Outline of Lecture 10

- Basics of First-order logic
- A sketch of Description Logics and the relation to First-order logic and Modal Logic
- Combining quantifiers with modalities
- Counterpart Theory

First-Order Logic (FOL)

- FOL is an expressive, general purpose language
- With historical roots in
  - Aristotelian Syllogisms
    - e.g. conclusions inferred from two (quantificational) premises
  - Boole’s logic
    - e.g. the basic algebraic rules governing conjunction, negation, etc. (1854)
  - Frege’s Begriffsschrift
    - a fully formal notation for logic encompassing modern first-order (1879)
  - Peirce’s logical investigations
    - e.g. the distinction between first- and second-order quantifier (1885)
- Important historically in axiomatising foundational theories in mathematics
First-Order Logic

- The ‘axiomatic method’:
  - completely abstract description of a domain by specification of what objects exist and what their properties are, e.g.:
    - Abstract Algebra (groups, fields, vector spaces) (e.g. Galois 1832)
    - Euclid’s Geometry as axiomatised by David Hilbert (1899)
    - Set Theory, e.g. ZFC (1908-1930) or NBG (1920s-1940s)
  - also applied in
    - specification of software
    - logic programming
    - axiomatisation of upper/foundational ontologies

FOL: More History

- Gödel’s completeness theorem, proved by Kurt Gödel in 1929
- Undecidability: Alonzo Church and Alan Turing in 1936 and 1937, respectively, gave a negative answer to the Entscheidungsproblem posed by David Hilbert in 1928.
- Expressivity: By the Löwenheim–Skolem theorem, there is no FOL theory that has as its unique model the natural numbers.

Many-Sorted First-Order Logic: Syntax

- Non-logical symbols (signatures)
  - a signature $\Sigma$ is a triple $<S, F, P>$ with
    - $S$ a set of sorts, and $S^*$ the set of words over $S$
    - for each $w \in S^*$, a subset $F_{w, s} \subseteq F$ of function symbols
    - for each $w \in S^*$, a subset $P_w \subseteq P$ of predicate symbols
    - constants of sort $s$ are the nullary functions in $F_{w,s}$
- Logical symbols
  - a set $X_s$ of variables for each sort $s$.
  - the Boolean operators, conjunction, negation, etc.
  - the identity symbol $=$ and the quantifiers ‘for all’ $\forall$ and ‘exists’ $\exists$.
- Formulae are constructed in the usual way respecting typing

Many-Sorted First-Order Logic: Semantics

- Given a signature $\Sigma = <S, F, P>$, a $\Sigma$-model $M$ consists of:
  - A carrier set $M_s \neq \emptyset$ for each sort $s \in S$
  - A function $f^M_{w,s} : M_{s_1} \times \ldots \times M_{s_n} \to M_s$ for each $f \in F_{w,s}$, where $w = s_1 \cdots s_n$.
    In particular, for a constant, this is just an element of $M_s$.
  - A relation $p^M_{w} \subseteq M_{s_1} \times \ldots \times M_{s_n}$ for each $p \in P_{w}$, $w = s_1 \cdots s_n$.
    ‘Standard’ FOL has simply just one sort (single-sorted).
Natural Language Examples

› “Every person who lives in Bremen lives in Germany”
› Pick a unary predicate Person
› Two constants Bremen and Germany
› A binary relation lives-in

› Formalise as:
› \( \forall x . \text{Person}(x) \land \text{lives-in}(x, \text{Bremen}) \rightarrow \text{lives-in}(x, \text{Germany}) \)
› Alternative formalisations:
› Axiomatise: Cities, Countries, Containment (Parthood)
› Limitations: Quantification over predicates, modalities, constructions such as ‘terribly small’, ‘walking quickly’, etc.

