Logik für Informatiker
Logic for computer scientists

Proof rules for quantifiers

Till Mossakowski

WiSe 2009/10

Universal Elimination ($\forall$ Elim)

$$
\begin{array}{l|l}
 & \forall x\, S(x) \\
 & \quad \vdots \\
\triangleright & S(c)
\end{array}
$$

Existential Introduction (∃ Intro)

$$
\begin{array}{c|l}
 & S(c) \\
 & \vdots \\
\triangleright & \exists x\, S(x)
\end{array}
$$

Example: ∀-**Elim** and ∃-**Intro**

$\forall x[\mathrm{Cube}(x) \rightarrow \mathrm{Large}(x)]$
$\forall x[\mathrm{Large}(x) \rightarrow \mathrm{LeftOf}(x, b)]$
$\mathrm{Cube}(d)$

$\exists x[\mathrm{Large}(x) \wedge \mathrm{LeftOf}(x, b)]$

Existential Elimination (∃ Elim):

$$
\begin{array}{l|l}
 & \exists x\, S(x) \\
 & \vdots \\
 & \quad \boxed{c}\ S(c) \\
 & \quad \vdots \\
 & \quad Q \\
\triangleright & Q
\end{array}
$$

Where c does not occur outside the subproof where it is introduced.

Example: $\exists$-**Elim**

$$\forall x[\text{Cube}(x) \rightarrow \text{Large}(x)]$$
$$\forall x[\text{Large}(x) \rightarrow \text{LeftOf}(x, b)]$$
$$\exists x\ \text{Cube}(x)$$

$$\exists x[\text{Large}(x) \land \text{LeftOf}(x, b)]$$

General Conditional Proof (∀ Intro):

$$\begin{array}{l|l} & \begin{array}{|l} \boxed{c}\ P(c) \\ \hline \vdots \\ Q(c) \end{array} \\ \triangleright & \forall x\,(P(x) \to Q(x)) \end{array}$$

Where c does not occur outside the subproof where it is introduced.

Example: General Conditional Proof

$\forall x[\mathrm{Cube}(x) \rightarrow \mathrm{Large}(x)]$
$\forall x[\mathrm{Large}(x) \rightarrow \mathrm{LeftOf}(x, b)]$

$\forall x[\mathrm{Cube}(x) \rightarrow \mathrm{LeftOf}(x, b)$

Universal Introduction (∀ Intro):

$$
\begin{array}{l|l}
 & \quad \begin{array}{|l} \boxed{c} \\ \hline \vdots \\ P(c) \end{array} \\
\triangleright & \forall x\, P(x)
\end{array}
$$

Where c does not occur outside the subproof where it is introduced.

Prenex normal form (reminder)

$$
\begin{array}{|l}
\exists x \text{Cube}(x) \rightarrow \forall y \text{Small}(y) \\
\hline
\forall x \forall y (\text{Cube}(x) \rightarrow \text{Small}(y))
\end{array}
$$

Example with multiple quantifiers

$$\exists y[\mathsf{Girl}(y) \land \forall x(\mathsf{Boy}(x) \rightarrow \mathsf{Likes}(x, y))]$$

$$\forall x[\mathsf{Boy}(x) \rightarrow \exists y(\mathsf{Girl}(y) \land \mathsf{Likes}(x, y))]$$

Example: de Morgan's Law

$$\neg\forall x\ P(x)$$

$$\exists x\ \neg P(x)$$

(is not valid in intuitionistic logic, only in classical logic)

Example: The Barber Paradox

$$\exists z\ \exists x\ [ManOf(x,z) \wedge \forall y\ (ManOf(y,z) \rightarrow (Shave(x,y) \leftrightarrow \neg Shave(y,y)))]$$

$$\bot$$