Enriching $\mathcal{EL}$-Concepts with Greatest Fixpoints

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Abstract. We investigate the expressive power and computational complexity of $\mathcal{EL}^\nu$, the extension of the lightweight description logic $\mathcal{EL}$ with concept constructors for greatest fixpoints. It is shown that $\mathcal{EL}^\nu$ has the same expressive power as $\mathcal{EL}$ extended with simulation quantifiers and that it can be characterized as a largest fragment of monadic second-order logic that is preserved under simulations and has finite minimal models. As in basic $\mathcal{EL}$, all standard reasoning problems for general TBoxes can be solved in polynomial time. $\mathcal{EL}^\nu$ has a range of very desirable properties that $\mathcal{EL}$ itself is lacking. Firstly, least common subsumers w.r.t. general TBoxes as well as most specific concepts always exist and can be computed in polynomial time. Secondly, $\mathcal{EL}^\nu$ shares with $\mathcal{EL}$ the Craig interpolation property and the Beth definability property, but in contrast to $\mathcal{EL}$ allows the computation of interpolants and explicit concept definitions in polynomial time.

1 INTRODUCTION

The well-known description logic (DL) $\mathcal{ALC}$ is usually regarded as the basic DL that comprises all Boolean concept constructors and from which more expressive DLs are derived by adding further expressive means. This fundamental role of $\mathcal{ALC}$ is largely due to its well-behavedness regarding logical, model-theoretic, and computational properties which can, in turn, be explained nicely by the fact that $\mathcal{ALC}$-concepts can be characterized as the bisimulation invariant fragment of first-order logic (FO): an FO formula is invariant under bisimulation if, and only if, it is equivalent to an $\mathcal{ALC}$-concept [22, 12, 16]. For example, invariance under bisimulation can explain the tree-model property of $\mathcal{ALC}$ and its favorable computational properties [24]. In the above characterization, the condition that $\mathcal{ALC}$ is a fragment of FO is much less important than its bisimulation invariance. In fact, $\mathcal{ALC}\mu$, the extension of $\mathcal{ALC}$ with fixpoint operators, is not a fragment of FO, but inherits almost all important properties of $\mathcal{ALC}$ [7, 11, 19]. Similar to $\mathcal{ALC}$, $\mathcal{ALC}\mu$'s fundamental role (in particular in its formulation as the modal mu-calculus) can be explained by the fact that $\mathcal{ALC}\mu$-concepts comprise exactly the bisimulation invariant fragment of monadic second-order logic (MSO) [14, 7]. Indeed, from a purely theoretical viewpoint it is hard to explain why $\mathcal{ALC}$ rather than $\mathcal{ALC}\mu$ forms the logical underpinning of current ontology language standards; the facts that mu-calculus concepts can be hard to grasp and that, despite the same theoretical complexity, efficient reasoning in $\mathcal{ALC}\mu$ is more challenging than in $\mathcal{ALC}$ are probably the main reasons.

In recent years, the development of very large ontologies and the use of ontologies to access instance data has led to a revival of interest in *tractable* DLs. The main examples are $\mathcal{EL}$ [4] and DL-Lite [8], the logical underpinnings of the OWL profiles OWL2 EL and OWL2 QL, respectively. In contrast to $\mathcal{ALC}$, a satisfactory characterization of the expressivity of such DLs is still missing, and a first aim of this paper is to fill this gap for $\mathcal{EL}$. To this end, we characterize $\mathcal{EL}$ as a maximal fragment of FO that is preserved under *simulations* and has *finite minimal models*. Note that preservation under simulations alone would characterize $\mathcal{EL}$ with disjunctions, and the existence of minimal models reflects the "Horn-logic character" of $\mathcal{EL}$.

The second and main aim of this paper, however, is to introduce and investigate two equi-expressive extensions of $\mathcal{EL}$ with greatest fixpoints, $\mathcal{EL}^\nu$ and $\mathcal{EL}^{\nu+}$, and to prove that they stand in a similar relationship to $\mathcal{EL}$ as $\mathcal{ALC}\mu$ to $\mathcal{ALC}$. To this end, we prove that $\mathcal{EL}^\nu$ (and therefore also $\mathcal{EL}^{\nu+}$, which admits mutual fixpoints and is exponentially more succinct than $\mathcal{EL}^\nu$) can be characterized as a maximal fragment of MSO that is preserved under *simulations* and has *finite minimal models*. Similar to $\mathcal{ALC}\mu$, $\mathcal{EL}^\nu$ and $\mathcal{EL}^{\nu+}$ inherit many good properties of $\mathcal{EL}$ such as its Horn-logic character and the crucial fact that *reasoning with general concept inclusions (GCIs) is still tractable*. In contrast to $\mathcal{ALC}\mu$, the development of practical decision procedures is thus no obstacle to using $\mathcal{EL}^{\nu+}$. Moreover, $\mathcal{EL}^{\nu+}$ has a number of very useful properties that $\mathcal{EL}$ and most of its extensions are lacking. To begin with, we show that in $\mathcal{EL}^{\nu+}$ *least common subsumers (lcs)* w.r.t. general TBoxes always exist and can be computed in polynomial time (for a bounded number of concepts). This result can be regarded as an extension of similar results for least common subsumers w.r.t. *classical* TBoxes in $\mathcal{EL}$ with greatest fixpoint semantics in [2]. Similarly, in $\mathcal{EL}^{\nu+}$ *most specific concepts* always exist and can be computed in linear time; a result which also generalizes [2]. Secondly, we show that $\mathcal{EL}^{\nu+}$ has the *Beth definability property* with explicit definitions being computable in polytime and of polynomial size. It has been convincingly argued in [21, 20] that this property is of great interest for structuring TBoxes and for ontology based data access. Another application of $\mathcal{EL}^{\nu+}$ is demonstrated in [15], where the succinct representations of definitions in $\mathcal{EL}^{\nu+}$ are used to develop polytime algorithms for decomposing certain general $\mathcal{EL}$-TBoxes.

To prove these result and provide a better understanding of the modeling capabilities of $\mathcal{EL}^{\nu+}$ we show that it has the same expressive power as extensions of $\mathcal{EL}$ by means of *simulation quantifiers*, a variant of second-order quantifiers that quantifies "modulo a simulation of the model"; in fact, the relationship between simulation quantifiers and $\mathcal{EL}^{\nu+}$ is somewhat similar to the relationship between $\mathcal{ALC}\mu$ and bisimulation quantifiers [10].

Most proofs are provided in an appendix.

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2 PRELIMINARIES

Let $\mathsf{N_C}$ and $\mathsf{N_R}$ be countably infinite and mutually disjoint sets of concept and role names. *$\mathcal{EL}$-concepts* are built according to the rule

$$C := \quad A \quad | \quad \top \quad | \quad \bot \quad | \quad C \sqcap D \quad | \quad \exists r.C,$$

where $A \in \mathsf{N_C}$, $r \in \mathsf{N_R}$, and C, D range over $\mathcal{EL}$-concepts[3]. An *$\mathcal{EL}$-concept inclusion* takes the form $C \sqsubseteq D$, where C, D are $\mathcal{EL}$-concepts. As usual, we use $C \equiv D$ to abbreviate $C \sqsubseteq D, D \sqsubseteq C$. A *general $\mathcal{EL}$-TBox $\mathcal{T}$* is a finite set of $\mathcal{EL}$-concept inclusions. An *ABox assertion* is an expression of the form $A(a)$ or $r(a, b)$, where a, b are from a countably infinite set of individual names $\mathsf{N_I}$, $A \in \mathsf{N_C}$, and $r \in \mathsf{N_R}$. An *ABox* is a finite set of ABox assertions. By $\mathsf{Ind}(\mathcal{A})$ we denote the set of individual names in $\mathcal{A}$. An *$\mathcal{EL}$-knowledge base (KB)* is a pair $(\mathcal{T}, \mathcal{A})$ that consists of an $\mathcal{EL}$-TBox $\mathcal{T}$ and an ABox $\mathcal{A}$.

The semantics of $\mathcal{EL}$ is based on interpretations $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$, where the *domain* $\Delta^\mathcal{I}$ is a non-empty set, and $\cdot^\mathcal{I}$ is a function mapping each concept name A to a subset $A^\mathcal{I}$ of $\Delta^\mathcal{I}$, each role name r to a binary relation $r^\mathcal{I} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}$, and each individual name a to an element $a^\mathcal{I}$ of $\Delta^\mathcal{I}$. The interpretation $C^\mathcal{I} \subseteq \Delta^\mathcal{I}$ of $\mathcal{EL}$-concepts C in an interpretation $\mathcal{I}$ is defined in the standard way [5], and so are models of TBoxes, ABoxes, and KBs. We will often make use of the fact that $\mathcal{EL}$-concepts can be regarded as formulas in FO (and, therefore, MSO) with unary predicates from $\mathsf{N_C}$, binary predicates from $\mathsf{N_R}$, and exactly one free variable [5]. We will often not distinguish between $\mathcal{EL}$-concepts and their translations into FO/MSO.

We now introduce $\mathcal{EL}^\nu$, the extension of $\mathcal{EL}$ with greatest fixpoints and the main language studied in this paper. *$\mathcal{EL}^\nu$-concepts* are defined like $\mathcal{EL}$-concepts, but additionally allow the greatest fixpoint constructor $\nu X.C$, where X is from a countably infinite set of *(concept) variables* $\mathsf{N_V}$ and C an $\mathcal{EL}^\nu$-concept. A variable is *free* in a concept C if it occurs in C at least once outside the scope of any ν-constructor that binds it. An $\mathcal{EL}^\nu$-concept is *closed* if it does not contain any free variables. An *$\mathcal{EL}^\nu$-concept inclusion* takes the form $C \sqsubseteq D$, where C, D are are closed $\mathcal{EL}^\nu$-concepts. The semantics of the greatest fixpoint constructor is as follows, where $\mathcal{V}$ is an *assignment* that maps variables to subsets of $\Delta^\mathcal{I}$ and $\mathcal{V}[X \mapsto W]$ denotes $\mathcal{V}$ modified by setting $\mathcal{V}(X) = W$:

$$(\nu X.C)^{\mathcal{I},\mathcal{V}} = \bigcup\{W \subseteq \Delta^\mathcal{I} \mid W \subseteq C^{\mathcal{I},\mathcal{V}[X\mapsto W]}\}$$

Example 1 For the concept $C = \nu X.(\exists \mathsf{has\_parent}.X)$, we have $e \in C^\mathcal{I}$ if, and only if, there is an infinite $\mathsf{has\_parent}$-chain starting at e in $\mathcal{I}$, i.e., there exist $e_0, e_1, e_2, \ldots$ such that $e = e_0$ and $(e_i, e_{i+1}) \in \mathsf{has\_parent}^\mathcal{I}$ for all $i \geq 0$.

We can now form the TBox $\mathcal{T} = \{\mathsf{Human\_being} \sqsubseteq C\}$ stating that every human being has an infinite chain of parents.

We will also consider an extended version of the ν-constructor that allows to capture mutual recursion. It has been considered e.g. in [9, 23] and used in a DL context in [19]; it can be seen as a variation of the fixpoint equations considered in [7]. The constructor has the form $\nu_i X_1 \cdots X_n.C_1, \ldots, C_n$ where $1 \leq i \leq n$. The semantics is defined by setting $(\nu_i X_1 \cdots X_n.C_1, \ldots, C_n)^{\mathcal{I},\mathcal{V}}$ to

$$\bigcup\{W_i \mid \quad \exists W_1, \ldots, W_{i-1}, W_{i+1}, \ldots, W_n \text{ s.t. for } 1 \leq j \leq n: \; W_j \subseteq C_j^{\mathcal{I},\mathcal{V}[X_1\mapsto W_1,\ldots,X_n\mapsto W_n]}\}$$

We use $\mathcal{EL}^{\nu+}$ to denote $\mathcal{EL}$ extended with this mutual greatest fixpoint constructor. Clearly, $\nu X.C \equiv \nu_1 X.C$, thus every $\mathcal{EL}^\nu$-concept is equivalent to an $\mathcal{EL}^{\nu+}$-concept. We now consider the converse direction. Firstly, the following proposition follows immediately from well known results on mutual fixpoint constructors [7].

[3] In the literature, $\mathcal{EL}$ is typically defined without $\bot$. The sole purpose of including $\bot$ here is to simplify the formulation of some results.