FOL Example: Mereology in Dolce

› Parthood is a partial order, i.e. a reflexive, antisymmetric, transitive binary relation
› Generic mereology for sort \( s \), and mereology for sorts \( T, S, PD \)

\[
\text{spec GenMereology} [\text{sort } s] = \\
\text{spec GenParthood} [\text{sort } s]
\]
then

\[
\text{preds } PP(x, y : s) \equiv P(x, y) \land \neg P(y, x); \\
O(x, y : s) \equiv \exists z : s \cdot P(z, x) \land P(z, y); \\
A(x : s) \equiv \neg \exists y : s \cdot PP(y, x);
\]
then

\[
\forall x, y : s \cdot \\
\neg P(x, y) \Rightarrow (\exists z : s \cdot P(z, x) \land \neg O(z, y)) \\
\exists z : s \cdot A(z) \land P(z, x)
\]
then %implies

\[
\forall x, y, z, z' : s \cdot \\
(\forall z' : s \cdot A(z') \Rightarrow P(z', x) \Rightarrow P(z', y)) \Rightarrow P(x, y)
\]
end

Description Logics

› DLs focus on the representation of terminological knowledge:
› formalise the basic terminology adopted in an application
› Terminologies are formalised as a collection of concepts and relations
› e.g. ‘Course’, ‘Lecturer’, and ‘gives_course’, ‘attends_lecture’
› DL knowledge bases define basic concepts and give relationships between them in the form of subsumptions

The Description Logic \( \mathcal{ALC} \)

› Atomic symbols:
› concept names (unary predicates): \( A, B, C, D, \ldots \)
› role names (binary predicates): \( R, S, T, \ldots \)
› Concept constructors:
› Top/Bottom \( \top, \bot \) \hspace{2cm} Manchester Syntax (HETS)
› negation \( \neg C \) \hspace{2cm} Thing, Nothing
› conjunction \( C \sqcap D \) \hspace{2cm} C and D
› disjunction \( C \sqcup D \) \hspace{2cm} C or D
› existential restriction \( \exists R.C \) \hspace{2cm} R some C
› value restriction \( \forall R.C \) \hspace{2cm} R only C
› Complex concepts:
› \( \neg (A \sqcup \exists R.(\forall S.B \sqsupset \neg C)) \)
› For example: Human \( \sqcap \exists \) Lives-in Bremen \( \sqsubseteq \exists \) Lives-in Germany
Semantics of ALC

- A model is a pair <W, I> where W is a set and I an interpretation function
  - assigning subsets of W to concept names
  - assigning subsets of W × W to role names
- Concept constructors:
  - W, empty set Ø
  - set complement
  - set intersection
  - set union
  - { v ∈ W | ∃ w ∈ W . v R w ∧ C(w) }:
    - ∃R.C
  - { v ∈ W | ∀ w ∈ W . v R w → C(w) }:
    - ∀R.C
- Definition 2.2.16. Model

Example: Pizzas

- Here is a small excerpt from the pizza ontology:
  VegetarianPizza  ⊑ Pizza
  MagheritaPizza  ⊑ Pizza
  TomatoTopping  ⊑ VegetableTopping
  MozzarellaTopping  ⊑ CheeseTopping
  VegetarianPizza  ⊑ ∃ hasTopping (VegetableTopping ∪ CheeseTopping)
  MagheritaPizza  ⊑ ∃ hasTopping MozzarellaTopping ∪
  ∃ hasTopping TomatoTopping ∪
  ∀ hasTopping (MozzarellaTopping ∪ TomatoTopping)

- It follows that the following is true in all models of this Tbox:
  MagheritaPizza  ⊑ VegetarianPizza

OWL: DL SROIQ

- The web ontology language OWL uses the logic SROIQ, adding e.g.:

  \[ \phi ::= \ldots R_1 \ldots R_n \subseteq R \quad \text{ObjectProperty: } R \quad R^T_1 \ldots R^T_n \subseteq R^T \]
  \[ \text{Dis}(R_1, R_2) \quad \text{Disjoint } R_1, R_2 \]
  \[ \text{Rn}(R) \quad \text{Reflexive } R \]
  \[ \text{IrR}(R) \quad \text{Irreflexive } R \]
  \[ \text{Asy}(R) \quad \text{Asymmetric } R \]

  where \( R \circ S = \{(x, z) \exists y (x, y) \in R, (y, z) \in S\} \)

  The new concept \( \exists R . \text{Self} \) with \( (\exists R . \text{Self})^T = \{ x | x \in \Delta^T, (x, x) \in R^T \} \)

  For more on DLs and OWL, see e.g.:
  - http://dl.kr.org/courses.html

Logic Translations

- How do we move from one logic to another?
  - change of syntax
  - change of semantics
- Requirements
  - preserve the meaning of the original formalisation
  - models of the original formulas should be ‘obtainable’ from the models of the translated formulas
Example: The Standard Translation

