exists x[Large(x) \land LeftOf(x, b)]
From $\exists x S(x)$, we can infer $S(c)$, if $c$ is a new name not used otherwise.
Example: Scotland Yard searched a serial killer. The did not know who he was, but for their reasoning, they called him “Jack the ripper”.
This would have been an unfair procedure if there had been a real person named Jack the ripper.
∀x[Cube(x) → Large(x)]
∀x[Large(x) → LeftOf(x, b)]
∃x Cube(x)

∃x[Large(x) ∧ LeftOf(x, b)]
If we introduce a new name $c$ that is not used elsewhere, and can prove $S(c)$, then we can also infer $\forall x S(x)$. 

Example:

*Theorem* Every even number greater zero is the sum of two odd numbers.

*Proof* Let $n > 0$ be even, i.e. $n = 2m$ with $m > 0$. If $m$ is odd, then $m + m = n$ does the job. If $m$ is even, consider $(m - 1) + (m + 1) = n$. 
Arguments involving multiple quantifiers

\[ \exists y [ \text{Girl}(y) \land \forall x (\text{Boy}(x) \rightarrow \text{Likes}(x, y))] \]

\[ \forall x [\text{Boy}(x) \rightarrow \exists y (\text{Girl}(y) \land \text{Likes}(x, y))] \]

\[ \forall x [\text{Boy}(x) \rightarrow \exists y (\text{Girl}(y) \land \text{Likes}(x, y))] \]

\[ \exists y [\text{Girl}(y) \land \forall x (\text{Boy}(x) \rightarrow \text{Likes}(x, y))] \]

\[ \exists y [\text{Girl}(y) \land \forall x (\text{Boy}(x) \rightarrow \text{Likes}(x, y))] \]
A (counter)example

![Diagram with relationships between Alex, Zoe, Eric, Rachel, Matt, Laura, Brad, Betsy, Tom, and Sarah. Each person is connected to others with lines, indicating relationships. A heart symbol indicates a special connection or preference.](image-url)
• strongly typed; types are declared using the sort keyword
  sort Blocks

• predicates have to be declared with their types
  preds Cube, Dodec, Tet : Blocks

• propositional variables = nullary predicates
  preds A, B, C : ()

• constants have to be declared with their types
  ops a, b, c : Blocks
spec Tarski1 = sort Blocks
preds Cube, Dodec, Tet, Small, Medium, Large : Blocks
ops a, b, c : Blocks
  . not a=b . not a=c . not b=c
  . Small(a) => Cube(a)  %(small_cube_a)%
  . Small(a) <=> Small(b)  %(small_a_b)%
  . Small(b) \ Medium(b)  %(small_medium_b)%
  . Medium(b) => Medium(c)  %(medium_b_c)%
  . Medium(c) => Tet(c)  %(medium_tet_c)%
  . not Tet(c)  %(not_tet_c)%
  . Cube(a)  %(cube_a)%, %implied
  . Cube(b)  %(cube_b)%, %implied
Exercises

- chapter 10: 10.20 to 10.31
- chapters 11 and 12
- additional exercise (grade 1): write a complete axiomatization of Tarski’s World in CASL.