Proposition 2 *For every $\mathcal{EL}^{\nu+}$-concept one can construct an equivalent $\mathcal{EL}^\nu$-concept.*

In this paper, we define the *length of a concept C* as the number of occurrences of symbols in it. Then the translation in Proposition 2 yields an exponential blow-up and one can show that indeed there is a sequence of $\mathcal{EL}^{\nu+}$-concepts $C_0, C_1, \ldots$ such that C_i is of length $p(i)$, p a polynomial, whereas the shortest $\mathcal{EL}^\nu$-concept equivalent to C_i is of length at least 2^i (see appendix).

By extending the translation of $\mathcal{EL}$-concepts into FO in the obvious way, one can translate closed $\mathcal{EL}^{\nu+}$-concepts into MSO formulas with one free first-order variable. We will often not distinguish between $\mathcal{EL}^{\nu+}$-concepts and their translation into MSO.

3 CHARACTERIZING $\mathcal{EL}$ USING SIMULATIONS

The purpose of this section is to provide a model-theoretic characterization of $\mathcal{EL}$ as a fragment of FO that is similar in spirit to the well-known characterization of $\mathcal{ALC}$ as the bisimulation-invariant fragment of FO. To this end, we first characterize $\mathcal{EL}^\sqcup$, the extension of $\mathcal{EL}$ with the disjunction constructor $\sqcup$, as the fragment of FO that is preserved under simulation. Then we characterize the fragment $\mathcal{EL}$ of $\mathcal{EL}^\sqcup$ using, in addition, the existence of minimal models. A *pointed interpretation* is a pair $(\mathcal{I}, d)$ consisting of an interpretation $\mathcal{I}$ and $d \in \Delta^\mathcal{I}$. A *signature* Σ is a set of concept and role names.

Definition 3 (Simulations) Let $(\mathcal{I}_1, d_1)$ and $(\mathcal{I}_2, d_2)$ be pointed interpretations and Σ a signature. A relation $S \subseteq \Delta^{\mathcal{I}_1} \times \Delta^{\mathcal{I}_2}$ is a *Σ-simulation* between $(\mathcal{I}_1, d_1)$ and $(\mathcal{I}_2, d_2)$, in symbols $S : (\mathcal{I}_1, d_1) \leq_\Sigma (\mathcal{I}_2, d_2)$, if $(d_1, d_2) \in S$ and the following conditions hold:

1. for all concept names $A \in \Sigma$ and all $(e_1, e_2) \in S$, if $e_1 \in A^{\mathcal{I}_1}$ then $e_2 \in A^{\mathcal{I}_2}$;
2. for all role names $r \in \Sigma$, all $(e_1, e_2) \in S$, and all $e_1' \in \Delta^{\mathcal{I}_1}$ with $(e_1, e_1') \in r^{\mathcal{I}_1}$, there exists $e_2' \in \Delta^{\mathcal{I}_2}$ such that $(e_2, e_2') \in r^{\mathcal{I}_2}$ and $(e_1', e_2') \in S$.

If such an S exists, then we also say that $(\mathcal{I}_2, d_2)$ *Σ-simulates* $(\mathcal{I}_1, d_1)$ and write $(\mathcal{I}_1, d_1) \leq_\Sigma (\mathcal{I}_2, d_2)$.

If $\Sigma = \mathsf{N_C} \cup \mathsf{N_R}$, then we omit Σ and use the term *simulation* to denote Σ-simulations and $(\mathcal{I}_1, d_1) \leq (\mathcal{I}_2, d_2)$ stands for $(\mathcal{I}_1, d_1) \leq_\Sigma (\mathcal{I}_2, d_2)$. It is well-known that the description logic $\mathcal{EL}$ is intimately related to the notion of a simulation, see for example [3, 17]. In particular, $\mathcal{EL}$-concepts are preserved under simulations in the sense that if $d_1 \in C^{\mathcal{I}_1}$ for an $\mathcal{EL}$-concept C and $(\mathcal{I}_1, d_1) \leq (\mathcal{I}_2, d_2)$, then $d_2 \in C^{\mathcal{I}_2}$. This observation, which clearly generalizes to $\mathcal{EL}^\sqcup$, illustrates the (limitations of the) modeling capabilities of $\mathcal{EL}$/$\mathcal{EL}^\sqcup$. We now strengthen it to an exact characterization of the expressive power of these logics relative to FO.

Let $\varphi(x)$ be an FO-formula (or, later, MSO-formula) with one free variable x. We say that $\varphi(x)$ is *preserved under simulations* if, and only if, for all $(\mathcal{I}_1, d_1)$ and $(\mathcal{I}_2, d_2)$, $\mathcal{I}_1 \models \varphi[d_1]$ and $(\mathcal{I}_1, d_1) \leq (\mathcal{I}_2, d_2)$ implies $\mathcal{I}_2 \models \varphi[d_2]$.

Theorem 4 *An FO-formula $\varphi(x)$ is preserved under simulations if, and only if, it is equivalent to an $\mathcal{EL}^\sqcup$-concept.*

To characterize $\mathcal{EL}$, we add a central property of Horn-logics on top of preservation under simulations. Let $\mathcal{L}$ be a set of FO (or, later, MSO) formulas, each with one free variable. We say that $\mathcal{L}$ *has (finite) minimal models* if, and only if, for every $\varphi(x) \in \mathcal{L}$ there exists a (finite) pointed interpretation $(\mathcal{I}, d)$ such that for all $\psi(x) \in \mathcal{L}$, we have $\mathcal{I} \models \psi[d]$ if, and only if, $\forall x.(\varphi(x) \rightarrow \psi(x))$ is a tautology.

Theorem 5 *The set of $\mathcal{EL}$-concepts is a maximal set of FO-formulas that is preserved under simulations and has minimal models (equivalently: has finite minimal models): if $\mathcal{L}$ is a set of FO-formulas that properly contains all $\mathcal{EL}$-concepts, then either it contains a formula not preserved under simulations or it does not have (finite) minimal models.*

We note that de Rijke and Kurtonina have given similar characterizations of various non-Boolean fragments of $\mathcal{ALC}$. In particular, Theorem 4 is rather closely related to results proved in [16] and would certainly have been included in the extensive list of characterizations given there had $\mathcal{EL}$ already been as popular as it is today. In contrast, the novelty of Theorem 5 is that it makes the Horn character of $\mathcal{EL}$ explicit through minimal models while the characterizations of disjunction-free languages in [16] are based on simulations that take sets (rather than domain-elements) as arguments.

4 SIMULATION QUANTIFIERS AND $\mathcal{EL}^\nu$

To understand and characterize the expressive power and modeling capabilities of $\mathcal{EL}^\nu$, we introduce three distinct types of simulation quantifiers and show that, in each case, the resulting language has the same expressive power as $\mathcal{EL}^\nu$.

Simulating interpretations. The first language $\mathcal{EL}^{si}$ extends $\mathcal{EL}$ by the concept constructor $\exists^{sim}(\mathcal{I}, d)$, where $(\mathcal{I}, d)$ is a finite pointed interpretation in which only finitely many $\sigma \in \mathsf{N_C} \cup \mathsf{N_R}$ have a non-empty interpretation $\sigma^\mathcal{I} \subseteq \Delta^\mathcal{I}$. The semantics of $\exists^{sim}(\mathcal{I}, d)$ is defined by setting for all interpretations $\mathcal{J}$ and $e \in \Delta^\mathcal{J}$,

$$e \in (\exists^{sim}(\mathcal{I}, d))^\mathcal{J} \text{ iff } (\mathcal{I}, d) \leq (\mathcal{J}, e).$$

Example 6 Let $\mathcal{I}$ be an interpretation such that $\Delta^\mathcal{I} = \{d\}$, $(d, d) \in \mathsf{has\_parent}^\mathcal{I}$, and $\sigma^\mathcal{I} = \emptyset$ for all remaining role and concept names σ. Then $\exists^{sim}(\mathcal{I}, d)$ is equivalent to the concept $\nu X.(\exists \mathsf{has\_parent}.X)$ from Example 1.

To attain a better understanding of the constructor $\exists^{sim}$, it is interesting to observe that *every $\mathcal{EL}^{si}$*-concept is equivalent to a concept of the form $\exists^{sim}(\mathcal{I}, d)$.

Lemma 7 *For every $\mathcal{EL}^{si}$-concept C one can construct, in linear time, an equivalent concept of the form $\exists^{sim}(\mathcal{I}, d)$.*

Proof By induction on the construction of C. If $C = A$ for a concept name A, then let $\mathcal{I} = (\{d\}, \cdot^\mathcal{I})$, where $A^\mathcal{I} = \{d\}$ and $\sigma^\mathcal{I} = \emptyset$ for all symbols distinct from A. Clearly, A and $\exists^{sim}(\mathcal{I}, d)$ are equivalent. For $C_1 = \exists^{sim}(\mathcal{I}_1, d_1)$ and $C_2 = \exists^{sim}(\mathcal{I}_2, d_2)$ assume that $\Delta^{\mathcal{I}_1} \cap \Delta^{\mathcal{I}_2} = \{d_1\} = \{d_2\}$. Then $\exists^{sim}(\mathcal{I}_1 \cup \mathcal{I}_2, d_1)$ is equivalent to $C_1 \sqcap C_2$, where $\Delta^{\mathcal{I}_1 \cup \mathcal{I}_2} = \Delta^{\mathcal{I}_1} \cup \Delta^{\mathcal{I}_2}$, and $\sigma^{\mathcal{I}_1 \cup \mathcal{I}_2} = \sigma^{\mathcal{I}_1} \cup \sigma^{\mathcal{I}_2}$ for all $\sigma \in \mathsf{N_C} \cup \mathsf{N_R}$. For $C = \exists r.\exists^{sim}(\mathcal{I}, d)$ construct a new interpretation $\mathcal{I}'$ by adding a new node e to $\Delta^\mathcal{I}$ and setting $(e, d) \in r^{\mathcal{I}'}$. Then $\exists^{sim}(\mathcal{I}', e)$ and C are equivalent.

We will show that there are polynomial translations between $\mathcal{EL}^{si}$ and $\mathcal{EL}^{\nu+}$. When using $\mathcal{EL}^{\nu+}$ in applications and to provide a translation from $\mathcal{EL}^{\nu+}$ to $\mathcal{EL}^{si}$, it is convenient to have available a "syntactic" simulation operator.

Simulating models of TBoxes. The second language $\mathcal{EL}^{st}$ extends $\mathcal{EL}$ by the concept constructor $\exists^{sim}\Sigma.(\mathcal{T}, C)$, where Σ is a finite signature, $\mathcal{T}$ a general TBox, and C a concept. To admit nestings of $\exists^{sim}$, the concepts of $\mathcal{EL}^{st}$ are defined by simultaneous induction; namely, $\mathcal{EL}^{st}$-concepts, concept inclusions, and general TBoxes are defined as follows:

- every $\mathcal{EL}$-concept, concept inclusion, and general TBox is an $\mathcal{EL}^{st}$-concept, concept inclusion, and general TBox, respectively;
- if $\mathcal{T}$ is a general $\mathcal{EL}^{st}$-TBox, C an $\mathcal{EL}^{st}$-concept, and Σ a finite signature, then $\exists^{sim}\Sigma.(\mathcal{T}, C)$ is an $\mathcal{EL}^{st}$-concept;
- if C, D are $\mathcal{EL}^{st}$-concepts, then $C \sqsubseteq D$ is a $\mathcal{EL}^{st}$-concept inclusion;
- a general $\mathcal{EL}^{st}$-TBox is a finite set of $\mathcal{EL}^{st}$-concept inclusions.

The semantics of $\exists^{sim}\Sigma.(\mathcal{T}, C)$ is as follows:

$d \in (\exists^{sim}\Sigma.(\mathcal{T}, C))^\mathcal{I}$ iff there exists $(\mathcal{J}, e)$ such that $\mathcal{J}$ is a model of $\mathcal{T}$, $e \in C^\mathcal{J}$ and $(\mathcal{J}, e) \leq_\Gamma (\mathcal{I}, d)$, where $\Gamma = (\mathsf{N_C} \cup \mathsf{N_R}) \setminus \Sigma$.