- The standard translation T from ALC to FOL maps
- Concepts names $A$ → unary predicates $P_A$
- Role names $R$ → binary predicates $P_R$
- Object names $a$ → constants $c_a$
- and uses the following translation rules for complex concepts:
  $$\begin{align*}
  T^x(A) &= P_A(x) \\
  T^x(\neg C) &= \neg T^x(C) \\
  T^x(C \sqcap D) &= T^x(C) \land T^x(D) \\
  T^x(C \sqcup D) &= T^x(C) \lor T^x(D) \\
  T^x(\exists R.C) &= \exists y. P_R(x, y) \land T^y(C) \\
  T^x(\forall R.C) &= \forall y. P_R(x, y) \rightarrow T^y(C)
  \end{align*}$$
- Concepts correspond to formulas with exactly one free variable.
- $T^y$ is just like $T^x$ with $x$ and $y$ interchanged
- Note that two variables are enough for the translation

Modal Logic and DL

- Modal (propositional) logic adds sentential operators like 'possibly', or 'necessarily', to propositional logic:
  $$\begin{align*}
  ALC &\cong \text{multimodal logic } K \\
  \text{concept } C &\text{ formula } \varphi \\
  \text{atomic concept } A &\text{ propositional variable } p \\
  C \sqcap D &\varphi \land \psi \\
  C \sqcup D &\varphi \lor \psi \\
  \neg C &\neg \varphi \\
  \exists R.C &\Diamond_R \varphi \\
  \forall R.C &\Box_R \varphi
  \end{align*}$$
- The correspondence on the right gives rise to a logic translation.

Heterogeneous Ontologies

- In order to systematically link ontological modules formulated in different formalisms we need to:
  - fix a logic graph
  - give logic translations (institution co-morphisms)

Modality and Quantification

Basic Problems

Barcan Formulae

Lambda Abstraction

Counterparts
Outline

- Modality, Quantification, and Identity (Aristotle, Modal Syllogistic)
- De Re/De Dicto, and Lambda Abstraction (Stalnaker, R. Thomason 1968, Fitting 1998)
- Kripke Semantics for QML (Kripke 1963)
- FOL Counterpart Theory as Semantics for QML (David Lewis 1968)

Combining Modality and Quantifiers

Example: combining modality, quantification, and identity
- All humans are necessarily mortal.
- The number of planets is necessarily 9.
- Necessarily, the number of planets is 9.
- The president of the US someday won’t be the president of the US.
- The president of the US might not have been Barack Obama.
- I could have been quite unlike what I in fact am.

Semantic complexities: necessary properties, de de, de dicto, identity across worlds, persistence over time, counterfactuality, dissimilar counterparts, etc.

Modality, Quantification, and Identity

Items that can be varied according to universal logic: signatures - grammar - models - satisfaction

- **Signatures:** (non-logical symbols) propositions; predicates; functions, constants, terms.
- **Grammar:** (logical symbols) variables and quantifiers; modalities; identity symbol; lambda abstraction; well-formed expressions (formulae); substitution.
- **Models:** possible world; domains of discourse; accessibility (counterpart relations); object (individual)
- **Satisfaction:** vary the truth conditions for quantifiers; vary conditions for identity statements, etc.

Garson’s Forest (1984)

A combination of 2 (or 3) logical theories
- modal predicate logic
- quantified modal logic
- first-order modal logic
- first-order intensional logic, etc.

Items that can be varied according to universal logic: signatures - grammar - models - satisfaction
Possible Objects and Quantification

Objects
- Worldbound or not
- Constant or Varying
- Objects/individual concepts/traces of objects/counterparts

Quantification
- Existence property: E(x)
- Vary the base logic: classical or free quantification, etc.
- Quantify over what entities?

