On the Number of Local Minima to the Point Feature Based SLAM Problem

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***Abstract*— Map joining is an efficient strategy for solving feature based SLAM problems. This paper demonstrates that joining of two 2D local maps, formulated as a nonlinear least squares problem has at most two local minima, when the associated uncertainties can be described using spherical covariance matrices. Necessary and sufficient condition for the existence of two minima is derived and it is shown that more than one minimum exists only when the quality of the local maps used for map joining is extremely poor. The analysis explains to some extent why a number of optimization based SLAM algorithms proposed in the recent literature that rely on local search strategies are successful in converging to the globally optimal solution from poor initial conditions, particularly when covariance matrices are spherical. It also demonstrates that the map joining problem has special properties that may be exploited to reliably obtain globally optimal solutions to the SLAM problem.**

I. INTRODUCTION

When SLAM problem is formulated as a nonlinear least squares problem, the dimension of the problem is very high because all feature positions and robot poses are present as variables. It can be expected that such high dimensional nonlinear optimization problem have a huge number of local minima and in general local search strategies are unlikely to be successful unless a very good initial guess is available. However, recent research shows that some methods based on local search can sometimes provide surprisingly good solutions to SLAM without being trapped into a local minimum.

For pose graph SLAM problems, the results presented in [1] surprised many SLAM researchers where stochastic gradient descent (SGD) is used to solve the optimization problem by dealing with each constraint individually and the algorithm can converge to the correct solution with poor initial values. Recently, a more efficient SLAM algorithm, tree-based network optimizer (TORO), was proposed in [2] where a tree structure is used on top of the SGD approach. Surprisingly, very large scale problems can be solved efficiently without the need of good initial values, especially when the covariance matrices of the relative poses are close to spherical [2].

Our initial investigation [3] into point feature based SLAM, formulated as a nonlinear least squares problem has also highlighted some interesting behavior. A simple Gauss-Newton algorithm can sometimes converge to the global optimal solution from random initial values, when used with the popular Victoria Park dataset [4]. This, however, occurs only when the covariances of observations and odometries are set to be identity matrices, although the resulting solution is very close to the true solution obtained using the correct sensor and motion models. A number of numerical experiments demonstrated that the chance of getting trapped in a local minimum from a random initial guess is only about 20%. The DLR-Spatial Cognition dataset [5] also exhibits similar behavior, when started from a zero initial guess.

These results indicate that the number of local minima present in the nonlinear least squares formulation of the SLAM problem is likely to be small if the covariance matrices are spherical. This observation is the main motivation for the work presented in this paper. In particular, we examine the problem of joining two maps as well as the special case where information gathered at two robot poses are combined to build a local map. We argue that any feature based SLAM problem can be decomposed into a sequence containing these two steps. It is theoretically proven that the nonlinear least squares optimization problems associated with both these scenarios have at most two local minima. It is experimentally demonstrated that (a) the two local minima occur only when the odometry and observation information are extremely inconsistent with each other, and (b) the solution to the approximate map joining problem using spherical covariance matrices is practically very close to the true solution to the map joining problem using the actual covariance matrices.

The paper is organized as follows. Section II formulates the least squares SLAM and map joining problems. Section III provides a lemma which underpins the proofs of the main results. Section IV analyzes the one-step SLAM problem while Section V examines the map joining problem. Experimental results to demonstrate the outcomes of the analysis is presented in Section VI. Section VII concludes the paper. Appendix A presents the proof of the lemma and Appendix B presents the proof of the main theorem of the paper. [1]

II. LEAST SQUARES SLAM AND MAP JOINING FORMULATION

A. Least Squares SLAM Formulation

Suppose a number of 2D point features $f_1, f_2, \cdots, f_N$ in the environments are observed from a sequence of 2D

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[1]The MATLAB source code for testing the results are available at http://services.eng.uts.edu.au/˜sdhuang/research.htm.

robot poses $r_0, r_1, r_2, \cdots, r_p$. The first robot pose (pose r_0) is chosen as the origin of the global coordinate frame.

We use $X_{f_j} = (x_{f_j}, y_{f_j})^T$ to denote the x, y position of feature f_j. $X_{r_i} = (x_{r_i}, y_{r_i})^T$ denotes the x, y position of robot pose r_i while ϕ_{r_i} denotes the orientation of pose r_i. R_{r_i} is the rotation matrix of pose r_i given by

$$R_{r_i} = R(\phi_{r_i}) = \begin{bmatrix} \cos\phi_{r_i} & -\sin\phi_{r_i} \\ \sin\phi_{r_i} & \cos\phi_{r_i} \end{bmatrix}. \tag{1}$$

The least squares SLAM formulation [6] is to use the odometry and the range and bearing observation information to estimate the state vector containing all the robot poses and all the feature positions

$$X = (X_{f_1}^T, \cdots, X_{f_N}^T, X_{r_1}^T, \phi_{r_1}, \cdots, X_{r_p}^T, \phi_{r_p})^T \tag{2}$$

and the SLAM problem is to minimize [6]

$$\begin{aligned} F(X) = &\sum_{i=1}^{p} (O_i^{i-1} - H^{Oi}(X))^T P_{Oi}^{-1} (O_i^{i-1} - H^{Oi}(X)) \\ &+ \sum_{i,j} (Z_j^i - H^{Z_j^i}(X))^T P_{Z_j^i}^{-1} (Z_j^i - H^{Z_j^i}(X)) \end{aligned} \tag{3}$$

where O_i^{i-1} ($1 \le i \le p$) are odometries, Z_j^i are observations, and P_{Oi} and $P_{Z_j^i}$ are the corresponding covariance matrices.

In the above least squares SLAM formulation, $H^{Z_j^i}(X)$ and $H^{Oi}(X)$ are the corresponding functions relating Z_j^i and O_i^{i-1} to the state X. The odometry is a function of two poses $(X_{r_{i-1}}^T, \phi_{r_{i-1}})^T$ and $(X_{r_i}^T, \phi_{r_i})^T$ and is given by

$$H^{Oi}(X) = \begin{bmatrix} R_{r_{i-1}}^T (X_{r_i} - X_{r_{i-1}}) \\ \phi_{r_i} - \phi_{r_{i-1}} \end{bmatrix}. \tag{4}$$

The range and bearing observation is a function of one pose $(X_{r_i}^T, \phi_{r_i})^T$ and one feature position X_{f_j} and is given by

$$H^{Z_j^i}(X) = R_{r_i}^T (X_{f_j} - X_{r_i}). \tag{5}$$

In particular, since $\phi_{r_0} = 0$ and $X_{r_0} = (0,0)^T$, the odometry function from robot r_0 to r_1 is given by

$$H^{O1}(X) = \begin{bmatrix} X_{r_1} \\ \phi_{r_1} \end{bmatrix} \tag{6}$$

and the observation function from robot r_0 to f_j is given by

$$H^{Z_j^0}(X) = X_{f_j}. \tag{7}$$

"One-step SLAM" problem is the special case where the number of robot poses is two, i.e. $p = 1$.