Example 8 Let $\mathcal{T} = \{A \sqsubseteq \exists \mathsf{has\_parent}.A\}$ and $\Sigma = \{A\}$. Then $\exists^{sim}\Sigma.(\mathcal{T}, A)$ is equivalent to the concept $\exists^{sim}(\mathcal{I}, d)$ defined in Example 6.

We will later exploit the fact that $\exists^{sim}\Sigma.(\mathcal{T}, C)$ is equivalent to $\exists^{sim}\Sigma \cup \{A\}.(\mathcal{T}', A)$, where A is a fresh concept name and $\mathcal{T}' = \mathcal{T} \cup \{A \sqsubseteq C\}$. Another interesting (but subsequently unexploited) observation is that we can w.l.o.g. restrict Σ to singleton sets since

$$\begin{aligned} \exists^{sim}(\{\sigma\} \cup \Sigma).(\mathcal{T}, C) &\equiv \exists^{sim}\{\sigma\}.(\emptyset, \exists^{sim}\Sigma.(\mathcal{T}, C)) \\ \exists^{sim}\emptyset.(\mathcal{T}, C) &\equiv \exists^{sim}\{B\}.(\mathcal{T}, C) \end{aligned}$$

where B is a concept name that does not occur in $\mathcal{T}$ and C.

Simulating models of KBs. The third language $\mathcal{EL}^{sa}$ extends $\mathcal{EL}$ by the concept constructor $\exists^{sim}\Sigma.(\mathcal{T}, \mathcal{A}, a)$, where a is an individual name in the ABox $\mathcal{A}$, $\mathcal{T}$ is a TBox, and Σ a finite signature. More precisely, we define $\mathcal{EL}^{sa}$-concepts, concept inclusions, general TBoxes, and KBs, by simultaneous induction as follows:

- every $\mathcal{EL}$-concept, concept inclusion, general TBox, and KB is an $\mathcal{EL}^{sa}$-concept, concept inclusion, general TBox, and KB, respectively;
- if $(\mathcal{T}, \mathcal{A})$ is a general $\mathcal{EL}^{sa}$-KB, a an individual name in $\mathcal{A}$, and Σ a finite signature, then $\exists^{sim}\Sigma.(\mathcal{T}, \mathcal{A}, a)$ is an $\mathcal{EL}^{sa}$-concept;
- if C, D are $\mathcal{EL}^{sa}$-concepts, then $C \sqsubseteq D$ is an $\mathcal{EL}^{sa}$-concept inclusion;
- a general $\mathcal{EL}^{sa}$-TBox is a finite set of $\mathcal{EL}^{sa}$-concept inclusions;
- an $\mathcal{EL}^{sa}$-KB is a pair $(\mathcal{T}, \mathcal{A})$ consisting of a general $\mathcal{EL}^{sa}$-TBox and an ABox.

The semantics of $\exists^{sim}\Sigma.(\mathcal{T}, \mathcal{A}, a)$ is as follows:

$d \in (\exists^{sim}\Sigma.(\mathcal{T}, \mathcal{A}, a))^\mathcal{I}$ iff there exists $d \in (\exists^{sim}\Sigma.(\mathcal{T}, \mathcal{A}, a))^\mathcal{I}$ iff there exists a model $\mathcal{J}$ of $(\mathcal{T}, \mathcal{A})$ such that $(\mathcal{J}, a^\mathcal{J}) \leq_\Gamma (\mathcal{I}, d)$, where $\Gamma = (\mathsf{N_C} \cup \mathsf{N_R}) \setminus \Sigma$.

Example 9 Let $\mathcal{T} = \emptyset$, $\mathcal{A} = \{\mathsf{has\_parent}(a, a)\}$, and $\Sigma = \emptyset$. Then $\exists^{sim}\Sigma.(\mathcal{T}, \mathcal{A}, a)$ is equivalent to the concept $\exists^{sim}(\mathcal{I}, d)$ defined in Example 6.

Let $\mathcal{L}_1, \mathcal{L}_2$ be sets of concepts. We say that $\mathcal{L}_2$ is *polynomially at least as expressive as* $\mathcal{L}_1$, in symbols $\mathcal{L}_1 \leq_p \mathcal{L}_2$, if for every $C_1 \in \mathcal{L}_1$ one can construct in polynomial time a $C_2 \in \mathcal{L}_2$ such that C_1 and C_2 are equivalent. We say that $\mathcal{L}_1, \mathcal{L}_2$ are *polynomially equivalent*, in symbols $\mathcal{L}_1 \equiv_p \mathcal{L}_2$, if $\mathcal{L}_1 \leq_p \mathcal{L}_2$ and $\mathcal{L}_2 \leq_p \mathcal{L}_1$.

Theorem 10 *The languages $\mathcal{EL}^{\nu+}$, $\mathcal{EL}^{si}$, $\mathcal{EL}^{st}$, and $\mathcal{EL}^{sa}$ are polynomially equivalent.*

We provide sketches of proofs of $\mathcal{EL}^{si} \leq_p \mathcal{EL}^{\nu+}$, $\mathcal{EL}^{\nu+} \leq_p \mathcal{EL}^{st}$, $\mathcal{EL}^{st} \leq_p \mathcal{EL}^{sa}$, and $\mathcal{EL}^{sa} \leq_p \mathcal{EL}^{si}$.

$\mathcal{EL}^{si} \leq_p \mathcal{EL}^{\nu+}$. By Lemma 7, considering $\mathcal{EL}^{si}$-concepts of the form $\exists^{sim}(\mathcal{I}, d)$ is sufficient. Each such concept is equivalent to the $\mathcal{EL}^{\nu+}$-concept $\nu_\ell d_1 \cdots d_n.C_1, \ldots, C_n$, where the domain $\Delta^\mathcal{I} = \{d_1, \ldots, d_n\}$ is regarded as a set of concept variables, $d = d_\ell$, and

$$C_i = \bigsqcap\{A \mid d_i \in A^\mathcal{I}\} \sqcap \bigsqcap\{\exists r.d_j \mid (d_i, d_j) \in r^\mathcal{I}\}.$$

$\mathcal{EL}^{\nu+} \leq_p \mathcal{EL}^{st}$. Let C be a closed $\mathcal{EL}^{\nu+}$-concept. An equivalent $\mathcal{EL}^{st}$-concept is constructed by replacing each subconcept of C of the form $\nu_\ell X_1, \ldots, X_n.C_1, \ldots, C_n$ with an $\mathcal{EL}^{st}$-concept, proceeding from the inside out. We assume that for every variable X that occurs in the original $\mathcal{EL}^{\nu+}$-concept C, there is a concept name A_X that does not occur in C. Now $\nu_\ell X_1, \ldots, X_n.C_1, \ldots, C_n$ (which potentially contains free variables) is replaced with the $\mathcal{EL}^{st}$-concept

$$\exists^{sim}\{A_{X_1}, \ldots, A_{X_n}\}.(\{A_{X_i} \sqsubseteq C_i^\downarrow \mid 1 \leq i \leq n\}, A_{X_\ell})$$

where $C_i^\downarrow$ is obtained from C_i by replacing every variable X with the concept name A_X.

$\mathcal{EL}^{st} \leq_p \mathcal{EL}^{sa}$. Let C be an $\mathcal{EL}^{st}$-concept. As already observed, we may assume that D is a concept name in all subconcepts $\exists^{sim}\Sigma.(\mathcal{T}, D)$ of C. Now replace each $\exists^{sim}\Sigma.(\mathcal{T}, A)$ in C, proceeding from the inside out, by $\exists^{sim}\Sigma.(\mathcal{T}, \mathcal{A}, a)$, where $\mathcal{A} = \{A(a)\}$. The resulting concept is equivalent to C.

$\mathcal{EL}^{sa} \leq_p \mathcal{EL}^{si}$. To prove this inclusion, we make use of *canonical models* for $\mathcal{EL}^{sa}$-KBs, and extension of the canonical models used for $\mathcal{EL}$ in [4]. In particular, canonical models for $\mathcal{EL}^{sa}$ can be constructed by an extension of the algorithm given in [4], see the appendix for details.

Theorem 11 (Canonical model) *For every consistent $\mathcal{EL}^{sa}$-KB $(\mathcal{T}, \mathcal{A})$, one can construct in polynomial time a model $\mathcal{I}_{\mathcal{T},\mathcal{A}}$ of $(\mathcal{T}, \mathcal{A})$ with $|\Delta^{\mathcal{I}_{\mathcal{T},\mathcal{A}}}|$ bounded by twice the size of $(\mathcal{T}, \mathcal{A})$ and such that for every model $\mathcal{J}$ of $(\mathcal{T}, \mathcal{A})$, we have $(\mathcal{I}_{\mathcal{T},\mathcal{A}}, a^{\mathcal{I}_{\mathcal{T},\mathcal{A}}}) \leq (\mathcal{J}, a^\mathcal{J})$ for all $a \in \mathsf{Ind}(\mathcal{A})$.*

To prove $\mathcal{EL}^{sa} \leq_p \mathcal{EL}^{si}$, it suffices to show that any outermost occurrence of a concept of the form $\exists^{sim}\Sigma.(\mathcal{T}, \mathcal{A}, a)$ in an $\mathcal{EL}^{sa}$-concept C can be replaced with the equivalent $\mathcal{EL}^{si}$-concept $\exists^{sim}(\mathcal{I}^\Sigma_{\mathcal{T},\mathcal{A}}, a)$, where $\mathcal{I}^\Sigma_{\mathcal{T},\mathcal{A}}$ denotes $\mathcal{I}_{\mathcal{T},\mathcal{A}}$ except that all $\sigma \in \Sigma$ are interpreted as empty sets. First let $d \in (\exists^{sim}\Sigma.(\mathcal{T}, \mathcal{A}, a))^\mathcal{J}$. Then there is a model $\mathcal{I}'$ of $(\mathcal{T}, \mathcal{A})$ such that $(\mathcal{I}', a^{\mathcal{I}'}) \leq_\Sigma (\mathcal{J}, d)$. By Theorem 11, $(\mathcal{I}_{\mathcal{T},\mathcal{A}}, a^{\mathcal{I}_{\mathcal{T},\mathcal{A}}}) \leq (\mathcal{I}', a^{\mathcal{I}'})$. Thus, by closure of simulations under composition, $(\mathcal{I}^\Sigma_{\mathcal{T},\mathcal{A}}, a) \leq_\Sigma (\mathcal{J}, d)$ as required. The converse direction follows from the condition that $\mathcal{I}_{\mathcal{T},\mathcal{A}}$ is a model of $(\mathcal{T}, \mathcal{A})$. This finishes our proof sketch for Theorem 10.

It is interesting to note that, as a consequence of the proofs of Theorem 10, for every $\mathcal{EL}^{\nu+}$-concept there is an equivalent $\mathcal{EL}^{\nu+}$-concept of polynomial size in which the greatest fixpoint constructor is not nested, and similarly for $\mathcal{EL}^{st}$, $\mathcal{EL}^{sa}$. An important consequence of the existence of canonical models, as granted by Theorem 11, is that reasoning in our family of extensions of $\mathcal{EL}$ is tractable. Recall that *KB consistency* is the problem of deciding whether a given KB has a model; *subsumption w.r.t. general TBoxes* is the problem of deciding whether a subsumption $C \sqsubseteq D$ follows from a general TBox $\mathcal{T}$ (in symbols, $\mathcal{T} \models C \sqsubseteq D$); and the *instance problem* is the problem of deciding whether an assertion $C(a)$ follows from a KB $(\mathcal{T}, \mathcal{A})$ (in symbols, $(\mathcal{T}, \mathcal{A}) \models C(a)$).

Theorem 12 (Tractable reasoning) *Let $\mathcal{L}$ be any of the languages $\mathcal{EL}^\nu$, $\mathcal{EL}^{\nu+}$, $\mathcal{EL}^{si}$, $\mathcal{EL}^{st}$, or $\mathcal{EL}^{sa}$. Then KB consistency, subsumption w.r.t. TBoxes, and the instance problem can be decided in* PTIME .

Proof (sketch) By Theorem 10, it suffices to concentrate on $\mathcal{L} = \mathcal{EL}^{sa}$. The PTIME decidability of KB consistency is proved in the appendix as part of the algorithm that constructs the canonical model. Subsumption w.r.t. general TBoxes can be polynomially reduced in the standard way to the instance problem. Finally, by Theorem 11, we can decide the instance problem as follows: to decide whether $(\mathcal{T}, \mathcal{A}) \models C(a)$, where we can w.l.o.g. assume that $C = A$ for a concept name A, we check whether $(\mathcal{T}, \mathcal{A})$ is inconsistent or $a^{\mathcal{I}_{\mathcal{T},\mathcal{A}}} \in A^{\mathcal{I}_{\mathcal{T},\mathcal{A}}}$. Both can be done in PTIME .

