From syllogism to common sense: a tour through the logical landscape

Conditionals

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Aim of today's lecture

- Interpreting the implication "$p \rightarrow q$" via the truth function

$$f(x, y) = 1 \text{ iff } x = 0 \text{ or } y = 1$$

  is a questionable choice
- This form of interpretation is often called material implication or material conditional
- We want to discuss its (in)adequacy and look at alternative forms of conditionals

And now . . .

What are conditionals?

Conditionals . . .

- relate two propositions, the antecedent and the consequent
- express that the consequent *depends in some sense* on the antecedent
- are expressed in English by 'if' constructions:

If the bough breaks, (then) the cradle will fall.[1]

The cradle will fall if the bough breaks.

The bough breaks only if the cradle falls.

If the bough were to break, the cradle would fall.

Were the bough to break, the cradle would fall.

[1] From ▸ Rock-a-bye Baby, a nursery rhyme and lullaby.

Grammatical cautions

- Grammar of conditional sentences restricts relations between tense and mood of antecedent and consequent

  tense: past, present, future
  mood: indicative, subjunctive *(Indikativ, Konjunktiv)*

- Tense and mood of stand-alone antecedents/consequents may differ:

  If he *takes* a plane, he will get there quicker.
  He *will take* a plane.
  Hence, he will get there quicker.

  If it *had rained*, the street *would have been* wet.
  It *did* rain.
  So, the street *is* wet.

If-sentences that are not conditionals

- Examples:

  If I may say so, you have a nice earring.

  (Even) if he was plump, he could still run fast.

  If you want a banana, there's one in the kitchen.

- Rough-and-ready test whether "if A, then B" is a conditional: Is it equivalent to "A implies B"?

The material conditional

- Connective $\rightarrow$ (sometimes $\supset$) is called material conditional or material implication
- From our truth-functional approach: $p \rightarrow q$ is
  - true iff p is false or q is true
  - logically equivalent to $\neg p \vee q$
- Consequences:

  $q \models p \rightarrow q$ (a true statement is implied by anything)
  $\neg p \models p \rightarrow q$ (a false statement implies everything)

  Sometimes called the paradoxes of material implication

Can English conditionals be represented by "$\rightarrow$"?

Are these sentences intuitively true?

If New York is in New Zealand, then $2 + 2 = 4$.

If New York is in the United States,
then World War II ended in 1945.

If World War II ended in 1941, then gold is an acid.

(Where is the dependency between antecedent and consequent?)

Another look at the previous three sentences

Communication is usually based on rules of conversation:

- Be relevant

  "How do you use this computer?"
  "There's a book over there." *(It's a computer manual.)*

- Assert the strongest claim you can make

  "Which is the oldest university in Germany?"
  "It's either Heidelberg or Köln." *(I don't know which.)*

Speakers violate the second rule when saying:

If New York is in New Zealand, then $2 + 2 = 4$.

If New York is in the United States,
then World War II ended in 1945.

If World War II ended in 1941, then gold is an acid.

Subjunctive Conditionals

Harder objection to material conditional:

Pairs of conditionals with the same antecedent and consequent, but different truth values

(1) If Oswald didn't shoot Kennedy, someone else did.

(2) If Oswald hadn't shot Kennedy, someone else would have.

(2) is called subjunctive or counterfactual conditional

- Named after grammatical mode of subjunctive (Konjunktiv) and after "counter to (against) the observed facts"
- Cannot be material: (2) is false despite its false antecedent

(1) is called indicative conditional

- Named after grammatical mode of indicative
- May be material: (1) is true; its antecedent is false

Reasons why indicative and subjunctive conditional differ

(1) If Oswald didn't shoot Kennedy, someone else did.

(2) If Oswald hadn't shot Kennedy, someone else would have.

