1. Supply the missing details for the proofs of soundness (Slide 61) and completeness (Lemma on Slide 64):
   a) Show that the six basic rules preserve the consequence relation. For example, for \((\land 2)\), show: if \(X \models \alpha \land \beta\), then \(X \models \alpha, \beta\).
   (This is almost trivial for (IS), (MR), \((\land 1)\), \((\land 2)\), and not difficult for \((\lnot 1)\), \((\lnot 2)\).)
   b) Show properties \(C^+\) and \(C^-\).

2. Show that every set can be totally ordered. That is, show the following proposition:

   For every set \(M\), there is an irreflexive, transitive, and connex relation \(<\) on \(M\).
   \((<\) is a connex relation on \(M\) if, for all \(a, b \in M\), it holds that \(a < b\) or \(b < a\) or \(a = b\).)

   a) As a warm-up, show the proposition for finite \(M\) via induction on \(n\).
   b) For infinite \(M\), proceed according to the following schedule, using the propositional compactness theorem.
   - For every pair \((a, b) \in M \times M\), introduce a new propositional variable \(p_{ab}\) that represents the statement \(a < b\).
   - Construct an infinite set \(X\) of propositional formulas that expresses irreflexivity, transitivity and connexity of \(<\).
     For example, reflexivity of \(<\) would be expressed by \(\{p_{aa} \mid a \in M\}\).
   - Explain how the models of \(X\) correspond exactly to the possible orders \(<\) on \(M\).
   - Show that every finite subset of \(X_0\) has a model.
   - Use the propositional compactness theorem to conclude that \(X\) is satisfiable.

3. Prove the following in the Hilbert calculus.
   a) \(\vdash \alpha \rightarrow (\beta \rightarrow \alpha)\)
   b) \(\vdash \top\)