B. Map Joining

Joining of multiple local maps obtained by solving the above least squares problem can also be formulated as an optimization problem [7][8]. Suppose that there are a sequence of k local maps and the end robot pose of local map j is the start robot pose of local map $j+1$. The state vector of the map joining problem considered in [7] contains all the feature positions and robot end poses of each local map:

$$X_{MJ} = (X_{r_{1e}}^T, \phi_{r_{1e}}, \cdots, X_{r_{ke}}^T, \phi_{r_{ke}}, X_{f_1}^T, \cdots, X_{f_N}^T)^T \tag{8}$$

where r_{je} is the robot end pose of local map j ($1 \le j \le k$).

Suppose local map j is defined by $(\hat{X}_j^L, P_j^L)$ where $\hat{X}_j^L$ is the state estimate and P_j^L is the associated covariance matrix. Also assume the features present in the local map j are $f_{j1}, \cdots, f_{jn_j}$. The local map state estimate $\hat{X}_j^L$ can be regarded as an observation of the true relative positions from the robot start pose $r_{(j-1)e}$ to the features $f_{j1}, \cdots, f_{jn_j}$ and the robot end pose r_{je}. That is,

$$\hat{X}_j^L = H_j(X_{MJ}) + w_j \tag{9}$$

where

$$H_j(X_{MJ}) = \begin{pmatrix} R_{r_{(j-1)e}}^T (X_{r_{je}} - X_{r_{(j-1)e}}) \\ \phi_{r_{je}} - \phi_{r_{(j-1)e}} \\ R_{r_{(j-1)e}}^T (X_{f_{j1}} - X_{r_{(j-1)e}}) \\ \vdots \\ R_{r_{(j-1)e}}^T (X_{f_{jn_j}} - X_{r_{(j-1)e}}) \end{pmatrix}$$

and w_j is the zero-mean Gaussian "observation noise" whose covariance matrix is P_j^L (when $j = 1$, $X_{r_{(j-1)e}} = [0,0]^T, \phi_{r_{(j-1)e}} = 0$).

So the problem of joining local maps 1 to k is to estimate the global state X_{MJ} using all the local map information (9) for $j = 1, \cdots, k$. This problem can be formulated as a least squares problem. That is, finding X_{MJ} such that

$$\sum_{j=1}^{k} (\hat{X}_j^L - H_j(X_{MJ}))^T (P_j^L)^{-1} (\hat{X}_j^L - H_j(X_{MJ})) \tag{10}$$

is minimized.

Most of the map joining algorithms such as sequential map joining [7] and divide-and-conquer strategy [9] combine two maps at a time. Furthermore, it can be seen that "one-step SLAM" problem defined in (3) with $p = 1$ is also a special case of joining two maps. Therefore, any feature based SLAM problem can be decomposed to a sequence of problems of joining two local maps.

III. A Useful Lemma

It will be shown in the following sections that the problem of joining two maps and its special case, one-step SLAM problem can both be reduced to a nonlinear equation constrained by a nonlinear inequality with one variable, when associated uncertainties can be described using spherical covariance matrices. The following lemma gives a special property of such problems.

Lemma 1: Assume that $a > 0$, $C_\phi \in [-\pi, \pi)$ are two constants. Consider the following two conditions:

$$f(\phi) = a\sin(\phi + C_\phi) + \phi = 0 \tag{11}$$

$$g(\phi) = a\cos(\phi + C_\phi) + 1 > 0 \tag{12}$$

There are at least one and at most two $\phi \in [-\pi, \pi)$ satisfying (11)-(12) simultaneously. Moreover, there are two solutions if and only if

$$a \geq 1 \tag{13}$$
$$-a \sin C_\phi - \pi \leq 0 \tag{14}$$
$$\sqrt{a^2-1} + \phi_1 \geq 0 \tag{15}$$
$$-\sqrt{a^2-1} + \phi_2 \leq 0 \tag{16}$$
$$-a \sin C_\phi + \pi \geq 0 \tag{17}$$
$$\phi_1 - \phi_2 \leq 0 \tag{18}$$

hold simultaneously. Here

$$\begin{aligned} \phi_1 &= wrap(\arccos(-\tfrac{1}{a}) - C_\phi) \\ \phi_2 &= wrap(-\arccos(-\tfrac{1}{a}) - C_\phi) \end{aligned} \tag{19}$$

where $wrap(\theta)$ is a function which wraps θ into $[-\pi, \pi)$.

Proof: See Appendix A. ■

Remark 1: Fig. 1 illustrates the conditions (13)-(18). The possible pair of a, C_ϕ when there are two solutions to satisfy conditions (11)-(12) simultaneously is shown in the shaded area. For example, it can be seen that if $a < 1$, there is only one solution. If $|C_\phi| < \arcsin(\frac{\pi}{\sqrt{\pi^2+1}}) = 1.2626$, there is also only one solution. Fig. 2 shows the functions $f(\phi)$ and $g(\phi)$ when $a = 3, C_\phi = 2$ and it is clear that there are two solutions to (11)-(12).

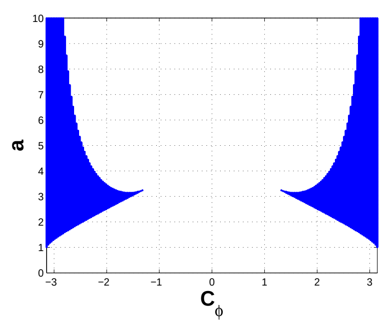


Fig. 1. Possible situations of having two solutions by satisfying conditions (13)-(18). The x-axis is C_ϕ, and y-axis is a. In the shaded area, there are two solutions to (11)-(12), in the other area, there is only one solution.

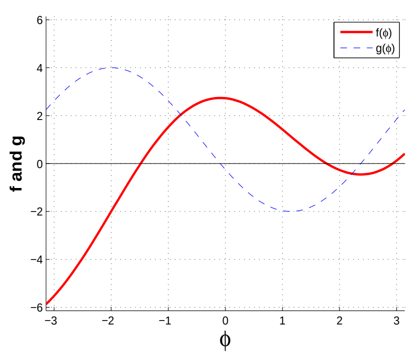


Fig. 2. An example of two solutions to (11)-(12): $a = 3, C_\phi = 2$.

IV. One-step SLAM

This section analyzes the number of local minima present in the one-step SLAM problem.

Suppose there are n features which are all observed by both pose r_0 and pose r_1, as shown in Fig. 3. Denote $X = (x_{f_1}, y_{f_1}, \ldots, x_{f_n}, y_{f_n}, x_r, y_r, \phi)^T$ where (x_r, y_r, ϕ) is robot pose r_1, and consider the case when the covariance matrices $P_{O_1}, P_{Z_i^0}, P_{Z_i^1}, i = 1, \cdots, n$ are all identity matrices.