- Tenses and/or moods of verbs involved differ;
  grammar of "if"s is identical (claims Priest)
- Different tenses cause us to evaluate truth values of (1), (2) from different points in time:

  (1) is evaluated from the present where K. has, *in fact,* been shot

  (2) asks us to evaluate

  (2') If Oswald doesn't shoot Kennedy, someone else will.

  from a point of time just before Kennedy was shot;

  is, in some sense, the past tense of (2')

Past tenses seem to be the main culprit

No difference if (1) and (2) are both in present tense:

(1) If I shoot you, you will die.

(2) If I were to shoot you, you would die.

(Caution:
"I were" is present tense subjunctive;
"I was" is past tense indicative)

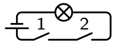
Question

Does only the subjunctive conditional behave non-material?

No! Indicative conditional cannot be material either

Following inferences are valid, but corresponding arguments aren't.

- $(p \land q) \to r \quad \vdash \quad (p \to r) \lor (q \to r)$



  If you close switches 1 and 2, then the light will go on.
  Hence, if you close one of switches 1, 2, the light will go on.

- $(p \to q) \land (p' \to q') \quad \vdash \quad (p \to q') \lor (p' \to q)$

  If John is in Paris, he is in France,
  and if John is in London, he is in England.
  Hence, either, if John is in Paris, he is in England,
  or, if John is in London, he is in France.

- $\neg(p \to q) \quad \vdash \quad p$

  It is not the case that, if there is a god,
  the prayers of evil people will be heard.
  Hence, there is a god.

Indicative conditional cannot be material (2)

Notice:

- All three previous conditionals are indicative.
- Appealing to the conversational rules cannot explain why the conclusions appear odd.

  For example, in

  If you close switches 1 and 2, then the light will go on.
  Hence, if you close one of switches 1, 2, the light will go on.

  we don't know which disjunct of the conclusion is true – *both* appear to be false.

Conclusion

English "if" is not understood as "$\to$".

Arguments *for* material implication

Why did anyone ever think that the English conditional is material?

1. Until 1960's, the only semantics was via Boolean functions
   "$\to$" is the only Boolean fct. that looks (remotely) plausible
2. Claim: "if p then q" is true iff "$p \to q$" is true.

   Proof:

"$\Rightarrow$" Suppose that "if p then q" is true.

Case 1: $\neg p$ is true.
Then, clearly, $\neg p \lor q$ is true.
Case 2: p is true.
Then, by *modus ponens* via assumption, q true.
Hence, $\neg p \lor q$ is true.

Thus, in either case, $p \to q$ is true.

Arguments *for* the material implication (2)

Claim: "if p then q" is true iff "$p \to q$" is true.

Proof:

"$\Leftarrow$" The following claim is plausible:

(*) If there is some true statement r such that $p, r \vdash q$, then "if p then q" is true.

E.g.: p = "Oswald didn't kill Kennedy",
q = "someone else killed Kennedy",
r = "someone killed Kennedy"

Now suppose that $\neg p \lor q$ is true. Use $\neg p \lor q$ as r.
Disjunctive syllogism: $p, \ \neg p \lor q \ \vdash \ q$
$\Rightarrow$ via (*): "if p then q" is true.

$\square$

Arguments *for* the material implication (3)

3. Material implication makes attractive patterns valid:
   - Contraposition
     "If p then q. Hence, if not q, then not p."
   - Hypothetical syllogism
     "If p then q. If q then r. Hence, if p then r."
   - Strengthening the antecedent
     "If p then q. Hence, if p and r, then q."

However, these patterns are problematic too:

> If he has made a mistake, then it is not a big one.
> Hence, if he has made a big mistake, he hasn't made a mistake.

Defence of contraposition:
Inference does not preserve assertability (see next section)

Interlude: Final Proof of the Non-Existence of God

Now, it is such a bizarrely improbable coincidence that anything so mind-bogglingly useful could have evolved purely by chance that some have chosen to see it as the final proof of the NON-existence of God. The argument goes something like this:

"I refuse to prove that I exist," says God, "for proof denies faith, and without faith I am nothing."

todsicherer Hinweis

"But," says Man, "the Babel fish is a dead giveaway, isn't it? It could not have evolved by chance. It proves that You exist, and so therefore, by Your own arguments, You don't. QED"

"Oh dear," says God, "I hadn't thought of that," and promptly vanishes in a puff of logic.

