Logik für Informatiker
Logic for computer scientists

Soundness and completeness

Till Mossakowski, Lutz Schröder

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Conditionals

P	Q	P → Q
T	T	**T**
T	F	**F**
F	T	**T**
F	F	**T**

Game rule: $P \to Q$ is replaced by $\neg P \vee Q$.

Formalisation of conditional sentences

- The following English constructions are all translated $P \to Q$: If P then Q; Q if P; P only if Q; and Provided P, Q.
- Unless P, Q and Q unless P are translated: $\neg P \to Q$.
- Q is a logical consequence of $P_1, \ldots, P_n$ if and only if the sentence $(P1 \wedge \cdots \wedge P_n) \to Q$ is a logical truth.

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Conditional Elimination ($\rightarrow$ Elim)

$$
\begin{array}{r|l}
 & P \rightarrow Q \\
 & \vdots \\
 & P \\
 & \vdots \\
\triangleright & Q
\end{array}
$$

Conditional Introduction ($\rightarrow$ Intro)

$$
\begin{array}{l|l}
 & \begin{array}{|l} P \\ \hline \vdots \\ Q \end{array} \\
\triangleright & P \rightarrow Q
\end{array}
$$

Biconditionals

P	Q	P $\leftrightarrow$ Q
T	T	**T**
T	F	**F**
F	T	**F**
F	F	**T**

Game rule: $P \leftrightarrow Q$ is replaced by $(P \to Q) \wedge (Q \to P)$.

Biconditional Elimination ($\leftrightarrow$ Elim)

$$
\begin{array}{l|l}
 & \mathsf{P} \leftrightarrow \mathsf{Q} \ (\text{or } \mathsf{Q} \leftrightarrow \mathsf{P}) \\
 & \vdots \\
 & \mathsf{P} \\
 & \vdots \\
\triangleright & \mathsf{Q}
\end{array}
$$

Biconditional Introduction ($\leftrightarrow$ Intro)

$$
\begin{array}{l|l}
 & \quad \begin{array}{|l} P \\ \hline \vdots \\ Q \end{array} \\
 & \quad \begin{array}{|l} Q \\ \hline \vdots \\ P \end{array} \\
\triangleright & P \leftrightarrow Q
\end{array}
$$

Reiteration (Reit)

$$\begin{array}{l|l} & P \\ & \vdots \\ \triangleright & P \end{array}$$

Object and meta theory

Object theory = reasoning within a formal proof system (e.g. Fitch)

Meta theory = reasoning about a formal proof system

Tautological consequence

A sentence S is a *tautological consequence* of a set of sentences $\mathcal{T}$, written

$$\mathcal{T} \models_T S,$$

if all valuations of atomic formulas with truth values that make all sentences in $\mathcal{T}$ true also make S true.

$\mathcal{T}$ is called *tt-satisfiable*, if there is a valuation making all sentences in $\mathcal{T}$ true. (Note: $\mathcal{T}$ may be infinite.)

Propositional proofs

S is $\mathcal{F}_T$-*provable* from $\mathcal{T}$, written

$$\mathcal{T} \vdash_T S,$$

if there is a formal proof of S with premises drawn from $\mathcal{T}$ using the elimination and introduction rules for $\vee, \wedge, \neg, \rightarrow, \leftrightarrow$ and $\bot$.

Again note: $\mathcal{T}$ may be infinite.

Consistency

A set of sentences $\mathcal{T}$ is called *formally inconsistent*, if

$$\mathcal{T} \vdash_T \bot.$$

Example: $\{A \vee B, \neg A, \neg B\}$.

Otherwise, $\mathcal{T}$ is called *formally consistent*.

Example: $\{A \vee B, A, \neg B\}$

Theorem 1. The proof calculus $\mathcal{F}_T$ is sound, i.e. if

$$\mathcal{T} \vdash_T S,$$

then

$$\mathcal{T} \models_T S.$$

Proof: Book: by contradiction, using the first invalid step.
Here: by induction on the length of the proof.

Completeness

Theorem 2 (Bernays, Post). The proof calculus $\mathcal{F}_T$ is complete, i.e. if

$$\mathcal{T} \models_T S,$$

then

$$\mathcal{T} \vdash_T S.$$

Theorem 2 follows from:

Theorem 3. Every formally consistent set of sentences is tt-satisfiable.

Lemma 4. $\mathcal{T} \cup \{\neg S\} \vdash_T \bot$ if and only if $\mathcal{T} \vdash_T S$.

Proof of Theorem 3

A set $\mathcal{T}$ is *formally complete*, if for any sentence S, either $\mathcal{T} \vdash_T S$ or $\mathcal{T} \vdash_T \neg S$.

Proposition 5. Every formally complete and formally consistent set of sentences is tt-satisfiable.

Proposition 6. Every formally consistent set of sentences can be expanded to a formally complete and formally consistent set of sentences.

Proof of Proposition 5

Lemma 7. Let $\mathcal{T}$ be formally complete and formally consistent. Then

1. $\mathcal{T} \vdash_T (R \wedge S)$ iff $\mathcal{T} \vdash_T R$ and $\mathcal{T} \vdash_T S$
2. $\mathcal{T} \vdash_T (R \vee S)$ iff $\mathcal{T} \vdash_T R$ or $\mathcal{T} \vdash_T S$
3. $\mathcal{T} \vdash_T (\neg S)$ iff $\mathcal{T} \nvdash_T S$
4. $\mathcal{T} \vdash_T (R \rightarrow S)$ iff $\mathcal{T} \nvdash_T R$ or $\mathcal{T} \vdash_T S$
5. $\mathcal{T} \vdash_T (R \leftrightarrow S)$ iff $(\mathcal{T} \vdash_T R$ iff $\mathcal{T} \vdash_T S)$

Proof of Proposition 6

Lemma 8. A set of sentences $\mathcal{T}$ is formally complete if and only if for any *atomic* sentence A,

$$\text{either } \mathcal{T} \vdash_T A \text{ or } \mathcal{T} \vdash_T \neg A.$$

Compactness Theorem

Theorem 9. Let $\mathcal{T}$ be any set of sentences. If every finite subset of $\mathcal{T}$ is tt-satisfiable, then $\mathcal{T}$ itself is satisfiable.