$\mathcal{ALC}_{\mathcal{ALC}}$: a Context Description Logic

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Abstract. We develop a novel description logic (DL) for representing and reasoning with contextual knowledge. Our approach descends from McCarthy's tradition of treating contexts as formal objects over which one can quantify and express first-order properties. As a foundation we consider several common product-like combinations of DLs with multimodal logics and adopt the prominent $(\mathbf{K}_n)_{\mathcal{ALC}}$. We then extend it with a second sort of vocabulary for describing contexts, i.e., objects of the second dimension. In this way, we obtain a *two-sorted, two-dimensional combination of a pair of DLs* $\mathcal{ALC}$, called $\mathcal{ALC}_{\mathcal{ALC}}$. As our main technical result, we show that the satisfiability problem in this logic, as well as in its proper fragment $(\mathbf{K}_n)_{\mathcal{ALC}}$ with global TBoxes and local roles, is 2ExpTime-complete. Hence, the surprising conclusion is that the significant increase in the expressiveness of $\mathcal{ALC}_{\mathcal{ALC}}$ due to adding the vocabulary comes for no substantial price in terms of its worst-case complexity.

1 Introduction

Over two decades ago John McCarthy introduced the AI community to a new paradigm of formalizing contexts in logic-based knowledge systems. This idea, presented in his Turing Award Lecture [1], was quickly picked up by others and by now has led to a significant body of work studying different implementations of the approach in a variety of formal frameworks and applications [2,3,4,5,6,7,8]. The great appeal of McCarthy's paradigm stems from the simplicity and intuitiveness of the three major postulates it is based on:

1. Contexts are formal objects. More precisely, a *context* is anything that can be denoted by a first-order term and used meaningfully in a statement of the form $ist(c, p)$, saying that proposition p *is true* in context c [1,5,6,2], e.g., $ist(Hamlet$, 'Hamlet is a prince.'). By adopting a strictly formal view on contexts, one can bypass unproductive debates on what they really are and instead take them as primitives underlying practical models of contextual reasoning.

2. Contexts are organized in relational structures. In the commonsense reasoning, contextual assumptions are dynamically and directionally altered [8,2]. Contexts are entered and then exited, accessed from other contexts or transcended to broader ones. Formally, we want to allow nestings of the form $ist(c, ist(c', p))$, e.g., $ist(France, ist(capital$, 'The city river is Seine.')).

3. Contexts have properties and can be described. As first-order objects, contexts can be in a natural way described in a first-order language [4,6]. This allows for addressing them generically through quantified formulas such as $\forall x(P(x) \rightarrow ist(x, p))$, expressing that p is true in every context of type P, e.g., $\forall x(barbershop(x) \rightarrow ist(x,$ 'Main service is a haircut.'$))$.

The goal of this work is to import McCarthy's paradigm into the framework of Description Logics (DLs), a popular family of knowledge representation formalisms, with many successful applications [9]. Although the importance of contexts in DLs has been generally acknowledged, the framework is still not supported with a dedicated, generic theory of accommodating contextual knowledge. The most common perspectives considered in this area are limited to: 1) integration of local ontologies [10,11], 2) modeling levels of abstraction as subsets of DL models [12,13], and 3) capturing dynamics of knowledge across a fixed modal dimension, most typically a temporal one [14,15,16].

The DL $\mathcal{ALC}_{\mathcal{ALC}}$, which we develop here, is a novel formalism for representing and reasoning with context-dependent knowledge. On the one hand, we systematically incorporate the three postulates of McCarthy, and thus, ground our proposal in a longstanding tradition of formalizing contexts in AI. On the other, we build on top of two-dimensional DLs [17], which provide $\mathcal{ALC}_{\mathcal{ALC}}$ with well-understood formal foundations. In this paper we present a thorough study of the formal properties of $\mathcal{ALC}_{\mathcal{ALC}}$, including its expressiveness, computational complexity and relationships to other formalisms. As our main technical result, we show that the satisfiability problem in $\mathcal{ALC}_{\mathcal{ALC}}$, as well as in its proper fragment $(\mathbf{K}_n)_{\mathcal{ALC}}$ with global TBoxes and local roles, is 2ExpTime-complete. This reveals that the jump in the complexity from ExpTime is essentially caused by the interaction of multiple $\mathbf{K}$-modalities with global TBoxes.

2 Overview

We start with an outline of the milestones for constructing and studying the logic $\mathcal{ALC}_{\mathcal{ALC}}$. Then, we recap the basic notions concerning the DL $\mathcal{ALC}$.

2.1 Roadmap

We introduce $\mathcal{ALC}_{\mathcal{ALC}}$ in a gradual way. First, in Section 3, we elaborate on some well-studied combinations of the DL $\mathcal{ALC}$ with modal logics, known as two-dimensional or modal DLs [18,17,19]. From our perspective, the two-dimensional semantics of such logics is very well suited for representing context objects and the relational structures they form. After some conceptual and computational evaluation we then adopt $(\mathbf{K}_n)_{\mathcal{ALC}}$ as the foundation for our context DL. Finally, we show that the migration from $\mathcal{ALC}$ to $(\mathbf{K}_n)_{\mathcal{ALC}}$ with global TBoxes and local roles rises the complexity from ExpTime to 2ExpTime.

Next, in Section 4, we extend $(\mathbf{K}_n)_{\mathcal{ALC}}$ with a second sort of vocabulary, which serves for describing contexts. Formally, we can see this extension as a shift from $(\mathbf{K}_n)_{\mathcal{ALC}}$ to $\mathcal{ALC}_{\mathcal{ALC}}$, i.e., a *two-sorted*, *two-dimensional* combination of a pair of DLs $\mathcal{ALC}$. Each sort in $\mathcal{ALC}_{\mathcal{ALC}}$ applies to its corresponding dimension and the two are allowed to interact in a controlled manner. Since such an extension is relatively uncommon, we then relate $\mathcal{ALC}_{\mathcal{ALC}}$ to the standard framework of products of modal logics and show that the departure is not radical. More interestingly, we also prove that the extension, although offering a lot of expressive flexibility, is not to be paid for in yet another increase of the worst-case complexity. Satisfiability in $\mathcal{ALC}_{\mathcal{ALC}}$ remains 2ExpTime-complete.

