3. Inefficiency of Equilibria

Motivations

Earlier examples showed us that the outcome resulting from rational behaviour by self-interested players can be very far from “optimal”.

Ex. in Prisoner’s dilemma, NE (-3,-3) is clearly worse than (-1,-1).

Natural question: how far from optimal can equilibria be ?

Purpose of this chapter is to define a measure of the inefficiency of equilibria, and to apply this notion to an interesting class of games.

Defining optimality

Need to define what it means for an outcome to be optimal.

Clearly this depends on the type of game we are considering.

Solution: introduce an objective function (to maximize / minimize)

Optimal outcomes = outcomes which optimize objective function

Commonly used objective functions:

- utilitarian: sum of the costs / payoffs for the players
  - if costs, want to minimize; if payoffs, want to maximize
- egalitarian: maximum cost / minimum payoff
  - want to minimize max cost, maximize min payoff

Quantifying inefficiency

General idea: take ratio between equilibrium and optimal outcome

$$\frac{\text{value of objective function at equilibrium}}{\text{optimal value of objective function}}$$

Will assume objective function non-negative. This means:

- if maximize, then ratio is always ≤ 1
- if minimize, then ratio is always ≥ 1

Quantifying inefficiency

General idea: take ratio between equilibrium and optimal outcome

$$\frac{\text{value of objective function at equilibrium}}{\text{optimal value of objective function}}$$

We need to decide upon:

- which objective function to use
- which type of equilibrium we want to study
- how to handle case of multiple equilibria

Quantifying inefficiency

General idea: take ratio between equilibrium and optimal outcome

$$\frac{\text{value of objective function at equilibrium}}{\text{optimal value of objective function}}$$

We need to decide upon:

- which objective function to use   utilitarian / egalitarian
- which type of equilibrium we want to study   Nash
- how to handle case of multiple equilibria

Price of anarchy and price of stability

Multiple equilibria: pessimistic and optimistic approaches

Price of anarchy (poa): consider worst-case

$$\frac{\text{value of objective function at worst equilibrium}}{\text{optimal value of objective function}}$$

Price of stability (pos): consider best-case

$$\frac{\text{value of objective function at best equilibrium}}{\text{optimal value of objective function}}$$

Another possibility: average case

Example: Prisoner's dilemma

Nash equilibrium

	confess	don't confess
confess	-3 , -3	0 , -4
don't confess	-4 , 0	-1 , -1

Optimal outcome

1. Utilitarian objective function = sum of years in prison

$$\text{poa} = \text{pos} = \frac{6}{2} = 3$$

2. Egalitarian objective function = maximum years in prison

$$\text{poa} = \text{pos} = \frac{3}{1} = 3$$

Example: Stag Hunt

	stag	hare
stag	4 , 4	0 , 3
hare	3 , 0	3 , 3

Utilitarian objective function: sum of payoffs

Price of anarchy?

Price of stability?

Example: Stag Hunt

	stag	hare
stag	4 , 4	0 , 3
hare	3 , 0	3 , 3

Utilitarian objective function: sum of payoffs

Price of anarchy?

$$\text{poa} = \frac{6}{8} = \frac{3}{4}$$

Price of stability?

Example: Stag Hunt

	stag	hare
stag	4 , 4	0 , 3
hare	3 , 0	3 , 3

Utilitarian objective function: sum of payoffs

Price of anarchy?

$$\text{poa} = \frac{6}{8} = \frac{3}{4}$$

Price of stability?

$$\text{pos} = \frac{8}{8} = 1$$

Price of anarchy unbounded in general

Nash equilibrium

	confess	don't confess
confess	-n , -n	0 , -n-1
don't confess	-n-1 , 0	-1 , -1

Optimal outcome

For utilitarian and egalitarian objective functions:

$$\text{poa} = \text{pos} = \frac{2n}{2} = n$$

Question: Can we find natural classes of games in which poa is bounded ?

Selfish routing games

Selfish routing games are an important class of games, which can be used to study how self-interested players route traffic in a congested network (e.g. vehicle traffic, internet routing).

A (non-atomic) selfish routing game consists of:

- a directed graph $G = (V, E)$
- a finite set $(s_1, d_1), ..., (s_k, d_k)$ of source-destination pairs
- a vector $r \in \mathbb{R}^k$, where r_i indicates the amount of traffic to be routed from s_i to d_i
- for each edge $e \in E$, a cost function $c_e : \mathbb{R}^+ \to \mathbb{R}^+$, describing the delay as a function of the traffic on the edge

Selfish routing games, continued

We define $\mathcal{P}_i$ as the set of paths in G from s_i to d_i.

We let $\mathcal{P} = \bigcup_{i=1}^{k} \mathcal{P}_i$.

A (feasible) flow is a non-negative vector indexed by $\mathcal{P}$ which specifies the amount of traffic on each path and is such that $\sum_{q \in \mathcal{P}_i} f_q = r_i$ for every $1 \leq i \leq k$.

We use f_e to denote the flow on edge e: $f_e = \sum_{q \in \mathcal{P}; e \in q} f_q$.

Cost of a path q is defined as follows: $c_q(f) = \sum_{e \in q} c_e(f_e)$.

Cost of a flow f: $C(f) = \sum_{q \in \mathcal{P}} c_q(f) \cdot f_q = \sum_{e \in E} c_e(f_e) \cdot f_e$

Optimal and equilibrium flows

We define an optimal flow to be a flow of minimal cost.

In order to study the price of anarchy in routing games, we also need to define an equilibrium notion for this setting.

A flow f is an equilibrium flow if for every source-destination pair (s_i, d_i) and every pair of $s_i - d_i$ paths $q, q' \in \mathcal{P}_i$ with $f_q > 0$, we have

$$c_q(f) \leq c_{q'}(f)$$

In other words, each path which is used in the equilibrium flow f must have minimum possible cost (w.r.t. f).

Same general idea as Nash equilibria.

Properties of equilibrium flows

Existence:

Every selfish routing problem has at least one equilibrium flow.

