From Syllogism to Common Sense

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Modal and Intuitionistic Logics and Semantics

Lecture 7

Summary Propositional Logic

- We introduced the basic problems, notions and ideas related to classical propositional logic, i.e. syntax and semantics, Boolean functions, Truth tables.
- The notion of proof, Hilbert systems, natural deduction
- The notion of entailment, truth in a model, satisfaction
- Completeness:
  - Satisfaction vs. Consistency
  - equivalence of **finitary** proof theoretic notion and semantic notion of satisfiability
  - Essential idea: models are built from **syntactic material**

Outline of Lectures 7 & 8

- Logical Pluralism in modern KR & AI
  - we sketch the line from Aristotle via Frege to logical pluralism
  - introduce intuitionistic propositional logic (**IPC**)
  - some standard propositional modal logics (**MLs**)
- Non-classical logics
  - we give in particular some Hilbert and other proof systems
  - and discuss translations between logics, the Glivenko and Gödel embeddings of intuitionistic logic.
- Admissible vs. Derivable Rules
  - we will introduce the notion of admissible rules and give examples

From Aristotle And Boole to Carnap's Pluralism

Or, From Syllogistic to Classical Logic to An Explosion of Logical Calculi in Modern KR and AI

The Rise of Classical Logic

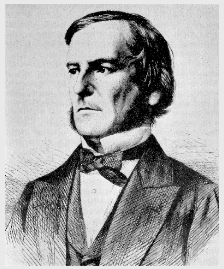
- Classical first-order logic (FOL) is an expressive, general purpose language
- Important historically in axiomatising foundational theories in mathematics
- With historical roots in
  - **Aristotelian** Syllogisms
    - e.g. conclusions inferred from two (quantificational) premises
  - **Boole's** logic
    - e.g. the basic algebraic rules governing conjunction, negation, etc. (1854)
  - **Frege's** Begriffsschrift
    - a fully formal notation for logic encompassing modern first-order (1879)
  - **Peirce's** logical investigations
    - e.g. the distinction between first- and second-order quantifier (1885)

Aristotle's Organon

Aristotle

- Aristotelian Syllogisms
  - e.g. conclusions inferred from two (quantificational) premises.
  - Modal logic, propositional and quantificational

Boolean Logic

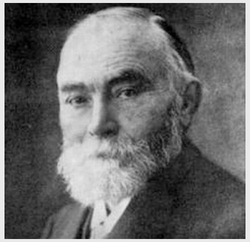


George Boole

Augustus De Morgan

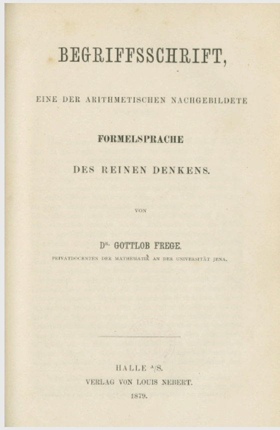
- the basic algebraic rules governing conjunction, negation, etc. (1854)
- De Morgan's Laws
- basis for modern propositional calculus

Frege's Begriffsschrift



Gottlob Frege

- 1879: Introduction of \`quantified variables'
- Essentially a formal system for classical, bivalent second-order logic with identity



BEGRIFFSSCHRIFT,

EINE DER ARITHMETISCHEN NACHGEBILDETE

FORMELSPRACHE

DES REINEN DENKENS.

VON

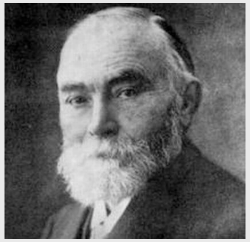
Dr. GOTTLOB FREGE.

HALLE a/S.

VERLAG VON LOUIS NEBERT.

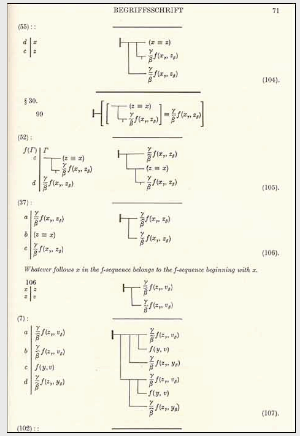
1879.