In Section 5, we present an example application of $\mathcal{ALC}_{\mathcal{ALC}}$. Finally, in Section 6, we conclude the paper and point to directions for future research.

2.2 Preliminaries: DL $\mathcal{ALC}$

A DL language is specified by a vocabulary $\Sigma = (N_I, N_C, N_R)$, where N_I is a set of *individual names*, N_C a set of *concept names*, N_R a set of *role names*, and a number of operators for constructing complex concept descriptions [9]. The $\mathcal{ALC}$ concept language L over Σ is the smallest set of concepts containing $\top$, all concept names from N_C and closed under the constructors:

$$\neg C \;\mid\; C \sqcap D \;\mid\; \exists r.C$$

where $C, D \in L$ and $r \in N_R$. Conventionally, we abbreviate $\neg\top$ with $\bot$, $\neg(\neg C \sqcap \neg D)$ with $C \sqcup D$ and $\neg\exists r.\neg C$ with $\forall r.C$. The semantics of L is given through *interpretations* of the form $\mathcal{I} = (\Delta, \cdot^{\mathcal{I}})$, where Δ is a non-empty *domain* of individuals, and $\cdot^{\mathcal{I}}$ is an *interpretation function*. The meaning of the vocabulary is fixed via mappings: $a^{\mathcal{I}} \in \Delta$ for every $a \in N_I$, $A^{\mathcal{I}} \subseteq \Delta$ for every $A \in N_C$ and $r^{\mathcal{I}} \subseteq \Delta \times \Delta$ for every $r \in N_R$, and $\top^{\mathcal{I}} = \Delta$. Then the function is inductively extended over L according to the fixed semantics of the constructors:

$$\begin{aligned}(\neg C)^{\mathcal{I}} &= \{x \in \Delta \mid x \notin C^{\mathcal{I}}\},\\ (C \sqcap D)^{\mathcal{I}} &= \{x \in \Delta \mid x \in C^{\mathcal{I}} \cap D^{\mathcal{I}}\},\\ (\exists r.C)^{\mathcal{I}} &= \{x \in \Delta \mid \exists y:\ \langle x, y\rangle \in r^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\}.\end{aligned}$$

A *knowledge base* (or an *ontology*) $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ consists of a TBox $\mathcal{T}$ and an ABox $\mathcal{A}$. The TBox contains general concept inclusion axioms (GCIs) $C \sqsubseteq D$, for arbitrary concepts $C, D \in L$. We write $C \equiv D$ whenever both $C \sqsubseteq D$ and $D \sqsubseteq C$ are in $\mathcal{T}$. The ABox consists of concept assertions $C(a)$ and role assertions $r(a, b)$, where $a, b \in N_I$, $C \in L$ and $r \in N_R$. An interpretation $\mathcal{I}$ *satisfies* an axiom in either of the following cases:

- $\mathcal{I} \models C \sqsubseteq D$ *iff* $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$,
- $\mathcal{I} \models C(a)$ *iff* $a^{\mathcal{I}} \in C^{\mathcal{I}}$,
- $\mathcal{I} \models r(a, b)$ *iff* $\langle a^{\mathcal{I}}, b^{\mathcal{I}}\rangle \in r^{\mathcal{I}}$.

Finally, $\mathcal{I}$ is a *model* of a DL knowledge base whenever it satisfies all its axioms.

3 Adding context structures: from $\mathcal{ALC}$ to $(\mathrm{K}_n)_{\mathcal{ALC}}$

In order to introduce *context structures* into the DL semantics, and thus account for the first two postulates of McCarthy, we move from $\mathcal{ALC}$ to its two-dimensional, multi-modal extensions.

3.1 Syntax and semantics

A two-dimensional, multi-modal concept language $L_{\mathcal{ALC}}$ over vocabulary Σ is the smallest set of concepts containing $\top$, concept names from N_C and closed under the $\mathcal{ALC}$ and the two new constructors:

$$\Diamond_i C \mid \Box_i C$$

where $C \in L_{\mathcal{ALC}}$ and $1 \leq i \leq n$ for some fixed $n \in \mathbb{N}$. It is assumed that $\Box_i$ abbreviates $\neg\Diamond_i\neg$. In our framework, every i is interpreted as a distinguished *contextualization operation*. The modal *context operators* associated with i enable a transition to the state of affairs holding in some ($\Diamond_i$) or all ($\Box_i$) contexts accessible from the current one through i. An interpretation of $L_{\mathcal{ALC}}$ is defined as a tuple $\mathfrak{M} = (\mathfrak{C}, \{R_i\}_{1\leq i\leq n}, \Delta, \{\cdot^{\mathcal{I}(c)}\}_{c\in\mathfrak{C}})$, where:

- $\mathfrak{C}$ is a non-empty *context domain*,
- $R_i \subseteq \mathfrak{C} \times \mathfrak{C}$ is an *accessibility relation* on $\mathfrak{C}$, associated with $\Diamond_i$ and $\Box_i$,
- Δ is a non-empty *object domain*,
- $\cdot^{\mathcal{I}(c)}$ is an *interpretation function* in context c.

