Logik für Informatiker
Logic for computer scientists

Multiple Quantifiers

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Multiple quantifiers

$$\forall x \exists y\ Likes(x, y)$$

is very different from

$$\exists y \forall x\ Likes(x, y)$$

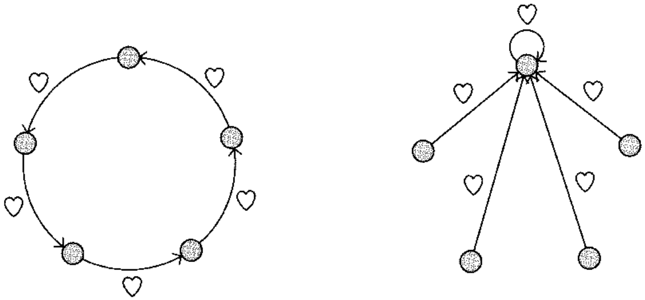


Figure 11.1: A circumstance in which $\forall x\, \exists y\, \mathsf{Likes}(x, y)$ holds versus one in which $\exists y\, \forall x\, \mathsf{Likes}(x, y)$ holds. It makes a big difference to someone!

Prenex Normal Form

Goal: shift all quantifiers to the top-level

Rules for conjunctions and disjunctions

$(\forall x P) \wedge Q \rightsquigarrow \forall x(P \wedge Q)$ $\qquad$ $(\exists x P) \wedge Q \rightsquigarrow \exists x(P \wedge Q)$

$P \wedge (\forall x Q) \rightsquigarrow \forall x(P \wedge Q)$ $\qquad$ $P \wedge (\exists x Q) \rightsquigarrow \exists x(P \wedge Q)$

$(\forall x P) \vee Q \rightsquigarrow \forall x(P \vee Q)$ $\qquad$ $(\exists x P) \vee Q \rightsquigarrow \exists x(P \vee Q)$

$P \vee (\forall x Q) \rightsquigarrow \forall x(P \vee Q)$ $\qquad$ $P \vee (\exists x Q) \rightsquigarrow \exists x(P \vee Q)$

Prenex Normal Form (cont'd)

Rules for negations, implications, equivalences

$\neg\forall x P \rightsquigarrow \exists x(\neg P)$ $\qquad$ $\neg\exists x P \rightsquigarrow \forall x(\neg P)$

$(\forall x P) \to Q \rightsquigarrow \exists x(P \to Q)$ $\qquad$ $(\exists x P) \to Q \rightsquigarrow \forall x(P \to Q)$

$P \to (\forall x Q) \rightsquigarrow \forall x(P \to Q)$ $\qquad$ $P \to (\exists x Q) \rightsquigarrow \exists x(P \to Q)$

$P \leftrightarrow Q \rightsquigarrow (P \to Q) \land (Q \to P)$

Prenex Normal Form: example

What is the prenex normal form of

$$\exists x Cube(x) \rightarrow \forall y Small(y)$$

Proof methods for quantifiers

Universal elimination
Universal statments can be instantiated to any object.

From $\forall x S(x)$, we may infer $S(c)$.

Existential introduction
If we have established a statement for an instance, we can also establish the corresponding existential statement.

From $S(c)$, we may infer $\exists x S(x)$.

Example

$$\forall x[\mathrm{Cube}(x) \rightarrow \mathrm{Large}(x)]$$
$$\forall x[\mathrm{Large}(x) \rightarrow \mathrm{LeftOf}(x, b)]$$
$$\mathrm{Cube}(d)$$

$$\exists x[\mathrm{Large}(x) \wedge \mathrm{LeftOf}(x, b)]$$

Existential elimination

From $\exists x S(x)$, we can infer things by assuming $S(c)$ in a subproof, if c is a new name not used otherwise.
Example: Scotland Yard searched a serial killer. The did not know who he was, but for their reasoning, they called him *"Jack the ripper"*.
This would have been an unfair procedure if there had been a real person named Jack the ripper.

Example

$$\forall x[\mathrm{Cube}(x) \to \mathrm{Large}(x)]$$
$$\forall x[\mathrm{Large}(x) \to \mathrm{LeftOf}(x, b)]$$
$$\exists x \mathrm{Cube}(x)$$

$$\exists x[\mathrm{Large}(x) \wedge \mathrm{LeftOf}(x, b)]$$

Universal generalization (introduction)

If we introduce a new name c that is not used elsewhere, and can prove $S(c)$, then we can also infer $\forall x S(x)$.

Example:

Theorem Every positive even number is the sum of two odd numbers.

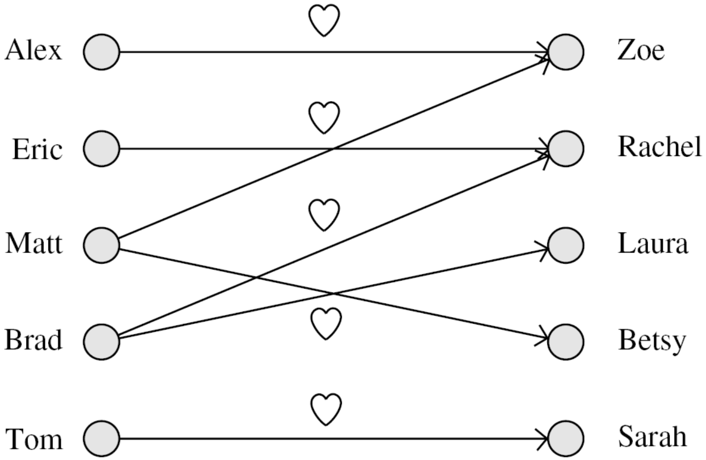
Proof Let $n > 0$ be even, i.e. $n = 2m$ with $m > 0$. If m is odd, then $m + m = n$ does the job. If m is even, consider $(m - 1) + (m + 1) = n$.

