Process Algebra CSP – A Technique to Model Concurrent Programs

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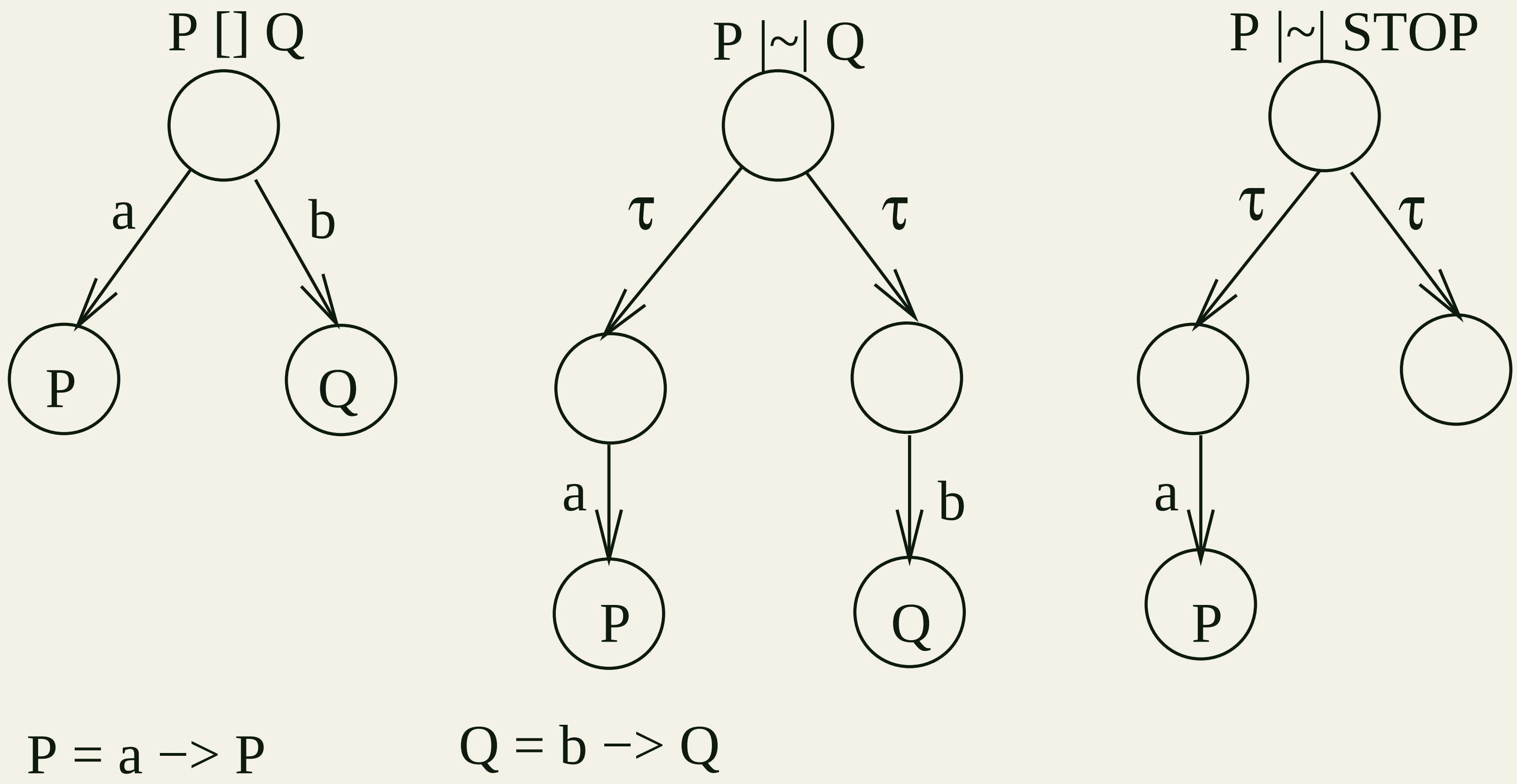
April 11, 2002

Contents

Denotational Semantics II



Example 1 (Uncompleteness of the traces model)

$$traces(P \,\Box\, Q) = traces(P \sqcap Q)$$

$$traces(P \sqcap Stop) = traces(P)$$

The divergences

- Definition

  $divergences(P)$ is a set of traces s of P on which P can diverge, in the sense that an infinite unbroken sequence of τ actions can occur after some $s' \leq s$.