For every $c \in \mathfrak{C}$, the interpretation function $\mathcal{I}(c)$ fixes the meaning of the language by extending the basic $\mathcal{ALC}$ interpretation rules with the additional:

$$(\Diamond_i C)^{\mathcal{I}(c)} = \{x \in \Delta \mid \exists d \in \mathfrak{C} : \ cR_i d \wedge x \in C^{\mathcal{I}(d)}\},$$
$$(\Box_i C)^{\mathcal{I}(c)} = \{x \in \Delta \mid \forall d \in \mathfrak{C} : \ cR_i d \rightarrow x \in C^{\mathcal{I}(d)}\}.$$

In what follows, we loosely refer to $\mathfrak{C}$ as the *context dimension* and to Δ as the *object dimension* of the combination (see example in Fig. 1). Generally, the semantic setup for multi-dimensional DLs allows several degrees of freedom regarding rigidity of names and domain assumptions [17]. Here, we pose the natural, rigid interpretation of individual names, i.e., $a^{\mathcal{I}(c)} = a^{\mathcal{I}(d)}$ for every $c, d \in \mathfrak{C}$, and local (non-rigid) interpretation of concepts. The interpretation of roles is discussed in the next paragraphs. We also assume that all contexts share the same object domain. Even if not suiting all applications, the constant domain assumption is known to be most universal, in the sense that the expanding/varying case can be always reduced to the constant one.

For a fixed language $L_{\mathcal{ALC}}$ the knowledge about the object dimension, now relative to contexts, can be expressed by means of usual axioms. In particular, a TBox $\mathcal{T}$ is a set of GCIs over concepts from $L_{\mathcal{ALC}}$. In this section it suffices to consider only the basic problem of *concept satisfiability* with respect to a global $\mathcal{T}$. The satisfaction relation for GCIs is defined with respect to an interpretation $\mathfrak{M}$ and a context $c \in \mathfrak{C}$:

- $(\mathfrak{M}, c) \models C \sqsubseteq D$ *iff* $C^{\mathcal{I}(c)} \subseteq D^{\mathcal{I}(c)}$.

We call $\mathfrak{M}$ a model of a global $\mathcal{T}$ whenever it satisfies all axioms in $\mathcal{T}$ in every $c \in \mathfrak{C}$. A concept C is satisfiable w.r.t. $\mathcal{T}$ *iff* there exists a model of $\mathcal{T}$ such that for some $c \in \mathfrak{C}$ and $d \in \Delta$ it is the case that $d \in C^{\mathcal{I}(c)}$.

It is not hard to see that without further constraints the resulting logic corresponds to the well-known product of multi-modal $\mathbf{K}_n$ with $\mathcal{ALC}$, denoted

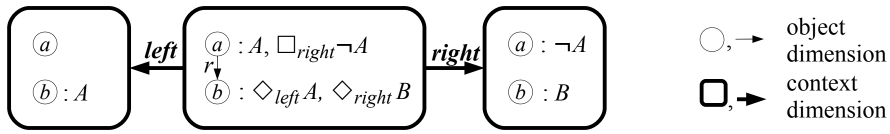

Fig. 1. A context structure modeling concept $A \sqcap \Box_{right}\neg A \sqcap \exists r.(\Diamond_{left}A \sqcap \Diamond_{right}B)$.

shortly as $(\mathbf{K}_n)_{\mathcal{ALC}}$ [18,20,17,19]. As for many other applications, also in the case of context DLs $(\mathbf{K}_n)_{\mathcal{ALC}}$ seems to provide the most natural and flexible foundation. Obviously, it is not difficult to further constrain accessibility relations in order to obtain context structures with more specific properties. Leaving a broader study of this subject for future research, let us just consider two such restrictions, sometimes evoked in the literature on contexts:

(quasi-functionality) $\forall c, d, e \in \mathfrak{C}\ (cRd \wedge cRe \rightarrow d = e)$,
(seriality) $\forall c \in \mathfrak{C}\ \exists d \in \mathfrak{C}\ (cRd)$.

Buvač's propositional logic of contexts [2,3] is a notational variant of $\mathbf{K}_n$, with $\Box_i\varphi$ written as $ist(i,\varphi)$. In Buvač's setting $\Box_i$ quantifies over possible interpretations of the context i. In our framework, where contexts are not modality indices but first-order objects, $\Box_i$ would quantify over possible contexts instead, which clearly distorts the intended behavior of *ist*. To avoid this, one might rather use $\Box_i$ of the logic $\mathbf{Alt}_n$, characterized by all quasi-functional Kripke frames [19]. In $\mathbf{Alt}_n$ there is at most one context accessible through each contextualization operation. Thus, $\Diamond_i\varphi \wedge \Diamond_i\psi$ semantically implies $ist(c, \varphi \wedge \psi)$ for some unique c. Nossum [8] pursues similar intuitions and advocates even stronger $\mathbf{DAlt}_n$, which is Kripke-complete w.r.t. all quasi-functional and serial frames. Such a semantics ensures that it is always possible to reach exactly one context through each accessibility relation. Since formally the two frame properties boil down to the functionality condition, it follows that the two operators $\Diamond_i, \Box_i$ collapse into a single $\bigcirc_i$. Finally $\mathbf{D}_n$, characterized by all serial frames, is used by Buvač [2,3] for verifying consistency of contextual knowledge. Since the seriality condition enforces existence of all potential contexts, the knowledge attributed to these contexts cannot be self-contradictory.