Zugabe

"Oh, that was easy," says Man, and for an encore goes on to prove that black is white and gets himself killed on the next zebra crossing.

From Douglas Adams, The Hitchhiker's Guide to the Galaxy ▸ YouTube

And now . . .

Assertability theory

- Instead of assigning a truth value to a statement A, use probability $P(A)$
- Statements with probability ≈ 1 are highly assertable, with probability ≈ 0 lowly assertable
- Probability can be subjective (according to assertor's beliefs) or objective (if measurable)
- Refinement: use conditional probability for conditionals

Conditional probabilities

- Acceptability or assertability of "if A then B" is equated with conditional probability of B given A
- $P(B \mid A) \approx 1\,(0) \rightsquigarrow$ "if A then B" is highly (lowly) assertable
- Example:

  If Wulff's back is covered by Merkel, he will remain in office.

  Assertability of this conditional is determined by
  P(W. remains in office | W.'s back is covered by M.)
- This theory explains why one says that
  assertors of "if A then B" and "if A then not B" *disagree*:

  $$P(B \mid A) = 1 - P(\overline{B} \mid A)$$

  i.e., if one is highly assertable, the other is lowly assertable

Conditional probabilities and the previous paradoxes

If New York is in New Zealand, then 2 + 2 = 4.
A B

$P(B \mid A) = \frac{P(B \cap A)}{P(A)}$ — undefined because $P(A) = 0$

If New York is in the United States,
then World War II ended in 1945.

$P(B \mid A) = 1/1 = 1$

If World War II ended in 1941, then gold is an acid.

$P(B \mid A)$ is undefined because $P(A) = 0$

$\rightsquigarrow$ Here, conditional is not automatically true when antecedent false or consequent true

Conditional probabilities and the previous paradoxes (2)

(1) If Oswald didn't shoot Kennedy, someone else did.
A B

$P(A)$: small, but nonzero
$P(B \cap A) = P(A)$: K. has been shot
$\Rightarrow P(B \mid A) = 1 \quad \rightsquigarrow$ highly assertable

(2) If Oswald hadn't shot Kennedy, someone else would have.

is the past tense of:

(2') If Oswald doesn't shoot Kennedy, someone else will.
A B

$P(B \cap A)$ small in relation to $P(A)$
$\Rightarrow P(B \mid A) \approx 0 \quad \rightsquigarrow$ lowly assertable

Conditional probabilities and the previous paradoxes (3)

Previous paradox of contraposition:

- Assume that this is highly assertable:

  If he has made a mistake, then it is not a big one.

  That is, P(non-big mistake | mistake) ≈ 1
  $\Rightarrow P$(non-big mistake) $\approx P$(mistake)
- *Independently,* this is lowly assertable:

  If he has made a big mistake, he hasn't made a mistake.

  Since every big mistake is a mistake, we have
  P(no mistake $\cap$ big mistake) $= 0$
  $\Rightarrow P$(no mistake | big mistake) $= 0$

Possible-worlds account: idea

- Possible world = a way things *may* be,
  as opposed to the actual world
- Idea: when reflecting on a conditional,
  we go through all possible situations in our head
  - If Merkel covers Wulff's back, then ...
  - If Merkel abandons Wulff, then ...
  - If Merkel shoots Wulff, then ...