Frege's Begriffsschrift



Gottlob Frege

- 1879: Introduction of \`quantified variables'
- Essentially a formal system for classical, bivalent second-order logic with identity
- From the modern viewpoint, using a highly unusual \`two-dimensional' notation.
- Hilbert systems = Frege systems



Peirce's Logic

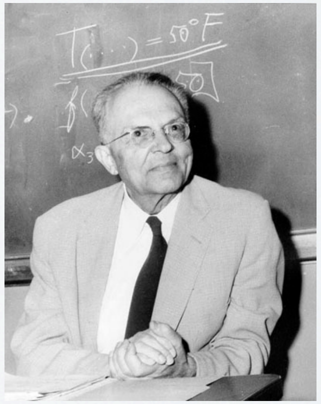
Charles Sanders Peirce

- e.g. the distinction between first- and second-order quantifier (1885)
- the distinction between modes of reasoning: abduction, deduction, induction

Logical Monism

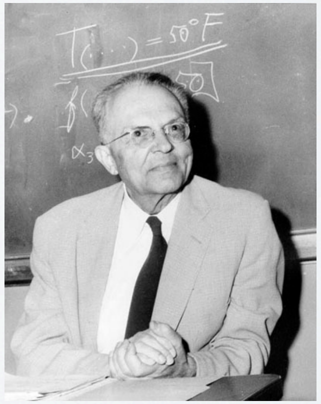
- Roughly: "There is only one true (correct/best/legitimate) logic".
- Candidates:
  - Classical Logic,
  - Intuitionistic Logic, ...
- Some problems:
  - Shouldn't a 'universal' logic contain semantic concepts, yielding paraconsistency?
  - What is the intuitionists metalogic to argue that it is the one true logic?

Carnap's Pluralism



- It is not our business to set up prohibitions, but to arrive at conventions. [...]
  In logic there are no morals. Everyone is at liberty to build up his own logic, i.e. his own language, as he wishes. All that is required of him is that, if he wishes to discuss it, he must state his methods clearly, and give syntactical rules instead of philosophical arguments.
- RUDOLF CARNAP
  The Logical Syntax of Language, 1934

Logical Pluralism ≈ no one true logic



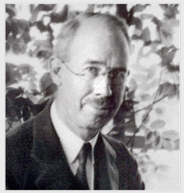
- Only later, when I became acquainted with the entirely different language forms of Principia Mathematica, the modal logic of C. I. Lewis, the intuitionistic logic of Brouwer and Heyting, and the typeless systems of Quine and others, did I recognise the infinite variety of possible language forms. On the one hand, I became aware of the problems connected with the finding of language forms suitable for given purposes; on the other hand, I gained the insight that one cannot speak of "the correct language form", because various forms have different advantages in different respects. The latter insight led me to the **principle of tolerance**.
- RUDOLF CARNAP, Intellectual Autobiography, 1963

Logical Pluralism ≈ a universe of logics

- There is a population explosion among the logical systems used in computing science. [...] However, it seems that many general results used in the applications are actually completely independent of what underlying logic is chosen.
  - JOSEPH A. GOGUEN AND ROD M. BURSTALL, *Institutions: Abstract Model Theory for Specification and Programming*, 1992
- [...] it is a fact of life that no single perspective, no single formalization or level of abstraction suffices to represent a system and reason about its behavior. [...] no logical formalism [...] will be best for all purposes. What exists is a space of possibilities (the universe of logics) in which careful choice of the formalisms that best suit some given purposes can be exercised.
  - JOSEPH MESEGUER and N. MARTÍ-OLIET, *From abstract data types to logical frameworks*, 1995

EARLY 20TH CENTURY: SYNTACTIC ERA

Jan Łukasiewicz

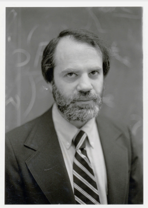


Clarence Irving Lewis

Luitzgen Egbertus Jan Brouwer

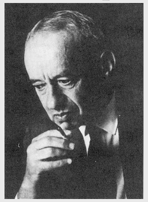
- The early 20th century saw an explosion of studies in non-classical logics
- Intuitionism, Modal Logic, Many-valued logic, etc.
- **Syntactic Era:** Logics were designed by studying axiomatic principles; methods employed mostly algebraic

Mid 20th Century: Semantic Era

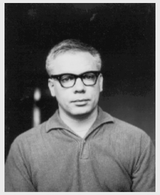


Saul Kripke

Jaakko Hintikka



Alfred Tarski



Stig Kanger

- Around 1960, the mid 20th century saw the development of "proper model theory" for non-classical logic, i.e. the advent of ***Kripke Semantics***
- **Semantic Era:** Logics were studied by applying purely semantic arguments, which led to a dramatic progress in understanding their formal properties.
- Other important figures: BJARNI JÓNSSON, ARTHUR PRIOR, RICHARD MONTAGUE.