3.2 Complexity

As it turns out, the choice between any of the characterizations discussed above is quite irrelevant from the computational perspective. In most cases the complexity results apply to all logics $L_{\mathcal{ALC}}$, for $L \in \{\mathbf{DAlt}_n, \mathbf{D}_n, \mathbf{Alt}_n, \mathbf{K}_n\}$. To ease the transfer of some of the observations we make below, we use the following reductions:

Proposition 1. *Concept satisfiability w.r.t. global TBoxes is polynomially reducible between the following logics (where $\mapsto$ means* reduces to*):*

$$(\mathbf{DAlt}_n)_{\mathcal{ALC}} \mapsto \{(\mathbf{D}_n)_{\mathcal{ALC}}, (\mathbf{Alt}_n)_{\mathcal{ALC}}\} \mapsto (\mathbf{K}_n)_{\mathcal{ALC}}.$$

To see that the reductions hold indeed, it is enough to notice that if $(C, \mathcal{T})$ is a problem of deciding whether a concept C is satisfiable w.r.t. a global TBox $\mathcal{T}$, then by simple transformations of C and $\mathcal{T}$ one can enforce only models that are bisimilar to those characterizing the respective frame conditions:

(quasi-functionality) W.l.o.g. assume that $C = \text{NNF}(C)$, where NNF stands for Negation Normal Form, and $\mathcal{T} = \{\top \sqsubseteq C_{\mathcal{T}}\}$, for some $C_{\mathcal{T}} = \text{NNF}(C_{\mathcal{T}})$. Let C' and $C'_{\mathcal{T}}$ be the result of replacing every subconcept $\Diamond_i B$ occurring in C and $C_{\mathcal{T}}$, respectively, with $(\Diamond_i \top) \sqcap (\Box_i B)$. Then, $(C, \mathcal{T})$ is satisfiable on a quasi-functional frame *iff* $(C', \{\top \sqsubseteq C'_{\mathcal{T}}\})$ is satisfiable.

(seriality) Let $\mathcal{T}' = \mathcal{T} \cup \{\top \sqsubseteq \Diamond_i \top \mid 1 \leq i \leq n\}$, where n is the number of all modalities occurring in $\mathcal{T}$ and C. Then, $(C, \mathcal{T})$ is satisfiable on a serial frame *iff* $(C, \mathcal{T}')$ is satisfiable.

Our first result is a negative one. It closes the option of using rigid roles, i.e., such that $r^{\mathcal{I}(c)} = r^{\mathcal{I}(d)}$ for every $c, d \in \mathfrak{C}$, or applying context operators to roles. Unfortunately, adding rigid roles leads to undecidability already for the strongest of the logics with just a single context operator.

Theorem 1. *Concept satisfiability in* $\mathbf{DAlt}_{\mathcal{ALC}}$ *w.r.t. global TBoxes and with a single rigid role is undecidable.*

The full proof, along the others from this paper, is included in the appendix of the accompanying technical report [21]. We notice that $\mathbf{DAlt}_{\mathcal{ALC}}$ corresponds to a fragment of $\text{LTL}_{\mathcal{ALC}}$ with the *next-time* operator, which is enough to construct a usual encoding of the undecidable $\mathbb{N} \times \mathbb{N}$ *tiling problem* [14]. Together with Proposition 1, the theorem immediately entails the following:

Theorem 2. *For any* $L \in \{\mathbf{DAlt}_n, \mathbf{D}_n, \mathbf{Alt}_n, \mathbf{K}_n\}$*, concept satisfiability in* $L_{\mathcal{ALC}}$ *w.r.t. global TBoxes with a single rigid role is undecidable.*

This result reveals an obvious limitation to the formalism, but a limitation one has to live with, considering that combinations of rigid roles with global TBoxes are rarely decidable unless the expressive power of the modal or the DL component is significantly reduced [19,14]. In the rest of this paper, we almost exclusively address the case of local (non-rigid) roles. To show decidability and the upper bound of the concept satisfiability problem in this setup,[3] we devise a quasistate elimination algorithm for $(\mathbf{K}_n)_{\mathcal{ALC}}$, similar to [19, Theorem 6.61]. As usual, the idea is to abstract from the domains $\mathfrak{C}$ and Δ and consider only a finite, in fact double exponential, number of quasistates which represent possible contexts inhabited by a finite number of possible types of individuals. Then, we iteratively eliminate all those that do not satisfy necessary conditions.

Theorem 3. *Deciding concept satisfiability in* $(\mathbf{K}_n)_{\mathcal{ALC}}$ *w.r.t. global TBoxes and only with local roles is in* 2ExpTime.

[3] Mind that the NExpTime-completeness result for concept satisfiability in $\mathbf{K}_{\mathcal{ALC}}$ [19, Theorem 15.15] applies to $\mathcal{ALC}$ with a single pair of $\mathbf{K}$ operators, full booleans on modalized formulas and no global TBoxes.

One could hope that at least some of the considered logics could be less complex than that. However, as the next theorem shows, this is not the case.

Theorem 4. *Deciding concept satisfiability in* $(\mathbf{DAlt}_n)_{\mathcal{ALC}}$ *w.r.t. global TBoxes and only with local roles is* 2ExpTime*-hard.*

For the proof we use a reduction of the word problem for exponentially bounded Alternating Turing Machines, which is known to be 2ExpTime-hard [22]. The increase in the complexity by one exponential, as compared to $\mathcal{ALC}$ alone (for which the problem is ExpTime-complete [9]), is notable and quite surprising. It could be expected that without rigid roles the satisfiability problem can be straightforwardly reduced to satisfiability in fusion models. This in turn should yield ExpTime upper bound by means of the standard techniques. However, as the following example for $(\mathbf{K}_n)_{\mathcal{ALC}}$ demonstrates, this strategy fails.

$$(\dagger)\ \Diamond_i C \sqcap \exists r.\Box_i \bot \qquad (\ddagger)\ \exists succ_i.C \sqcap \exists r.\forall succ_i.\bot$$

Although (†) clearly does not have a model, its reduction (‡) to a fusion language, where context operators are translated to restrictions on fresh $\mathcal{ALC}$ roles, is satisfiable. The reason is that while in the former case the information about the structure of the **K**-frame is global for all individuals, in the latter it becomes local. The r-successor in (‡) is simply not 'aware' that it should actually have a $succ_i$-successor.[4] This effect, amplified by presence of multiple modalities and global TBoxes (which can enforce infinite **K**-trees), makes the reasoning harder.