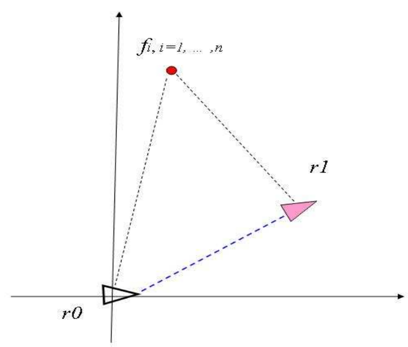


Fig. 3. One-step SLAM problem with n feature

Suppose the odometry between pose r_0 and pose r_1 is $O_1^0 = (z_{x_r}, z_{y_r}, z_\phi)^T$, the observation of feature f_i from r_0 is $Z_i^0 = (z_{x_{f_i}}, z_{y_{f_i}})^T$ and the observation of f_i from r_1 is $Z_i^1 = (z_{x^r_{f_i}}, z_{y^r_{f_i}})^T$, then the SLAM problem is to minimize

$$\begin{aligned} F(X) &= (O_1^0 - H^{O_1}(X))^T P_{O_1}^{-1} (O_1^0 - H^{O_1}(X)) \\ &+ \sum_{i=1}^{n} (Z_i^0 - H^{Z_i^0}(X))^T P_{Z_i^0}^{-1} (Z_i^0 - H^{Z_i^0}(X)) \\ &+ \sum_{i=1}^{n} (Z_i^1 - H^{Z_i^1}(X))^T P_{Z_i^1}^{-1} (Z_i^1 - H^{Z_i^1}(X)) \\ &= (z_{x_r} - x_r)^2 + (z_{y_r} - y_r)^2 + (z_\phi - \phi)^2 \\ &+ \sum_{i=1}^{n} [(z_{x_{f_i}} - x_{f_i})^2 + (z_{y_{f_i}} - y_{f_i})^2] \\ &+ \sum_{i=1}^{n} (Z_i^1 - R(\phi)^T \delta_i)^T (Z_i^1 - R(\phi)^T \delta_i) \end{aligned} \tag{20}$$

where

$$\delta_i = X_{f_i} - X_r = \begin{bmatrix} x_{f_i} - x_r \\ y_{f_i} - y_r \end{bmatrix} \tag{21}$$

Note that

$$\begin{aligned} &(Z_i^1 - R(\phi)^T \delta_i)^T (Z_i^1 - R(\phi)^T \delta_i) \\ =&|Z_i^1 - R(\phi)^T \delta_i|^2 \\ =&|R(\phi) Z_i^1 - \delta_i|^2 \end{aligned} \tag{22}$$

Thus the objective function (20) can be converted into

$$
\begin{aligned}
F(X) = &(z_{x_r} - x_r)^2 + (z_{y_r} - y_r)^2 + (z_\phi - \phi)^2 \\
&+ \sum_{i=1}^{n} [(z_{x_{f_i}} - x_{f_i})^2 + (z_{y_{f_i}} - y_{f_i})^2] \\
&+ \sum_{i=1}^{n} [(A_i - (x_{f_i} - x_r))^2 + (B_i - (y_{f_i} - y_r))^2]
\end{aligned} \tag{23}
$$

where

$$
A_i = z_{x^r_{f_i}} c_\phi - z_{y^r_{f_i}} s_\phi, \;\; B_i = z_{x^r_{f_i}} s_\phi + z_{y^r_{f_i}} c_\phi. \tag{24}
$$

Here c_ϕ, s_ϕ denote $\cos\phi$ and $\sin\phi$, respectively. Note that A_i and B_i satisfy the following equations

$$
\frac{dA_i}{d\phi} = -B_i, \;\; \frac{dB_i}{d\phi} = A_i, \;\; A_i^2 + B_i^2 = z^2_{x^r_{f_i}} + z^2_{y^r_{f_i}}. \tag{25}
$$

The number of local minima of objective function (23) is given by Theorem 1.

Theorem 1: The one-step SLAM problem with n features has at least one local minimum and at most two local minima. Moreover, there are two local minima if and only if conditions (13)-(18) hold with

$$
a = \sqrt{p^2 + (d+q)^2}, \;\; C_\phi = atan2(p, d+q) \tag{26}
$$

where $atan2(y, x)$ denotes the arc tangent of y, x and

$$
\begin{aligned}
d = &\frac{1}{2} \sum_{1 \le i \le n} [(z_{x_{f_i}} - z_{x_r})^2 + (z_{y_{f_i}} - z_{y_r})^2] \\
&- \frac{1}{2(n+2)} \sum_{1 \le i,j \le n} [(z_{x_{f_j}} - z_{x_r})(z_{x_{f_i}} - z_{x_r}) \\
&+ (z_{y_{f_j}} - z_{y_r})(z_{y_{f_i}} - z_{y_r})]
\end{aligned} \tag{27}
$$

$$
\begin{aligned}
p &= \delta_a c_{z_\phi} + \delta_b s_{z_\phi} \\
q &= -\delta_a s_{z_\phi} + \delta_b c_{z_\phi}
\end{aligned} \tag{28}
$$

with

$$
\begin{aligned}
\delta_a &= \sum_{j=1}^{n} \left(\begin{bmatrix} \Delta z_{x^r_{f_j}} \\ \Delta z_{y^r_{f_j}} \end{bmatrix}^T \begin{bmatrix} \frac{1}{n+2} z_{y_r} - \frac{1}{2} z_{y_{f_j}} + \frac{1}{2(n+2)} \sum_{i=1}^{n} z_{y_{f_i}} \\ \frac{-1}{n+2} z_{x_r} + \frac{1}{2} z_{x_{f_j}} - \frac{1}{2(n+2)} \sum_{i=1}^{n} z_{x_{f_i}} \end{bmatrix} \right) \\
\delta_b &= \sum_{j=1}^{n} \left(\begin{bmatrix} \Delta z_{x^r_{f_j}} \\ \Delta z_{y^r_{f_j}} \end{bmatrix}^T \begin{bmatrix} \frac{-1}{n+2} z_{x_r} + \frac{1}{2} z_{x_{f_j}} - \frac{1}{2(n+2)} \sum_{i=1}^{n} z_{x_{f_i}} \\ \frac{-1}{n+2} z_{y_r} + \frac{1}{2} z_{y_{f_j}} - \frac{1}{2(n+2)} \sum_{i=1}^{n} z_{y_{f_i}} \end{bmatrix} \right)
\end{aligned} \tag{29}
$$

and

$$
\begin{aligned}
\Delta z_{x^r_{f_i}} &= z_{x^r_{f_i}} - [(z_{x_{f_i}} - z_{x_r}) c_{z_\phi} + (z_{y_{f_i}} - z_{y_r}) s_{z_\phi}] \\
\Delta z_{y^r_{f_i}} &= z_{y^r_{f_i}} - [-(z_{x_{f_i}} - z_{x_r}) s_{z_\phi} + (z_{y_{f_i}} - z_{y_r}) c_{z_\phi}]
\end{aligned} \tag{30}
$$

for $i = 1, \cdots, n$. Here c_{z_ϕ} and s_{z_ϕ} denote $\cos(z_\phi)$ and $\sin(z_\phi)$, respectively.