$$traces_{\perp}(P) = traces(P) \cup \{s \mid s \in divergences(P)\}$$

Example 2 (A divergent producer/consumer system)

$$\mathit{BUFF}(<>) = \mathit{left}?x \rightarrow \mathit{BUFF}(<x>)$$

$$\mathit{BUFF}(s\,\hat{}\,<a>) = ((\mathit{left}?x \rightarrow \mathit{BUFF}(<x>\,\hat{}\,s\,\hat{}\,<a>)) \sqcap \mathit{Stop}) \;\Box\; (\mathit{right}.a \rightarrow \mathit{BUFF}(s))$$

$$\mathit{PROD\_INIT} = \mathit{produce}.0 \rightarrow \mathit{PROD}(0)$$

$$\mathit{PROD}(x) = \mathit{left}.x \rightarrow \mathit{PROD}(x \oplus 1)$$

$$\mathit{CONS} = \mathit{right}?x \rightarrow \mathit{CONS}$$

$$\mathit{PC\_SYS} = ((\mathit{PROD\_INIT} \;|||\; \mathit{CONS}) \underset{\{\mathit{left},\mathit{right}\}}{\|} \mathit{BUFF}(<>)) \setminus \{\mathit{left}, \mathit{right}\}$$

pc_live.fdr2

Start FDR2

- Calculating the divergences of processes
  - Foundamental operators

$$divergences(Stop) = \{\}$$

$$divergences(Skip) = \{\}$$

$$\begin{aligned} divergences(e \rightarrow P) = \{\langle a \rangle {}^\frown s \mid\ & a \in \mathsf{comms}(e) \\ & \wedge s \in divergences(\mathsf{subs}(a, e, P))\} \end{aligned}$$

$$divergences(P \mathbin{\Box} Q) = divergences(P) \cup divergences(Q)$$

$$divergences(P \sqcap Q) = divergences(P) \cup divergences(Q)$$

$$\begin{aligned} divergences(P \not< b \not> Q) = \ & \textbf{if } b \textbf{ then } divergences(P) \\ & \textbf{else } divergences(Q) \end{aligned}$$

- Parallel operators

$$divergences(P \underset{X}{\parallel} Q) = \{u \hat{}\, v \mid \exists\, s \in traces_{\perp}(P), t \in traces_{\perp}(Q). u \in (s \underset{X}{\parallel} t) \cap \Sigma^{*} \wedge (s \in divergences(P) \vee t \in divergences(Q))\}$$

- Hiding and renaming

$$\begin{aligned} divergences(P \setminus X) = & \{(s \setminus X)\hat{\ }t \mid s \in divergences(P)\} \\ & \cup \{(u \setminus X)\hat{\ }t \mid u \text{ infinit} \wedge (u \setminus X) \text{ finit} \\ & \qquad \wedge \forall s < u.s \in traces_{\perp}(P)\} \end{aligned}$$

$$divergences(P \llbracket R \rrbracket) = \{s'\hat{\ }t \mid \exists s \in divergences(P).sRs'\}$$

- Sequential operator

$$\begin{aligned} divergences(P;\ Q) = & \ divergences(P) \cup \\ & \{s\hat{\ }t \mid s\hat{\ }\langle \checkmark \rangle \in traces_{\perp}(P) \wedge t \in divergences(Q)\} \end{aligned}$$