The two complexity bounds from Theorem 3 and 4, together with the reductions established in Proposition 1, provide us with the completeness result.

Theorem 5. *For any* $L \in \{\mathbf{DAlt}_n, \mathbf{D}_n, \mathbf{Alt}_n, \mathbf{K}_n\}$*, deciding concept satisfiability in* $L_{\mathcal{ALC}}$ *w.r.t. global TBoxes and only with local roles is* 2ExpTime*-complete.*

The theorem is quite robust under changes of domain assumptions and holds already in the case of expanding/varying domains in $(\mathbf{Alt}_n)_{\mathcal{ALC}}$. The only exception applies to $(\mathbf{DAlt}_n)_{\mathcal{ALC}}$ and $(\mathbf{D}_n)_{\mathcal{ALC}}$ with expanding/varying domains, where reduction to $\mathcal{ALC}$ is still possible.

What follows from this analysis, is that by sacrificing the generality of $\mathbf{K}_n$-frames one does not immediately obtain a better computational behavior as long as multiple context operators are permitted. For this reason, we adopt $(\mathbf{K}_n)_{\mathcal{ALC}}$ as the baseline for $\mathcal{ALC}_{\mathcal{ALC}}$, leaving for now the option of restricting context structures as an open problem.

4 Describing contexts: from $(\mathbf{K}_n)_{\mathcal{ALC}}$ to $\mathcal{ALC}_{\mathcal{ALC}}$

We are now ready to define the target logic $\mathcal{ALC}_{\mathcal{ALC}}$, which additionally to $(\mathbf{K}_n)_{\mathcal{ALC}}$ offers a second sort of vocabulary for directly describing contexts. This extension addresses the third postulate of McCarthy.

[4] Demonstrating the corresponding phenomenon in $(\mathbf{DAlt}_n)_{\mathcal{ALC}}$ is not that straightforward due to the seriality condition, as then the global information concerns only the existence of $succ_i$-predecessors. Thus, one needs role inverses in the fusion language to observe the loss of such information.

4.1 Syntax and semantics

We start by introducing the context component of the language and then suitably revise the object component.

The *context language* L_C is an $\mathcal{ALC}$ concept language over vocabulary $\Gamma = (M_I, M_C, M_R)$, where M_I is a set of (*context*) *individual names*, M_C is a set of (*context*) *concept names*, and M_R is a set of (*context*) *role names*. For disambiguation, we use ***bold font*** when writing names from the context vocabulary and we denote the elements of L_C as *c-concepts*. The semantics is defined in the usual manner (as presented in Section 2.2), in terms of an interpretation function $\cdot^{\mathcal{J}}$ ranging over the context domain $\mathfrak{C}$. The *context knowledge base* $\mathcal{C}$ consists of TBox and ABox axioms over Γ and L_C, also with the usual satisfaction conditions. Thus, $\mathcal{C}$ is in fact a standard $\mathcal{ALC}$ ontology with standard models of the form $(\mathfrak{C}, \cdot^{\mathcal{J}})$.

The interpretations of the context language are incorporated in the full $\mathcal{ALC}_{\mathcal{ALC}}$ interpretations of the form $\mathfrak{M} = (\mathfrak{C}, \cdot^{\mathcal{J}}, \Delta, \{\cdot^{\mathcal{I}(c)}\}_{c \in \mathfrak{C}})$, where:

- $\mathfrak{C}$ is a non-empty *context domain*,
- $\cdot^{\mathcal{J}}$ is an *interpretation function* of the context language,
- Δ is a non-empty *object domain*,
- $\cdot^{\mathcal{I}(c)}$ is an *interpretation function* of the object language in c.

The divergence from the original $(\mathbf{K}_n)_{\mathcal{ALC}}$ interpretations is minor. Basically, the accessibility relations over $\mathfrak{C}$ become now redundant, as their function can be taken over by context roles. For every contextualization operation i we can assume an implicit correspondence $R_i = \boldsymbol{r}_i^{\mathcal{J}}$, for some $\boldsymbol{r}_i \in M_R$. Note that given the broadened take on the context dimension, we might be now less strict about the informal reading of some of the components of the framework. Arguably, not all context roles have to be necessarily seen as 'contextualization operations' and not all elements of $\mathfrak{C}$ as genuine 'contexts'. Sometimes they can be just entities needed for describing contexts. Nevertheless, we keep using the context-object nomenclature to avoid potential confusions.

Although one can already express rich knowledge about contexts, such knowledge remains 'invisible' from the object level. In order to render it more accessible, and so gain better control over the interaction between the dimensions, we need to suitably internalize context descriptions in the object language.

Let $\Sigma = (N_I, N_C, N_R)$ be the *object vocabulary* disjoint from Γ. The *object language* L_O over Σ and the context language L_C is the smallest set of concepts, called *o-concepts*, containing $\top$, concept names from N_C and closed under the $\mathcal{ALC}$ and the following two constructors:

$$\langle \boldsymbol{C} \rangle_{\boldsymbol{r}} D \ \mid \ [\boldsymbol{C}]_{\boldsymbol{r}} D$$

where $\boldsymbol{C} \in L_C$ and $\boldsymbol{r} \in M_R$. Again, $[\cdot]_{\boldsymbol{r}}$ abbreviates $\neg\langle\cdot\rangle_{\boldsymbol{r}}\neg$. Intuitively, $\langle \boldsymbol{C} \rangle_{\boldsymbol{r}} D$ denotes all objects which are D in *some* context which is $\boldsymbol{C}$ and is accessible through $\boldsymbol{r}$. Similarly, $[\boldsymbol{C}]_{\boldsymbol{r}} D$ denotes all objects which are D in *every* context which is $\boldsymbol{C}$ and is accessible through $\boldsymbol{r}$. Overall, the syntax of the object language diverges from the one of $(\mathbf{K}_n)_{\mathcal{ALC}}$ only in that the indices appearing by $\Diamond_i, \Box_i$

