$\mathcal{I} = (\mathrm{Sign}, \mathrm{Sen}, \mathrm{Mod}, \models, \Vdash)$

- $\mathrm{Sen}(\Sigma)$ – basic sentences
- Colimits of signatures and models,
- Initial models for signatures
- $\mathcal{I}$ has elementary diagrams ι with representable elementary extensions
- $\mathcal{I}$ is semiexact
- representable substitutions
- $\mathcal{D} \xhookrightarrow{\text{subcategory}}$ Representable signatures morphisms
  - $\mathcal{D}$ stable to pushouts
  - $\mathcal{D}$ has a set of nonisomorphic representants (co well powered)
- For every $x \in \mathcal{D}$, $\forall M$, $\forall M' \upharpoonright_x = M$

$$\begin{tikzcd} \Sigma \arrow[r, "x"] \arrow[d, "\varphi_\Sigma(M)"'] & \Sigma' \arrow[d, "\theta_M"] \\ \Sigma_M \arrow[r, "\theta"'] & \Sigma'' \end{tikzcd}$$

$\exists\, \rho_{M'}$ st

– $\rho_{M'}$ is basic $\in \mathrm{Sen}(\Sigma'')$

– $M_M \otimes M' \models \rho_{M'}$

– if $N' \upharpoonright_x = M$ & $M_M \otimes N' \models \rho_{M'}$ then $M' = N'$

- M_x is projective w.r.t $0_\Sigma \longrightarrow M_E$

where – M_x is a model that represents $x : \Sigma \longrightarrow \Sigma' \in \mathcal{D}$

– M_E is the basic model of E

We can construct:

$$\mathcal{J}^{\forall,\neg,\wedge} := (Sign, Sen^{\forall,\neg,\wedge}, Mod, \models)$$

- $Sen^{\forall,\neg,\wedge}(\Sigma) \supseteq Sen(\Sigma)$
- closed to $\neg$, $\wedge$ and $\forall x$

(x is in the set of representants of $\mathcal{D}$)

Entailment system for $\mathcal{J}^{\forall,\neg,\wedge}$ is complete

Basic deduction:

$$\Vdash \subseteq \vdash$$

Entailment system:

$$E \vdash E' \quad E' \vdash E'' \Rightarrow E \vdash E''$$

$$E \subseteq E' \Rightarrow E' \vdash E$$

$$\Gamma \vdash E \quad \Gamma \vdash E' \Rightarrow \Gamma \vdash E \cup E'$$

Translation rules

$$E \vdash E' \Rightarrow \varphi(E) \vdash \varphi(E') \qquad \forall \varphi$$

$$\varphi(E) \vdash \varphi(E') \Rightarrow E \vdash E' \qquad \forall \varphi \in \mathcal{D}$$

(Soundness of this rule ?)

Conjunction

$$\{\rho, \rho'\} \dashv\vdash \rho \wedge \rho'$$

Negation

$$E \vdash \rho \qquad E \vdash \neg\rho \Rightarrow E \vdash \rho'$$

$$\{E, \rho\} \vdash \rho' \quad \{E, \neg\rho\} \vdash \rho' \Rightarrow E \vdash \rho'$$

Quantification

$$E \vdash (\forall x)\rho' \Leftrightarrow x(E) \vdash \rho' \qquad x \in \mathcal{D}$$

Substitutivity

$$E \vdash (\forall x)\rho' \Rightarrow E \vdash \psi^{Sen}(\rho') \text{ where}$$

$\psi : x \to 1_\Sigma$ is a substitution

$$\begin{tikzcd} \Sigma \arrow[r, "x"] \arrow[dr] & \Sigma' \arrow[d, dotted, "\psi"] \\ & \Sigma \end{tikzcd} \quad (\psi^{Mod}, \psi^{Sen}) \qquad M_x \xrightarrow{h} 0_\Sigma$$

basic sentences

E is basic if $\exists$ M_E such that

$M \models E$ iff $\exists\, h : M_E \to M$

In concrete institutions = atomic sentences

Sen preserves inductive colimits

$$\begin{tikzcd} \Sigma_0 \arrow[r] & \Sigma_1 \arrow[r] & \cdots \arrow[r] & \Sigma' \end{tikzcd}$$

$$\Downarrow$$

$$\begin{tikzcd} Sen(\Sigma_0) \arrow[r] & Sen(\Sigma_1) \arrow[r] & \cdots \arrow[r] & Sen(\Sigma') \end{tikzcd}$$

usually if equations have a finite number of symbols

Elementary diagrams

$\Sigma \xrightarrow{\iota_\Sigma(M)} \Sigma_M$ (elementary extension)