Pluralism in Scope and Purpose

- Simple 'domain' / 'application' ontologies: Pizza, Family, FOAF, etc.
- NCI Thesaurus
  - about 34.000 concepts arranged in 20 taxonomic trees, reference terminology for cancer research, sub-Boolean description logic EL.
- Galen
  - medical domain ontology, relatively large, but also relatively complex axiomatisation in a more expressive DL, namely OWL-DL.
- Dolce, GFO, BFO, GUM
  - Foundational ontologies, first-order, higher-order, first-order modal logic being used. Complex axiomatisations.

Pluralism in Reasoning

- Various reasoning 'modes' and scenarios
  - **deductive:** consistency, entailment, instance checking, etc.
  - **inductive:** concept learning, etc.
  - **abductive:** explanation of (desired) entailment, etc.
  - **'modal':** temporal, spatial, epistemic extensions, etc.
  - **non-monotonic:** closed-world reasoning; defaults; rules, etc.
  - **para-consistent:** reasoning over inconsistent data, etc.
  - **fuzzy/probabilistic/uncertain:** vague concept membership, etc.

Logical Pluralism in AI & KR

- Knowledge Representation in general is a prime example for logical pluralism: all kinds of (non-classical) logics or reasoning are being used in different areas of KR, being hand-tailored to the task at hand.
  - Modal and Temporal Logics
  - Fuzzy and many-valued logics
  - Non-monotonic logics and abduction, etc.
  - DLs of various expressivity, FOL and various extensions thererof
  - in general, combining closed and open world reasoning
- **Universal logic** is to logic what universal algebra is to algebra: it is concerned with studying the most general features of logics or classes of logics.

Logical Pluralism Today

- JC Beall and G. Restall's: 'Cases' and generalized Tarski thesis (2000):
  - **V**: A conclusion *A* follows from premises, ∑, if and only if any case in which each premise in ∑ is true is also a case in which *A* is true.
- logis-as-modelling view (Cook/Shapiro): "there can be multiple, incompatible, competing models of the same phenomenon"
- Carnapian pluralism -> pragmatic, **no** pluralism within a fixed 'framework'; incompatible with substantial pluralism?
- Logical Pluralism conference in Tartu, 2008;
  St. Andrews Arché Course 'Logic or Logics', 2010, etc.

Universal Logic

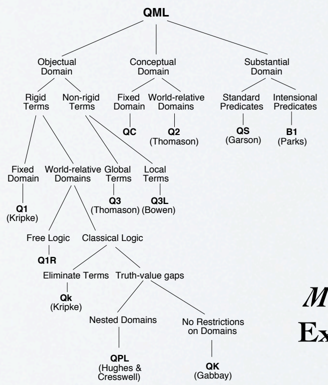
Items that can be varied according to universal logic:

- **Signatures**: (non-logical symbols) propositions; predicates; functions, constants, terms.
- **Grammar**: (logical symbols) variables and quantifiers; modalities; identity symbol; substitution.
- **Models**: possible world; domains of discourse; accessibility (counterpart relations) ; object (individual)
- **Satisfaction**: vary the truth conditions for quantifiers; Booleans; Modalities; vary conditions for identity statements, etc.

Benefits: Borrowing and combination of logics and reasoners, structuring, etc.

Modality, Quantification, and Identity

Garson (1984)



A combination of 2 (or 3) logical theories

- modal predicate logic
- quantified modal logic
- first-order modal logic
- first-order intensional logic,
- free logic, etc.

Modify: **Models / Syntax & Grammar / Satisfaction**
Extension (Modality) vs. Restriction (Intuitionism)

Intuitionistic Logic

Motivation and Kripke Semantics

Overview of this Part

- Kripke semantics for intuitionistic logic
  - Hilbert System
- Kripke semantics for modal logic and correspondence theory
  - Tableaux Calculus
- Logic translations
  - Relations between **IPC** and **CPC**, and **ML**
  - transfer of properties between logics.
  - Glivenko's Theorem and the Gödel translation.