Besides of the canonical model of a KB from Theorem 11, we also require the canonical model $\mathcal{I}_{\mathcal{T},C}$ of a general $\mathcal{EL}^{\nu+}$-TBox $\mathcal{T}$ and concept C which is defined by taking the reduct not interpreting A of the canonical model $\mathcal{I}_{\mathcal{T}',\mathcal{A}}$ for $\mathcal{T}' = \mathcal{T} \cup \{A \sqsubseteq C\}$ and $\mathcal{A} = \{A(a)\}$ (A a fresh concept). We set $d_C = a^{\mathcal{I}_{\mathcal{T}',\mathcal{A}}}$. $\mathcal{I}_{\mathcal{T},C}$ is a model of $\mathcal{T}$ with $d_C \in C^{\mathcal{I}_{\mathcal{T},C}}$ such that $(\mathcal{I}_{\mathcal{T},C}, d_C) \leq (\mathcal{J}, e)$ for all models $\mathcal{J}$ of $\mathcal{T}$ with $e \in C^\mathcal{J}$.

5 CHARACTERIZING $\mathcal{EL}^\nu$ USING SIMULATIONS

When characterizing $\mathcal{EL}$ as a fragment of first-order logic in Theorem 5, our starting point was the observation that $\mathcal{EL}$-concepts are preserved under simulations and that $\mathcal{EL}$ is a Horn logic, thus having finite minimal models. The same is true for $\mathcal{EL}^\nu$: first, $\mathcal{EL}^\nu$-concepts are preserved under simulations, as $\mathcal{EL}^{si}$ is obviously preserved under simulations and, by Theorem 10, every $\mathcal{EL}^\nu$-concept is equivalent to an $\mathcal{EL}^{si}$-concept. And second, a finite minimal model of an $\mathcal{EL}^\nu$-concept C is given by the canonical model $(\mathcal{I}_{\mathcal{T},C}, d_C)$ defined above for $\mathcal{T} = \emptyset$. However, $\mathcal{EL}^\nu$ is clearly not a fragment of FO. Instead, it relates to *MSO* in exactly the way that $\mathcal{EL}$ related to FO.

Theorem 13 *The set of $\mathcal{EL}^\nu$-concepts is a maximal set of MSO-formulas that is preserved under simulations and has finite minimal models: if $\mathcal{L}$ is a set of MSO-formulas that properly contains all $\mathcal{EL}^\nu$-concepts, then either it contains a formula not preserved under simulations or is does not have finite minimal models.*

Proof Assume that $\mathcal{L} \supseteq \mathcal{EL}^\nu$ is preserved under simulations and has finite minimal models. Let $\varphi(x) \in \mathcal{L}$. We have to show that $\varphi(x)$ is equivalent to an $\mathcal{EL}^\nu$-concept. To this end, take a finite minimal model of φ, i.e., an interpretation $\mathcal{I}$ and a $d \in \Delta^\mathcal{I}$ such that for all $\psi(x) \in \mathcal{L}$ we have that $\forall x.(\varphi(x) \rightarrow \psi(x))$ is valid iff $\mathcal{I} \models \psi[d]$. We will show that φ is equivalent to (the MSO translation of) $\exists^{sim}(\mathcal{I}, d)$. We may assume that $\exists^{sim}(\mathcal{I}, d) \in \mathcal{L}$. Since $d \in (\exists^{sim}(\mathcal{I}, d))^\mathcal{I}$, we thus have that $\forall x.(\varphi(x) \rightarrow \exists^{sim}(\mathcal{I}, d)(x))$ is valid. Conversely, assume that $d' \in (\exists^{sim}(\mathcal{I}, d))^\mathcal{J}$ for some interpretation $\mathcal{J}$. Then $(\mathcal{I}, d) \leq (\mathcal{J}, d')$. We have $(\mathcal{I}, d) \models \varphi[d]$. Thus, by preservation of $\varphi(x)$ under simulations, $\mathcal{J} \models \varphi[d']$. Thus $\forall x.(\exists^{sim}(\mathcal{I}, d)(x) \rightarrow \varphi(x))$ is also valid. This finishes the proof.

A number of closely related characterizations remain open. For example, we conjecture that an extension of Theorem 4 holds for $\mathcal{EL}^{\nu,\sqcup}$ and MSO (instead of $\mathcal{EL}$ and FO). Also, it is open whether Theorem 13 still holds if finite minimal models are replaced by arbitrary minimal models.

6 APPLICATIONS

The μ-calculus is considered to be extremely well-behaved regarding its expressive power and logical properties. The aim of this section is to take a brief look at the expressive power of its $\mathcal{EL}$-analogues $\mathcal{EL}^{\nu}$ and $\mathcal{EL}^{\nu+}$. In particular, we show that $\mathcal{EL}^{\nu+}$ is more well-behaved than $\mathcal{EL}$ in a number of respects. Throughout this section, we will not distinguish between the languages previously proved polynomially equivalent.

To begin with, we construct the *least common subsumer* (LCS) of two concepts w.r.t. a general $\mathcal{EL}^{\nu+}$-TBox (the generalization to more than two concepts is straightforward). Given a general $\mathcal{EL}^{\nu+}$-TBox $\mathcal{T}$ and concepts C_1, C_2, a concept C is called a *LCS of* C_1, C_2 *w.r.t.* $\mathcal{T}$ *in* $\mathcal{EL}^{\nu+}$ if $\mathcal{T} \models C_i \sqsubseteq C$ for $i = 1, 2$; if $\mathcal{T} \models C_i \sqsubseteq D$ for $i = 1, 2$ and D a $\mathcal{EL}^{\nu+}$-concept, then $\mathcal{T} \models C \sqsubseteq D$. It is known [2] that in $\mathcal{EL}$ the LCS does not always exist.

Example 14 In $\mathcal{EL}$, the LCS of A, B w.r.t.

$$\mathcal{T} = \{A \sqsubseteq \exists \mathsf{has\_parent}.A, B \sqsubseteq \exists \mathsf{has\_parent}.B\}$$

does not exist. In $\mathcal{EL}^{\nu}$, however, the LCS of A,B w.r.t. $\mathcal{T}$ is given by $\nu X.\exists \mathsf{has\_parent}.X$ (see Example 1).

To construct the LCS in $\mathcal{EL}^{\nu+}$, we adopt the product construction used in [2] for the case of classical TBoxes with a fixpoint semantics. For interpretations $\mathcal{I}_1$ and $\mathcal{I}_2$, the *product* $\mathcal{I}_1 \times \mathcal{I}_2$ is defined by setting $\Delta^{\mathcal{I}_1 \times \mathcal{I}_2} = \Delta^{\mathcal{I}_1} \times \Delta^{\mathcal{I}_2}$, $(d_1, d_2) \in A^{\mathcal{I}_1 \times \mathcal{I}_2}$ iff $d_i \in A^{\mathcal{I}_i}$ for $i = 1, 2$, and $((d_1, d_2), (d_1', d_2')) \in r^{\mathcal{I}_1 \times \mathcal{I}_2}$ iff $(d_i, d_i') \in r^{\mathcal{I}_i}$ for $i = 1, 2$.

Theorem 15 (LCS) *Let* $\mathcal{T}$ *be a general* $\mathcal{EL}^{\nu+}$*-TBox and* C_1 *and* C_2 *be* $\mathcal{EL}^{\nu+}$*-concepts. Then* $\exists^{sim}(\mathcal{I}_{\mathcal{T},C_1} \times \mathcal{I}_{\mathcal{T},C_2}, (d_{C_1}, d_{C_2}))$ *is the LCS of* C_1, C_2 *w.r.t.* $\mathcal{T}$ *in* $\mathcal{EL}^{\nu+}$.

The same product construction has been used in [2] for the case of classical TBoxes with a fixpoint semantics, which, however, additionally require a notion of conservative extension (see Section 7).

Our second result concerns the most specific concept, which plays an important role in the bottom-up construction of knowledge bases and has received considerable attention in the context of $\mathcal{EL}$ [2, 6]. Formally, a concept C is the *most specific concept (MSC)* for an individual a in a knowledge base $(\mathcal{T}, \mathcal{A})$ in $\mathcal{EL}^{\nu+}$ if $(\mathcal{T}, \mathcal{A}) \models C(a)$ and for every $\mathcal{EL}^{\nu+}$-concept D with $(\mathcal{T}, \mathcal{A}) \models D(a)$, we have $\mathcal{T} \models C \sqsubseteq D$. In $\mathcal{EL}$, the MSC need not exist, as is witnessed by the knowledge base $(\emptyset, \{\mathsf{has\_parent}(a, a)\})$, where the MSC for a is non-existent.

Theorem 16 (MSC) *In* $\mathcal{EL}^{\nu+}$*, the MSC always exists for any* a *in any KB* $(\mathcal{T}, \mathcal{A})$ *and is given as* $\exists^{sim}\emptyset.(\mathcal{T}, \mathcal{A}, a)$.

In [2], the MSC in $\mathcal{EL}$-KBs based on classical TBoxes with a fixpoint semantics is defined. The relationship between $\mathcal{EL}^{\nu+}$ and fixpoint TBoxes is discussed in more detail in Section 7.

We now turn our attention to issues of definability and interpolation. From now on, we use $\mathsf{sig}(C)$ to denote the set of concept and role names used in the concept C. A concept C is a Σ*-concept* if $\mathsf{sig}(C) \subseteq \Sigma$. Let $\mathcal{T}$ be a general $\mathcal{EL}^{\nu+}$-TBox, C an $\mathcal{EL}^{\nu+}$-concept and Γ a finite signature.

We start with considering the fundamental notion of a Γ-definition. The question addressed here is whether a given concept can be expressed in an equivalent way by referring only to the symbols in a given signature Γ [21, 20]. Formally, a Γ-concept D is an *explicit* Γ*-definition* of a concept C w.r.t. a TBox $\mathcal{T}$ if, and only if, $\mathcal{T} \models C \equiv D$ (i.e., C and D are equivalent w.r.t. $\mathcal{T}$). Clearly, explicit Γ-definitions do not always exist in any of the logics studied in this paper: for example, there is no explicit $\{A\}$-definition of B w.r.t. the TBox $\{A \sqsubseteq B\}$. However, it is not hard to show the following using the fact that $\exists^{sim}\Sigma.(\mathcal{T}, C)$ is the most specific Γ-concept that subsumes C w.r.t. $\mathcal{T}$.

Proposition 17 *Let* C *be an* $\mathcal{EL}^{\nu+}$*-concept,* $\mathcal{T}$ *a general* $\mathcal{EL}^{\nu+}$*-TBox and* Γ *a signature. There exists an explicit* Γ*-definition of* C *w.r.t.* $\mathcal{T}$ *iff* $\exists^{sim}\Sigma.(\mathcal{T}, C)$ *is such a definition (*$\Sigma = \mathsf{sig}(\mathcal{T}, C) \setminus \Gamma$*).*

It is interesting to note that if $\mathcal{T}$ happens to be a general $\mathcal{EL}$-TBox and C an $\mathcal{EL}$-concept and there exists an explicit Γ-definition of C w.r.t. $\mathcal{T}$, then the concept $\exists^{sim}\Sigma.(\mathcal{T}, C)$ from Proposition 17 is equivalent w.r.t. $\mathcal{T}$ to an $\mathcal{EL}$-concept over Γ. This follows from the fact that $\mathcal{EL}$ has the Beth definability property (see below for a definition) which follows immediately from interpolation results proved for $\mathcal{EL}$ in [15]. The advantage of giving explicit Γ-definitions in $\mathcal{EL}^{\nu+}$ even when $\mathcal{T}$ and C are formulated in $\mathcal{EL}$ is that Γ-definitions in $\mathcal{EL}^{\nu+}$ are of polynomial size while the following example shows that they may be exponentially large in $\mathcal{EL}$.