E.g.: in free logic:
- Define actualist quantifiers from possibilist ones:
  \[ P(a) \not\equiv \exists x.P(x) \]
- but
  \[ P(a), E(a) \models \exists x.P(x) \]

(Simplest) Kripke Semantics for MPL

- Standard frames: a set W of possible worlds, an accessibility R between worlds
- A domain D of (possible individuals), the same for each w in W
- Valuation \( \text{val}(x) \) assigns (rigidly) values in D to variables x
- Standard quantification theory over D, but interpretations of predicates can vary between worlds
- Formulae are constructed in the obvious way combining FOL grammar with modalities as new sentence operators

Models are: \( M = \langle W, R, D, \text{val} \rangle \)

Satisfaction is

\[ \langle M, v \rangle \models \forall x. \phi(x) \text{ iff for all } d \in D \text{ we have } \langle M(d/x), v \rangle \models \phi(x) \]

\[ \langle M, v \rangle \models \Box \phi(x) \text{ iff for all } w \in W \text{ such that } vRw : \langle M, w \rangle \models \phi(x) \]

De Re and De Dicto

- Basic distinction in modal predicate logic since medieval times:
  - de re: res = object:
    - a property is true of an object necessarily:
    - example: the number 9 is necessarily odd
  - de dicto: dicto = statement:
    - a proposition is necessarily true:
    - example: necessarily, \( 2 + 2 = 4 \)

Semantics vs Syntax

- Modelling insufficiencies can be addressed by both semantic and syntactic modifications:
- Syntax examples: lambda abstraction to disambiguate scope of terms (next slide);
- Semantic examples: changing truth definition, e.g. interpret objects as wordlines (individual concepts) with traces etc.
Lambda Abstraction

Skolemisation of \( \Box(\exists x)\phi(x) \)

necessitates non-rigid constants in \( \Box\phi(c) \)

Skolemisation of both

\( \Box(\exists x)\phi(x) \)

\( (\exists x)\Box\phi(x) \)

yields \( \Box\phi(c) \)

Distinguish:

\( \Box(\lambda x.\phi(x))(c) \)

\( (\lambda x.\Box\phi(x))(c) \)

Lambda Abstraction

Example: Quine 53, Three Grades of Modal Involvement

Necessarily true: ‘9 greater or equal 7’

Contingently true: ‘number of planets’ is greater or equal 7

- \( (\lambda x.\Box(x \geq 7))(t) \)
- \( \Box(\lambda x.x \geq 7)(t) \)

Alternatively: make 9 rigid, but ‘number of planets’ non-rigid

Assymetry between variables and constants

The Barcan Formulae

\( \forall x.\Box P(x) \rightarrow \Box \forall x. P(x) \)

- If everything is necessarily \( P \), then it is necessary that everything is \( P \).

By duality this is equivalent to

\( \Box \exists x. P(x) \rightarrow \exists x. P(x) \)

- Validity of BF implies that all objects which exist in every possible world (accessible to the actual world) exist in the actual world, i.e. that domains cannot grow. This thesis is sometimes known as actualism, i.e. that there are no merely possible individuals.

The Converse Barcan Formulae

\( \Box \forall x. P(x) \rightarrow \forall x. \Box P(x) \)

- If it is necessary that everything is \( P \), then everything is necessarily \( P \).

By duality this is equivalent to

\( \exists x. \Box P(x) \rightarrow \Box \exists x. P(x) \)

BF: nothing comes into existence

CBF: nothing goes out of existence
**Existence, Symmetry and Barcan**

- In the constant domain Kripke semantics, both BF and CBF are valid schemas.
- But we can introduce an ‘existence predicate’ and simulate varying domains (and falsify BF/CBF).
- In symmetric frames, BF and CBF are equivalent.
- Need to introduce ‘varying domain semantics’.
- Tutorial next week

**Standard CT as Semantics for MPL**

**Postulates**

- **P1** $\forall x \forall y (I(x, y) \rightarrow W(y))$
  (“Nothing is in anything except a world”)
- **P2** $\forall x \forall y \forall z (I(x, y) \land I(x, z) \rightarrow y = z)$
  (“Nothing is in two worlds”)
- **P3** $\forall x \forall y (C(x, y) \rightarrow \exists z I(x, z))$
  (“Whatever is a counterpart is in a world”)

**The actual world**

$\@ := \exists x. \forall y (I(y, x) \leftrightarrow A(y))$.

