Logik für Informatiker
Logic for computer scientists

Gödel's completeness theorem

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WiSe 2011/12

Completeness of propositional logic

Theorem (Completeness theorem). Let $\mathcal{T}$ be a set of propositional sentences, and S be a propositional sentence. If S is a tautological consequence of $\mathcal{T}$ ($\mathcal{T} \models S$), then $\mathcal{T} \vdash S$.

Recall: $\mathcal{T} \models S$ means that each valuation satisfying $\mathcal{T}$ also satisfies S.

$\mathcal{T} \vdash S$ means that S is derivable from $\mathcal{T}$ using the rules of Fitch.

Proof of propositional completeness

Call a set of sentences $\mathcal{T}$ *formally inconsistent*, if $\mathcal{T} \vdash \bot$.
Example: $\{A \vee B, \neg A, \neg B\}$.
Otherwise, $\mathcal{T}$ is called *formally consistent*.
Example: $\{A \vee B, A, \neg B\}$
Lemma 1 $\mathcal{T} \cup \{\neg S\} \vdash \bot$ iff $\mathcal{T} \vdash S$.
Reformulation of completeness: Every formally consistent set of sentences is tt-satisfiable.

Completeness of formally complete sets

A set $\mathcal{T}$ is *formally complete*, if for any sentence S in the same language as $\mathcal{T}$, either $\mathcal{T} \vdash S$ or $\mathcal{T} \vdash \neg S$.

*Lemma 2 (***Truth Lemma***)* For $\mathcal{T}$ formally complete and consistent,

1. $\mathcal{T} \vdash R \wedge S$ iff $\mathcal{T} \vdash R$ and $\mathcal{T} \vdash S$.
2. $\mathcal{T} \vdash R \vee S$ iff $\mathcal{T} \vdash R$ or $\mathcal{T} \vdash S$.
3. $\mathcal{T} \vdash \neg S$ iff not $\mathcal{T} \vdash S$.
4. $\mathcal{T} \vdash R \rightarrow S$ iff $\mathcal{T} \nvdash R$ or $\mathcal{T} \vdash S$.
5. $\mathcal{T} \vdash R \leftrightarrow S$ iff ($\mathcal{T} \vdash R$ iff $\mathcal{T} \vdash S$).

Finishing the proof

Lemma 3 Every formally complete and consistent set is tt-satisfiable.

Lemma 4 A set of sentence is formally complete iff for every atomic sentence A, either $\mathcal{T} \vdash A$ or $\mathcal{T} \vdash \neg A$.

Lemma 5 Every formally consistent set of sentences can be expanded to a formally consistent, formally complete set of sentences.

Gödel's completeness theorem

Theorem (Completeness theorem). Let $\mathcal{T}$ be a set of sentences of a first-order language L and S be a sentence of the same language. If S is a first-order consequence of $\mathcal{T}$ ($\mathcal{T} \models S$), then $\mathcal{T} \vdash S$.

Recall: $\mathcal{T} \models S$ means that each first-order structure satisfying $\mathcal{T}$ also satisfies S.

$\mathcal{T} \vdash S$ means that S is derivable from $\mathcal{T}$ using the rules of Fitch.

Henkin's proof of Gödel's theorem

Adding witnessing constants L is enriched to L_H by adding infinitely many constants, the *witnessing constants*.

The Henkin theory Isolate a theory $\mathcal{H}$ in the enriched language L_H, which contains *Henkin witnessing axioms*.

The Elimination theorem For any proof with sentences in L or from $\mathcal{H}$ as premises, and a sentence in L as conclusion, we can eliminate the premises from $\mathcal{H}$ from the proof.

The Henkin construction For every truth assignment h assinging TRUE to all formulas in $\mathcal{H}$, there is a first-order structure $\mathfrak{M}_h$ such that $\mathfrak{M}_h$ makes true all first-order sentence that h makes true.

How to obtain Gödel's theorem

Assume that $\mathcal{T} \models S$.

1. Hence, there is no model of $\mathcal{T} \cup \{\neg S\}$.
2. By the Henkin construction, there is no assignment h satisfying $\mathcal{T} \cup \mathcal{H} \cup \{\neg S\}$.
3. Hence, S is a tautological consequence of $\mathcal{T} \cup \mathcal{H}$.
4. By completeness of propositional logic, $\mathcal{T} \cup \mathcal{H} \vdash S$.
5. By the elimination theorem, $\mathcal{T} \vdash S$.

Adding witnessing constants

For each well-formed formula P of L with exactly one free variable, add a new constant c_P, P's *witnessing constant*.
E.g. for $Small(x) \wedge Cube(x)$, we introduce $c_{Small(x) \wedge Cube(x)}$.
This process has to be iterated:

$$L = L_0 \subseteq L_1 \subseteq L_2 \subseteq \ldots$$

E.g. for $Smaller(y, c_{Small(x) \wedge Cube(x)})$, we introduce $c_{Smaller(y, c_{Small(x) \wedge Cube(x)})}$.
Each witnessing constant appears in some L_n for the first time. n is called its *date of birth*.