Uniqueness:

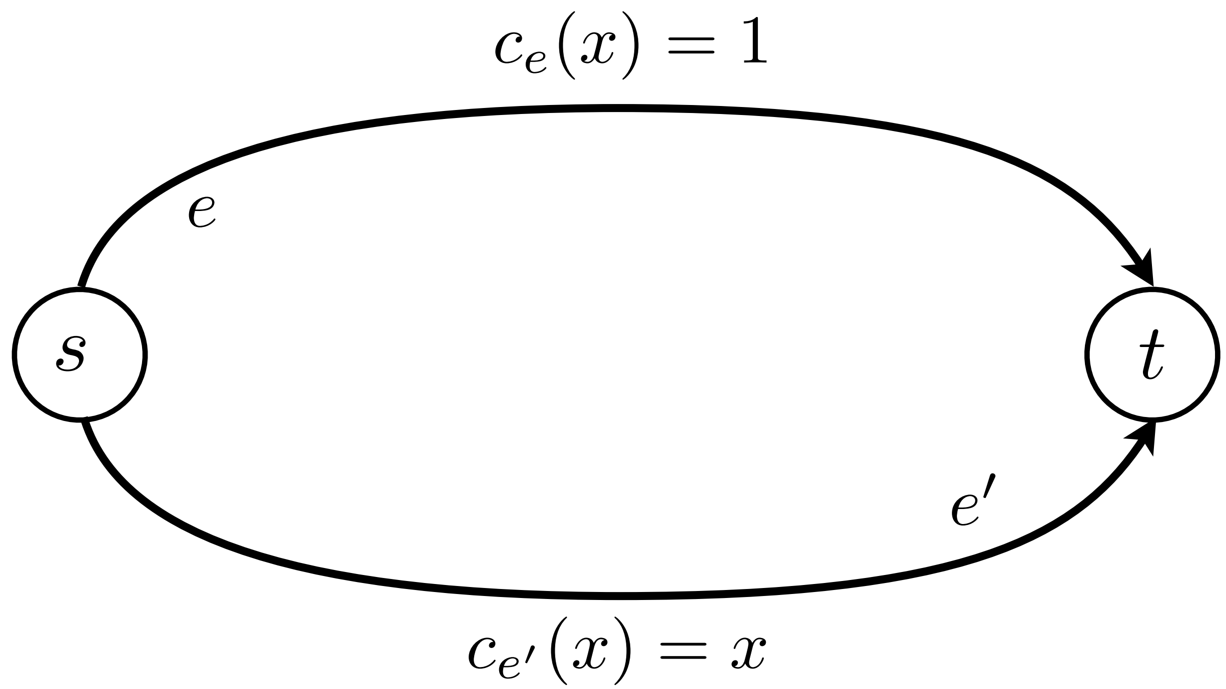
All equilibrium flows have the same cost.

More formally:

If f and g are both equilibrium flows, then $c_e(f) = c_e(g)$ for every edge e.

Pigou's example

There is a single source-destination pair (s, t) with 1 unit of traffic to be routed.



What are the flow equilibria in this routing game ?

Pigou's example

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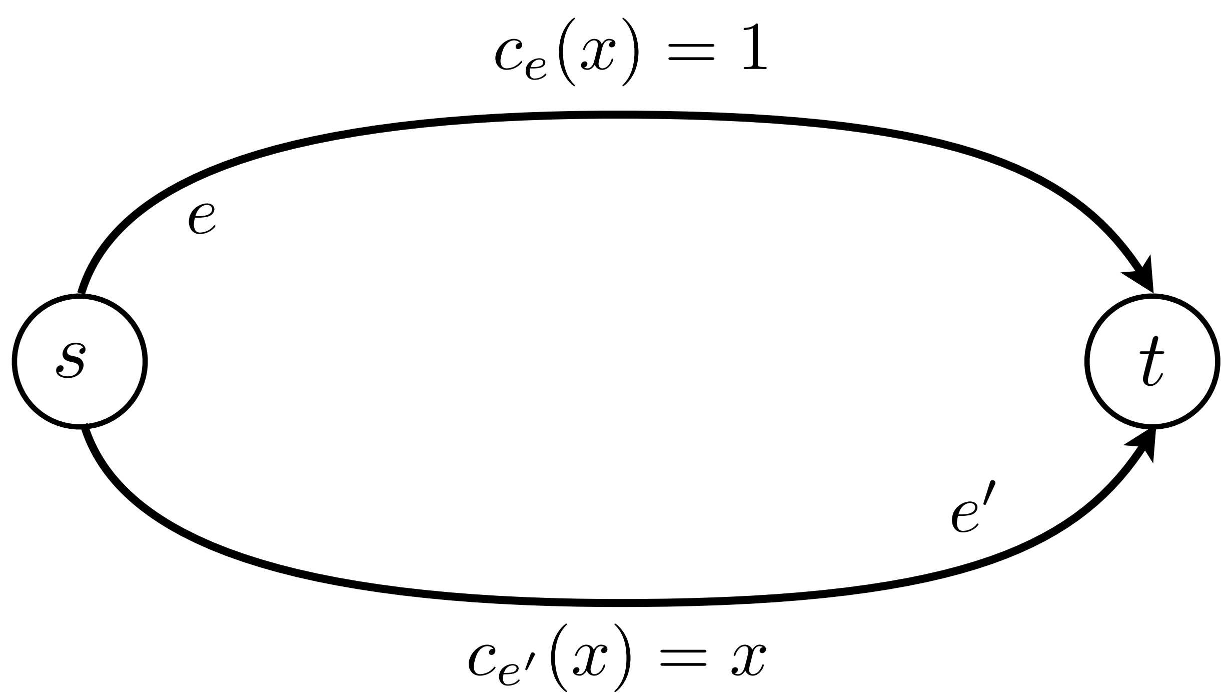

What are the flow equilibria in this routing game ?

<u>A single flow equilibrium in which all traffic is routed via bottom edge</u>.

If any traffic uses upper edge, then delay on lower edge will be < 1, so traffic on the upper edge will experience non-minimal cost.