Example 18 *Let* $\mathcal{T}$ *consist of* $A_i \equiv \exists r_i.A_{i+1} \sqcap \exists s_i.A_{i+1}$ *for* $0 \leq i < n$*, and* $A_n \equiv \top$*. Let* $\Gamma = \{r_0, \ldots, r_{n-1}, s_0, \ldots, s_{n-1}\}$*. Then* A_0 *has an explicit* Γ*-definition w.r.t.* $\mathcal{T}$ *in* $\mathcal{EL}$*, namely* C_0*, where* $C_i = \exists r_i.C_{i+1} \sqcap \exists s_i.C_{i+1}$ *and* $C_n = \top$*. This definition is of exponential size and it is easy to see that there is no shorter* Γ*-definition of* A_0 *w.r.t.* $\mathcal{T}$ *in* $\mathcal{EL}$*.*

Say that a concept C is *implicitly* Γ*-defined w.r.t.* $\mathcal{T}$ iff $\mathcal{T} \cup \mathcal{T}_\Gamma \models C \equiv C_\Gamma$, where $\mathcal{T}_\Gamma$ and C_Γ are obtained from $\mathcal{T}$ and C, respectively, by replacing each $\sigma \notin \Gamma$ by a fresh symbol σ'. The Beth definability property, which was studied in a DL context in [21, 20], ensures that concepts that are implicity Γ-defined have an explicit Γ-definition.

Theorem 19 (Beth Property) $\mathcal{EL}^{\nu+}$ *has the polynomial Beth definability property: for every general* $\mathcal{EL}^{\nu+}$*-TBox* $\mathcal{T}$*, concept* C*, and signature* Γ *such that* C *is implicitly* Γ*-defined w.r.t.* $\mathcal{T}$*, there is an explicit* Γ*-definition w.r.t.* $\mathcal{T}$*, namely* $\exists^{sim}(\mathsf{sig}(\mathcal{T}, C) \setminus \Gamma).(\mathcal{T}, C)$.

The proof of Theorem 19 relies on $\mathcal{EL}^{\nu}$ having a certain interpolation property. Say that two general TBoxes $\mathcal{T}_1$ and $\mathcal{T}_2$ are Δ*-inseparable* w.r.t. $\mathcal{EL}^{\nu}$ if $\mathcal{T}_1 \models C \sqsubseteq D$ iff $\mathcal{T}_2 \models C \sqsubseteq D$ for all $\mathcal{EL}^{\nu}$-inclusions $C \sqsubseteq D$.

Theorem 20 (Interpolation) *Let* $\mathcal{T}_1 \cup \mathcal{T}_2 \models C \sqsubseteq D$ *and assume that* $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *are* Δ*-inseparable w.r.t.* $\mathcal{EL}^{\nu}$ *for* $\Delta = \mathsf{sig}(\mathcal{T}_1, C) \cap \mathsf{sig}(\mathcal{T}_2, D)$*. Then the* Δ*-concept* $F = \exists^{sim}\Sigma.(\mathcal{T}_1, C)$*,* $\Sigma = \mathsf{sig}(\mathcal{T}_1, C) \setminus \Delta$*, is an interpolant of* C, D *w.r.t.* $\mathcal{T}_1, \mathcal{T}_2$*; i.e.* $\mathcal{T}_1 \models C \sqsubseteq F$ *and* $\mathcal{T}_2 \models F \sqsubseteq D$*.*

We show how Theorem 19 follows from Theorem 20. Assume that $\mathcal{T} \cup \mathcal{T}_\Gamma \models C \equiv C_\Gamma$, where $\mathcal{T}, \mathcal{T}_\Gamma, C, C_\Gamma$ satisfy the conditions of Theorem 19. Then $\mathcal{T}$ and $\mathcal{T}_\Gamma$ are Γ-inseparable and $\Gamma \supseteq \mathsf{sig}(\mathcal{T}, C) \cap$

$\mathsf{sig}(\mathcal{T}_\Gamma, C_\Gamma)$. Thus, by Theorem 20, $\mathcal{T} \models \exists^{sim}\Sigma.(\mathcal{T}_\Gamma, C_\Gamma) \sqsubseteq C$ for $\Sigma = \mathsf{sig}(\mathcal{T}_\Gamma, C_\Gamma) \setminus \Gamma$. Now Theorem 19 follows from the fact that $\exists^{sim}\Sigma.(\mathcal{T}_\Gamma, C_\Gamma)$ is equivalent to $\exists^{sim}\Sigma'.(\mathcal{T}, C)$ for $\Sigma' = \mathsf{sig}(\mathcal{T}, C) \setminus \Gamma$.

In [15], it is shown that $\mathcal{EL}$ also has this interpolation property. However, the advantage of using $\mathcal{EL}^{\nu+}$ is that interpolants are of polynomial size. The decomposition algorithm for $\mathcal{EL}$ given in [15] crucially depends on this property of $\mathcal{EL}^{\nu+}$.

7 RELATION TO TBOXES WITH FIXPOINT SEMANTICS

There is a tradition of considering DLs that introduce fixpoints at the TBox level instead of at the concept level [18, 19, 1]. In [3], Baader proposes and analyzes such a DL based on $\mathcal{EL}$ and greatest fixpoints. This DL, which we call $\mathcal{EL}^{\mathsf{gfp}}$ here, differs from $\mathcal{EL}^{\nu}$ in that (i) TBoxes are classical TBoxes rather than sets of GCIs $C \sqsubseteq D$, i.e., sets of expressions $A \equiv C$ with $A \in \mathsf{N_C}$ and C a concept (cycles are allowed) and (ii) the ν-concept constructor is not present; instead, a greatest fixpoint semantics is adopted for TBoxes.

On the concept level, $\mathcal{EL}^{\nu}$ is clearly strictly more expressive than $\mathcal{EL}^{\mathsf{gfp}}$: since fixpoints are introduced at the TBox level, concepts of $\mathcal{EL}^{\mathsf{gfp}}$ coincide with $\mathcal{EL}$-concepts, and thus there is no $\mathcal{EL}^{\mathsf{gfp}}$-concept equivalent to the $\mathcal{EL}^{\nu}$-concept $\nu X.\exists r.X$. In the following, we show that $\mathcal{EL}^{\nu}$ is also more expressive than $\mathcal{EL}^{\mathsf{gfp}}$ on the TBox level, even if we restrict $\mathcal{EL}^{\nu}$-TBoxes to (possibly cyclic) concept definitions, as in $\mathcal{EL}^{\mathsf{gfp}}$. We use the standard notion of logical equivalence, i.e., two TBoxes $\mathcal{T}$ and $\mathcal{T}'$ are *equivalent* iff $\mathcal{T}$ and $\mathcal{T}'$ have precisely the same models. As observed by Schild in the context of $\mathcal{ALC}$ [19], every $\mathcal{EL}^{\mathsf{gfp}}$-TBox $\mathcal{T} = \{A_1 \equiv C_1, \ldots, A_n \equiv C_n\}$ is equivalent in this sense to the $\mathcal{EL}^{\nu+}$-TBox $\{A_i \equiv \nu_i X_1, \ldots, X_n.C_1', \ldots, C_n' \mid 1 \leq i \leq n\}$, where each C_i' is obtained from C_i by replacing each A_j with X_j, $1 \leq j \leq n$. Note that since we are translating to mutual fixpoints, the size of the resulting TBox is polynomial in the size of the original one. In the converse direction, there is no equivalence-preserving translation.

Lemma 21 *For each $\mathcal{EL}^{\mathsf{gfp}}$-TBox, there is an equivalent $\mathcal{EL}^{\nu+}$-TBox of polynomial size, but no $\mathcal{EL}^{\mathsf{gfp}}$-TBox is equivalent to the $\mathcal{EL}^{\nu}$-TBox $\mathcal{T}_0 = \{A \equiv P \sqcap \nu X.\exists r.X\}$.*

Proof It is not difficult to show that for every $\mathcal{EL}^{\mathsf{gfp}}$-TBox $\mathcal{T}$, defined concept name A in $\mathcal{T}$, and role name r, at least one of the following holds:

- there is an $m \geq 0$ such that $\mathcal{T} \models A \sqsubseteq \exists r^n.\top$ implies $n \leq m$ or
- $\mathcal{T} \models A \sqsubseteq \exists r^n.B$ for some $n > 0$ and defined concept name B.

Since neither of these is true for $\mathcal{T}_0$, $\mathcal{T}$ is not equivalent to $\mathcal{T}_0$.

Restricted to classical TBoxes, $\mathcal{EL}^{\mathsf{gfp}}$ and $\mathcal{EL}^{\nu}$ become equi-expressive if the strict notion of equivalence used above is replaced with one based on conservative extensions, thus allowing the introduction of new concept names that are suppressed from logical equivalence.

8 Conclusion

We have introduced and investigated the extensions $\mathcal{EL}^{\nu}$ and $\mathcal{EL}^{\nu+}$ of $\mathcal{EL}$ with greatest fixpoint operators. The main result of this paper is that $\mathcal{EL}^{\nu+}$ can be regarded as a completion of $\mathcal{EL}$ regarding its expressive power in which reasoning is still tractable, but where many previously non-existent concepts (such as the LCS and MCS) exist and/or can be expressed more succinctly (such as interpolants and explicit concept definitions). Interestingly, the alternative extension of $\mathcal{EL}$ by smallest rather than greatest fixpoints is much less well-behaved. For example, even the addition of transitive closure to $\mathcal{EL}$ leads to non-tractable reasoning problems [13].

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A Proofs for Section 2

We develop a variant of $\mathcal{EL}^{si}$ that is polynomially equivalent to $\mathcal{EL}^{\nu}$ and use it to prove that $\mathcal{EL}^{\nu+}$ is exponentially more succinct than $\mathcal{EL}^{\nu}$.

Definition 22 *A* enriched tree interpretation *is a tuple* $(\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}, \rightarrow_{\varepsilon})$*, where* $\Delta^{\mathcal{I}}$ *and* $\cdot^{\mathcal{I}}$ *are as in an interpretation and* $\rightarrow_{\varepsilon} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ *such that the following conditions are satisfied:*

- *the graph* $G_{\mathcal{I}} := (\Delta^{\mathcal{I}}, \bigcup_{r \in \mathsf{N_R}} r^{\mathcal{I}})$ *is a tree and* $r \neq s$ *implies* $r^{\mathcal{I}} \cap s^{\mathcal{I}} = \emptyset$*;*
- *if* $d \rightarrow_{\varepsilon} e$*, then* e *is an ancestor of* d *in* $G_{\mathcal{I}}$*.*

A pointed enriched tree interpretation *is a pair* $(\mathcal{I}, d)$ *with* $\mathcal{I}$ *an enriched tree interpretation and* $d \in \Delta^{\mathcal{I}}$*. Let* $(\mathcal{I}_1, d_1)$ *be a pointed enriched tree interpretation,* $(\mathcal{I}_2, d_2)$ *a pointed interpretation, and* Σ *a signature. A* Σ-simulation *between* $(\mathcal{I}_1, d_1)$ *and* $(\mathcal{I}_2, d_2)$ *is a* Σ*-simulation* S *between* $((\Delta^{\mathcal{I}_1}, \cdot^{\mathcal{I}_1}), d_1)$ *and* $(\mathcal{I}_2, d_2)$ *such that*

- $(e_1, e_2) \in S$ *and* $e_1 \rightarrow_{\varepsilon} e_1'$ *implies* $(e_1', e_2) \in S$*.*

The logic $\mathcal{EL}^{tsi}$ is defined as $\mathcal{EL}^{si}$, but requires the interpretation $\mathcal{I}$ inside the constructor $\exists^{sim}(\mathcal{I}, d)$ to be an enriched tree interpretation with root d. We will now show that concepts can be translated back and forth between $\mathcal{EL}^{\nu}$ and $\mathcal{EL}^{tsi}$ such that equivalence is preserved and only a polynomial blowup occurs.