Proof: See Appendix B. ■

Remark 2: In Theorem 1, the $\Delta z_{x^r_{f_i}}, \Delta z_{y^r_{f_i}}$ defined in (30) are the differences between the observation from pose r_1 and the observation from pose r_0 combined with the odometry. Thus δ_a, δ_b, p, q as well as C_ϕ represent the level of inconsistency between the observation data from pose r_1 and the data obtained through observation from pose r_0 combined with the odometry. In the ideal case when the data are completely consistent (e.g. when all the sensors are perfect), we have $\Delta z_{x^r_{f_i}} = 0, \Delta z_{y^r_{f_i}} = 0, \; i = 1, \cdots, n$. Then $p = 0, q = 0$ and thus $C_\phi = 0$. In this case, it is evident that (18) does not hold anymore since $\frac{\pi}{2} < \arccos(-\frac{1}{a}) < \pi$, thus the problem has only one minimum (this can also be seen from Fig. 1 when $C_\phi = 0$).

Numerical Illustration. Consider the special case when only one feature f is observed from the two poses. Assume the odometry is given by $(z_{x_r}, z_{y_r}, z_\phi)^T = (2, 2, 0.5)^T$ and the observation from pose r_0 to f is $(z_{x_f}, z_{y_f})^T = (0, 3)^T$. We consider different observation from r_1 to f, $(z_x, z_y)^T$, by varying $\Delta z_x, \Delta z_y$. From (30), we have $z_x = -1.2757 + \Delta z_x$, and $z_y = 1.8364 + \Delta z_y$ because

$$
\begin{aligned}
(z_{x_f} - z_{x_r}) c_{z_\phi} + (z_{y_f} - z_{y_r}) s_{z_\phi} &= -1.2757, \\
-(z_{x_f} - z_{x_r}) s_{z_\phi} + (z_{y_f} - z_{y_r}) c_{z_\phi} &= 1.8364.
\end{aligned}
$$

Given any pair of $(\Delta z_x, \Delta z_y)$, the numerical values for $\delta_a, \delta_b, d, p, q, a, C_\phi$ can all be computed, and conditions (13)-(18) can be evaluated. Fig. 4 shows the number of minima that exist for different $(\Delta z_x, \Delta z_y)$. The shaded area corresponds to the case where there are two local minima, while the remaining space corresponds to the conditions where there is only one minimum.

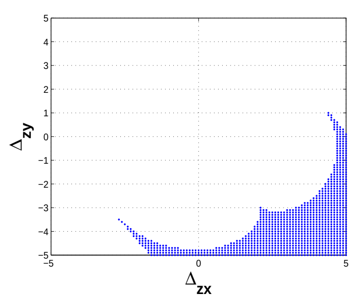


Fig. 4. Number of local minima to the one-step one-feature SLAM problem as a function of $\Delta z_x, \Delta z_y$. When $|\Delta z_x| \le 3, |\Delta z_y| \le 3$, there is only one local minimum. Normally, one cannot expect 3m measurement error with measurement values within 2m. So it is very unlikely to have two local minima unless the data association is wrong.

V. Joining Two Local Maps

This section demonstrates that the least squares optimization problem of joining two local maps, also has at most two local minima.

Consider the two local maps shown in Fig. 5 where r_0 is the start pose of local map 1 which is the origin of the global map, r_1 is the end pose in local map 1 as well as the start pose of local map 2, r_2 is the end pose in local map 2. Assume $(z_{x_{r_1}}, z_{y_{r_1}}, z_{\phi_{r_1}})^T$ is the estimate of pose r_1 in local map 1, $(z_{x^{r_1}_{r_2}}, z_{y^{r_1}_{r_2}}, z_{\phi^{r_1}_{r_2}})^T$ is the estimate of pose r_2 in local map 2. Suppose n is the number of features that appear

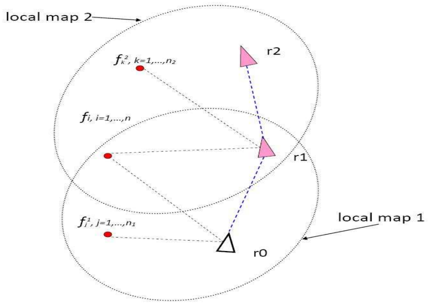


Fig. 5. Joining of two local maps

in both map 1 and map 2, and $z_{x_{f_i}}, z_{y_{f_i}}, z_{x^{r_1}_{f_i}}, z_{y^{r_1}_{f_i}}, i = 1, \cdots, n$ are the estimated positions of these features in local map 1 and local map 2. Suppose n_1 is the number of features that appear only in map 1, and $z_{x_{f^1_j}}, z_{y_{f^1_j}}, j = 1, \cdots, n_1$ are the estimated positions of these features in local map 1. Suppose n_2 is the number of features that appear only in map 2 and $z_{x^{r_1}_{f^2_k}}, z_{y^{r_1}_{f^2_k}}, k = 1, \cdots, n_2$ are the estimated positions of these features in local map 2.

Consider the case when the covariance matrices of the local maps are both identity, similar to (23), the map joining problem becomes minimizing

$$
\begin{aligned}
&F(X_{MJ}) = (z_{x_{r_1}} - x_{r_1})^2 + (z_{y_{r_1}} - y_{r_1})^2 + (z_{\phi_{r_1}} - \phi_{r_1})^2 \\
&+ (A_r - (x_{r_2} - x_{r_1}))^2 + (B_r - (y_{r_2} - y_{r_1}))^2 + (z_{\phi^{r_1}_{r_2}} \\
&- (\phi_{r_2} - \phi_{r_1}))^2 + \sum_{j=1}^{n_1} [(z_{x_{f^1_j}} - x_{f^1_j})^2 + (z_{y_{f^1_j}} - y_{f^1_j})^2] \\
&+ \sum_{i=1}^{n} [(z_{x_{f_i}} - x_{f_i})^2 + (z_{y_{f_i}} - y_{f_i})^2] \\
&+ \sum_{i=1}^{n} [(A_i - (x_{f_i} - x_{r_1}))^2 + (B_i - (y_{f_i} - y_{r_1}))^2] \\
&+ \sum_{k=1}^{n_2} [(C_k - (x_{f^2_k} - x_{r_1}))^2 + (D_k - (y_{f^2_k} - y_{r_1}))^2] \quad (31)
\end{aligned}
$$

where state X_{MJ} contains $(x_{r_1}, y_{r_1}, \phi_{r_1})^T$, $(x_{r_2}, y_{r_2}, \phi_{r_2})^T$, and all the feature positions, and

$$
\begin{aligned}
A_r &= c_{\phi_{r_1}} z_{x^{r_1}_{r_2}} - s_{\phi_{r_1}} z_{y^{r_1}_{r_2}}, B_r = s_{\phi_{r_1}} z_{x^{r_1}_{r_2}} + c_{\phi_{r_1}} z_{y^{r_1}_{r_2}} \\
A_i &= c_{\phi_{r_1}} z_{x^{r_1}_{f_i}} - s_{\phi_{r_1}} z_{y^{r_1}_{f_i}}, B_i = s_{\phi_{r_1}} z_{x^{r_1}_{f_i}} + c_{\phi_{r_1}} z_{y^{r_1}_{f_i}} \\
C_k &= c_{\phi_{r_1}} z_{x^{r_1}_{f^2_k}} - s_{\phi_{r_1}} z_{y^{r_1}_{f^2_k}}, D_k = s_{\phi_{r_1}} z_{x^{r_1}_{f^2_k}} + c_{\phi_{r_1}} z_{y^{r_1}_{f^2_k}}
\end{aligned}
$$