Pigou’s example

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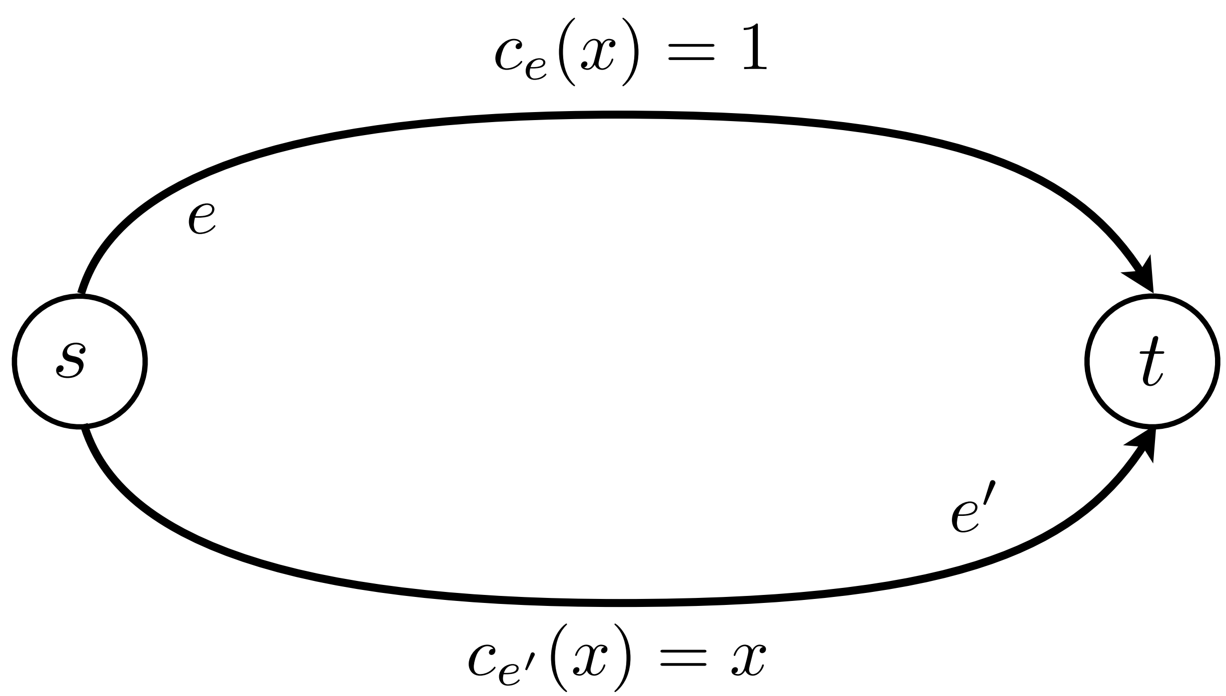
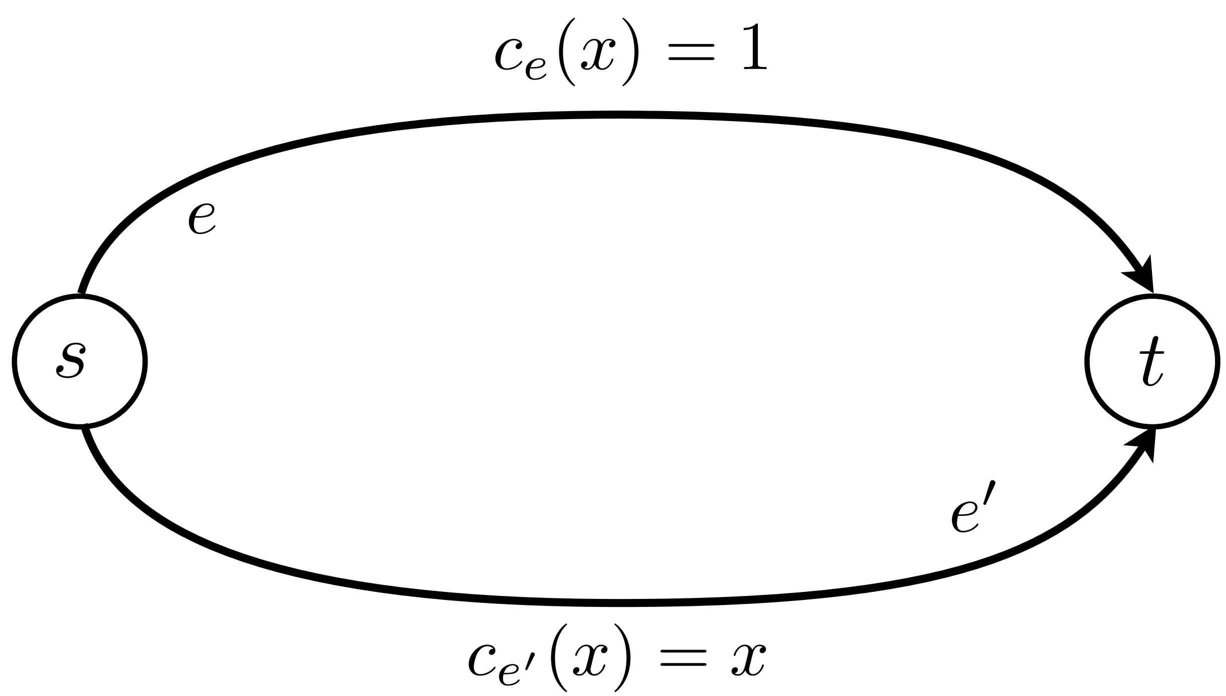


What is the optimal cost in this routing game ?

Pigou's example

There is a single source-destination pair (s, t) with 1 unit of traffic to be routed.



What is the optimal cost in this routing game ?

Split traffic evenly between upper and lower edges.