Intuitionistic Logic as Calculus

- Intuitionistic propositional logic **IPC** in an attempt to provide a formal explication of Luitzen Egbertus Jan Brouwer's philosophy of intuitionism (1907/8).
- One of Brouwer's main positions was a rejection of the **tertium**:
  - [To the Intuitionist] the dogma of the universal validity of the principle of excluded third is a phenomenon in the history of civilisation, like the former belief in the rationality of π, or in the rotation of the firmament about the earth. (Brouwer 1952)
- The propositional calculus was devised by Kolmogorov (1925), Orlov (1928), and Glivenko (1929). The first-order version by Arend Heyting (1930)
- A main idea in Heyting's formalisation was to preserve not **truth** (as in classical logic), but **justifications.**

Hilbert system for Intuitionistic Logic

Axioms

$$p_1 \to (p_2 \to p_1)$$
$$(p_1 \to p_2) \to (p_1 \to (p_2 \to p_3)) \to (p_1 \to p_3)$$
$$p_1 \to p_1 \vee p_2$$
$$p_2 \to p_1 \vee p_2$$
$$(p_1 \to p_3) \to (p_2 \to p_3) \to (p_1 \vee p_2 \to p_3)$$
$$\bot \to p_1$$
$$p_1 \wedge p_2 \to p_1$$
$$p_1 \wedge p_2 \to p_2$$
$$p_1 \to p_2 \to p_1 \wedge p_2$$

- A Frege system for **IPC**

Modus Ponens
$$\frac{p_1 \qquad p_1 \to p_2}{p_2}$$

- An important property is the disjunction property (which does not hold classically). It can be read in a constructive fashion:
  - for every proof of a disjunction $A \vee B$
  - there exists a proof of either A or B

Non-Validities in IPC

$$p \vee \neg p$$
$$\neg\neg p \to p$$
$$(p \to q) \to (\neg p \vee q)$$
$$\neg(p \wedge q) \to (\neg p \vee \neg q)$$
$$(\neg p \to q) \to (\neg q \to p)$$
$$(\neg p \to \neg q) \to (q \to p)$$
$$((p \to q) \to p) \to p$$

Kripke Semantics for IPC: Intuition

- One of the main principles of intuitionism is that the truth of a statement can only be established by giving a **constructive proof**.
- When reading intuitionistic formulae, it is therefore instructive to read the connectives in terms of `proofs' or `constructions' as follows:

- A proof of a proposition $\varphi \wedge \psi$ consists of a proof of φ and a proof of ψ.
- A proof of $\varphi \vee \psi$ is given by presenting either a proof of φ or a proof ψ.
- A proof of $\varphi \rightarrow \psi$ is a construction which, given a proof of φ, returns a proof of ψ.
- $\perp$ has no proof and a proof of $\neg\varphi$ is a construction which, given a proof of φ, would return a proof of $\perp$.

- The **law of excluded middle** is clearly not valid in this interpretation.

Kripke Semantics for IPC: Intuition

- The Kripke semantics for **IPC** can be understood to interpret this intuition in an **epistemic way** as follows: (see Chagrov & Zakharyaschev 1997)
  - possible worlds are '*states of knowledge*'
  - moving from one world to the next ***preserves*** the current knowledge
  - a proposition not true now ***can become true*** at a later stage

- $\varphi \wedge \psi$ is true at a state x if both φ and ψ are true at x.
- $\varphi \vee \psi$ is true at x if either φ or ψ is true at x.
- $\varphi \rightarrow \psi$ is true at a state x if, for every subsequent possible state y, in particular x itself, φ is true at y only if ψ is true at y.
- $\perp$ is true nowhere.

Kripke Semantics for IPC

- With this idea in mind, intuitionistic propositional logic **IPC** can now be elegantly characterised via Kripke semantics by modifying the notion of **satisfaction**:

A Kripke frame for **IPC** is a frame $\langle W, \leq \rangle$, where $\leq$ is a partial order (i.e. reflexive, antisymmetric, and transitive). The notions of valuation however is different:

Upward Closed Valuations

$\beta(p) \subseteq W$ such that: for every $x \in \beta(p)$ and $y \in W$ with xRy: we have $y \in \beta(p)$

$$
\begin{array}{rcl}
M_x \not\models \perp & & \\
M_x \models p \wedge q & \iff & M_x \models p \text{ and } M_x \models q \\
M_x \models p \vee q & \iff & M_x \models p \text{ or } M_x \models q \\
M_x \models p \rightarrow q & \iff & \text{for any } y \geq x : \text{if } M_y \models p \text{ then } M_y \models q \\
M_x \models \neg p & \iff & \text{for no } y \geq x : M_y \models p \; (\iff M_x \models p \rightarrow \perp)
\end{array}
$$