are now replaced with context roles, while both operators embrace a single c-concept, which additionally qualifies the accessed contexts. Consequently, the changes in the semantics affect only the contextualized concepts:

$$(\langle \boldsymbol{C} \rangle_{\boldsymbol{r}} D)^{\mathcal{I}(c)} = \{x \in \Delta \mid \exists d \in \mathfrak{C}: \ \langle c, d \rangle \in \boldsymbol{r}^{\mathcal{J}} \wedge d \in \boldsymbol{C}^{\mathcal{J}} \wedge x \in D^{\mathcal{I}(d)}\},$$
$$([\boldsymbol{C}]_{\boldsymbol{r}} D)^{\mathcal{I}(c)} = \{x \in \Delta \mid \forall d \in \mathfrak{C}: \ \langle c, d \rangle \in \boldsymbol{r}^{\mathcal{J}} \wedge d \in \boldsymbol{C}^{\mathcal{J}} \rightarrow x \in D^{\mathcal{I}(d)}\}.$$

To grant maximum flexibility in expressing the knowledge about the object dimension we first define the set of possible *object formulas*, i.e., formulas which can meaningfully hold in individual contexts:

$$B \sqsubseteq D \ \mid \ a : D \ \mid \ s(a, b) \ \mid \ \neg\varphi \ \mid \ \varphi \wedge \psi \ \mid \ \langle \boldsymbol{C} \rangle_{\boldsymbol{r}} \varphi \ \mid \ [\boldsymbol{C}]_{\boldsymbol{r}} \varphi$$

where B, D are o-concepts, $a, b \in N_I$, $s \in N_R$, $\boldsymbol{C}$ is a c-concept and $\boldsymbol{r} \in M_R$. Object formulas are satisfied by $\mathfrak{M}$ in context $c \in \mathfrak{C}$ in the following cases:

- $(\mathfrak{M}, c) \models B \sqsubseteq D$ *iff* $B^{\mathcal{I}(c)} \subseteq D^{\mathcal{I}(c)}$,
- $(\mathfrak{M}, c) \models a : D$ *iff* $a^{\mathcal{I}(c)} \in D^{\mathcal{I}(c)}$,
- $(\mathfrak{M}, c) \models s(a, b)$ *iff* $\langle a^{\mathcal{I}(c)}, b^{\mathcal{I}(c)} \rangle \in s^{\mathcal{I}(c)}$,
- $(\mathfrak{M}, c) \models \neg\varphi$ *iff* $(\mathfrak{M}, c) \not\models \varphi$,
- $(\mathfrak{M}, c) \models \varphi \wedge \psi$ *iff* $(\mathfrak{M}, c) \models \varphi$ and $(\mathfrak{M}, c) \models \psi$,
- $(\mathfrak{M}, c) \models \langle \boldsymbol{C} \rangle_{\boldsymbol{r}} \varphi$ *iff* $(\mathfrak{M}, d) \models \varphi$ for some $d \in \mathfrak{C}$ s.t. $\langle c, d \rangle \in \boldsymbol{r}^{\mathcal{J}}$ and $d \in \boldsymbol{C}^{\mathcal{J}}$,
- $(\mathfrak{M}, c) \models [\boldsymbol{C}]_{\boldsymbol{r}} \varphi$ *iff* $(\mathfrak{M}, d) \models \varphi$ for every $d \in \mathfrak{C}$ s.t. $\langle c, d \rangle \in \boldsymbol{r}^{\mathcal{J}}$ and $d \in \boldsymbol{C}^{\mathcal{J}}$.

Then we define an *object knowledge base* $\mathcal{O}$ as a set of axioms of two forms:

$$\boldsymbol{a} : \varphi \ \mid \ \boldsymbol{C} : \varphi$$

where $\boldsymbol{a} \in M_I$, $\boldsymbol{C}$ is a c-concept and φ is an object formula. Such axioms have a straightforward reading: φ is true in context $\boldsymbol{a}$; and φ is true in every context which is $\boldsymbol{C}$. Formally, we specify those conditions as follows:

- $\mathfrak{M} \models \boldsymbol{a} : \varphi$ *iff* $(\mathfrak{M}, c) \models \varphi$ for $c = \boldsymbol{a}^{\mathcal{J}}$,
- $\mathfrak{M} \models \boldsymbol{C} : \varphi$ *iff* $(\mathfrak{M}, c) \models \varphi$ for every $c \in \boldsymbol{C}^{\mathcal{J}}$.

A pair $\mathcal{K} = (\mathcal{C}, \mathcal{O})$ is called an $\mathcal{ALC}_{\mathcal{ALC}}$ *knowledge base*. An interpretation $\mathfrak{M}$ is a *model* of $\mathcal{K}$ whenever all axioms in $\mathcal{K}$ are satisfied. A small example of an $\mathcal{ALC}_{\mathcal{ALC}}$ knowledge base is presented in Section 5.