  Ask ourselves how things are in (hypothetical) situations
  where a certain conditional is true

Possible-worlds account: strict implication

One way to make this idea more precise:

- "if p then q" is true at the actual world w iff
  of the possible worlds where p is true,
  those closest or most similar to w make q true
- Reformulation (D. Lewis):
  all closest possible worlds make material $p \to q$ true

$\rightsquigarrow$ Strict implication: $\Box(p \to q)$ *see also modal logic lecture*

- There are many ways to define closeness/similarity;
  Kripke models: based on Kripke frames (W, R)
  digraphs with nodes = worlds (W), edges = closeness (R)
- Special case: intuitionistic implication *see previous lecture*
  (Caution: intuitionistic logic has a different negation)

Paradoxes avoided by strict implication

- Some of the above paradoxes are avoided:

  $q \not\models \Box(p \to q)$

  $\neg p \not\models \Box(p \to q)$

  (This one *can't* be avoided with intuitionistic implication.)

  $\Box((p \land q) \to r) \not\models \Box(p \to r) \lor \Box(q \to r)$

  $\Box(p \to q) \land \Box(p' \to q') \not\models \Box(p \to q') \lor \Box(p' \to q)$

  $\neg\Box(p \to q) \not\models p$

- For each of them, one can construct a model
  that satisfies the left-hand formula but not the right-hand one.

Paradoxes *not* avoided by strict implication

- Some paradoxes cannot be avoided with strict implication:

  $\models \Box((p \land \neg p) \to q)$

  $\models \Box(p \to \Box(q \to q))$

  $\models \Box(p \to (q \lor \neg q))$

  (This one *can* be avoided
  with intuitionistic implication + negation.)

  $\Box(p \to q) \models \Box(\neg q \to \neg p)$ (contraposition)

  $\Box(p \to q), \Box(q \to r) \models \Box(p \to r)$ (hypoth. syllogism)

  $\Box(p \to q) \models \Box(p \land r \to q)$ (strength. antecedent)

  Reason: strict implication just lifts material implication to
  closest worlds

Relevant logics (also: relevance logic)

- A propositional logic is relevant if:
  $\models p \to q$ only if p, q have some prop. variable in common
- Ensures the antecedent being "relevant" for the consequent
- Avoids paradoxes

$$\models \Box((p \wedge \neg p) \to q)$$
$$\models \Box(p \to \Box(q \to q))$$
$$\models \Box(p \to (q \vee \neg q))$$

$\Rightarrow$ Material, strict, intuitionistic conditionals are not relevant

- Semantics: Kripke frames with ternary relation

$$w \models p \to q \;\Leftrightarrow\; \forall v, u \text{ with } Rwvu(v \not\models p \text{ or } u \models q)$$

Generalises strict implication – just set $v = u$

Meta-linguistic account

- Between material and possible-worlds interpretation
- "if p then q" true iff
  exists statement r satisfying some condition φ
  such that $p, r \models q$
- Meta-linguistic theory explains how to specify φ
- Can be seen as precursor of possible-worlds account:

$$p, r \models q \Leftrightarrow q \text{ true in in every possible world that satisfies } p, r$$
$$\Leftrightarrow \forall w \text{ with } w \models r \; (w \models p \Rightarrow w \models q)$$

$\rightsquigarrow$ w with $w \models r$ are "closest" worlds
$\rightsquigarrow$ specifying φ on $r \mathrel{\hat{=}}$ deciding which worlds are closest

Summary

- Material implication suffers from several paradoxes (unintuitive validities or consequences)
- Still, there is justification for material implication
- Alternative accounts ...
  - include assertability theory, conditional probabilities, strict implication, intuitionistic implication, relevant logics, meta-linguistic theory
  - usually cannot avoid all paradoxes
  - often require more complex semantics (possible worlds, binary/ternary relations, ...)

Literature

Contents is taken from:

- G. Priest: *An Introduction to Non-Classical Logic*, 2nd ed., Cambridge, 2011. (Sections 1.6–1.10, 4.5–4.7, 6.6, 9.7)
  Not available at SUB; ask us if you want to read it.
- F. Jackson: *Conditionals*, Oxford, 1991.
  (Introduction for overview;
  individual essays for deeper insights)
  Available at SUB