Cost of this flow: $\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}$

Pigou's example

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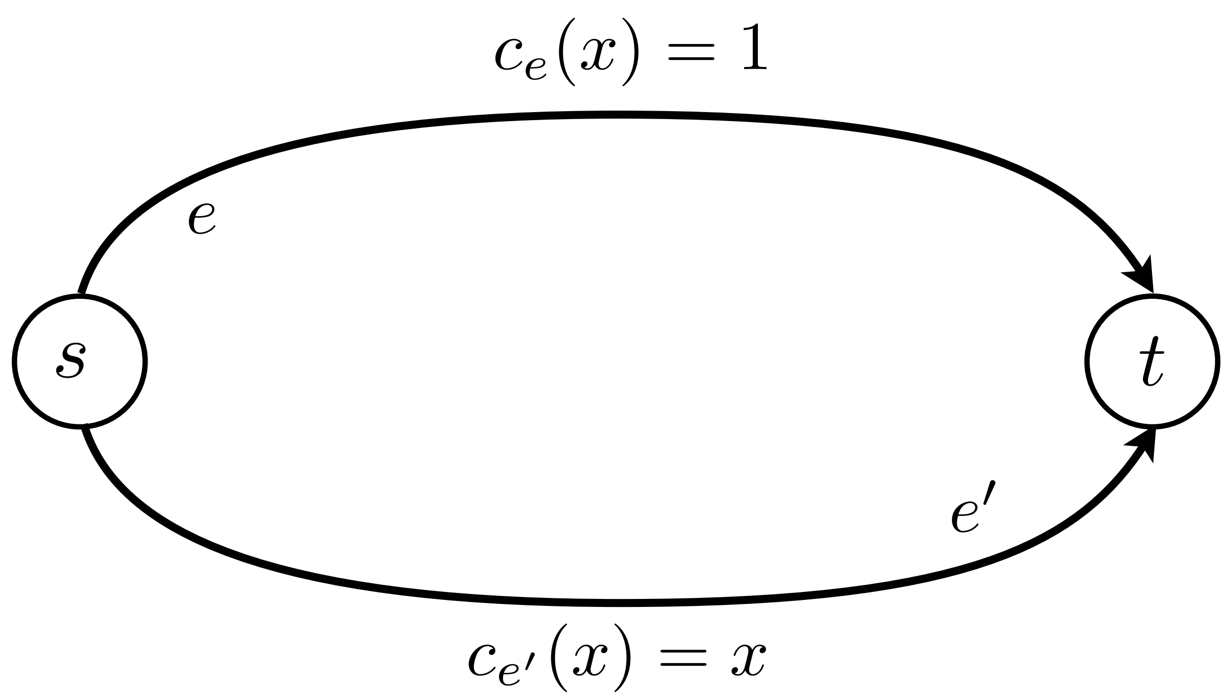
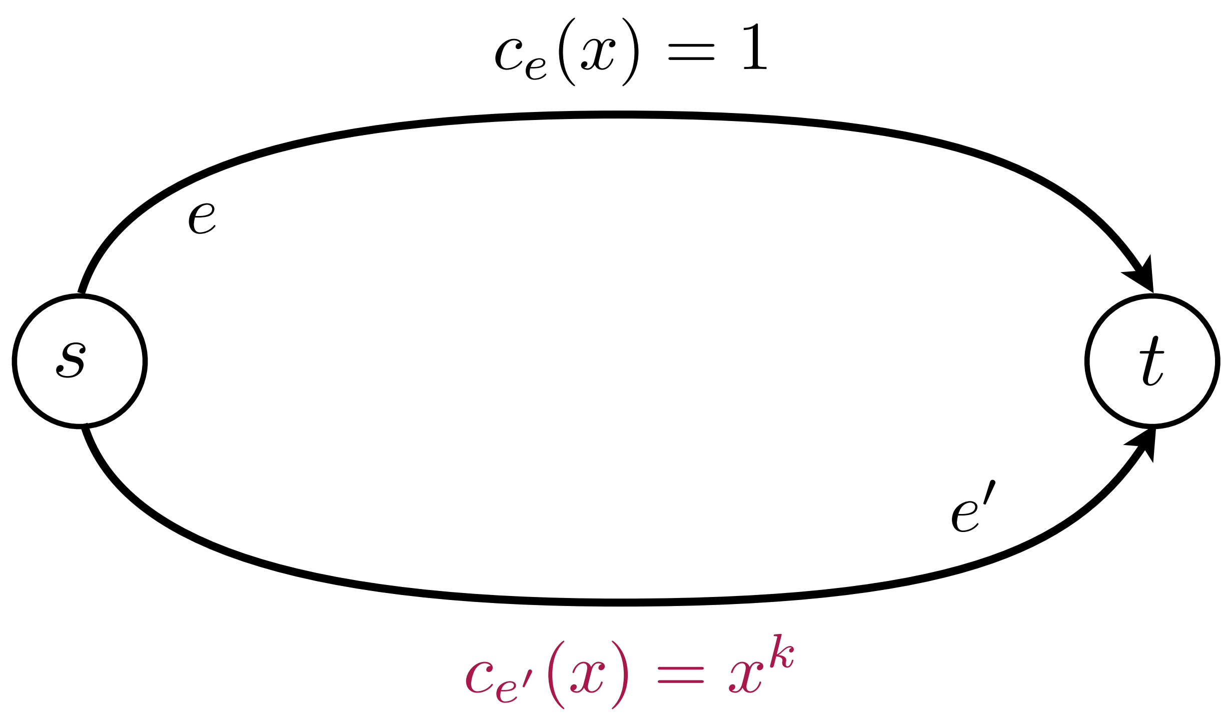


What is the price of anarchy in this routing game ?

$$\frac{\text{Cost of equilibrium flow}}{\text{Cost of optimal flow}} = \frac{1}{\frac{3}{4}} = \frac{4}{3}$$

Variant of Pigou's example

There is a single source-destination pair (s, t) with 1 unit of traffic to be routed.



