Logik für Informatiker
Logic for computer scientists

Logical Consequence and Formal Proofs

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Logical consequence

A sentence B is a logical consequence of A_1, ..., A_n, if all circumstances that make A_1, ..., A_n true also make B true.
In symbols: $A_1, \ldots, A_n \models B$.
$A_1, \ldots, A_n$ are called premises, B is called conclusion.
In this case, it is a valid argument to infer B from $A_1, \ldots A_n$. If also $A_1, \ldots A_n$ are true, then the valid argument is sound.

Logical consequence — examples

- All men are mortal. Socrates is a man. So, Socrates is mortal. (valid, sound)
- All rich actors are good actors. Brad Pitt is a rich actor. So he must be a good actor. (valid, but not sound)
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Fitch notation

All men are mortal
Socrates is a man

So, Socrates is mortal

A_1
...
A_n

B

Premise_1
...
Premise_n

Conclusion

Methods for showing (in)validity of arguments

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Validity To show that an argument is valid, we must provide a proof. A proof consists of a sequence of proof steps, each of which must be valid.

- In propositional logic, we also can use truth tables to show validity. This it not possible in first-order logic.

Invalidity An argument can shown to be invalid by finding a counterexample (model), i.e. a circumstance where the premises are true, but the conclusion is false.

Informal and formal proofs

- informal reasoning is used in everyday life
- semi-formal reasoning is used in mathematics and theoretical computer science
  - balance between readability and precision
- formal proofs:
  - follow some specific rule system,
  - and are entirely rigorous
  - and can be checked by a computer

An informal proof

- Since Socrates is a man and all men are mortal, it follows that Socrates is mortal.
- But all mortals will eventually die, since that is what it means to be mortal.
- So Socrates will eventually die.
- But we are given that everyone who will eventually die sometimes worries about it.
- Hence Socrates sometimes worries about dying.

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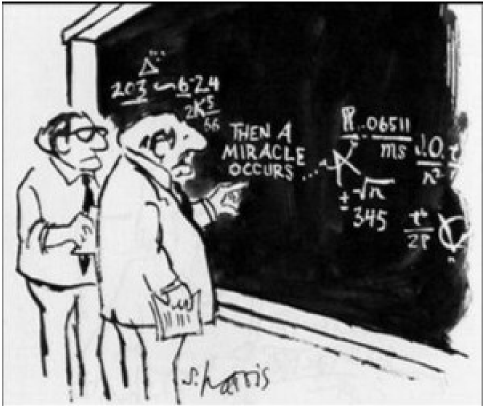
The need for formal proofs



"I think you should be more explicit here in step two."

A formal proof

1. Cube(c)	
2. c = b	
3. Cube(b)	**= Elim:** 1,2

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Four principles for the identity relation

1. $=$**Elim**: If $b = c$, then whatever holds of b holds of c (indiscernibility of identicals).
2. $=$**Intro**: $b = b$ is always true in FOL (reflexivity of identity).
3. Symmetry of Identity: If $b = c$, then $c = b$.
4. Transitivity of Identity: If $a = b$ and $b = c$, then $a = c$.

The latter two principles follow from the first two.

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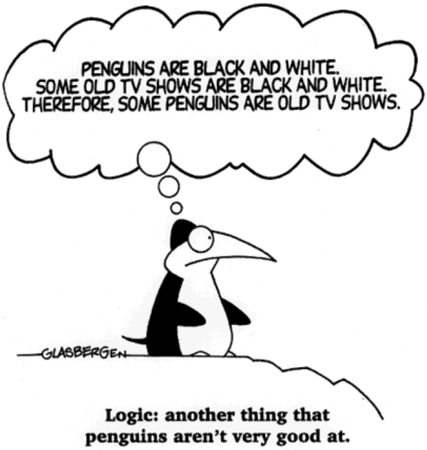
Transitivity . . .



Informal proof of symmetry of identity

- Suppose that $a = b$.
- We know that $a = a$, by the reflexivity of identity.
- Now substitute the name b for the first use of the name a in $a = a$, using the indiscernibility of identicals.
- We come up with $b = a$, as desired.

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Formal proofs

P	
Q	
R	
S_1	Justification 1
...	
...	
S_n	Justification n
S	Justification n+1

Formal proof of symmetry of identity

1. $a = b$	
2. $a = a$	$=$**Intro:**
3. $b = a$	$=$**Elim:** 2,1

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Fitch rule: Identity introduction

Identity Introduction (= Intro):

$\triangleright$ | $\mathsf{n} = \mathsf{n}$

Fitch rule: Identity elimination

Identity Elimination (= Elim):

$$
\begin{array}{l|l}
 & P(n) \\
 & \vdots \\
 & n = m \\
 & \vdots \\
\triangleright & P(m)
\end{array}
$$

Fitch rule: Reiteration

Reiteration (Reit):

	P
	⋮
▷	P

Formal proofs in Fitch

P	
Q	
R	
S_1	Justification 1
...	
...	
S_n	Justification n
S	Justification n+1

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Fitch rule: Reiteration

Reiteration (Reit):

$$\begin{array}{r|l} & P \\ & \vdots \\ \triangleright & P \end{array}$$

Example proof in fitch