Lemma 23 $\mathcal{EL}^{\nu} \leq_p \mathcal{EL}^{tsi}$*.*

Proof. Let C be a closed $\mathcal{EL}^{\nu}$-concept and $\mathfrak{V}$ the set of variables used in C. With every subconcept D of C, we associate an enriched tree interpretation $\mathcal{I}_C$ and a map $\pi_C : \Delta^{\mathcal{I}_C} \rightarrow 2^{\mathfrak{V}}$, proceeding by induction on the structure of D:

- if D is a concept name, then $\Delta^{\mathcal{I}_D} = \{d\}$, $D^{\mathcal{I}_D} = \{d\}$, $\sigma^{\mathcal{I}_D} = \emptyset$ for all symbols σ distinct from D, $\rightarrow_{\varepsilon} = \emptyset$, and $\pi_C(d) = \emptyset$;
- if D is a variable, then $\Delta^{\mathcal{I}_D} = \{d\}$, $\sigma^{\mathcal{I}_D} = \emptyset$ for all symbols σ, $\rightarrow_{\varepsilon} = \emptyset$, and $\pi_C(d) = \{D\}$;
- if $D = D_1 \sqcap D_2$, assume w.l.o.g. that $\mathcal{I}_{D_1}$ and $\mathcal{I}_{D_2}$ have the same root and $\Delta^{\mathcal{I}_{D_1}} \cap \Delta^{\mathcal{I}_{D_2}}$ is a singleton that contains that root. Then set $\mathcal{I}_D = \mathcal{I}_{D_1} \cup \mathcal{I}_{D_2}$ and $\pi_D = \pi_{D_1} \cup \pi_{D_2}$;
- if $D = \exists r.E$, construct $\mathcal{I}_D$ from $\mathcal{I}_E$ by adding a new node e and setting $(e, d) \in r^{\mathcal{I}_D}$; extend π_E to π_D by setting $\pi_D(e) = \emptyset$;
- if $D = \nu X.E$, then construct $\mathcal{I}_D$ from $\mathcal{I}_E$ by setting

  $$\rightarrow_{\varepsilon} = \rightarrow_{\varepsilon} \cup \{(d, e) \mid X \in \pi_E(d) \text{ and } e \text{ is the root of } \mathcal{I}_E\}$$

  and π_D from π_E by setting $\pi_D(d) = \pi_E(d) \setminus \{X\}$ for all $d \in \Delta^{\mathcal{I}_D}$.

It can be shown that $\exists^{sim}(\mathcal{I}_c, d)$, with d the root of $\mathcal{I}_c$, is equivalent to C.

Lemma 24 $\mathcal{EL}^{tsi} \leq_p \mathcal{EL}^{\nu}$*.*

Proof. Since the proof of Lemma 7 works also for $\mathcal{EL}^{tsi}$ and does not increase the size of concepts, it suffices to show that every concept $\exists^{sim}(\mathcal{I}, d_0)$ with $\mathcal{I}$ an enriched tree interpretation, can be translated to an $\mathcal{EL}^{\nu}$-concept with only a polynomial blowup. To this end, define $W = \{d \in \Delta^{\mathcal{I}} \mid e \rightarrow_{\varepsilon} d \text{ for some } e\}$, fix a variable X_d for each $d \in W$, and define a concept C_d for each $d \in \Delta^{\mathcal{I}}$ by proceeding bottom-up:

- if $d \in \Delta^{\mathcal{I}}$ is a leaf, then $C_d = \bigsqcap\{A \in \mathsf{N_C} \mid d \in A^{\mathcal{I}}\} \sqcap \bigsqcap_{d \rightarrow_{\varepsilon} e} X_e$;
- if $d \in \Delta^{\mathcal{I}} \setminus W$ has successors $d_1, \dots, d_k$ in $\mathcal{I}$, where $(d, d_i) \in r_i^{\mathcal{I}}$ for $1 \leq i \leq k$, then $C_d = \bigsqcap\{A \in \mathsf{N_C} \mid d \in A^{\mathcal{I}}\} \sqcap \bigsqcap_{d \rightarrow_{\varepsilon} e} X_e \sqcap \bigsqcap_{1 \leq i \leq k} \exists r_i.C_{d_i}$;
- if $d \in W$ has successors $d_1, \dots, d_k$ in $\mathcal{I}$, where $(d, d_i) \in r_i^{\mathcal{I}}$ for $1 \leq i \leq k$, then $C_d = \nu X_d.\big(\bigsqcap\{A \in \mathsf{N_C} \mid d \in A^{\mathcal{I}}\} \sqcap \bigsqcap_{d \rightarrow_{\varepsilon} e} X_e \sqcap \bigsqcap_{1 \leq i \leq k} \exists r_i.C_{d_i}\big)$.

It can be shown that C_{d_0} is equivalent to $\exists^{sim}(\mathcal{I}, d_0)$.

We have thus established the following.

Theorem 25 $\mathcal{EL}^{\nu}$ *and* $\mathcal{EL}^{tsi}$ *are polynomially equivalent.*

Based on Theorem 25, we can use $\mathcal{EL}^{tsi}$ instead of $\mathcal{EL}^{\nu+}$ when proving that $\mathcal{EL}^{\nu+}$ is exponentially more succinct than $\mathcal{EL}^{\nu}$. This is what we do in the following.

Theorem 26 *Every* $\mathcal{EL}^{\nu}$*-concept that is equivalent to the* $\mathcal{EL}^{\nu+}$*-concept*

$$\nu_0 X_0, \dots, X_n.C_0, \dots, C_n, \text{ with } C_i = A_i \sqcap \exists r.X_{i+1} \sqcap \exists s.X_{i+1}$$

(and where we set $X_{n+1} := X_0$*) has length at least* 2^n*.*

Proof. We use E_i to denote the concept given in Theorem 26. It suffices to show that in every $\mathcal{EL}^{tsi}$-concept of the form $\exists^{sim}(\mathcal{I}, d)$ that is equivalent to E_i, we have $|\Delta^{\mathcal{I}}| \geq 2^n$. Thus, let $\exists^{sim}(\mathcal{I}, d)$ be equivalent to E_i. To start with, it is easy to use E_i, the semantics of $\mathcal{EL}^{\nu+}$, and the definition of simulations to show that we must have $d \in A_0^{\mathcal{I}}$ and $d \notin A_i^{\mathcal{I}}$ for $i \neq 0$. Using the same arguments, it is clear that there must be $d_r \in \Delta^{\mathcal{I}}$ and $d_s \in \Delta^{\mathcal{I}}$ such that $(d, d_r) \in r^{\mathcal{I}}$, $(d, d_s) \in s^{\mathcal{I}}$, $d_r, d_s \in A_1^{\mathcal{I}}$ and $d_r, d_s \notin A_i^{\mathcal{I}}$ for $i \neq 1$. In particular, we have $d \neq d_r$ and $d \neq d_s$ due to the concept memberships and $d_r \neq d_s$ since $\mathcal{I}$ is an enriched tree interpretation. We can repeat this argument for the next level of the tree interpretation, getting additional (and distinct) domain elements d_{rr}, d_{rs}, d_{ss}, and d_{sr}. Clearly, this argument can be repeated n times (but not more often since then, the concept names A_i start repeating on each path), thus yielding a binary tree of depth n, which has $> 2^n$ nodes.

Note that the concept given in Theorem 26 can be represented by an $\mathcal{EL}^{\nu}$-concept that has only polynomially many different subconcepts, i.e., structure sharing can be used to avoid the blowup identified in Theorem 26. We leave it as an open problem whether this holds for all $\mathcal{EL}^{\nu}$-concepts. This is not a trivial question unless the number of variables that can be bound simultaneously in a mutual fixpoint operator is bounded by a constant.

B Proofs for Section 3

In this section, we prove Theorems 4 and 5. We require some basic and well-known operations on interpretations. First, for an interpretation $\mathcal{I}$ and $d \in \Delta^{\mathcal{I}}$, we denote by $\mathcal{I}_d$ the *tree-unraveling* of $\mathcal{I}$ in d: the domain $\Delta^{\mathcal{I}_d}$ consists of all words

$$d r_1 d_1 r_2 \cdots r_n d_n$$

such that $n \geq 0$ and $(d_i, d_{i+1}) \in r_{i+1}^{\mathcal{I}}$ for all $i \geq 0$ (we set $d_0 = d$). We let $\sigma \cdot d' \in A^{\mathcal{I}_d}$ iff $d' \in A^{\mathcal{I}}$ and $r^{\mathcal{I}_d}$ consists of all pairs $(\sigma, \sigma r d') \in \Delta^{\mathcal{I}_d} \times \Delta^{\mathcal{I}_d}$.

Secondly, for $\ell \geq 0$, we denote with $\mathcal{I}_d^{\ell}$ the subinterpretation of $\mathcal{I}_d$ induced by all elements that are reachable in at most ℓ-many steps from the root d of the tree $\mathcal{I}_d$.

Definition 27 (ℓ-local) *A concept C is called ℓ-local iff there is some $\ell \geq 0$ such that for all pointed interpretations $(\mathcal{I}, d)$ we have $d \in C^{\mathcal{I}}$ iff $d \in C^{\mathcal{I}^\ell_d}$.*

Let Σ be a finite signature. A *pointed Σ-interpretation* $(\mathcal{I}, d)$ is a pointed interpretation in which $\sigma^{\mathcal{I}} = \emptyset$ for all $\sigma \notin \Sigma$.

Definition 28 *Let Σ be finite, and $(\mathcal{I}, d)$ a pointed Σ-interpretation. The ℓ-characteristic concept $X^\ell(\mathcal{I}, d)$ of $(\mathcal{I}, d)$ is recursively defined as follows:*

$X^0(\mathcal{I}, d) := \bigsqcap\{A \in \mathsf{N_C} \mid d \in A^{\mathcal{I}}\}$
$X^{\ell+1}(\mathcal{I}, d) := X^0(\mathcal{I}, d) \sqcap \bigsqcap_{r \in \mathsf{N_R}} \bigsqcap\{\exists r.X^\ell(\mathcal{I}, d') \mid (d, d') \in r^{\mathcal{I}}\}$

Observe that for $X^{\ell+1}(\mathcal{I}, d)$ to be well-defined, we have to ensure that there are only finitely many conjuncts in X when forming a conjunction $\bigsqcap_{C \in X} C$. This can be easily proved by induction. In fact, one can easily prove that, up to logical equivalence, there exist only finitely many ℓ-characteristic concepts for each finite signature.

Observation 1 *For each finite signature Σ and each $\ell \geq 0$ there are, up to logical equivalence, finitely many ℓ-characteristic concepts of pointed Σ-interpretations.*

Observation 2 *$e \in (X^\ell(\mathcal{I}, d))^{\mathcal{J}}$ iff $(\mathcal{I}^\ell_d, d) \leq (\mathcal{J}, e)$.*

Proof We start with the direction from left to right. The proof is by induction on ℓ. For $\ell = 0$, the claim is trivial. Now assume that it has been proved for ℓ and let $e \in (X^{\ell+1}(\mathcal{I}, d))^{\mathcal{J}}$. We may assume that $\mathcal{I}$ is already a tree-structure. Then for all $r \in \mathsf{N_R}$ and every d' with $(d, d') \in r^{\mathcal{I}}$, we have $e \in (\exists r.X^\ell(\mathcal{I}, d'))^{\mathcal{J}}$. So there is some e' with $(e, e') \in r^{\mathcal{J}}$ such that $e' \in (X^\ell(\mathcal{I}, d'))^{\mathcal{J}}$. The induction hypothesis yields that there is a simulation $S_{d'} : (\mathcal{I}^\ell_{d'}, d') \leq (\mathcal{J}, e')$. We define

$$S_d := \{(d, e)\} \cup \bigcup_{r \in \Sigma} \bigcup_{(d,d') \in r^{\mathcal{I}}} S_{d'}$$

and show that S_d is a simulation.

Condition 1 of Definition 3 holds for all $(d', e') \neq (d, e)$ in S_d as they already belong to simulations. Additionally, $e \in (X^0(\mathcal{I}, d))^{\mathcal{J}}$ and so $d \in A^{\mathcal{I}}$ implies $e \in A^{\mathcal{J}}$ for all concept names $A \in \Sigma$, too.

Condition 2 is met for (d, e): By definition of S_d, for all $r \in \Sigma$ and every r-successor d' of d there is an r-successor e' of e such that $(d', e') \in S_d$.

Let $(d_1, e_1) \in S$ and let $(d, e) \neq (d_1, e_1)$. So $(d_1, e_1) \in S_{d'}$ for some r-successor d' of d. Now let $s \in \Sigma$ and d_2 be an s-successor of d_1 in $\mathcal{I}^{\ell+1}_d$. It is to show, that there is some s-successor e_2 of e_1 such that $(d_2, e_2) \in S_d$.

