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From syllogism to common sense ...

Exercise Sheet 8: Modal logic

To be discussed on 26 January 2012

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1. Consider the frame $F = (W, R)$ with $W = \{x_1, \ldots, x_5\}$ and $R = \{(x_i, x_j) \mid i = 1, 2, 3, 4\}$, and the valuation β with $\beta(p) = \{x_2, x_3\}$, $\beta(q) = \{x_1, \ldots, x_5\}$, and $\beta(r) = \emptyset$. Let $M = (F, \beta)$. Which of the following claims hold, which don't? Use the definition of "$\models$" on Slide 8.

a) $M_{x_1} \models \Diamond\Box p$

b) $M_{x_1} \models \Diamond\Box p \to p$

c) $M_{x_2} \models \Diamond(p \wedge \neg r)$

d) $M_{x_1} \models q \wedge \Diamond(q \wedge \Diamond(q \wedge \Diamond(p \wedge \Diamond q)))$

2. a) Show that $\Box\varphi$ is equivalent to $\neg\Diamond\neg\varphi$, for any formula φ.

That is, use the definition of "$\models$" on Slide 8 to show: for all pointed models M_x, it holds that $M_x \models \Box\varphi$ if and only if $M_x \models \neg\Diamond\neg\varphi$.

b) Show: $\Box(\varphi\vee\psi) \to (\Box\varphi\vee\Box\psi)$ is equivalent to $\neg\big(\Box(\varphi\vee\psi)\wedge\Diamond\neg\varphi\wedge\Diamond\neg\psi\big)$.

Use only known propositional equivalences and the equivalence in a).

3. Show that the necessitation rule preserves validity, i.e., $\models \varphi$ implies $\models \Box\varphi$ for all formulas φ.

Remember the "four layers" of satisfaction/validity:

- $M_x \models \varphi$, for pointed models M_x, is given on Slide 8.
- $M \models \varphi$, for a model $M = (W, R, \beta)$: for all $x \in W$, $M_x \models \varphi$.
- $F \models \varphi$, for a frame $F = (W, R)$: for all models $M = (W, R, \beta)$, $M \models \varphi$.
- $\models \varphi$: for all frames F, $F \models \varphi$.

4. Use the tableau method (Slides 10–15) to show that the following formula is unsatisfiable: $p \to \big(\Diamond(p \wedge q) \wedge \Diamond(p \wedge \neg q) \wedge \Box(p \to q)\big)$

5. Show that each of the following formulas is not valid, using tableaux.

a) $\Box\bot$ (as in propositional logic, $\bot$ abbreviates $p \wedge \neg p$)

b) $\Diamond p \to \Box p$

c) $p \to \Box\Diamond p$

d) $\Diamond\Box p \to \Box\Diamond p$

For each formula, find a non-empty set of frames on which it is valid.