$M / Mod(\Sigma) \overset{i_{\Sigma,M}}{\cong} Mod(\Sigma_M, E_M)$ $\qquad M_M := 0_{E_M}$

In FOL

$(S, F) \xrightarrow{\iota_\Sigma(M)} (S, F \cup M)$

$E_M := \{ \sigma(\underline{m}) = M_\sigma(\underline{m}) \mid \forall \sigma \in F, \forall \underline{m} \in M \}$

$$\begin{tikzcd} (S,F) \arrow[r, "X"] \arrow[d, "\iota_\Sigma(M)"'] & (S, F \cup X) \arrow[d, "\theta_M"] \\ (S, F \cup M) \arrow[r, "\theta"'] & (S, F \cup M \cup X) \end{tikzcd}$$

$\rho_{M'} = \wedge \{ x = M_x \}$

finite set of variables

Representable morphisms

$\Sigma \xrightarrow{\chi} \Sigma'$ is representable $\Longleftrightarrow$ $M_\chi / Mod(\Sigma) \simeq Mod(\Sigma')$

In FOL $\quad T_F(X) / Mod(S,F) \simeq Mod(S, F \cup X)$

Representable substitutions

$\psi : \chi_1 \to \chi_2$

$$\begin{tikzcd} & \Sigma_1 \arrow[dd, dashed, "(\psi^{Mod}, \psi^{Sen})"] \\ \Sigma \arrow[ur, "\chi_1"] \arrow[dr, "\chi_2"'] & \\ & \Sigma_2 \end{tikzcd}$$

$h : M_{\chi_1} \to M_{\chi_2}$

$T_F(X) \to T_F(Y)$ $\Longrightarrow$ $\chi_1 \to \chi_2$

$(\Sigma \cup X) \to (\Sigma \cup Y)$

Schetch of proof

Definition

E is a Henkin theory if

1. $E \nvdash \rho \iff E \vdash \neg\rho$ (maximality)
2. $E \vdash (\forall x)\rho' \iff \forall x' : \Sigma' \to \Sigma$
   $x ; x' = 1_\Sigma$
   $\Rightarrow E \vdash x'(\rho')$

• The classical formulation of 2

$E \vdash (\forall x)\rho' \iff E \vdash \rho' \frac{c}{x}$ for all constants c in F

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1. Every consistent theory can be extended to a Henkin theory

$(\Sigma, E) \xrightarrow{f} (\Sigma', E')$

$E \nvdash \perp$ (consistent) $\Rightarrow$

$\exists f$ s.t. $f(E) \subseteq E'$ and E' is Henkin

2. Any Henkin theory has a model

3. Any consistent theory has a model

4. $E \models \rho \Rightarrow E \vdash \rho$

(Assume $E \nvdash \rho \Rightarrow \{E, \neg\rho\} \nvdash \perp \Rightarrow$
$\{E, \neg\rho\}$ has a model $\Rightarrow$ contradiction)

Construction of Henkin theory

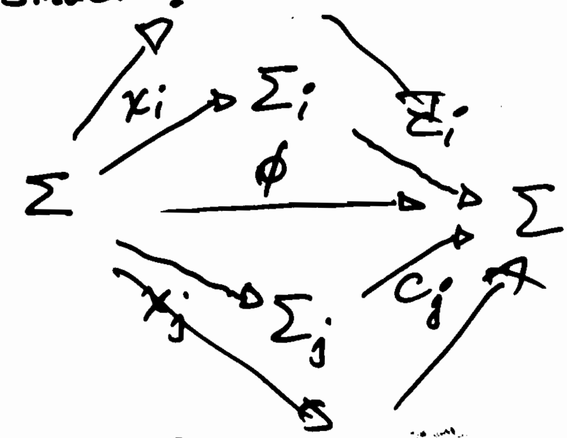
- In classical proof :

For each $(\forall x)\rho'$ we add a constant $c_{x,\rho'}$

$$(S, F) \longrightarrow (S, F \cup \{c_{x,\rho'} \mid x, \rho' \text{ in } (\forall x)\rho'\})$$

$$E \cup \{(\exists x)\neg\rho' \longrightarrow \neg\rho' \tfrac{c_{x,\rho'}}{x}\}$$

- Abstract :



$$E \cup \{\phi((\exists x_i)\neg\rho'_i) \to \Xi_i(\rho'_i)\}$$

$$\Sigma \to \Sigma_1 \cdots \to \Sigma'$$

Construction of Henkin model

E Henkin theory

$$E_B := \{\rho \mid E \vdash \rho, \ \rho \text{ basic}\}$$

We can prove inductively over the structure of ρ

$$M_{E_B} \models \rho \iff E \vdash \rho$$