4.2 Complexity and expressiveness

Obviously, the expressiveness of $\mathcal{ALC}_{\mathcal{ALC}}$ properly subsumes that of $(\mathbf{K}_n)_{\mathcal{ALC}}$. In particular, the following relationship holds:

Proposition 2. *Concept satisfiability problem in $(\mathbf{K}_n)_{\mathcal{ALC}}$ w.r.t. global TBoxes is polynomially reducible to knowledge base satisfiability in $\mathcal{ALC}_{\mathcal{ALC}}$.*

To see this is indeed the case suppose $(C, \mathcal{T})$ is the problem of deciding whether concept C is satisfiable w.r.t. global TBox $\mathcal{T}$. Let C' and $\mathcal{T}'$ be the results of replacing every $\Diamond_i$ with $\langle\top\rangle_{\boldsymbol{r}_i}$ and every $\Box_i$ with $[\top]_{\boldsymbol{r}_i}$ in C and $\mathcal{T}$, respectively, where for $i \neq j$ we have $\boldsymbol{r}_i \neq \boldsymbol{r}_j$. Further define $\mathcal{C} = \emptyset$ and $\mathcal{O} = \{\boldsymbol{c} : a : C'\} \cup \{\top : C \sqsubseteq D \mid C \sqsubseteq D \in \mathcal{T}'\}$. It clearly follows that C is satisfiable w.r.t. $\mathcal{T}$ in $(\mathbf{K}_n)_{\mathcal{ALC}}$ *iff* the knowledge base $\mathcal{K} = (\mathcal{C}, \mathcal{O})$ is satisfiable in $\mathcal{ALC}_{\mathcal{ALC}}$. Note, that the reduction holds even when object roles are interpreted rigidly.

This naturally means that the 2ExpTime lower bound established in Theorem 5 transfers immediately to $\mathcal{ALC}_{\mathcal{ALC}}$. But can it get even higher? Quite surprisingly, the answer is negative. Despite the increase of expressiveness, satisfiability problem in $\mathcal{ALC}_{\mathcal{ALC}}$ remains in 2ExpTime.

Theorem 6. *Deciding satisfiability of an $\mathcal{ALC}_{\mathcal{ALC}}$ knowledge base in which object roles are interpreted locally is* 2ExpTime*-complete.*

The proof of the upper bound is based on quasimodel elimination technique, which extends the one used for Theorem 3. In particular, every quasistate has to carry now also the type of the context which it represents and the set of object formulas which are satisfied in it.

To give a final insight into the expressiveness of the formalism, in more traditional terms of products of modal logics, we show that $\mathcal{ALC}_{\mathcal{ALC}}$ (with rigid roles) is equally expressive to the full $\mathcal{ALC}$ language over the union of two vocabularies interpreted in product models.

Let L_1 and L_2 be two $\mathcal{ALC}$ concept languages over disjoint vocabularies $\Gamma = (M_C, M_R, \emptyset)$ and $\Sigma = (N_C, N_R, \emptyset)$, respectively. Now, let $L_{1\times 2}$ be the $\mathcal{ALC}$ concept language over vocabulary $\Theta = (M_C \cup N_C, M_R \cup N_R, \emptyset)$. The semantics for $L_{1\times 2}$ is given through *product interpretations* $\mathcal{P} = (\mathfrak{C} \times \Delta, \cdot^{\mathcal{P}})$, which align every $r \in N_R$ along the 'vertical' dimension and every $\boldsymbol{p} \in M_R$ along the 'horizontal' one. Thus, $r^{\mathcal{P}}, \boldsymbol{p}^{\mathcal{P}} \subseteq (\mathfrak{C} \times \Delta) \times (\mathfrak{C} \times \Delta)$ and for every $u, v, w \in \mathfrak{C}$ and $x, y, z \in \Delta$:

$$\langle (u,x),(v,y)\rangle \in r^{\mathcal{P}} \rightarrow u = v \;\&\; \langle (w,x),(w,y)\rangle \in r^{\mathcal{P}},$$
$$\langle (u,x),(v,y)\rangle \in \boldsymbol{p}^{\mathcal{P}} \rightarrow x = y \;\&\; \langle (u,z),(v,z)\rangle \in \boldsymbol{p}^{\mathcal{P}}.$$

All concepts are interpreted as subsets of $\mathfrak{C} \times \Delta$. Additionally, we force every $\boldsymbol{A} \in M_C$ to be interpreted rigidly across the 'vertical' dimension, i.e., for every $v \in \mathfrak{C}$ and $x, y \in \Delta$ we assume:

$$(*) \quad (v,x) \in \boldsymbol{A}^{\mathcal{I}} \rightarrow (v,y) \in \boldsymbol{A}^{\mathcal{I}}$$

Finally, $\cdot^{\mathcal{P}}$ is extended inductively as usual. A concept $C \in L_{1\times 2}$ is satisfiable *iff* for some product model $\mathcal{P} = (\mathfrak{C} \times \Delta, \cdot^{\mathcal{P}})$ it is the case that $C^{\mathcal{P}} \neq \emptyset$. On the contrary to the others, the condition $(*)$ is rather uncommon in the realm of products of modal logics. Nevertheless, it captures precisely the difference between the semantics of the two sorts of concepts. Without it the sorts collapse into one, while the whole logic turns into a notational variant of $(\mathbf{K}_n)_{\mathcal{ALC}}$. It turns out that the following claim holds:

Theorem 7. *The language $L_{1\times 2}$ interpreted in product models is exactly as expressive as the concept language of $\mathcal{ALC}_{\mathcal{ALC}}$ interpreted in models with rigid interpretations of object roles.*

What follows from Theorem 7 is that the syntactic constraints of $\mathcal{ALC}_{\mathcal{ALC}}$, which make the logic more intuitive and well-behaved, by no means lead to loss of expressiveness. Moreover, it shows that $\mathcal{ALC}_{\mathcal{ALC}}$ (at least in its concept component) does not seriously deviate from the usual products of modal logics. In principle, the only feature distinguishing it from $(\mathbf{K}_n)_{\mathcal{ALC}}$ (both with and without rigid roles) is the condition (*) imposed on the interpretations of selected concepts, which in $\mathcal{ALC}_{\mathcal{ALC}}$ we simply happen to call context concepts.

5 Contextual ontologies — example

One of the designated applications of $\mathcal{ALC}_{\mathcal{ALC}}$ is construction of *contextual ontologies*. The distinguishing feature of such ontologies is that they allow for varying the characterization of concepts according to contexts. Hence, $\mathcal{ALC}_{\mathcal{ALC}}$ can provide a good formal support for exchanging and integrating information in DL. Moreover, as the context knowledge base can be created independently from the object component, the framework encourages reuse of existing ontologies.

