Logik für Informatiker
Logic for computer scientists

Description Logics and First-Order Logic

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Description Logics: Signatures

A *DL-signature* $\Sigma = (\mathbf{C}, \mathbf{R}, \mathbf{I})$ consists of

- a set $\mathbf{C}$ of concept names,
- a set $\mathbf{R}$ of role names,
- a set $\mathbf{I}$ of individual names,

Description Logics: Concepts

For a signature $\Sigma = (\mathbf{C}, \mathbf{R}, \mathbf{I})$ the set of $\mathcal{ALC}$-*concepts* over Σ is defined by the following grammar:

		(Hets) Manchester syntax
$C ::=$	A for $A \in \mathbf{C}$	a concept name
	$\mid \top$	`Thing`
	$\mid \bot$	`Nothing`
	$\mid \neg C$	`not C`
	$\mid C \sqcap C$	`C and C`
	$\mid C \sqcup C$	`C or C`
	$\mid \exists R.C$ for $R \in \mathbf{R}$	`R some C`
	$\mid \forall R.C$ for $R \in \mathbf{R}$	`R only C`

$\mathcal{ALC}$ stands for "attributive language with complement"

Description Logic: Sentences

The set of $\mathcal{ALC}$-Sentences over Σ ($\mathsf{Sen}(\Sigma)$) is defined as

- $C \sqsubseteq D$, where C and D are $\mathcal{ALC}$-concepts over Σ.

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  Class: C SubclassOf: D
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- $a : C$, where $a \in \mathbf{I}$ and C is a $\mathcal{ALC}$-concept over Σ.

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  Individual: a Types: C
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- $R(a_1, a_2)$, where $R \in \mathbf{R}$ and $a_1, a_2 \in \mathbf{I}$.

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  Individual: a1 Facts: R a2
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Description Logic: Models

Given $\Sigma = (\mathbf{C}, \mathbf{R}, \mathbf{I})$, a Σ-*model* is of form $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where

- $\Delta^{\mathcal{I}}$ is a non-empty set
- $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ for each $A \in \mathbf{C}$
- $R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ for each $R \in \mathbf{R}$
- $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$ for each $a \in \mathbf{I}$

We can extend $\cdot^{\mathcal{I}}$ to all concepts as follows:

$$
\begin{array}{lcl}
\top^{\mathcal{I}} & = & \Delta^{\mathcal{I}} \\
\bot^{\mathcal{I}} & = & \emptyset \\
(\neg C)^{\mathcal{I}} & = & \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}} \\
(C \sqcap D)^{\mathcal{I}} & = & C^{\mathcal{I}} \cap D^{\mathcal{I}} \\
(C \sqcup D)^{\mathcal{I}} & = & C^{\mathcal{I}} \cup D^{\mathcal{I}} \\
(\exists R.C)^{\mathcal{I}} & = & \{x \in \Delta^{\mathcal{I}} | \exists y \in \Delta^{\mathcal{I}}.(x,y) \in R^{\mathcal{I}}, y \in C^{\mathcal{I}}\} \\
(\forall R.C)^{\mathcal{I}} & = & \{x \in \Delta^{\mathcal{I}} | \forall y \in \Delta^{\mathcal{I}}.(x,y) \in R^{\mathcal{I}} \Rightarrow y \in C^{\mathcal{I}}\}
\end{array}
$$

Description Logic: Satisfaction of sentences in a model

$$\begin{array}{lll} \mathcal{I} \models C \sqsubseteq D & \text{iff} & C^{\mathcal{I}} \subseteq D^{\mathcal{I}}. \\ \mathcal{I} \models a : C & \text{iff} & a^{\mathcal{I}} \in C^{\mathcal{I}}. \\ \mathcal{I} \models R(a_1, a_2) & \text{iff} & (a_1^{\mathcal{I}}, a_2^{\mathcal{I}}) \in R^{\mathcal{I}}. \end{array}$$

Translating ALC to FOL: Signatures

$\phi((\mathbf{C}, \mathbf{R}, \mathbf{I})) = (F, P)$ with

- $S = \{\texttt{Thing}\}$ (one sort = single-sorted)
- $F = \{a : \texttt{Thing} | a \in \mathcal{I}\}$ (constants)
- $P = \{A : \texttt{Thing} | A \in \mathbf{C}\} \cup \{R : \texttt{Thing} \times \texttt{Thing} | R \in \mathbf{R}\}$ (predicate symbols)

Translating ALC to FOL: Concepts

- $\alpha_x(A) = A(x : \texttt{Thing})$
- $\alpha_x(\neg C) = \neg\alpha_x(C)$
- $\alpha_x(C \sqcap D) = \alpha_x(C) \wedge \alpha_x(D)$
- $\alpha_x(C \sqcup D) = \alpha_x(C) \vee \alpha_x(D)$
- $\alpha_x(\exists R.C) = \exists y : \texttt{Thing}.(R(x, y) \wedge \alpha_y(C))$
- $\alpha_x(\forall R.C) = \forall y : \texttt{Thing}.(R(x, y) \to \alpha_y(C))$

Sentence translation

- $\alpha_\Sigma(C \sqsubseteq D) = \forall x : \texttt{Thing}. (\alpha_x(C) \rightarrow \alpha_x(D))$
- $\alpha_\Sigma(a : C) = \alpha_x(C)[a/x]$[1]
- $\alpha_\Sigma(R(a, b)) = R(a, b)$

Model translation (FOL-models are translated to $\mathcal{ALC}$-models!)

- For $M' \in \mathrm{Mod}^{FOL}(\phi\Sigma)$ define $\beta_\Sigma(M') := (\Delta, \cdot^I)$ with $\Delta = M'_{\texttt{Thing}}$ and $A^I = M'_A, a^I = M'_a, R^I = M'_R$.

[1] Replace x by a.

Translating ALC to FOL: Correctness

Theorem 1: $C^{\mathcal{I}} = \left\{m \in M'_{\mathtt{Thing}} | M' + \{x \mapsto m\} \models \alpha_x(C)\right\}$
Proof: By Induction over the structure of C.

- $A^{\mathcal{I}} = M'_A = \left\{m \in M'_{\mathtt{Thing}} | M' + \{x \mapsto m\} \models A(x)\right\}$
- $(\neg C)^{\mathcal{I}} = \Delta \setminus C^{\mathcal{I}}$
  $=^{I.H.} \Delta \setminus \{m \in M'_{\mathtt{Thing}} | M' + \{x \mapsto m\} \models \alpha_x(C)\}$
  $= \{m \in M'_{\mathtt{Thing}} | M' + \{x \mapsto m\} \models \neg\alpha_x(C)\}$

Theorem 2: (Satisfaction condition)

$$\beta(M) \models \varphi \text{ iff } M \models \alpha(\varphi)$$

Theorem 3: (Logical consequence coincides)

$$\Gamma \models \varphi \text{ (in } \mathcal{ALC}\text{) iff } \alpha(\Gamma) \models \alpha(\varphi) \text{ (in FOL)}$$