The Henkin witnessing axioms

$$\exists x\ P(x) \rightarrow P(c_{P(x)})$$

intuitve idea:
if there is something that satisfies $P(x)$, then the object named by $c_{P(x)}$ provides an example

The Henkin theory

H1 $\exists x\ P(x) \to P(c_{P(x)})$ (witnessing axioms, $\hat{=}$ $\exists$**Elim**)

H2 $P(c) \to \exists x\ P(x)$ ($\hat{=}$ $\exists$**Intro**)

H3 $\neg\forall x\ P(x) \leftrightarrow \exists x\ \neg P(x)$ (reduces $\forall$ to $\exists$)

H4 $c = c$ ($\hat{=}$ $=$**Intro**)

H5 $(P(c) \wedge c = d) \to P(d)$ ($\hat{=}$ $=$**Elim**)

Independence lemma

If c_P and c_Q are two witnessing constants and the date of birth of c_P is less than or equal to that of c_Q, then c_Q does not appear in the witnessing axiom of c_P.

Elimination Theorem

For any proof with sentences in L or from $\mathcal{H}$ as premises, and a sentence in L as conclusion, we can eliminate the premises from $\mathcal{H}$ from the proof.

Deduction Theorem If $\mathcal{T} \cup \{P\} \vdash Q$, then $\mathcal{T} \vdash P \to Q$.

Lemma 1 If $\mathcal{T} \vdash P \to Q$ and $\mathcal{T} \vdash \neg P \to Q$, then $\mathcal{T} \vdash Q$.

Lemma 2 If $\mathcal{T} \vdash (P \to Q) \to R$,
then $\mathcal{T} \vdash \neg P \to R$, and $\mathcal{T} \vdash Q \to R$.

Lemma 3 Let $\mathcal{T}$ be a set of sentences of some first-order language L, and Q be a sentence. Let $P(x)$ be a formula of L with one free variable and which does not contain c. If $\mathcal{T} \vdash P(c) \to Q$ and c does not appear in $\mathcal{T}$ or Q, then $\mathcal{T} \vdash \exists x\ P(x) \to Q$.

Lemma 4 Let $\mathcal{T}$ be a set of sentences of some first-order language L, and Q be a sentence of L. Let $P(x)$ be a formula of L with one free variable and which does not contain c. If $\mathcal{T} \cup \{\exists x\ P(x) \rightarrow P(c)\} \vdash Q$ and c does not appear in $\mathcal{T}$ or Q, then $\mathcal{T} \vdash Q$.

Lemma 5 Let $\mathcal{T}$ be a set of first-order sentences, let $P(x)$ be a formula with one free variable, and let c and d be constant symbols. The following are provable in $\mathcal{F}$:

$$P(c) \to \exists x\ P(x)$$
$$\neg\forall x\ P(x) \leftrightarrow \exists x\ \neg P(x)$$
$$c = c$$
$$(P(c) \wedge c = d) \to P(d)$$

For any proof with sentences in L or from $\mathcal{H}$ as premises, and a sentence in L as conclusion, we can eliminate the premises from $\mathcal{H}$ from the proof.

For every truth assignment h assinging $\textsc{true}$ to all formulas in $\mathcal{H}$, there is a first-order structure $\mathfrak{M}_h$ such that $\mathfrak{M}_h$ makes true all first-order sentence that h makes true.
Given h, construct $\mathfrak{M}$ as follows:

- $D^{\mathfrak{M}}$: set of constant symbols in L_H
- $\mathfrak{M}(c)$ is just c
- $\mathfrak{M}(R)$ is $\{\langle c_1, \ldots c_n\rangle \mid h(R(c_1, \ldots c_n)) = \textsc{true}\}$

$$c \equiv d \text{ if and only if } h(c = d) \; = \; \textsc{true}$$

$$\mathfrak{M}_h = \mathfrak{M}/ \equiv$$

The truth lemma

For any sentence S of L_H, $\mathfrak{M}_h \models S$ iff $h(S) = \mathit{true}$.

Consequences of the Completeness Theorem

Theorem (Compactness theorem) Let $\mathcal{T}$ be a set of first-order sentences in a first-order language L. If every finite subset of $\mathcal{T}$ is first order-satisfiable, then $\mathcal{T}$ itself is first-order satisfiable.

Non-standard models of Peano arithmetic

Let L be the language of Peano arithmetic. Then there exists a first-order structure $\mathfrak{M}$ such that

1. $\mathfrak{M}$ contains all the natural numbers in its domain,
2. $\mathfrak{M}$ also contains elements greater than all the natural numbers, but
3. $\mathfrak{M}$ makes true exactly the same sentences of L as are true about the natural numbers.

Exercises

- 17.4-17.16, 19.1-19.23
- deadline: 5.1.2010