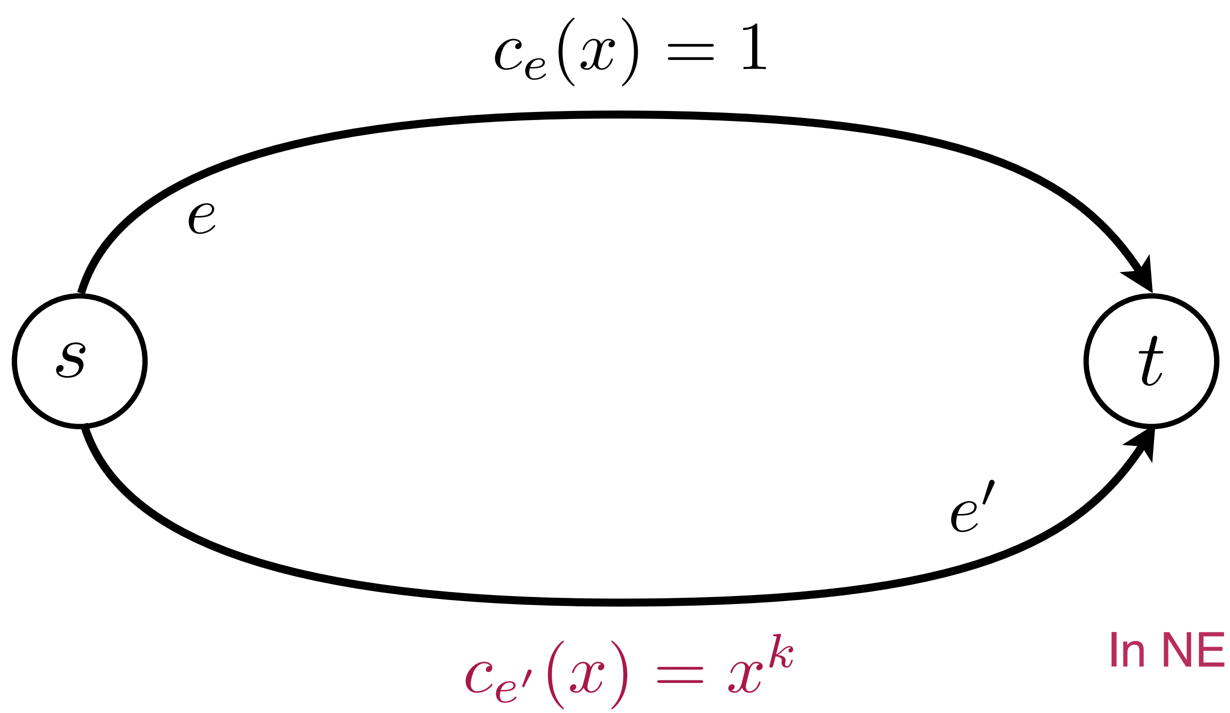
What is the optimal cost in this routing game ?

Cost approaches 0 as k approaches infinity

(put ϵ on top edge, $1 - \epsilon$ on bottom)

Variant of Pigou's example

There is a single source-destination pair (s, t) with 1 unit of traffic to be routed.



In NE, all players choose lower edge

What is the price of anarchy in this routing game ?

poa tends to infinity as k grows

Positive result

A linear cost function is of the form $c_e(x) = ax + b$ (a, b constants).

Theorem

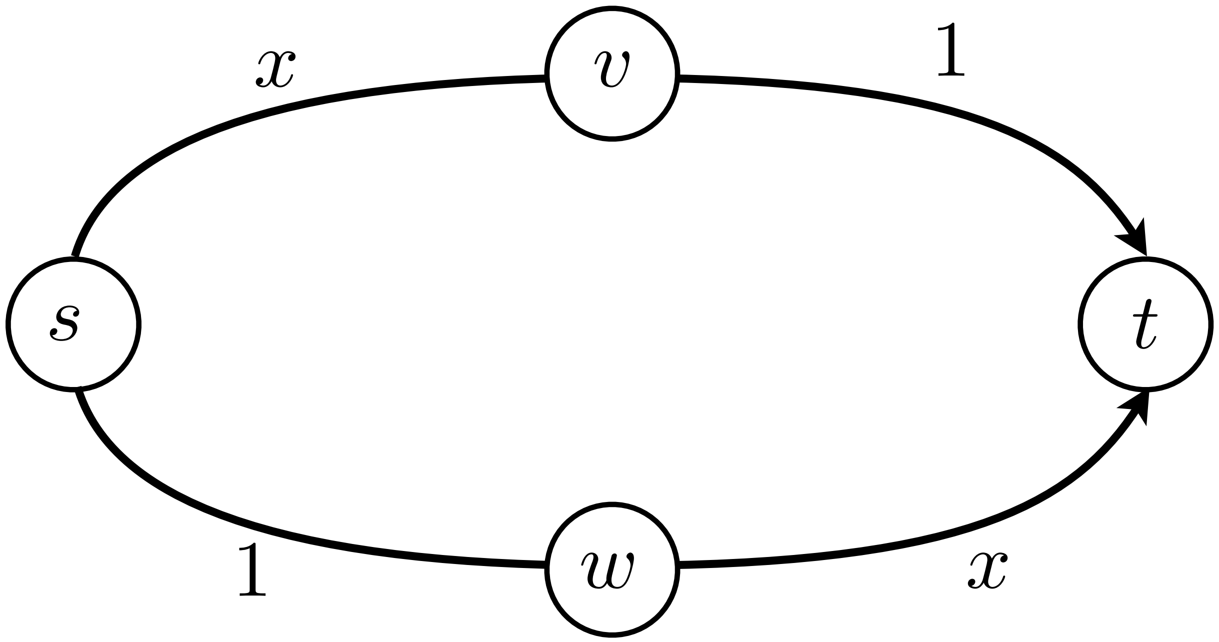
For every selfish routing problems with linear cost functions:

$$\text{cost of Nash flow} \leq \frac{4}{3} \text{ cost of optimum flow}$$

i.e. poa at most $\frac{4}{3}$

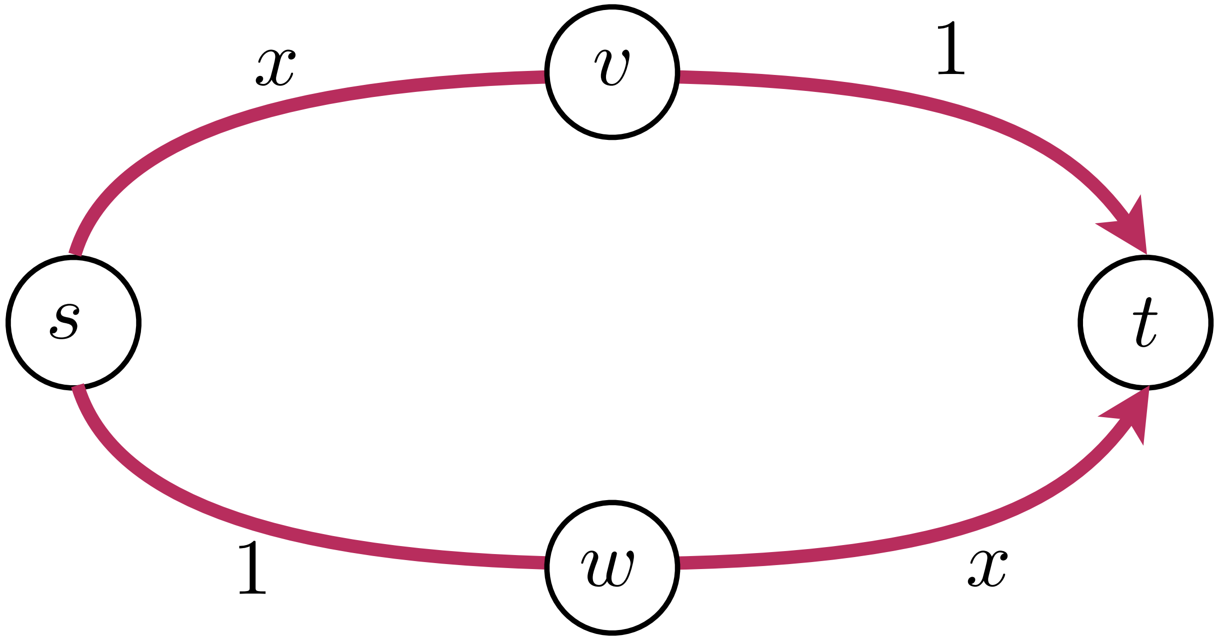
Nice thing is the network structure can be as complicated as we like, as long as we use only linear cost functions for edges.

Braess' paradox



What are the flow equilibria in this game ?

Braess' paradox

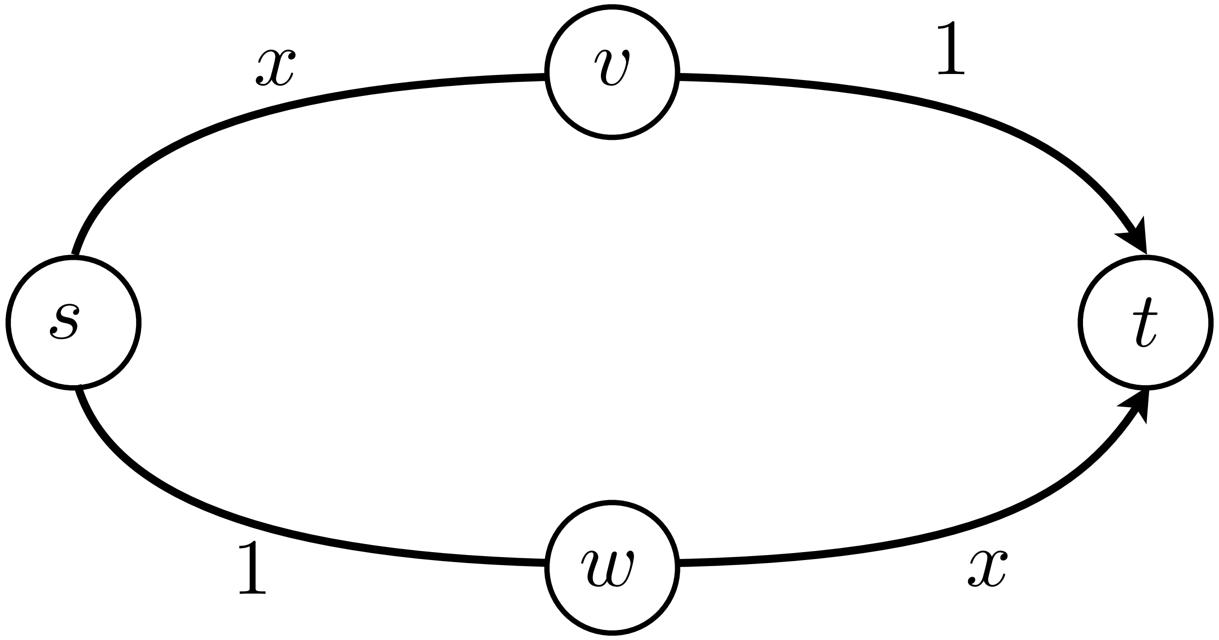


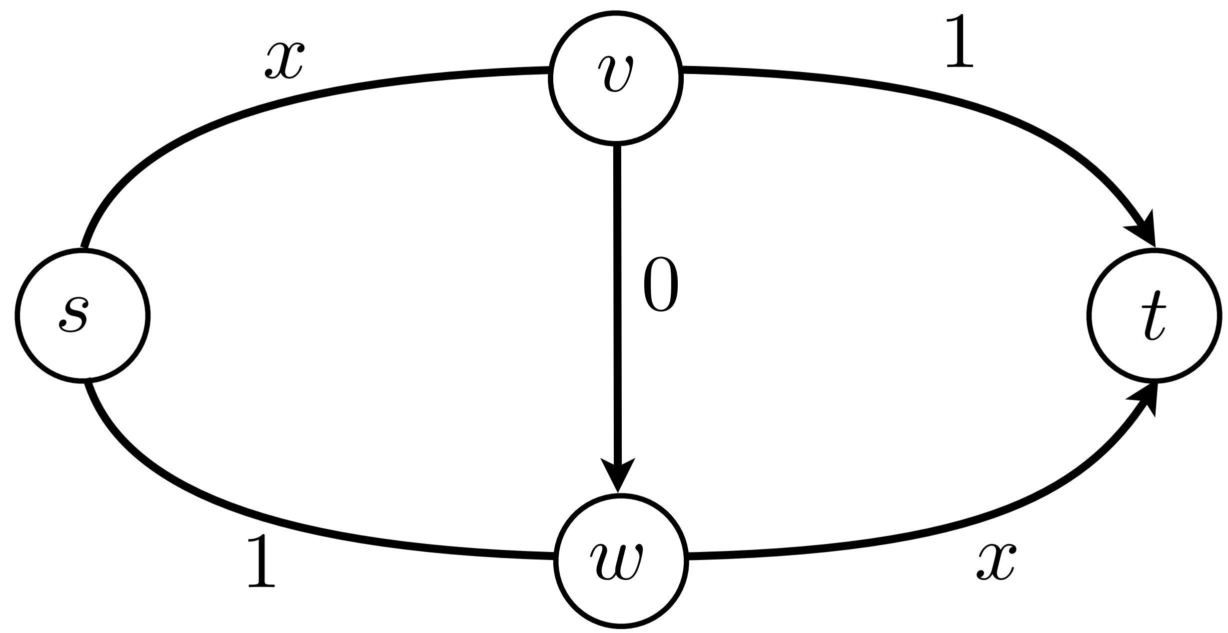
What are the flow equilibria in this game ?

