

Logik für Informatiker
Logic for computer scientists
Soundness and completeness

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P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Game rule: $P \rightarrow Q$ is replaced by $\neg P \vee Q$.

Formalisation of conditional sentences

- The following English constructions are all translated $P \rightarrow Q$:
If P then Q ; Q if P ; P only if Q ; and Provided P , Q .
- Unless P , Q and Q unless P are translated: $\neg P \rightarrow Q$.
- Q is a logical consequence of P_1, \dots, P_n if and only if the sentence $(P_1 \wedge \dots \wedge P_n) \rightarrow Q$ is a logical truth.

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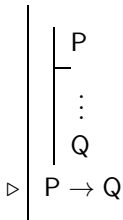
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Conditional Elimination (\rightarrow Elim)

$$\begin{array}{l|l} & P \rightarrow Q \\ & \vdots \\ & P \\ & \vdots \\ \triangleright & Q \end{array}$$

Conditional Introduction (\rightarrow Intro)



Biconditionals

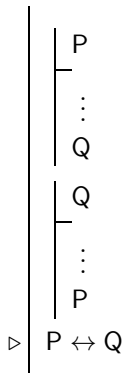
P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Game rule: $P \leftrightarrow Q$ is replaced by $(P \rightarrow Q) \wedge (Q \rightarrow P)$.

Biconditional Elimination (\leftrightarrow Elim)

		$P \leftrightarrow Q$ (or $Q \leftrightarrow P$)
		\vdots
		P
		\vdots
\triangleright		Q

Biconditional Introduction (\leftrightarrow Intro)



Reiteration (Reit)

$$\begin{array}{|l} P \\ \vdots \\ P \end{array}$$

Object and meta theory

Object theory = reasoning **within** a formal proof system (e.g. Fitch)

Meta theory = reasoning **about** a formal proof system

Tautological consequence

A sentence S is a *tautological consequence* of a set of sentences \mathcal{T} , written

$$\mathcal{T} \models_{\mathcal{T}} S,$$

if all valuations of atomic formulas with truth values that make all sentences in \mathcal{T} true also make S true.

\mathcal{T} is called *tt-satisfiable*, if there is a valuation making all sentences in \mathcal{T} true. (Note: \mathcal{T} may be infinite.)

S is $\mathcal{F}_{\mathcal{T}}$ -provable from \mathcal{T} , written

$$\mathcal{T} \vdash_{\mathcal{T}} S,$$

if there is a formal proof of S with premises drawn from \mathcal{T} using the elimination and introduction rules for $\vee, \wedge, \neg, \rightarrow, \leftrightarrow$ and \perp .

Again note: \mathcal{T} may be infinite.

A set of sentences \mathcal{T} is called *formally inconsistent*, if

$$\mathcal{T} \vdash_{\mathcal{T}} \perp.$$

Example: $\{A \vee B, \neg A, \neg B\}$.

Otherwise, \mathcal{T} is called *formally consistent*.

Example: $\{A \vee B, A, \neg B\}$

Theorem 1. The proof calculus \mathcal{F}_T is sound, i.e. if

$$\mathcal{T} \vdash_T S,$$

then

$$\mathcal{T} \models_T S.$$

Proof: Book: by contradiction, using the first invalid step.
Here: by induction on the length of the proof.

Theorem 2 (Bernays, Post). The proof calculus $\mathcal{F}_{\mathcal{T}}$ is complete, i.e. if

$$\mathcal{T} \models_{\mathcal{T}} S,$$

then

$$\mathcal{T} \vdash_{\mathcal{T}} S.$$

Theorem 2 follows from:

Theorem 3. Every formally consistent set of sentences is tt-satisfiable.

Lemma 4. $\mathcal{T} \cup \{\neg S\} \vdash_{\mathcal{T}} \perp$ if and only if $\mathcal{T} \vdash_{\mathcal{T}} S$.

Proof of Theorem 3

A set \mathcal{T} is *formally complete*, if for any sentence S , either $\mathcal{T} \vdash_{\mathcal{T}} S$ or $\mathcal{T} \vdash_{\mathcal{T}} \neg S$.

Proposition 5. Every formally complete and formally consistent set of sentences is tt-satisfiable.

Proposition 6. Every formally consistent set of sentences can be expanded to a formally complete and formally consistent set of sentences.

Proof of Proposition 5

Lemma 7. Let \mathcal{T} be formally complete and formally consistent.
Then

- 1 $\mathcal{T} \vdash_{\mathcal{T}} (R \wedge S)$ iff $\mathcal{T} \vdash_{\mathcal{T}} R$ and $\mathcal{T} \vdash_{\mathcal{T}} S$
- 2 $\mathcal{T} \vdash_{\mathcal{T}} (R \vee S)$ iff $\mathcal{T} \vdash_{\mathcal{T}} R$ or $\mathcal{T} \vdash_{\mathcal{T}} S$
- 3 $\mathcal{T} \vdash_{\mathcal{T}} (\neg S)$ iff $\mathcal{T} \not\vdash_{\mathcal{T}} S$
- 4 $\mathcal{T} \vdash_{\mathcal{T}} (R \rightarrow S)$ iff $\mathcal{T} \not\vdash_{\mathcal{T}} R$ or $\mathcal{T} \vdash_{\mathcal{T}} S$
- 5 $\mathcal{T} \vdash_{\mathcal{T}} (R \leftrightarrow S)$ iff $(\mathcal{T} \vdash_{\mathcal{T}} R \text{ iff } \mathcal{T} \vdash_{\mathcal{T}} S)$

Proof of Proposition 6

Lemma 8. A set of sentences \mathcal{T} is formally complete if and only if for any *atomic* sentence A ,

either $\mathcal{T} \vdash_{\mathcal{T}} A$ or $\mathcal{T} \vdash_{\mathcal{T}} \neg A$.

Theorem 9. Let \mathcal{T} be any set of sentences. If every finite subset of \mathcal{T} is tt-satisfiable, then \mathcal{T} itself is satisfiable.