The tree interpretation $(\mathcal{I}^{\ell+1}_d, d)$ is composed of d, connected by edges to the roots of the subtrees $\mathcal{I}^\ell_{d'}$, obtained from its successors d'. So every s-successor of d_1 in $\mathcal{I}^{\ell+1}_d$ is an s-successor of d_1 in $\mathcal{I}^\ell_{d'}$. As $S_{d'}$ is a simulation, there must be some s-successor e_2 of e_1 such that $(d_2, e_2) \in S_{d'}$; hence $(d_2, e_2) \in S_d$.

For the converse direction, assume that $(\mathcal{I}^\ell_d, d) \leq (\mathcal{J}, e)$. Note that $d \in (X^\ell(\mathcal{I}, d))^{\mathcal{I}_d}$ and that $X^\ell(\mathcal{I}, d)$ has nesting-depth not exceeding ℓ and is therefore ℓ-local. Hence $d \in (X^\ell(\mathcal{I}, d))^{\mathcal{I}^\ell_d}$, which, as $\mathcal{EL}$-concepts are preserved by simulation, yields $e \in (X^\ell(\mathcal{I}, d))^{\mathcal{J}}$.

Proof of Theorem 4 Let Σ denote the signature of a FO-formula $\varphi(x)$ what is preserved under simulation. It follows that $\varphi(x)$ is invariant under bisimulations, and so it can be regarded as an $\mathcal{ALC}$-concept F. We may assume that F contains symbols from Σ only. Clearly, every $\mathcal{ALC}$-concept is ℓ-local for ℓ the nesting-depth of existential and value restrictions in F. We define

$$X_F := \{X^\ell(\mathcal{I}^\ell_d, d) \mid d \in F^{\mathcal{I}}, \mathcal{I} \text{ a } \Sigma\text{-interpretation }\}.$$

According to Observation 1, on each level ℓ, there are up to logical equivalence only finitely many ℓ-characteristic concepts for Σ; thus, we may assume that X_F is finite and $\bigsqcup_{C \in X_F} C$ is well defined. It remains to be shown that F and $\bigsqcup_{C \in X_F} C$ are equivalent. To this end, it is sufficient to show that $e \in F^{\mathcal{J}}$ iff $e \in (\bigsqcup_{C \in X_F} C)^{\mathcal{J}}$ for all pointed Σ-interpretations $(\mathcal{J}, e)$. Let $e \in F^{\mathcal{J}}$. We may assume that $X^\ell(\mathcal{J}^\ell_e, e) \in X_F$. As $(\mathcal{J}^\ell_e, e) \leq (\mathcal{J}, e)$, Observation 2 yields $e \in (X^\ell(\mathcal{J}^\ell_e, e))^{\mathcal{J}}$, which entails $e \in (\bigsqcup_{C \in X_F} C)^{\mathcal{J}}$.

Conversely, let $e \in (\bigsqcup_{C \in X_F} C)^{\mathcal{J}}$. Then $e \in (X^\ell(\mathcal{I}^\ell_d, d))^{\mathcal{J}}$ for some $X^\ell(\mathcal{I}^\ell_d, d) \in X_F$. From Observation 2, we obtain $(\mathcal{I}^\ell_d, d) \leq (\mathcal{J}, e)$. Due to the ℓ-locality of F, we have $d \in F^{\mathcal{I}^\ell_d}$ and as F is simulation preserved, $e \in F^{\mathcal{J}}$.

Proof of Theorem 5 Assume that $\mathcal{L}$ is a fragment of FO containing $\mathcal{EL}$ that is preserved under simulations and has (finite) minimal models. Let $\varphi(x) \in \mathcal{L}$. We show $\varphi(x)$ is equivalent to an $\mathcal{EL}$-concept. $\varphi(x)$ is logically equivalent to $\bigsqcup_{C \in X_F} C$, where X_F is as introduced in the proof of Theorem 4. Note that X_F contains $\mathcal{EL}$-concepts only. Thus, it is sufficient to show that $\bigsqcup_{C \in X_F} C$ is logically equivalent to one of its disjuncts.

Assume otherwise. Then $\forall x(\varphi(x) \rightarrow C(x))$ is not a tautology for any $C \in X_F$, where $C(x)$ denotes the translation of C into FO. Then $d \notin C^{\mathcal{I}}$ for any $C \in X_F$, where $(\mathcal{I}, d)$ is the (finite) minimal model of $\varphi(x)$. Thus $d \notin (\bigsqcup_{C \in X_F} C)^{\mathcal{I}}$ and we have derived a contradiction.

C Proofs for Section 4

We use the abbreviations

$$\vec{X} = X_1 \cdots X_n, \quad \vec{C} = C_1 \cdots C_n.$$

To prove the claims for the translations to and from $\mathcal{EL}^{\nu+}$ we remind the reader of the following well-known characterization of the greatest fixpoint operator. For a fixpoint concept $\nu X.C$, we write $C(\nu X.C)$ to denote the result of unfolding the fixpoint once, i.e., replacing every occurrence of X in C with $(\nu X.C)$. Similarly, the expression $C_i(\nu_i X_1 \cdots X_n.C_1, \ldots, C_n)$ denotes the result of replacing every occurrence of X_j in C_i, $1 \leq j \leq n$, with $\nu_j X_1 \cdots X_n.C_1, \ldots, C_n$.

We will consider concepts $\nu^\alpha X.C$ and $\nu^\alpha_i X_1 \cdots X_n.C_1, \ldots, C_n$ for every ordinal α. The semantics is defined by transfinite induction as

$$\begin{aligned} (\nu^0_i \vec{X}.\vec{C})^{\mathcal{I},\mathcal{V}} &= \Delta^{\mathcal{I}} \\ (\nu^{\alpha+1}_i \vec{X}.\vec{C})^{\mathcal{I},\mathcal{V}} &= (C_i(\nu^\alpha_i \vec{X}.\vec{C}))^{\mathcal{I},\mathcal{V}} \\ (\nu^\lambda_i \vec{X}.\vec{C})^{\mathcal{I},\mathcal{V}} &= \bigcap_{\alpha<\lambda} (\nu^\alpha_i \vec{X}.\vec{C})^{\mathcal{I},\mathcal{V}} \end{aligned}$$

where λ is a limit ordinal. It is standard to show that for all interpretations, $d \in \Delta^{\mathcal{I}}$, and assignments $\mathcal{V}$, we have that $d \in (\nu_i X_1 \cdots X_n.C_1, \ldots, C_n)^{\mathcal{I},\mathcal{V}}$ if, and only if, $d \in (\nu^\alpha_i X_1 \cdots X_n.C_1, \ldots, C_n)^{\mathcal{I},\mathcal{V}}$ for all $\alpha < |\Delta^{\mathcal{I}}|$.

Lemma 29 $\mathcal{EL}^{si} \leq_p \mathcal{EL}^{\nu+}$.

Proof We show that the claim that the concept $\exists^{sim}(\mathcal{I}, d)$ is equivalent to the $\mathcal{EL}^{\nu+}$-concept $\nu_\ell d_1 \cdots d_n.C_1, \ldots, C_n$, where $\Delta^\mathcal{I} = \{d_1, \ldots, d_n\}$ is regarded as a set of concept variables, $d = d_\ell$, and

$$C_i = \bigsqcap_{A \in \mathsf{N_C}} \{A \mid d_i \in A^\mathcal{I}\} \sqcap \bigsqcap_{r \in \mathsf{N_R}} \{\exists r.d_j \mid (d_i, d_j) \in r^\mathcal{I}\},$$

is correct.

Let $\mathcal{J}$ be an interpretation and $e \in (\exists^{sim}(\mathcal{I}, d))^\mathcal{J}$. Then $(\mathcal{I}, d) \leq (\mathcal{J}, e)$. We show by transfinite induction on α that for all $f \in \Delta^\mathcal{J}$ and $d_i \in \Delta^\mathcal{I}$ with $(\mathcal{I}, d_i) \leq (\mathcal{J}, f)$, we have $f \in (\nu_i^\alpha d_1 \cdots d_n.C_1, \ldots, C_n)^\mathcal{J}$. Clearly, this yields $e \in (\nu_\ell d_1 \cdots d_n.C_1, \ldots, C_n)^\mathcal{J}$ as required. Since the induction start is trivial, we concentrate on the induction step and the transfinite step. Let $f \in \Delta^\mathcal{J}$ and $d_i \in \Delta^\mathcal{I}$ with $(\mathcal{I}, d_i) \leq (\mathcal{J}, f)$.

- $f \in (\nu_i^{\alpha+1} \ell d_1 \cdots d_n.C_1, \ldots, C_n)^\mathcal{J}$.
  Let $(r_1, d_{i_1}), \ldots, (r_m, d_{i_m})$ be those elements of $\mathsf{N_R} \times \Delta^\mathcal{I}$ such that $(d_i, d_{i_j}) \in r_j^\mathcal{I}$ for $1 \leq j \leq m$. Since $(\mathcal{I}, d_i) \leq (\mathcal{J}, f)$, we find $f_1, \ldots, f_m \in \Delta^\mathcal{J}$ such that $(f, f_j) \in r_j^\mathcal{J}$ and $(\mathcal{I}, d_{i_j}) \leq (\mathcal{J}, f_j)$ for $1 \leq j \leq m$. The induction hypothesis yields $f_j \in (\nu_{i_j}^\alpha d_1 \cdots d_n.C_1, \ldots, C_n)^\mathcal{J}$. By definition of the $C_1, \ldots, C_n$ and the semantics of $\nu_i^{\alpha+1} d_1 \cdots d_n.C_1, \ldots, C_n$, this yields $f \in (\nu_i^{\alpha+1} d_1 \cdots d_n.C_1, \ldots, C_n)^\mathcal{J}$ as required.
- $f \in (\nu_i^\lambda d_1 \cdots d_n.C_1, \ldots, C_n)^\mathcal{J}$, λ a limit ordinal.
  By IH, we have that $f \in (\nu_i^\alpha d_1 \cdots d_n.C_1, \ldots, C_n)^\mathcal{J}$ for all $\alpha < \lambda$, thus it remains to use the semantics of $\nu_i^\lambda d_1 \cdots d_n.C_1, \ldots, C_n$.

Conversely, let $\mathcal{J}$ be an interpretation and $e \in (\nu_\ell d_1 \cdots d_n.C_1, \ldots, C_n)^\mathcal{J}$. We construct a sequence $S_0 \subseteq S_1 \subseteq \cdots$ of relations on $\Delta^\mathcal{I} \times \Delta^\mathcal{J}$ such that

$$(d_k, f) \in S_i \text{ implies } f \in (\nu_k d_1 \cdots d_n.C_1, \ldots, C_n)^\mathcal{J}. \qquad (*)$$

Start with putting $S_0 = \{(d_\ell, e)\}$. For the induction step, first set $S_{i+1} = S_i$. Then further extend S_{i+1} by considering all $(d_k, f) \in S_i \setminus S_{i-1}$ (where $S_{-1} := \emptyset$). Let $(r_1, d_{i_1}), \ldots, (r_m, d_{i_m})$ be those elements of $\mathsf{N_R} \times \Delta^\mathcal{I}$ such that $(d_k, d_{i_j}) \in r_j^\mathcal{I}$ for $1 \leq j \leq m$. We have $f \in (\nu_k d_1 \cdots d_n.C_1, \ldots, C_n)^\mathcal{J}$, thus also $f \in (C_k(\nu_k d_1 \cdots d_n.C_1, \ldots, C_n))^\mathcal{J}$. By definition of C_k, we thus find $f_1, \ldots, f_m \in \Delta^\mathcal{J}$ such that $(f, f_j) \in r_j$ and $f_j \in (\nu_{i_j} d_1 \cdots d_n.C_1, \ldots, C_n)^\mathcal{J}$ for $1 \leq j \leq m$. Add (f_j, d_{i_j}) to S_{i+1} for $1 \leq j \leq m$. Clearly, $(*)$ is satisfied.

Finally, set $S := \bigcup_{i \geq 0} S_i$. Using the definition of the S_i, $(*)$, and the definition of the concepts $C_1, \ldots, C_n$, it is straightforward to show that S is a simulation between $(\mathcal{I}, d_\ell)$ and $(\mathcal{J}, e)$. Thus, $e \in (\exists^{sim}(\mathcal{I}, d_\ell))^\mathcal{J}$ as required.