Arguments involving multiple quantifiers

$$\exists y[\mathrm{Girl}(y) \land \forall x(\mathrm{Boy}(x) \rightarrow \mathrm{Likes}(x,y))]$$
$$\vdash \forall x[\mathrm{Boy}(x) \rightarrow \exists y(\mathrm{Girl}(y) \land \mathrm{Likes}(x,y))]$$

$$\forall x[\mathrm{Boy}(x) \rightarrow \exists y(\mathrm{Girl}(y) \land \mathrm{Likes}(x,y))]$$
$$\vdash \exists y[\mathrm{Girl}(y) \land \forall x(\mathrm{Boy}(x) \rightarrow \mathrm{Likes}(x,y))]$$

A (counter)example



Common Algebraic Specification Language

- strongly typed; types are declated using the *sort* keyword
  sort Blocks
- predicates have to be declared with their types
  preds Cube, Dodec, Tet : Blocks
- propositional variables = nullary predicates
  preds A,B,C : ()
- constants have to be declared with their types
  ops a,b,c : Blocks

Example CASL specification: blocks

```
spec Tarski1 =     sort Blocks
  preds Cube, Dodec, Tet, Small, Medium, Large : Blocks
  ops a,b,c : Blocks
  . not a=b . not a=c  . not b=c
  . Small(a) => Cube(a)    %(small_cube_a)%
  . Small(a) <=> Small(b)  %(small_a_b)%
  . Small(b) \/ Medium(b)  %(small_medium_b)%
  . Medium(b) => Medium(c) %(medium_b_c)%
  . Medium(c) => Tet(c)    %(medium_tet_c)%
  . not Tet(c)             %(not_tet_c)%
  . Cube(a)                %(cube_a)% %implied
  . Cube(b)                %(cube_b)% %implied
```

Universal Elimination (∀ Elim)

$$
\begin{array}{l|l}
 & \forall x\, S(x) \\
 & \vdots \\
\triangleright & S(c)
\end{array}
$$

Existential Introduction (∃ Intro)

$$\begin{array}{l|l} & S(c) \\ & \vdots \\ \triangleright & \exists x\, S(x) \end{array}$$

Example: ∀-Elim and ∃-Intro

$$\forall x[\mathrm{Cube}(x) \rightarrow \mathrm{Large}(x)]$$
$$\forall x[\mathrm{Large}(x) \rightarrow \mathrm{LeftOf}(x, b)]$$
$$\mathrm{Cube}(d)$$

$$\exists x[\mathrm{Large}(x) \land \mathrm{LeftOf}(x, b)]$$

Existential Elimination (∃ Elim):

$$\begin{array}{l|l} & \exists x\, S(x) \\ & \vdots \\ & \quad \boxed{c}\ S(c) \\ & \quad \vdots \\ & \quad Q \\ \triangleright & Q \end{array}$$

Where c does not occur outside the subproof where it is introduced.

Example: ∃-Elim

$$\forall x[\mathrm{Cube}(x) \rightarrow \mathrm{Large}(x)]$$
$$\forall x[\mathrm{Large}(x) \rightarrow \mathrm{LeftOf}(x, b)]$$
$$\exists x\ \mathrm{Cube}(x)$$

$$\exists x[\mathrm{Large}(x) \wedge \mathrm{LeftOf}(x, b)]$$

General Conditional Proof (∀ Intro):

$$\begin{array}{l|l} & \begin{array}{|l} \boxed{c}\ P(c) \\ \hline \vdots \\ Q(c) \end{array} \\ \triangleright & \forall x\,(P(x) \to Q(x)) \end{array}$$

Where c does not occur outside the subproof where it is introduced.

Example: General Conditional Proof

$\forall x[\text{Cube}(x) \to \text{Large}(x)]$
$\forall x[\text{Large}(x) \to \text{LeftOf}(x, b)]$

$\forall x[\text{Cube}(x) \to \text{LeftOf}(x, b)$

Universal Introduction (∀ Intro):

$$\begin{array}{l|l} & \begin{array}{|l} \boxed{c} \\ \hline \vdots \\ P(c) \end{array} \\ \triangleright & \forall x\, P(x) \end{array}$$

Where c does not occur outside the subproof where it is introduced.

Prenex normal form (reminder)

$$\begin{array}{|l} \exists x \text{Cube}(x) \rightarrow \forall y \text{Small}(y) \\ \hline \forall x \forall y (\text{Cube}(x) \rightarrow \text{Small}(y)) \end{array}$$

Example with multiple quantifiers

$$\exists y[\text{Girl}(y) \land \forall x(\text{Boy}(x) \rightarrow \text{Likes}(x,y))]$$
$$\vdash$$
$$\forall x[\text{Boy}(x) \rightarrow \exists y(\text{Girl}(y) \land \text{Likes}(x,y))]$$

Example: de Morgan's Law

$$\frac{\neg\forall x\, P(x)}{\exists x\, \neg P(x)}$$

(is not valid in intuitionistic logic, only in classical logic)

Example: The Barber Paradox

$$\exists z\, \exists x\, [ManOf(x,z) \land \forall y\, (ManOf(y,z) \rightarrow (Shave(x,y) \leftrightarrow \neg Shave(y,y)))]$$

$$\bot$$