**Standard CT as Semantics for MPL**

**Primitive Predicates**

- $W(x)$ (x is a possible world)
- $I(x, y)$ (x lies in the possible world y)
- $A(x)$ (x is the actual world)
- $C(x, y)$ (x is a counterpart of y)

David Lewis 1968:
Counterpart Theory and Quantified Modal Logic

**Standard CT as Semantics for MPL**

**The standard translation of:** $\diamond \phi(\alpha_1, \ldots, \alpha_n)$ true at beta

\[ \diamond \phi(\alpha_1, \ldots, \alpha_n) \]

\[ \exists \beta_1 \exists \gamma_1 \ldots \exists \gamma_n (W(\beta_1) \land I(\gamma_1, \beta_1) \land C(\gamma_1, \alpha_1) \land \ldots \land I(\gamma_n, \beta_1) \land C(\gamma_n, \alpha_n) \land \phi^{\beta_1}(\gamma_1, \ldots, \gamma_n)) \]
**Standard CT as Semantics for MPL**

Non valid principles:  
Valid:
1. $\Box \phi \rightarrow \Box \Box \phi$ ("Becker’s Prinzip") 
2. $\phi \rightarrow \Box \Box \phi$ ("Brouwer’s Prinzip")
3. $(x \equiv y) \rightarrow \Box (x \equiv y)$ (x verschieden von y)
4. $(x \neq y) \rightarrow \Box (x \neq y)$ (x verschieden von y)
5. $\forall x \Box \phi(x) \rightarrow \Box \forall x \phi(x)$ (Barcan Formeln)
6. $\exists x \Box \phi(x) \rightarrow \Box \exists x \phi(x)$
7. $\Box \exists x \phi(x) \rightarrow \exists x \Box \phi(x)$

---

**Failure of Box Distribution in CT**

Simultaneous quantification over worlds and individuals:

(i) $\mathcal{M} \models (\Box (\phi(x) \land \forall x \phi(x)))^m$
(ii) $\mathcal{M} \models (\forall \exists x \Box \phi(x))^m$, d.h.
(iii) $\mathcal{M} \models (\neg \Box \forall x \phi(x))^m$.

Stand. Trans. for $\Box (\phi(x))^m$ is $\forall v \forall y (W(v) \land I(y, v) \land C(y, x) \rightarrow \chi^v(y))$.

Tautology: $(\phi(x) \land \forall x \phi(x)) \rightarrow \forall x \phi(x)$

Box distribution applied: $\Box (\phi(x) \land \forall x \phi(x)) \rightarrow \Box \forall x \phi(x)$

---

**Generalised Semantics**

Standard Kripke Semantics

<table>
<thead>
<tr>
<th>Constant Domains</th>
<th>Possibility Quantifiers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Counterpart Theory</td>
<td>Lewis 1968</td>
</tr>
<tr>
<td>Functor Semantics</td>
<td>Chihara 2001</td>
</tr>
<tr>
<td>(General) Counterpart Frames</td>
<td>Kecht/Kutz 2000</td>
</tr>
<tr>
<td>(General) Coherence Frames</td>
<td>Kecht/Kutz 2001</td>
</tr>
</tbody>
</table>

Varying Domains

<table>
<thead>
<tr>
<th>Actualist Quantifiers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kripke Bundle Semantics</td>
</tr>
<tr>
<td>Cartesian Metaframes</td>
</tr>
<tr>
<td>Modal Metaframes</td>
</tr>
<tr>
<td>Hyperdoctrinal Semantics/General Metaframes</td>
</tr>
</tbody>
</table>

---

**General Completeness: Balanced Standard Kripke Semantics**

- Extend the standard Kripkean semantics for MPL and constant domains as follows:
- Add an equivalence relation $E$ such that:
  - $vEw$ implies $P(x_1,...,x_n)$ is interpreted in the same way (world-mirrors).
  - if $vRv1$ and $vEw$, then there is a $w1$ such that $wRw1$ and $v1Ew1$ (mirrored worlds upwards indistinguishable).
  - interpret identity as equivalence: equivalent objects must be indistinguishable (equivalence).
### Competing Ontologies

| Cost of Generality: |
|-------------------|-------------------|
| poly-counterpart   | balanced standard |
| accessibility     | relations between worlds |
| objects           | global universe of objects |
| identity          | equivalence between objects |
| Predication       | globally (but admissible) |
| Haecceitism       | Yes, world-mirrors |

### Literature