Logik für Informatiker
Logic for computer scientists

First-order resolution

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First-order resolution

- generalises propositional resolution to first-order logic
- is a proof system that is well-suited for efficient implementation
- many automated first-order provers are based on resolution: SPASS, Prover9, Vampire
- also interactive provers for higher-order logic are based on resolution: Isabelle, HOL, HOL-light

Satisfiability and logical consequence

Logical consequence can be reduced to (un)satisfiability:

> *The logical consequence* $\mathcal{T} \models S$ *holds*
> *if and only if*
> $\mathcal{T} \cup \{\neg S\}$ *is unsatisfiable.*

Note: Resolution is about satisfiability.

Skolemization

The sentence

$$\forall x \exists y Neighbor(x, y)$$

is logically equivalent to the second-order sentence

$$\exists f \forall x Neighbor(x, f(x))$$

In first-order logic, we have the *Skolem normal form*

$$\forall x Neighbor(x, f(x))$$

Theorem about Skolem normal form

Theorem

A sentence $S \equiv \forall x \exists y P(x, y)$ is satisfiable iff its Skolem normal form $\forall x P(x, f(x))$ is.

Every structure satisfying the Skolem normal form also satisfies S. Moreover, every structure satisfying S can be turned into one satisfying the Skolem normal form. This is done by interpreting f by a function which picks out, for any object b in the domain, some object c such that they satisfy $P(x, y)$.

Unification of terms

$$\{P(f(a)), \forall x\ \neg P(f(g(x)))\}$$

is satisfiable, but

$$\{P(f(g(a))), \forall x\ \neg P(f(x))\}$$

is not. This can be seen with *unification*.
Terms $t_1, \ldots, t_n$ are *unifiable*, if there is a substitution of terms for some or all the variables in $t_1, \ldots, t_n$ such that the terms that result from the substitution are syntactically identical terms.

Example

$$f(g(z), x), \quad f(y, x), \quad f(y, h(a))$$

are unifiable by substituting $h(a)$ for x and $g(z)$ for y.

Prenex Normal Form

Goal: shift all quantifiers to the top-level

$$(\forall x P) \land Q \leadsto \forall x(P \land Q) \qquad (\exists x P) \land Q \leadsto \exists x(P \land Q)$$
$$P \land (\forall x Q) \leadsto \forall x(P \land Q) \qquad P \land (\exists x Q) \leadsto \exists x(P \land Q)$$
$$(\forall x P) \lor Q \leadsto \forall x(P \lor Q) \qquad (\exists x P) \lor Q \leadsto \exists x(P \lor Q)$$
$$P \lor (\forall x Q) \leadsto \forall x(P \lor Q) \qquad P \lor (\exists x Q) \leadsto \exists x(P \lor Q)$$
$$\neg \forall x P \leadsto \exists x(\neg P) \qquad \neg \exists x P \leadsto \forall x(\neg P)$$
$$(\forall x P) \rightarrow Q \leadsto \exists x(P \rightarrow Q) \qquad (\exists x P) \rightarrow Q \leadsto \forall x(P \rightarrow Q)$$
$$P \rightarrow (\forall x Q) \leadsto \forall x(P \rightarrow Q) \qquad P \rightarrow (\exists x Q) \leadsto \exists x(P \rightarrow Q)$$

$$P \leftrightarrow Q \leadsto (P \rightarrow Q) \land (Q \rightarrow P)$$

Alpha-renaming (change of bound variables)

The Prenex normal form algorithm assumes that all variables in a formula are distinct. This can be achieved by α-renaming:
$\forall x P(x) \rightsquigarrow \forall y P(y)$
$\exists x P(x) \rightsquigarrow \exists y P(y)$

Resolution for FOL

Suppose that we have a set $\mathcal{T}$ of sentences an want to show that they are not simultaneously first-order satisfiable.

1. Put each sentence in $\mathcal{T}$ into prenex form, say
$$\forall x_1 \exists y_1 \forall x_2 \exists y_2 \dots P(x_1, y_1, x_2, y_2, \dots)$$
2. Skolemize each of the resulting sentences, say
$$\forall x_1 \forall x_2 \dots P(x_1, f_1(x_1), x_2, f_2(x_1, x_2), \dots)$$
using different Skolem functions for different sentences.
3. Put each quantifier free matrix P into conjunctive normal form, say
$$P_1 \wedge P_2 \wedge \dots \wedge P_n$$
where each P_i is a disjunction of literals.
4. Distribute the universal quantifiers in each sentence across the conjunctions and drop the conjunction signs, ending with a set of sentences of the form
$$\forall x_1 \forall x_2 \dots P_i$$

5. Change the bound variables in each of the resulting sentences so that no variable appears in two of them.

6. Turn each of the resulting sentences into a set of literals by dropping the universal quantifiers and disjunction signs. In this way we end up with a set of resolution clauses.