As an example of a contextual ontology, we present a simple representation of knowledge about the food domain contextualized with respect to geographic locations. Consider the (context) geographic knowledge base $\mathcal{C} = (\mathcal{T}, \mathcal{A})$, where $\mathcal{T}$ is a TBox and $\mathcal{A}$ an ABox.

$$
\begin{aligned}
\mathcal{T} = \{ \; & (1)\, \mathbf{Country} \sqsubseteq \exists \mathbf{location}.\mathbf{Europe} \sqcup \exists \mathbf{location}.\mathbf{America}\;, \\
& (2)\, \mathbf{Region} \sqsubseteq \exists \mathbf{part\_of}.\mathbf{Country}, \\
& (3)\, \mathbf{City} \sqsubseteq \exists \mathbf{has\_part}.\mathbf{Neighborhood} \; \} \\
\mathcal{A} = \{ \; & (4)\, \mathit{US} : \mathbf{Country}, \\
& (5)\, \mathit{SanFrancisco} : \mathbf{City}, \\
& (6)\, \mathit{California} : \mathbf{Region}, \\
& (7)\, \mathbf{part\_of}(\mathit{California}, \mathit{US}), \\
& (8)\, \mathit{France} : \mathbf{Country} \sqcap \exists \mathbf{location}.\mathbf{Europe} \; \}
\end{aligned}
$$

Now, we define an (object) food ontology $\mathcal{O}$, contextualized with $\mathcal{C}$.

$$
\begin{aligned}
\mathcal{O} = \{ \; & (a)\, \top : \mathsf{Food} \equiv \mathsf{Meat} \sqcup \mathsf{Beverages} \sqcup \mathsf{Sea\_Food} \sqcup \mathsf{Grains} \\
& (b)\, \top : \mathsf{Wine} \equiv \mathsf{WhiteWine} \sqcup \mathsf{RedWine} \\
& (c)\, \top : (\mathtt{SauvignonBlanc} : \mathsf{WhiteWine}) \\
& (d)\, \mathbf{Country} : [\mathbf{Europe}]_{\mathbf{location}}(\mathsf{WhiteWine} \sqsubseteq \mathsf{Popular\_Beverage}) \\
& (e)\, \mathit{California} : \mathsf{WhiteWine} \sqsubseteq [\mathbf{Country}]_{\mathbf{part\_of}}\mathsf{Popular\_Wine} \\
& (f)\, \mathit{US} : \mathsf{Popular\_Wine} \sqsubseteq \neg\mathsf{Popular\_Beverage} \\
& (g)\, \mathit{SanFrancisco} : [\top]_{\mathbf{has\_part}}(\mathsf{WhiteWine} \sqsubseteq \neg\mathsf{Popular\_Wine}) \; \}
\end{aligned}
$$

Let us shortly highlight the intuition behind $\mathcal{O}$ by explaining some of the axiom definitions and the inferences they sanction. First, axioms (a)-(c) present geographic-independent terminology of the food domain. For example, by (c), `SauvignonBlanc` is a WhiteWine in any part of the world. Then, (d)-(g) characterize WhiteWine as Popular_Wine or Popular_Beverage according to different territories. We explain (d)-(g) in terms of `SauvignonBlanc`. By (d), in any **Country** that has as a **location Europe** (e.g., *France*) `SauvignonBlanc` is a

Popular_Beverage. However, by (e)-(f), SauvignonBlanc is *not* a Popular_Beverage in *US*. This, is explained as follows: (e) establishes that SauvignonBlanc is a Popular_Wine in any **Country** of which *California* is part of, namely *US*. Then, by (f), in the *US* any Popular_Wine is not a Popular_Beverage. Hence, SauvignonBlanc is not a Popular_Beverage in *US*. Although SauvignonBlanc is a Popular_Wine in *US*, this is does not necessarily transfer to more specific contexts. For instance, by (g), in every part of *SanFrancisco*, SauvignonBlanc is not in fact a Popular_Wine. In particular, by (3), there is at least one such **Neighborhood** in which this happens.

6 Conclusions and future work

We have presented a novel DL $\mathcal{ALC}_{\mathcal{ALC}}$ for representing and reasoning with contextual knowledge. Our approach is derived from McCarthy's conception of contexts as first-order objects which are describable in a first-order language. Formally, the logic extends the well-known $(\mathbf{K}_n)_{\mathcal{ALC}}$ with another sort of 'context' vocabulary interpreted over the **K**-dimension. The surprising conclusion is that the increase of the expressiveness of the logic due to this addition comes for no substantial price in terms of the worst-case complexity. The jump to 2ExpTime-completeness stems from the interaction of multiple modalities with global TBoxes and is inherent already to the underlying two-dimensional DLs.

We believe that with this work we have set the stage for a promising future research on similar combinations of DLs. Clearly, there are three major determinants of such formalisms which deserve a careful study: 1) *the expressiveness of the context language*, 2) *the expressiveness of the object language*, 3) *the level of interaction between the two*. Finding a proper balance between them is the key to identifying well-behaved and potentially useful fragments. One of the first directions, which we want to investigate, is to reduce the interaction between the languages by employing only **S5**-like operators. Such operators, e.g., $\langle \boldsymbol{C} \rangle \varphi$, would state that there exists a context of type C in which φ holds, without involving context roles. This modification should result in a better computational behavior and a somewhat simpler conceptual design of the language.

On the applied side, it could be interesting to consider a restricted fragment of the framework (a finite number of named contexts) for the task of ontology integration on the Semantic Web. Arguably, such fragment is sufficient to provide a logical underpinning for the ongoing endeavor of describing and linking OWL/RDFS knowledge sources in a context-sensitive manner.

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