Theorem 2: The map joining problem with two locals maps has at least one local minimum and at most two local minima. Moreover, there are two local minima if and only if conditions (13)-(18) hold with a, C_ϕ defined in (26), $d, p, q, \delta_a, \delta_b$ defined similar to (27),(28), and (29) with

$$
\begin{aligned}
\Delta z_{x^{r_1}_{f_i}} &= z_{x^{r_1}_{f_i}} - [(z_{x_{f_i}} - z_{x_{r_1}}) c_{z_{\phi_{r_1}}} + (z_{y_{f_i}} - z_{y_{r_1}}) s_{z_{\phi_{r_1}}}] \\
\Delta z_{y^{r_1}_{f_i}} &= z_{y^{r_1}_{f_i}} - [-(z_{x_{f_i}} - z_{x_{r_1}}) s_{z_{\phi_{r_1}}} + (z_{y_{f_i}} - z_{y_{r_1}}) c_{z_{\phi_{r_1}}}] \\
& i = 1, \cdots, n
\end{aligned}
$$

Proof: The proof follows similar arguments to those used for proving Theorem 1. It should be noted that some data including the numbers n_1, n_2 do not affect the results. Detailed proof is omitted. ∎

Remark 3: Theorems 1 and 2 can be extended to the case where covariance matrices for each observation/odometry (feature/pose) are all spherical (diagonal with the x, y elements being the same) but different from each other.

Remark 4: It is easy to see that when combining two local maps each containing more than two robot poses (e.g. Tectonic SAM map joining [8]), the same results hold as long as the covariances are spherical.

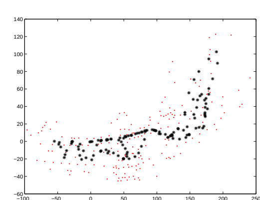
Remark 5: If the robot poses are not available, map joining problem reduces to the problem of finding the relative transformation between two coordinate frames given two corresponding point sets. When the covariances of the uncertainty associated with feature locations are assumed to be spherical, it is known that the problem has a closed-form solution [10].

VI. Experimental Results

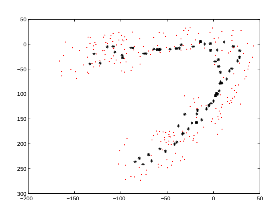
In this section, we use publicly available experimental datasets to demonstrate that the problem of joining two local maps has only one local minima in practice.

A. Results using Victoria Park dataset

The Victoria Park dataset was divided into two parts to build two local maps which are shown in Fig. 6. The covariance matrices of the two local maps were set to identity matrices. Using the local map data to compute the values of a, C_ϕ in Theorem 2, we obtain $a = 203660, C_\phi = -0.0029$. Obviously they do not satisfy the conditions (13)-(18), meaning that the map joining problem only has one local minimum. To check the result, Gauss Newton algorithm is used to solve the map joining problem. In an experiment with more than 100 trials, the algorithm always converged to the solution shown in Fig. 7(b) from arbitrary initial guesses to the robot poses and feature locations. Example initial guess is shown in Fig. 7(a).



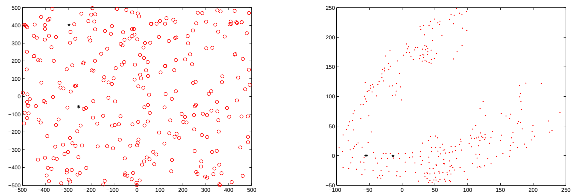
(a) Local map 1.



(b) Local map 2.

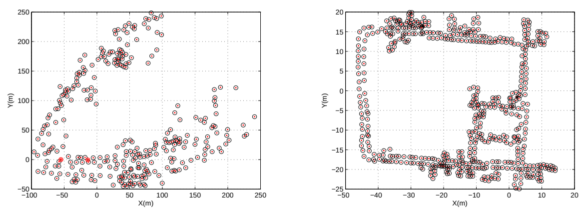
Fig. 6. Local maps 1 and 2 of Victora Park dataset. The black stars denote the robot position, the red dots denote the feature positions.

Fig. 8(a) compares the map joining result using identity covariance matrices with that using the original covariance matrices of the two local maps, the differences due to the use of spherical covariance matrices is negligible.



(a) Example initial guess to robot and feature locations. (b) Result of map joining.

Fig. 7. Result of map joining for the Victoria Park dataset. The black stars denote the robot position, the red circles denote the feature positions.

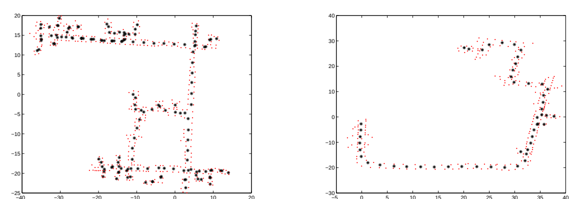


(a) Victoria Park dataset (b) DLR dataset

Fig. 8. Comparison of the map joining results using spherical covariance matrix and original covariances. The two results are almost identical. This is probably due to the high quality of the local maps.

B. *Results using DLR-Spatial Cognition dataset*

The experiments described in Section VI-A was repeated using the DLR dataset. Fig. 9 shows the two local maps obtained. An example initial guess and the map joining results are shown in Fig. 10(a) and Fig. 10(b). Fig. 8(b) compares the map joining result using identity covariance matrices with that using the original covariance matrices.

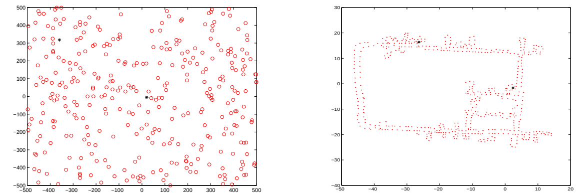


(a) Local map 1. (b) Local map 2.

Fig. 9. Local maps 1 and 2 of DLR dataset. The black stars denote the robot position, the red dots denote the feature positions.

VII. Conclusions and Future Work

This paper proves that nonlinear least squares problems associated with joining two 2D maps with spherical covariance matrices have at most two local minima. Moreover, it is demonstrated that two local minima exist only if the quality of local maps are much poorer than what is practically achievable. The necessary and sufficient condition for the existence of two local minima can be evaluated using the data from the two local maps. This makes it possible to guarantee that the globally optimum solution has been reached leading to the possibility of obtaining robust solutions to the SLAM problem even when the initial guess is unreliable.



(a) Example initial guess to robot and feature locations. (b) Result of map joining.

Fig. 10. Result of map joining for the DLR dataset. The black stars denote the robot position, the red circles denote the feature positions.

The results in this paper shows that the joining of two maps with spherical covariance matrices is equivalent to the solving of a one dimensional problem. Given the argument that all SLAM problems can be decomposed to that of joining two maps, it may be possible to use simple techniques such as bisection to obtain a solution to SLAM very efficiently. However, further work is needed to evaluate the impact of the assumption of spherical covariance matrices.