7. Use resolution and unification to resolve this set of clauses

$$\frac{\{C_1, \ldots, C_m\},\ \{\neg D_1, \ldots, D_n\}}{\{C_2\theta, \ldots C_m\theta, D_2\theta, \ldots, D_n\theta\}}$$

   if $C_1\theta = D_1\theta$ (θ is a unifier of C_1 and D_1)

Example I

Is the following argument valid?

$$\begin{array}{|l} \forall x(P(x,b) \lor Q(x)) \\ \forall y(\neg P(f(y),b) \lor Q(y)) \\ \hline \forall y(Q(y) \lor Q(f(y)) \end{array}$$

Reformulated: is the following set unsatisfiable?

$$\forall x(P(x,b) \lor Q(x))$$
$$\forall y(\neg P(f(y),b) \lor Q(y))$$
$$\neg\forall y(Q(y) \lor Q(f(y))$$

Step 1: Prenex normal form

$$\forall x(P(x, b) \vee Q(x))$$
$$\forall y(\ \neg P(f(y), b) \vee Q(y))$$
$$\exists y \neg (Q(y) \vee Q(f(y))$$

Step 2: Skolemization

$$\forall x(P(x, b) \vee Q(x))$$
$$\forall y(\neg P(f(y), b) \vee Q(y))$$
$$\neg(Q(c) \vee Q(f(c)))$$

Since the existential quantifier was not preceeded by any universal quantifier, we need a 0-ary function symbol, that is, an individual constant c.

Step 3: Conjunctive normal form

$$\forall x(P(x,b) \vee Q(x))$$
$$\forall y(\neg P(f(y),b) \vee Q(y))$$
$$\neg Q(c) \wedge \neg Q(f(c))$$

Step 4: Drop conjunctions

$$\forall x(P(x,b) \vee Q(x))$$
$$\forall y(\neg P(f(y),b) \vee Q(y))$$
$$\neg Q(c)$$
$$\neg Q(f(c))$$

Step 5: change bound variables: nothing to do.

Step 6: Drop universal quantifiers and disjunctions, and step 7: do resolution

1. $\{P(x, b),\ Q(x)\}$
2. $\{\neg P(f(y), b),\ Q(y)\}$
3. $\{\neg Q(c)\}$
4. $\{\neg Q(f(c))\}$

5. $\{Q(y), Q(f(y))\}$ 1,2 with $f(y)$ for x
6. $\{Q(f(c))\}$ 3,5 with c for y
7. $\square$ 4,6

Example II

Is the following argument valid?

From

"Everyone admires someone who admires them unless they admire Quaid."

we can infer

"There are people who admire each other, at least one of whom admires Quaid."

The formalization

$$\begin{array}{l|l} & \forall x[\neg A(x,q) \rightarrow \exists y(A(x,y) \wedge A(y,x))] \\ \hline & \exists x \exists y[A(x,q) \wedge A(x,y) \wedge A(y,x)] \end{array}$$

Reformulated: is the following set unsatisfiable?

$$\forall x[\neg A(x,q) \rightarrow \exists y(A(x,y) \wedge A(y,x))]$$
$$\neg\exists x \exists y[A(x,q) \wedge A(x,y) \wedge A(y,x)]$$

Step 1: Prenex normal form

$$\forall x \exists y[\neg A(x,q) \to (A(x,y) \wedge A(y,x))]$$
$$\forall x \forall y \neg[A(x,q) \wedge A(x,y) \wedge A(y,x)]$$

Step 2: Skolemization

$$\forall x[\neg A(x,q) \to (A(x,f(x)) \wedge A(f(x),x))]$$
$$\forall x \forall y \neg[A(x,q) \wedge A(x,y) \wedge A(y,x)]$$

Step 3: Conjunctive normal form

$$\forall x[(A(x,q) \vee A(x,f(x))) \wedge (A(x,q) \vee A(f(x),x))]$$
$$\forall x \forall y[\neg A(x,q) \vee \neg A(x,y) \vee \neg A(y,x)]$$

Step 4: Drop conjunctions

$$\forall x(A(x,q) \vee A(x,f(x)))$$
$$\forall x(A(x,q) \vee A(f(x),x))$$
$$\forall x \forall y[\neg A(x,q) \vee \neg A(x,y) \vee \neg A(y,x)]$$

Step 5: change bound variables.

$$\forall x(A(x,q) \vee A(x,f(x)))$$
$$\forall y(A(y,q) \vee A(f(y),y))$$
$$\forall z \forall w[\neg A(z,q) \vee \neg A(z,w) \vee \neg A(w,z)]$$

Step 6: Drop universal quantifiers and disjunctions, and step 7: do resolution

1. $\{A(x,q), A(x,f(x))\}$
2. $\{A(y,q), A(f(y),y)\}$
3. $\{\neg A(z,q), \neg A(z,w), \neg A(w,z)\}$
4. ... [homework: fill in the rest]