A single flow equilibrium with traffic evenly split between the two routes.

Cost of equilibrium flow = 1.5

Braess' paradox

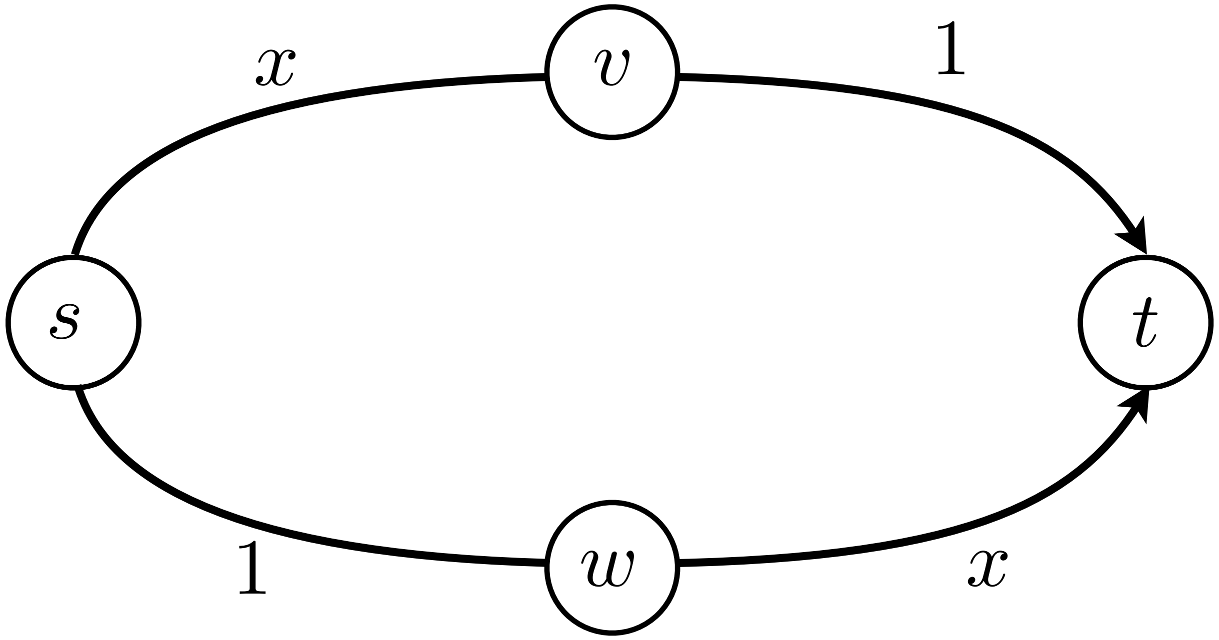


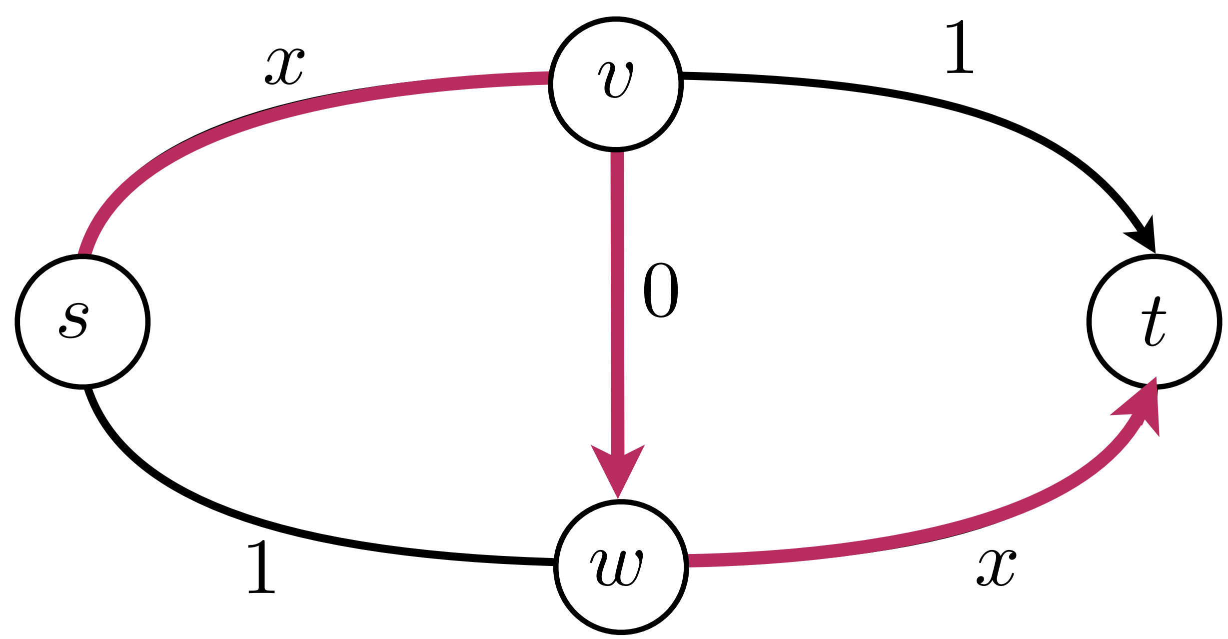


Cost of equilibrium flow = 1.5

What are the flow equilibria in this game ?