The results presented in this paper clearly show that SLAM is a very special optimization problem, and goes someway towards explaining the success of some of the recent techniques for SLAM that rely on local search strategies yet lead to good solutions. Further work on the analysis of these algorithms, for example TORO, may lead to even better and more efficient solutions to SLAM.

Furthermore, the extension of the results to multi-step SLAM, the joining of multiple maps, 2D bearing-only or range-only SLAM, and 3D SLAM are all non-trivial. Work in all these directions has the potential to enhance the understanding of this important robotics problem and lead to more reliable and efficient solutions to robot navigation.

Appendix

This appendix provides the proofs of Lemma 1 and Theorem 1.

A. *Proof of Lemma 1*

It is easy to see that $g(\phi) = \frac{df(\phi)}{d\phi}$.

First consider the case when $a < 1$. Since $\cos(\phi + C_\phi) \geq -1$, $g(\phi) > 0$ for any ϕ. That is $f(\phi)$ is monotone increasing. Since we have

$$f(-\pi) = a\sin(-\pi + C_\phi) - \pi < a - \pi < 1 - \pi < 0,$$
$$f(\pi) = a\sin(\pi + C_\phi) + \pi > -a + \pi > \pi - 1 > 0,$$

there is one and only one $\phi \in [-\pi, \pi)$ satisfying (11)-(12) simultaneously.

Now consider the case when $a \geq 1$. $g(\phi) = 0$ has two solutions in $[-\pi, \pi)$ which are $\phi_1 = wrap(\arccos(-\frac{1}{a}) - C_\phi)$, $\phi_2 = wrap(-\arccos(-\frac{1}{a}) - C_\phi)$.

First consider the case when $\phi_1 \leq \phi_2$, since $g(\phi_1) = 0, g(\phi_2) = 0$, ϕ_1, ϕ_2 divide interval $[-\pi, \pi)$ into three intervals where f is monotone in each of the intervals, i.e., $[-\pi, \phi_1]$, $[\phi_1, \phi_2]$, and $[\phi_2, \pi)$, there are at most two

ϕ satisfying (11)-(12) simultaneously, which belong to intervals $[-\pi, \phi_1]$ and $[\phi_2, \pi)$ provided that $f(\phi)$ is monotone increasing in these two intervals.

Since $|\phi| \leq \pi$, the number of solutions to satisfy (11)-(12) simultaneously can be analyzed by observing the four values $f(-\pi), f(\phi_1), f(\phi_2), f(\pi)$.

Note that $f(-\pi) = -a \sin C_\phi - \pi$, $f(\pi) = -a \sin C_\phi + \pi$, $f(\phi_1) = \sqrt{a^2-1} + \phi_1$, $f(\phi_2) = -\sqrt{a^2-1} + \phi_2$, it is impossible to have $f(-\pi) > 0$ and $f(\pi) < 0$ simultaneously, thus, there are 12 cases which are stated in Table I.

We will now show that the two cases

$$f(-\pi) < 0, f(\phi_1) < 0, f(\phi_2) < 0, f(\pi) < 0,$$
$$f(-\pi) > 0, f(\phi_1) > 0, f(\phi_2) > 0, f(\pi) > 0$$

could never happen.

For the case $f(-\pi) < 0, f(\phi_1) < 0, f(\phi_2) < 0, f(\pi) < 0$ it can be obtained by $f(\pi) < 0$ that

$$\sin C_\phi > \frac{\pi}{a} > 0 \tag{32}$$

Thus $C_\phi \in (0, \pi)$. Since $a > 0$, we have $\arccos(-\frac{1}{a}) \in (\pi/2, \pi]$, thus we have $\arccos(-\frac{1}{a}) - C_\phi \in (-\pi/2, \pi)$, since $\arccos(-\frac{1}{a}) - C_\phi \in [-\pi, \pi)$, we have $\phi_1 = wrap(\arccos(-\frac{1}{a}) - C_\phi) = \arccos(-\frac{1}{a}) - C_\phi \in (-\pi/2, \pi)$. Moreover, by $f(\pi) < 0$ we also have $a > \pi$. Thus

$$f(\phi_1) = \sqrt{a^2-1} + \phi_1 > \sqrt{\pi^2-1} - \pi/2 > 0.$$

This contradicts with $f(\phi_1) < 0$.

Similarly, for the case $f(-\pi) > 0, f(\phi_1) > 0, f(\phi_2) > 0, f(\pi) > 0$, it can be obtained by $f(-\pi) > 0$ that

$$\sin C_\phi < -\frac{\pi}{a} < 0 \tag{33}$$

Thus $C_\phi \in (-\pi, 0)$. Since $a > 0$, we have $\arccos(-\frac{1}{a}) \in (\pi/2, \pi]$ and $-\arccos(-\frac{1}{a}) \in [-\pi, -\pi/2)$, thus we have $-\arccos(-\frac{1}{a}) - C_\phi \in (-\pi, \pi/2)$. Since $-\arccos(-\frac{1}{a}) - C_\phi \in [-\pi, \pi)$, we have $\phi_2 = wrap(-\arccos(-\frac{1}{a}) - C_\phi) \in (-\pi, \pi/2)$. Moreover, by $f(-\pi) > 0$ we also have $a > \pi$. Thus $f(\phi_2) = -\sqrt{a^2-1} + \phi_2 < -\sqrt{\pi^2-1} + \pi/2 < 0$. This contradicts with $f(\phi_2) > 0$.

Thus, when $\phi_1 \leq \phi_2$, only 10 out of the 12 cases listed in Table I may happen. So there are at least one and at most two solutions ϕ satisfying conditions (11)-(12) simultaneously, and there are two solutions if and only if the following conditions $f(-\pi) \leq 0, f(\phi_1) \geq 0, f(\phi_2) \leq 0, f(\pi) \geq 0$ hold simultaneously, which are equivalent to conditions in (14)-(17).

Now consider the case when $\phi_1 > \phi_2$. Following the same lines as that of the case when $\phi_1 \leq \phi_2$, we have that to ensure that there are two ϕ satisfying conditions (11)-(12) simultaneously, the following conditions $f(-\pi) \leq 0, f(\phi_2) \geq 0, f(\phi_1) \leq 0, f(\pi) \geq 0$ need to hold simultaneously, which

TABLE I

ANALYSIS OF SOLUTION TO (11)-(12).

f($-\pi$)	-	-	-	-	-	-	-	-	+	+	+	+
f(ϕ_1)	-	-	-	-	+	+	+	+	-	-	+	+
f(ϕ_2)	-	-	+	+	-	-	+	+	-	+	-	+
f(π)	-	+	-	+	-	+	-	+	+	+	+	+
No.	0	1	1	1	1	2	1	1	1	1	1	0

'-' denotes '<0', '+' denotes '>0', '0, 1, 2' denote the number of solution to satisfy (11)-(12) simultaneously.

are

$$-a \sin C_\phi - \pi \leq 0 \tag{34}$$
$$\sqrt{a^2-1} + \phi_1 \leq 0 \tag{35}$$
$$-\sqrt{a^2-1} + \phi_2 \geq 0 \tag{36}$$
$$-a \sin C_\phi + \pi \geq 0 \tag{37}$$
$$\phi_1 > \phi_2 \tag{38}$$

However, from (35), we have $\phi_1 \leq -\sqrt{a^2-1} < 0$, and from (36), we have $\phi_2 \geq \sqrt{a^2-1} > 0$, which means $\phi_1 < \phi_2$ and contradicts with (38). Thus it is impossible for (34)-(38) to hold simultaneously.

