

# Logik für Informatiker

## Formal proofs for propositional logic

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WiSe 2011/12

# The truth table method

- A sentence is a tautology if and only if it evaluates to **TRUE** in all rows of its complete truth table.
- Truth tables can be constructed with the program **Boole**.

# Tautological equivalence and consequence

- Two sentences  $P$  and  $Q$  are **tautologically equivalent**, if they evaluate to the same truth value in all valuations (rows of the truth table).
- $Q$  is a **tautological consequence** of  $P_1, \dots, P_n$  if and only if every row that assigns TRUE to each of  $P_1, \dots, P_n$  also assigns TRUE to  $Q$ .
- If  $Q$  is a tautological consequence of  $P_1, \dots, P_n$ , then  $Q$  is also a **logical consequence** of  $P_1, \dots, P_n$ .
- Some logical consequences are not tautological ones.

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## de Morgan's laws and double negation

$$\neg(P \wedge Q) \Leftrightarrow (\neg P \vee \neg Q)$$

$$\neg(P \vee Q) \Leftrightarrow (\neg P \wedge \neg Q)$$

$$\neg\neg P \Leftrightarrow P$$

Note:  $\neg$  binds stronger than  $\wedge$  and  $\vee$ . Brackets are needed to override this.

# Negation normal form

- **Substitution of equivalents:** If  $P$  and  $Q$  are logically equivalent:  $P \Leftrightarrow Q$  then the results of substituting one for the other in the context of a larger sentence are also logically equivalent:  $S(P) \Leftrightarrow S(Q)$
- A sentence is in **negation normal form** (NNF) if all occurrences of  $\neg$  apply directly to atomic sentences.
- Any sentence built from atomic sentences using just  $\wedge$ ,  $\vee$ , and  $\neg$  can be **put into negation normal form** by repeated application of the de Morgan laws and double negation.



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# Distributive laws

For any sentences  $P$ ,  $Q$ , and  $R$ :

- **Distribution of  $\wedge$  over  $\vee$ :**

$$P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R).$$

- **Distribution of  $\vee$  over  $\wedge$ :**

$$P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R).$$

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# Conjunctive and disjunctive normal form

- A sentence is in **conjunctive normal form** (CNF) if it is a conjunction of one or more disjunctions of one or more literals.
- Distribution of  $\vee$  over  $\wedge$  allows you to **transform** any sentence in negation normal form into conjunctive normal form.

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# Disjunctive normal form

- A sentence is in **disjunctive normal form** (DNF) if it is a disjunction of one or more conjunctions of one or more literals.
- Distribution of  $\wedge$  over  $\vee$  allows you to **transform** any sentence in negation normal form into disjunctive normal form.
- Some sentences are in both CNF and DNF.

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- Some sentences are in both CNF and DNF.

# Logical consequence

- $Q$  is a **logical consequence** of  $P_1, \dots, P_n$ , if all worlds that make  $P_1, \dots, P_n$  true also make  $Q$  true.
- $Q$  is a **tautological consequence** of  $P_1, \dots, P_n$ , if all valuations of atomic formulas with truth values that make  $P_1, \dots, P_n$  true also make  $Q$  true.
- $Q$  is a **TW-logical consequence** of  $P_1, \dots, P_n$ , if all worlds from Tarski's world that make  $P_1, \dots, P_n$  true also make  $Q$  true.

# Proofs

- With proofs, we try to show (tauto)logical consequence
- Truth-table method can lead to very large tables, proofs are often shorter
- Proofs are also available for consequence in full first-order logic, not only for tautological consequence

# Limits of the truth-table method

- 1 truth-table method leads to **exponentially growing** tables
  - 20 atomic sentences  $\Rightarrow$  more than 1.000.000 rows
- 2 truth-table method cannot be extended to **first-order logic**
  - **model checking** can overcome the first limitation (up to 1.000.000 atomic sentences)
  - **proofs** can overcome both limitations

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# Proofs

- A proof consists of a sequence of **proof steps**
- Each proof step is known to be valid and should
  - be significant but easily understood, in **informal** proofs,
  - follow some **proof rule**, in **formal** proofs.
- Some valid patterns of inference that generally go unmentioned in informal (but not in formal) proofs:
  - From  $P \wedge Q$ , infer  $P$ .
  - From  $P$  and  $Q$ , infer  $P \wedge Q$ .
  - From  $P$ , infer  $P \vee Q$ .

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# Proof by cases (disjunction elimination)

To prove  $S$  from  $P_1 \vee \dots \vee P_n$ , prove  $S$  from each of  $P_1, \dots, P_n$ .