Braess' paradox





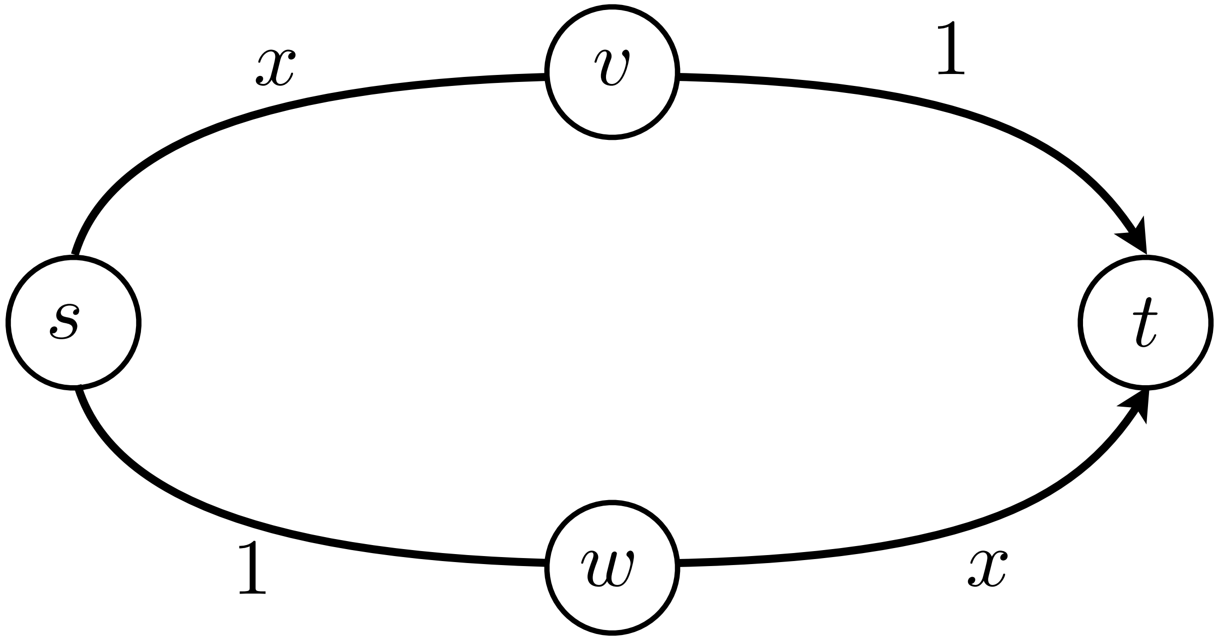
Cost of equilibrium flow = 1.5

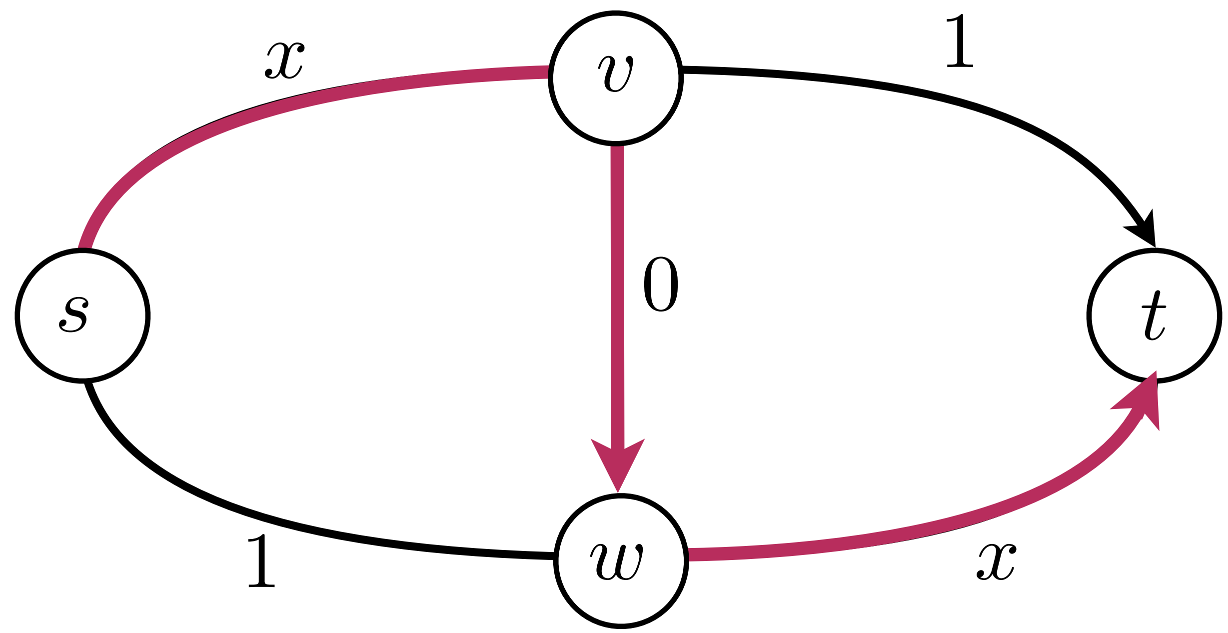
What are the flow equilibria in this game ?

<u>A single flow equilibrium with all traffic using the new path.</u>

Cost of equilibrium flow = 2

Braess' paradox





Cost of equilibrium flow = 1.5

Cost of equilibrium flow = 2

Adding new edges can worsen congestion !

Reducing cost of selfish routing

Saw on the previous slide that sometimes extra edges can worsen the situation.

Therefore, one might want to be able to test whether a network could be improved by removing some of its edges.

Unfortunately, this turns out to be a hard problem !

Theorem

It is NP-complete to decide whether there exists a set of edges whose removal would reduce the social cost of the equilibrium flow in a selfish routing problem.