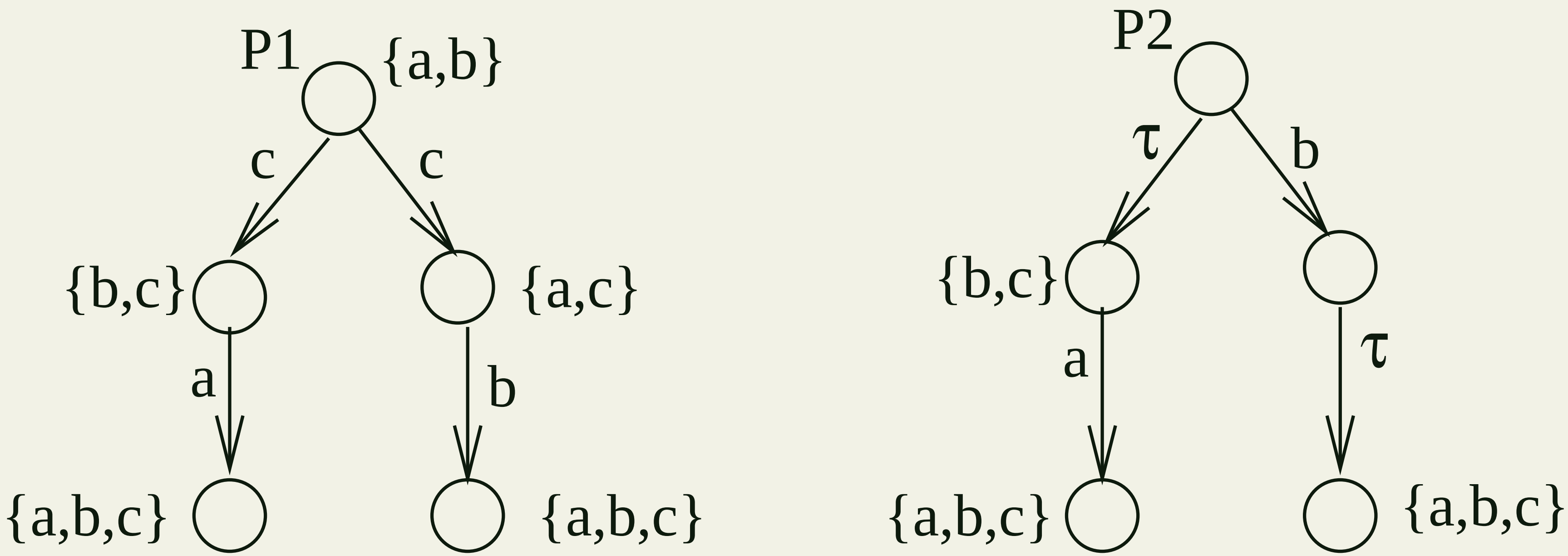
The failures model ($\mathcal{F}$)

- Definitions

$$\textit{refusals}(P) = \{a \mid \nexists\, s \in \Sigma^*. a\hat{\ }s \in \textit{traces}(P)\}$$

$\textit{failures}(P)$ consists of all pairs of (s, X) where s is a trace of P and X a set of actions P can refuse in some stable state (unable to perform τ or $\checkmark$) after s, or results from a state after s which can perform $\checkmark$ and $X \subseteq \Sigma$.

$$\textit{failures}_\perp(P) = \textit{failures}(P) \cup \{(s, X) \mid s \in \textit{divergences}(P)\}$$



Example 3 (Refusal sets of some transition systems)

$$P_1 = (c \to a \to Stop) \,\square\, (c \to b \to Stop)$$

$$P_2 = ((c \to a \to Stop) \,\square\, (b \to c \to Stop)) \setminus \{c\}$$

- Calculating the failures semantics of processes
  - Foundamental operators

$$failures(Stop) = \{(\langle\rangle, X) \mid X \subseteq \Sigma^{\checkmark}\}$$

$$failures(Skip) = \{(\langle\rangle, X) \mid X \subseteq \Sigma\} \cup \{(\langle \checkmark \rangle, X \mid X \subseteq \Sigma^{\checkmark}\}$$

$$\begin{aligned} failures(e \rightarrow P) = & \{(\langle\rangle, X) \mid \mathsf{comms}(e) \cap X = \{\}\} \\ & \cup \{(\langle a \rangle ^\frown s, X) \mid a \in \mathsf{comms}(e) \\ & \quad \wedge (s, X) \in failures(\mathsf{subs}(a, e, P))\} \end{aligned}$$

$$failures(P \sqcap Q) = failures(P) \cup failures(Q)$$

$$failures(P \not< b \not> Q) = \textbf{if } b \textbf{ then } failures(P) \textbf{ else } failures(Q)$$

$$failures(P \mathrel{\Box} Q) =$$
$$\{(\langle\rangle, X) \mid (\langle\rangle, X) \in failures(P) \cap failures(Q)\}$$
$$\cup \{(s, X) \mid (s, X) \in failures(P) \cup failures(Q) \wedge s \neq \langle\rangle\}$$
$$\cup \{(\langle\rangle, X) \mid X \subseteq \Sigma \wedge \langle \checkmark \rangle \in traces(P) \cup traces(Q)\}$$

- Parallel Operators

$$failures(P \underset{X}{\parallel} Q) =$$
$$\{(u, Y \cup Z) \mid Y \setminus (X \cup \checkmark) = Z \setminus (X \cup \checkmark)$$
$$\wedge \exists\, s, t.(s, Y) \in failures(P)$$
$$\wedge (t, Z) \in failures(Q) \wedge u \in s \underset{X}{\parallel} t\}$$

- Hiding and renaming

$$failures(P \setminus X) = \{(s \setminus X, Y) \mid (s, Y \cup X) \in failures(P)\}$$

$$failures(P \llbracket R \rrbracket) =$$
$$\{(s', X) \mid \exists\, s . sRs' \wedge (s, R^{-1}(X)) \in failures(P)\}$$