In summary, there are two solutions if and only if conditions in (14)-(17) hold simultaneously together with $\phi_1 \leq \phi_2$ and $a \geq 1$. This completes the proof.

B. Proof of Theorem 1

We prove the theorem by showing that for the objective function $F(X)$ in (23), its gradient $\nabla F(X) = 0$ is equivalent to a nonlinear equation with only one variable ϕ. The key reason for this is that the $2n+2$ equations in $\nabla F(X) = 0$ are linear when ϕ is fixed and thus all the other $2n+2$ variables can be expressed by ϕ.

In fact, by (23) and (25)

$$\nabla F(X) = 2 \begin{bmatrix} -(z_{x_{f_1}} - x_{f_1}) - (A_1 - x_{f_1} + x_r) \\ -(z_{y_{f_1}} - y_{f_1}) - (B_1 - y_{f_1} + y_r) \\ -(z_{x_{f_2}} - x_{f_2}) - (A_2 - x_{f_2} + x_r) \\ -(z_{y_{f_2}} - y_{f_2}) - (B_2 - y_{f_2} + y_r) \\ \vdots \\ -(z_{x_{f_n}} - x_{f_n}) - (A_n - x_{f_n} + x_r) \\ -(z_{y_{f_n}} - y_{f_n}) - (B_n - y_{f_n} + y_r) \\ -z_{x_r} + x_r + \sum_{i=1}^{n} (A_i - x_{f_i} + x_r) \\ -z_{y_r} + y_r + \sum_{i=1}^{n} (B_i - y_{f_i} + y_r) \\ \phi - z_\phi + \sum_{i=1}^{n} [B_i(x_{f_i} - x_r) - A_i(y_{f_i} - y_r)] \end{bmatrix}$$

Let $\nabla F(X) = 0$. The first $2n+2$ equations can be expressed by

$$M_n X_n = N_n \tag{39}$$

where X_n is the vector of the first $2n+2$ elements of state variable X

$$X_n = \begin{bmatrix} x_{f_1} & y_{f_1} & \cdots & x_{f_n} & y_{f_n} & x_r & y_r \end{bmatrix}^T$$

and

$$M_n = \begin{bmatrix} A & C^T \\ C & D \end{bmatrix} \tag{40}$$

$$N_n = \begin{bmatrix} z_{x_{f_1}} + A_1 \\ z_{y_{f_1}} + B_1 \\ \vdots \\ z_{x_{f_n}} + A_n \\ z_{y_{f_n}} + B_n \\ z_{x_r} - \sum_{i=1}^{n} A_i \\ z_{y_r} - \sum_{i=1}^{n} B_i \end{bmatrix} \tag{41}$$

with

$$\begin{aligned} A &= 2I_{2n\times 2n}, \\ C &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ \vdots & \vdots \\ -1 & 0 \\ 0 & -1 \end{bmatrix}^T_{2n\times 2} \\ D &= \begin{bmatrix} n+1 & 0 \\ 0 & n+1 \end{bmatrix}, \end{aligned}$$

Applying Matrix Inversion Lemma, we have

$$M_n^{-1} = \begin{bmatrix} Y & U^T \\ U & V \end{bmatrix}$$

where

$$\begin{aligned} Y &= \frac{1}{2}\begin{bmatrix} Y_1 & Y_2 & Y_2 & \cdots & Y_2 \\ Y_2 & Y_1 & Y_2 & \cdots & Y_2 \\ Y_2 & Y_2 & Y_1 & \cdots & Y_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Y_2 & Y_2 & Y_2 & \cdots & Y_1 \end{bmatrix}_{2n\times 2n} \\ U &= \begin{bmatrix} Y_2 & \cdots & Y_2 \end{bmatrix}_{2\times 2n}, \\ V &= \begin{bmatrix} \frac{2}{n+2} & 0 \\ 0 & \frac{2}{n+2} \end{bmatrix} \end{aligned} \tag{42}$$

with

$$\begin{aligned} Y_1 &= \begin{bmatrix} 1+\frac{1}{n+2} & 0 \\ 0 & 1+\frac{1}{n+2} \end{bmatrix}, \\ Y_2 &= \begin{bmatrix} \frac{1}{n+2} & 0 \\ 0 & \frac{1}{n+2} \end{bmatrix}. \end{aligned} \tag{43}$$

Then from

$$X_n = M_n^{-1} N_n, \tag{44}$$

we have

$$\begin{aligned} x_{f_j} - x_r =& -\frac{1}{n+2} z_{x_r} + \frac{1}{2} z_{x_{f_j}} - \frac{1}{2(n+2)} \sum_{i=1}^{n} z_{x_{f_i}} \\ & + \frac{1}{2} A_j + \frac{1}{2(n+2)} \sum_{i=1}^{n} A_i \\ y_{f_j} - y_r =& -\frac{1}{n+2} z_{y_r} + \frac{1}{2} z_{y_{f_j}} - \frac{1}{2(n+2)} \sum_{i=1}^{n} z_{y_{f_i}} \\ & + \frac{1}{2} B_j + \frac{1}{2(n+2)} \sum_{i=1}^{n} B_i \\ j =& 1, \ldots, n \end{aligned} \tag{45}$$

Combining the above equations with the last equation in $\nabla F(X) = 0$, we have

$$\begin{aligned} &\phi - z_\phi + \sum_{j=1}^{n} [B_j(-\frac{1}{n+2} z_{x_r} + \frac{1}{2} z_{x_{f_j}} - \frac{1}{2(n+2)} \sum_{i=1}^{n} z_{x_{f_i}})] \\ &\quad - \sum_{j=1}^{n} [A_j(-\frac{1}{n+2} z_{y_r} + \frac{1}{2} z_{y_{f_j}} - \frac{1}{2(n+2)} \sum_{i=1}^{n} z_{y_{f_i}})] \\ &\quad = 0 \end{aligned} \tag{46}$$

which is equivalent to

$$\begin{aligned} &\tilde{\phi} + \sum_{j=1}^{n} \{ \begin{bmatrix} z_{x_{f_j}} - z_{x_r} \\ z_{y_{f_j}} - z_{y_r} \end{bmatrix}^T \begin{bmatrix} s_{\tilde{\phi}} & -c_{\tilde{\phi}} \\ c_{\tilde{\phi}} & s_{\tilde{\phi}} \end{bmatrix} [\frac{1}{2} \begin{bmatrix} z_{x_{f_j}} - z_{x_r} \\ z_{y_{f_j}} - z_{y_r} \end{bmatrix} \\ &\quad + \frac{1}{2(n+2)} \sum_{i=1}^{n} \begin{bmatrix} z_{x_r} - z_{x_{f_i}} \\ z_{y_r} - z_{y_{f_i}} \end{bmatrix}]\} \\ &\quad + \sum_{j=1}^{n} \{ \begin{bmatrix} \Delta z_{x^r_{f_j}} \\ \Delta z_{y^r_{f_j}} \end{bmatrix}^T \begin{bmatrix} s_\phi & -c_\phi \\ c_\phi & s_\phi \end{bmatrix} \\ &\quad \times \begin{bmatrix} -\frac{1}{n+2} z_{x_r} + \frac{1}{2} z_{x_{f_j}} - \frac{1}{2(n+2)} \sum_{i=1}^{n} z_{x_{f_i}} \\ -\frac{1}{n+2} z_{y_r} + \frac{1}{2} z_{y_{f_j}} - \frac{1}{2(n+2)} \sum_{i=1}^{n} z_{y_{f_i}} \end{bmatrix} \} \\ &\quad = 0 \end{aligned}$$

which can be denoted as

$$d \sin\tilde{\phi} + \delta_a \cos\phi + \delta_b \sin\phi + \tilde{\phi} = 0 \tag{47}$$

where $\tilde{\phi} = \phi - z_\phi$, and d is defined in (27) and δ_a, δ_b are defined in (29).