Non-Validities in IPC

$$p \vee \neg p$$
$$\neg\neg p \rightarrow p$$
$$(p \rightarrow q) \rightarrow (\neg p \vee q)$$
$$\neg(p \wedge q) \rightarrow (\neg p \vee \neg q)$$
$$(\neg p \rightarrow q) \rightarrow (\neg q \rightarrow p)$$
$$(\neg p \rightarrow \neg q) \rightarrow (q \rightarrow p)$$
$$((p \rightarrow q) \rightarrow p) \rightarrow p$$

- **FOR THE FIRST:** Assume a point x where p does not hold, and a later point y where it becomes known that p.
- **FOR THE SECOND:** Assume a point **x** which falsifies the implicatioin. Then at some point **y** the double implication will hold, but p does not hold. Then not p will never hold later, but at some later point **z** (after y) it becomes known that p.

Kripke Semantics for IPC

- **Generation Theorem:** The truth of a formula at a point x depends only on that part of the model that can "be seen" from x.
- Need the notions of (-> whiteboard):
  - **frame / subframe / root / generated subframe / submodel**
- **Compare:** Coincidence Lemma: truth depends only on the variables that appear in a formula:
- v1, v2, v3 ... b1, ... b2,... v34,...
- **Proof: ...**
- **Corollaries:**
  - **(1)** Truth is invariant under the formation of disjoint unions.
  - **(2)** It always suffices to consider generated (rooted) submodels.

Logic Translations

- How do we move from one logic to another?
  - change of syntax
  - change of semantics
- Requirements
  - preserve the meaning of the original formalisation
  - models of the original formulas should be 'obtainable' from the models of the translated formulas

Translating IPC: Glivenko's Theorem

- We can embed **CPC** into **IPC** by simply adding a double negation:
- The following is called **Glivenko's Theorem**

$$\text{For every formula } \varphi \in \mathsf{CPC} \iff \neg\neg\varphi \in \mathsf{IPC}.$$

- Such embeddings from $\mathbf{L}_1$ to $\mathbf{L}_2$ have several useful features, e.g.:
  - (1) logical connectives in $\mathbf{L}_1$ can be understood in terms of those of $\mathbf{L}_1$.
  - (2) various properties of logics may be preserved along an embedding, e.g.:
    - if $\mathbf{L}_2$ is a decidable logic, then so is $\mathbf{L}_1$.

Example

$$p \vee \neg p \in CPC \iff \neg\neg(p \vee \neg p) \in IPC$$

Glivenko's Theorem: Proof

- **Glivenko's Theorem.** For every formula $\varphi \in \mathsf{CPC} \iff \neg\neg\varphi \in \mathsf{IPC}$.
- **Proof.** (Easy direction)

Suppose $\neg\neg\varphi \in \mathbf{IPC}$.

Then $\neg\neg\varphi \in \mathbf{CPC}$.
Thus, by the classical law of double negation, i.e.

$$\neg\neg\varphi \longleftrightarrow \varphi \in \mathbf{CPC}$$

we obtain $\varphi \in \mathbf{CPC}$.

Glivenko's Theorem: Proof

- **Glivenko's Theorem.** For every formula $\varphi \in \mathsf{CPC} \iff \neg\neg\varphi \in \mathsf{IPC}$.
- **Proof.** (Not so easy direction)

By contraposition, assume $\neg\neg\varphi \notin \mathbf{IPC}$.
Then there are a **finite model** M and a point x in M such that $M_x \not\models \neg\neg\varphi$.
Hence there is a $y \in x\uparrow$ for which $y \models \neg\varphi$.
Let z be some **final point** in the set $y\uparrow$.

Because truth is **propagated upwards**, we have: $z \models \neg\varphi$ and so $z \not\models \neg\neg\varphi$.
Let M' be the submodel of M generated by z, i.e., $M'_z \models p \iff M_z \models p$, for every variable p.
According to the **generation theorem**, M' refutes $\neg\neg\varphi$. But since this model contains only one point, it follows that $\neg\neg\varphi \notin \mathbf{CPC}$, which, by the law of double negation, implies $\varphi \notin \mathbf{CPC}$.

Literature

- Alexander Chagrov & Michael Zakharyaschev, **Modal Logic,** *Oxford Logic Guides, Volume 35, 1997.*
- Melvin Fitting, **Proof Methods for Modal and Intuitionistic Logic,** *Reidel, 1983.*
- Till Mossakowski, Andrzej Tarlecki, Razvan Diaconescu, **What is a logic translation?,** *Logica Universalis, 3(1), pp. 95–124, 2009.*