- Sequential operator

$$failures(P;\ Q) =$$
$$\{(s, X) \mid (s, X \cup \{\checkmark\}) \in failures(P)\}$$
$$\cup \{(s\,\hat{}\,t, X) \mid s\,\hat{}\,\langle \checkmark \rangle \in traces(P) \wedge (t, X) \in failures(Q)\}$$

The failures/divergences model ($\mathcal{FD}$)

- failures/divergences model: $(failures_{\bot}(P),\ divergences(P))$
- Process relations
  - Equivalences

$$
\begin{array}{ll}
P =_{\mathcal{T}} Q & traces(Q) = traces(P) \\
P =_{\mathcal{F}} Q & failures(Q) = failures(P) \\
P =_{\mathcal{FD}} Q & failures_{\bot}(Q) = failures_{\bot}(P) \text{ and} \\
 & divergences(Q) = divergences(P)
\end{array}
$$

  - Refinements

$$
\begin{array}{ll}
P \sqsubseteq_{\mathcal{T}} Q & traces(Q) \subseteq traces(P) \\
P \sqsubseteq_{\mathcal{F}} Q & failures(Q) \subseteq failures(P) \\
P \sqsubseteq_{\mathcal{FD}} Q & failures_{\bot}(Q) \subseteq failures_{\bot}(P) \text{ and} \\
 & divergences(Q) \subseteq divergences(P)
\end{array}
$$

- The properties of the failures/divergences model

F1 $traces_{\perp}(P) = \{s \mid (s, X) \in failures_{\perp}(P)\}$ is non-empty and prefix closed.

F2 $(s, X) \in failures_{\perp}(P) \wedge Y \subseteq X \Longrightarrow (s, Y) \in failures_{\perp}(P)$

F3 $(s, X) \in failures_{\perp}(P) \wedge \forall a \in Y . s\hat{}\langle a \rangle \notin traces_{\perp}(P)$
$\Longrightarrow (s, X \cup Y) \in failures_{\perp}(P)$

F4 $s\hat{}\langle \checkmark \rangle \in traces_{\perp}(P) \Longrightarrow (s, \Sigma) \in failures_{\perp}(P)$

D1 $s \in divergences(P) \cap \Sigma^* \wedge t \in \Sigma^{*\checkmark}$
$\Longrightarrow s\hat{}t \in divergences(P)$

D2 $s \in divergences(P) \Longrightarrow (s, X) \in failures_{\perp}(P)$

D3 $s\hat{}\langle \checkmark \rangle \in divergences(P) \Longrightarrow s \in divergences(P)$

Example 4 (Prove the following (in)equivalences)

$P \ = \mathsf{div}$ (e.g. $P = (a \rightarrow P) \setminus \{a\}$)

$P' = \mathsf{div} \ ||| \ (a \rightarrow Stop)$

$P =_{\mathcal{FD}} P\prime$ $P \neq_{\mathcal{T}} P\prime$

$Q \ = \ Stop$

$Q' = \ Stop \sqcap \mathsf{div}$

$Q =_{\mathcal{T}} Q\prime$ $Q =_{\mathcal{F}} Q\prime$ $Q \neq_{\mathcal{FD}} Q\prime$

Example 5 (Process Equivalence Some algebraic laws)

$$P \,\Box\, P = P \qquad \langle \Box\ \text{–idem} \rangle$$

$$P \sqcap P = P \qquad \langle \sqcap\ \text{–idem} \rangle$$

$$P \,\Box\, Q = Q \,\Box\, P \qquad \langle \Box\ \text{–sym} \rangle$$

$$P \sqcap Q = Q \sqcap P \qquad \langle \sqcap\ \text{–sym} \rangle$$

$$P \,\Box\, (Q \,\Box\, R) = (P \,\Box\, Q) \,\Box\, R \qquad \langle \Box\ \text{–assoc} \rangle$$

$$P \sqcap (Q \sqcap R) = (P \sqcap Q) \sqcap R \qquad \langle \sqcap\ \text{–assoc} \rangle$$

$$P \,\Box\, (Q \sqcap R) = (P \,\Box\, Q) \sqcap (P \,\Box\, R) \qquad \langle \Box\ \text{–dist} \rangle$$

$$P \sqcap (Q \,\Box\, R) = (P \sqcap Q) \,\Box\, (P \sqcap R) \qquad \langle \sqcap - \Box\ \text{–dist} \rangle$$