Let $k = \delta_a \cos\phi + \delta_b \sin\phi$, we have

$$\begin{aligned} k &= \delta_a \cos(\tilde{\phi} + z_\phi) + \delta_b \sin(\tilde{\phi} + z_\phi) \\ &= (\delta_a \cos(z_\phi) + \delta_b \sin(z_\phi)) \cos\tilde{\phi} \\ &\quad + (-\delta_a \sin(z_\phi) + \delta_b \cos(z_\phi)) \sin\tilde{\phi} \end{aligned}$$

Then, we have that (47) becomes

$$d \sin\tilde{\phi} + \tilde{\phi} + p\cos\tilde{\phi} + q\sin\tilde{\phi} = 0$$

where p, q are defined in (28). Furthermore, it can be rewritten as

$$(d+q)\sin\tilde{\phi} + p\cos\tilde{\phi} + \tilde{\phi} = 0. \tag{48}$$

Since $x_r, y_r, x_{f_i}, y_{f_i}, i = 1, \ldots, n$ depend only on variable ϕ, the number of the critical points of the objective function $F(X)$ depends on the number of solution $\tilde{\phi}$ to (48).

Furthermore, we have

$$\nabla^2 F(X) = 2\begin{bmatrix} M_n & L_n^T \\ L_n & 1 + \sum_{i=1}^{n}[A_i(x_{f_i} - x_r) + B_i(y_{f_i} - y_r)] \end{bmatrix}$$

where M_n is the same as stated in (40), and

$$L_n = \begin{bmatrix} B_1 & -A_1 & \cdots & B_n & -A_n & -\sum_{i=1}^{n} B_i & \sum_{i=1}^{n} A_i \end{bmatrix}$$

applying Schur Complement [11], we have $\nabla^2 F(X) > 0$ is equivalent to

$$1 + \sum_{i=1}^{n}[A_i(x_{f_i} - x_r) + B_i(y_{f_i} - y_r)] - L_n M_n^{-1} L_n^T > 0$$

which is

$$1 + \sum_{j=1}^{n}[A_j(x_{f_j} - x_r) + B_j(y_{f_j} - y_r)] - \frac{1}{2}\sum_{j=1}^{n}(A_j^2 + B_j^2) - \frac{1}{2(n+2)}\sum_{j=1}^{n}\sum_{i=1}^{n}(A_j A_i + B_j B_i) > 0$$

Submit (45) into the above inequality, we have

$$1 + \sum_{j=1}^{n}(\begin{bmatrix} A_j \\ B_j \end{bmatrix}^T \begin{bmatrix} \frac{-1}{n+2} z_{x_r} + \frac{1}{2} z_{x_{f_j}} - \frac{1}{2(n+2)} \sum_{i=1}^{n} z_{x_{f_i}} \\ \frac{-1}{n+2} z_{y_r} + \frac{1}{2} z_{y_{f_j}} - \frac{1}{2(n+2)} \sum_{i=1}^{n} z_{y_{f_i}} \end{bmatrix}) > 0 \tag{49}$$

which is equivalent to

$$\begin{aligned} 1 &+ \sum_{j=1}^{n}\{\begin{bmatrix} z_{x_{f_j}} - z_{x_r} \\ z_{y_{f_j}} - z_{y_r} \end{bmatrix}^T \begin{bmatrix} c_{\tilde{\phi}} & s_{\tilde{\phi}} \\ -s_{\tilde{\phi}} & c_{\tilde{\phi}} \end{bmatrix} [\frac{1}{2}\begin{bmatrix} z_{x_{f_j}} - z_{x_r} \\ z_{y_{f_j}} - z_{y_r} \end{bmatrix} \\ &+ \frac{1}{2(n+2)}\sum_{i=1}^{n}\begin{bmatrix} z_{x_r} - z_{x_{f_i}} \\ z_{y_r} - z_{y_{f_i}} \end{bmatrix}]\} \\ &+ \sum_{j=1}^{n}\{\begin{bmatrix} \Delta z_{x_{f_j}^r} \\ \Delta z_{y_{f_j}^r} \end{bmatrix}^T \begin{bmatrix} c_\phi & s_\phi \\ -s_\phi & c_\phi \end{bmatrix} \\ &\times \begin{bmatrix} -\frac{1}{n+2} z_{x_r} + \frac{1}{2} z_{x_{f_j}} - \frac{1}{2(n+2)} \sum_{i=1}^{n} z_{x_{f_i}} \\ -\frac{1}{n+2} z_{y_r} + \frac{1}{2} z_{y_{f_j}} - \frac{1}{2(n+2)} \sum_{i=1}^{n} z_{y_{f_i}} \end{bmatrix}\} > 0 \end{aligned}$$

which is

$$d\cos\tilde{\phi} - \delta_a \sin\phi + \delta_b \cos\phi + 1 > 0. \tag{50}$$

Since (50) is equivalent to

$$(d+q)\cos\tilde{\phi} - p\sin\tilde{\phi} + 1 > 0 \tag{51}$$

It is evident that the number of minima to objective function $F(X)$ equals to the number of solutions to satisfy conditions (48) and (51) simultaneously.

Note that (48) and (51) are equivalent to

$$a(\cos C_\phi \sin\tilde{\phi} + \sin C_\phi \cos\tilde{\phi}) + \tilde{\phi} = 0 \tag{52}$$

$$a(\cos C_\phi \cos\tilde{\phi} - \sin C_\phi \sin\tilde{\phi}) + 1 > 0 \tag{53}$$

where $a = \sqrt{p^2 + (d+q)^2}$ and

$$\sin C_\phi = \frac{p}{a}, \quad \cos C_\phi = \frac{d+q}{a} \tag{54}$$

Furthermore, (52) and (53) are equivalent to

$$a\sin(\tilde{\phi} + C_\phi) + \tilde{\phi} = 0 \tag{55}$$

$$a\cos(\tilde{\phi} + C_\phi) + 1 > 0 \tag{56}$$

Applying Lemma 1, it can be obtained that if conditions (13)-(18) hold with a, C_ϕ defined in (26), there are two minima to SLAM problem, else, there is only one minimum. This completes the proof.

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