**Claim:** there are irrational numbers  $b$  and  $c$  such that  $b^c$  is rational.

**Proof:**  $\sqrt{2}^{\sqrt{2}}$  is either rational or irrational.

**Case 1:** If  $\sqrt{2}^{\sqrt{2}}$  is rational: take  $b = c = \sqrt{2}$ .

**Case 2:** If  $\sqrt{2}^{\sqrt{2}}$  is irrational: take  $b = \sqrt{2}^{\sqrt{2}}$  and  $c = \sqrt{2}$ .

$$\text{Then } b^c = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{(\sqrt{2} \cdot \sqrt{2})} = \sqrt{2}^2 = 2.$$

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# Proof by contradiction

To prove  $\neg S$ , assume  $S$  and prove a contradiction  $\perp$ .  
( $\perp$  may be inferred from  $P$  and  $\neg P$ .)

Assume  $Cube(c) \vee Dodec(c)$  and  $Tet(b)$ .

**Claim:**  $\neg(b = c)$ .

**Proof:** Let us assume  $b = c$ .

**Case 1:** If  $Cube(c)$ , then by  $b = c$ , also  $Cube(b)$ , which contradicts  $Tet(b)$ .

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# Arguments with inconsistent premises

A proof of a contradiction  $\perp$  from premises  $P_1, \dots, P_n$  (without additional assumptions) shows that the premises are **inconsistent**. An argument with inconsistent premises is always **valid**, but more importantly, always **unsound**.

Home(max)  $\vee$  Home(claire)

$\neg$ Home(max)

$\neg$ Home(claire)

Home(max)  $\wedge$  Happy(carl)

# Arguments without premises

A proof without any premises shows that its conclusion is a **logical truth**.

Example:  $\neg(P \wedge \neg P)$ .

# Formal proofs in Fitch

- Well-defined set of **formal proof rules**
- Formal proofs in Fitch can be **mechanically checked**
- For each connective, there is
  - an **introduction rule**, e.g. “from  $P$ , infer  $P \vee Q$ ”.
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## Conjunction Elimination ( $\wedge$ Elim)

$$\triangleright \left| \begin{array}{l} P_1 \wedge \dots \wedge P_i \wedge \dots \wedge P_n \\ \vdots \\ P_i \end{array} \right.$$

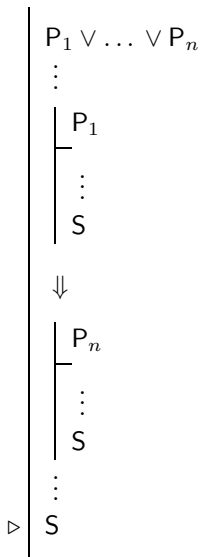
## Conjunction Introduction ( $\wedge$ Intro)

$$\begin{array}{l} | \\ | \\ P_1 \\ | \\ \Downarrow \\ | \\ P_n \\ | \\ \vdots \\ | \\ \triangleright P_1 \wedge \dots \wedge P_n \end{array}$$

## Disjunction Introduction ( $\vee$ Intro)

$$\triangleright \left| \begin{array}{l} P_i \\ \vdots \\ P_1 \vee \dots \vee P_i \vee \dots \vee P_n \end{array} \right.$$

## Disjunction Elimination ( $\vee$ Elim)



## The proper use of subproofs

1. $(B \wedge A) \vee (A \wedge C)$	
2. $B \wedge A$	
3. $B$	$\wedge$ <b>Elim</b> : 2
4. $A$	$\wedge$ <b>Elim</b> : 2
5. $A \wedge C$	
6. $A$	$\wedge$ <b>Elim</b> : 5
7. $A$	$\vee$ <b>Elim</b> : 1, 2–4, 5–6
8. $A \wedge B$	$\wedge$ <b>Intro</b> : 7, 3

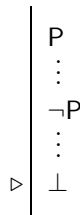
# The proper use of subproofs (cont'd)

- In justifying a step of a subproof, you may cite any **earlier step** contained in the main proof, or in any subproof whose assumption is **still in force**. You may **never** cite individual steps inside a subproof that has **already ended**.
- Fitch enforces this automatically by not permitting the citation of individual steps inside subproofs that have ended.

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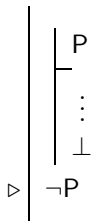
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## $\perp$ Introduction ( $\perp$ Intro)





## Negation Introduction ( $\neg$ Intro)



## Negation Elimination ( $\neg$ Elim)

$$\begin{array}{|l} \neg\neg P \\ \vdots \\ P \end{array}$$

## $\perp$ Elimination ( $\perp$ Elim)

$$\begin{array}{c|c} & \perp \\ & \vdots \\ \triangleright & P \end{array}$$