We call an $\mathcal{EL}^{\nu+}$-concept C with free variables $X_1, \ldots, X_k$ *equivalent* to an $\mathcal{EL}^{st}$-concept D if for all interpretations $\mathcal{I}$, $d \in \Delta^\mathcal{I}$, and assignments $\mathcal{V}$, we have $d \in C^{\mathcal{I},\mathcal{V}}$ iff $d \in D^\mathcal{J}$, where $\mathcal{J}$ is obtained from $\mathcal{I}$ by setting $A_X^\mathcal{J} = \mathcal{V}(X)$ for all variables X.

Lemma 30 $\mathcal{EL}^{\nu+} \leq_p \mathcal{EL}^{st}$.

Proof The concept $C^\sharp$ is produced by starting with C and then replacing each subconcept of the form $\nu_\ell X_1, \ldots, X_n.C_1, \ldots, C_n$ with an $\mathcal{EL}^{st}$-concept, proceeding from the inside out. We assume that for every variable X that occurs in the original $\mathcal{EL}^{\nu+}$-concept C, there is a concept name A_X that does not occur in C.

We replace each subconcept $\nu_\ell X_1, \ldots, X_n.C_1, \ldots, C_n$ (which potentially contains free variables) with the $\mathcal{EL}^{st}$-concept

$$\exists^{sim}\{A_{X_1}, \ldots, A_{X_n}\}.(\{A_{X_i} \sqsubseteq C_i^\downarrow \mid 1 \leq i \leq n\}, A_{X_\ell})$$

where $C_i^\downarrow$ is obtained from C_i by replacing every variable X with the concept name A_X. It thus remains to show that the above $\mathcal{EL}^s$-concept is equivalent to $\nu_\ell X_1, \ldots, X_n.C_1, \ldots, C_n$ (in the above sense).

Let $\mathcal{I}$ be an interpretation, $\mathcal{V}$ an assigment, and $d \in (\nu_\ell X_1, \ldots, X_n.C_1, \ldots, C_n)^{\mathcal{I},\mathcal{V}}$. We obtain the interpretation $\mathcal{J}$ from $\mathcal{I}$ by setting $A_{X_i}^\mathcal{J} = (\nu_i X_1, \ldots, X_n.C_1, \ldots, C_n)^{\mathcal{I},\mathcal{V}}$ for $1 \leq i \leq n$ and $A_Y^\mathcal{J} = \mathcal{V}(Y)$ for all concept variables $Y \notin \{X_1, \ldots, X_n\}$. Thus, $d \in A_{X_\ell}^\mathcal{J}$. Since $\nu_i X_1 \cdots X_n.C_1, \ldots, C_n \equiv C_i(\nu_i X_1 \cdots X_n.C_1, \ldots, C_n)$ for $1 \leq i \leq n$ and by definition of $\mathcal{J}$, $\mathcal{J}$ satisfies $A_{X_i} \sqsubseteq C_i^\downarrow$ for $1 \leq i \leq n$. By definition of $\mathcal{J}$, the identity map on $\Delta^\mathcal{I}$ is an $\Sigma \setminus \{A_{X_1}, \ldots, A_{X_n}\}$-simulation from $(\mathcal{J}, d)$ to $(\mathcal{I}, d)$. It follows that $d \in \exists^{sim}\{A_{X_1}, \ldots, A_{X_n}\}.(\{A_{X_i} \sqsubseteq C_i^\downarrow \mid 1 \leq i \leq n\}, A_{X_\ell})$ as required.

Let $\mathcal{I}$ be an interpretation with $d \in (\exists^{sim}\{A_{X_1}, \ldots, A_{X_n}\}.(\{A_{X_i} \sqsubseteq C_i^\downarrow \mid 1 \leq i \leq n\}, A_{X_\ell})^\mathcal{I}$. Then there exists a model $\mathcal{J}$ of $\{A_{X_i} \sqsubseteq C_i^\downarrow \mid 1 \leq i \leq n\}$ and an $e \in A_{X_\ell}^\mathcal{J}$ such that $(\mathcal{J}, e) \leq_{\Sigma \setminus \{A_{X_1}, \ldots, A_{X_n}\}} (\mathcal{I}, d)$. Define an assignment $\mathcal{V}_\mathcal{I}$ by setting $\mathcal{V}_\mathcal{I}(X) = A_X^\mathcal{I}$ for all variables X, and similarly for $\mathcal{V}_\mathcal{J}$. We have to show that $d \in (\nu_\ell X_1, \ldots, X_n.C_1, \ldots, C_n)^{\mathcal{I},\mathcal{V}_\mathcal{I}}$. To do this, we establish two claims.

Claim 1. For all ordinals α and $1 \leq i \leq n$, $\mathcal{V}_\mathcal{J}(X_i) \subseteq (\nu_i^\alpha X_1, \ldots, X_n.C_1, \ldots, C_n)^{\mathcal{J},\mathcal{V}_\mathcal{J}}$.

The induction start is trivial, hence it remains to consider the induction step and the transfinite step:

- $\mathcal{V}_\mathcal{J}(X_i) \subseteq (\nu_i^{\alpha+1} X_1, \ldots, X_n.C_1, \ldots, C_n)^{\mathcal{J},\mathcal{V}_\mathcal{J}}$.
  Let $e' \in \mathcal{V}_\mathcal{J}(X_i)$. As required for $\mathcal{J}$ we have $A_{X_j}^\mathcal{J} = \mathcal{V}_\mathcal{J}(X_j)$ f.a. $1 \leq j \leq n$. The induction hypothesis claims $\mathcal{V}_\mathcal{J}(X_j) \subseteq (\nu_j^\alpha X_1, \ldots, X_n.C_1, \ldots, C_n)^{\mathcal{J},\mathcal{V}_\mathcal{J}}$ for all $1 \leq j \leq n$. Hence $(C_i^\downarrow)^{\mathcal{J},\mathcal{V}_\mathcal{J}} \subseteq (C_i(\nu_i^\alpha X_1, \ldots, X_n.C_1, \ldots, C_n))^{\mathcal{J},\mathcal{V}_\mathcal{J}}$. Since $\mathcal{J}$ is a model of $A_{X_i} \sqsubseteq C_i^\downarrow$ it follows that
  $$e' \in (C_i(\nu_i^\alpha \vec{X}.\vec{C}))^{\mathcal{J},\mathcal{V}_\mathcal{J}} = (\nu_i^{\alpha+1} \vec{X}.\vec{C})^{\mathcal{J},\mathcal{V}_\mathcal{J}}.$$
- $\mathcal{V}_\mathcal{J}(X_i) \subseteq (\nu_i^\lambda X_1, \ldots, X_n.C_1, \ldots, C_n)^{\mathcal{J},\mathcal{V}_\mathcal{J}}$, λ a limit ordinal.
  Let $e' \in \mathcal{V}_\mathcal{J}(X_i)$. By IH, $e' \in (\nu_i^\alpha X_1, \ldots, X_n.C_1, \ldots, C_n)^{\mathcal{J},\mathcal{V}_\mathcal{J}}$ for all $\alpha < \lambda$ and thus
  $$e' \in \bigcap_{\alpha < \lambda} (\nu_i^\alpha \vec{X}.\vec{C})^{\mathcal{J},\mathcal{V}_\mathcal{J}} = (\nu_i^\lambda \vec{X}.\vec{C})^{\mathcal{J},\mathcal{V}_\mathcal{J}}.$$

The second claim is also proved by transfinite induction on α.

Claim 2. For $1 \leq i \leq n$ we have: if $e' \in (\nu_i^\alpha X_1, \ldots, X_n.C_1, \ldots, C_n)^{\mathcal{J},\mathcal{V}_\mathcal{J}}$ and $(\mathcal{J}, e') \leq_{\Sigma \setminus \{A_{X_1}, \ldots, A_{X_n}\}} (\mathcal{I}, d')$ then $d' \in (\nu_i^\alpha X_1, \ldots, X_n.C_1, \ldots, C_n)^{\mathcal{I},\mathcal{V}_\mathcal{I}}$.

Again, the induction start is trivial and we concentrate on the induction step and transfinite step:

- $d' \in (\nu_i^{\alpha+1} X_1, \ldots, X_n.C_1, \ldots, C_n)^{\mathcal{I},\mathcal{V}_\mathcal{I}}$.
  Let $(r_1, e_{i_1}), \ldots, (r_m, e_{i_m})$ be those elements of $\mathsf{N_R} \times \Delta^\mathcal{J}$ such that $(e', e_{i_j}) \in r_j^\mathcal{I}$ for $1 \leq j \leq m$. Since $(\mathcal{J}, e') \leq (\mathcal{I}, d')$, we find $d_1, \ldots, d_m \in \Delta^\mathcal{I}$ such that $(d', d_j) \in r_j^\mathcal{I}$ and $(\mathcal{J}, e_{i_j}) \leq (\mathcal{I}, d_j)$ for $1 \leq j \leq m$. The induction hypothesis yields $d_j \in (\nu_{i_j}^\alpha d_1 \cdots d_n.C_1, \ldots, C_n)^{\mathcal{I},\mathcal{V}_\mathcal{I}}$. By definition of the $C_1, \ldots, C_n$ and the semantics of $\nu_i^{\alpha+1} d_1 \cdots d_n.C_1, \ldots, C_n$, this yields $d' \in (\nu_i^{\alpha+1} d_1 \cdots d_n.C_1, \ldots, C_n)^{\mathcal{I},\mathcal{V}_\mathcal{I}}$ as required.
- $d' \in (\nu_i^\lambda X_1, \ldots, X_n.C_1, \ldots, C_n)^{\mathcal{I},\mathcal{V}_\mathcal{I}}$, λ a limit ordinal.
  If $e' \in (\nu_i^\alpha X_1, \ldots, X_n.C_1, \ldots, C_n)^{\mathcal{I},\mathcal{V}_\mathcal{I}}$ for all $\alpha < \lambda$ then, by the induction hypothesis, $d' \in (\nu_i^\alpha X_1, \ldots, X_n.C_1, \ldots, C_n)^{\mathcal{I},\mathcal{V}_\mathcal{I}}$ for all $\alpha < \lambda$. Thus $d' \in (\nu_i^\lambda X_1, \ldots, X_n.C_1, \ldots, C_n)^{\mathcal{I},\mathcal{V}_\mathcal{I}}$.

It remains to argue that Claims 1 and 2 imply $d \in (\nu_\ell X_1,\ldots,X_n.C_1,\ldots,C_n)^{\mathcal{I},\mathcal{V}_\mathcal{I}}$: Since $e \in A^{\mathcal{J}}_{X_\ell}$, we have $e \in \mathcal{V}(X_\ell)$. By Claim 1, $e \in (\nu^\alpha_\ell X_1,\ldots,X_n.C_1,\ldots,C_n)^{\mathcal{J},\mathcal{V}_\mathcal{J}}$ for all ordinals α. By Claim 2, $d \in (\nu^\alpha_\ell X_1,\ldots,X_n.C_1,\ldots,C_n)^{\mathcal{J},\mathcal{V}_\mathcal{I}}$ for all ordinals α. Thus, $d \in (\nu_\ell X_1,\ldots,X_n.C_1,\ldots,C_n)^{\mathcal{I},\mathcal{V}_\mathcal{I}}$ as required.

We are now going to prove Theorem 11. First, we require the following folklore result.

Lemma 31 *It can be checked in poly-time whether there exists a (Σ-)simulation between two finite pointed interpretations.*

We give a slightly modified formulation from which Theorem 11 follows directly.

Theorem 32 *It is decidable in polynomial time whether an $\mathcal{EL}^{sa}$-KB is satisfiable. Moreover, given a satisfiable $\mathcal{EL}^{sa}$-KB $(\mathcal{T},\mathcal{A})$, one can construct in polynomial time an interpretation $\mathcal{I}_{\mathcal{T},\mathcal{A}}$ with $|\Delta^{\mathcal{I}_{\mathcal{T},\mathcal{A}}}|$ bounded by twice the size of $(\mathcal{T},\mathcal{A})$ and the following property:*

(CAN) $\mathcal{I}_{\mathcal{T},\mathcal{A}}$ is a model of $(\mathcal{T},\mathcal{A})$ with $a^{\mathcal{I}_{\mathcal{T},\mathcal{A}}} = a$ for all $a \in \mathsf{Ind}(\mathcal{A})$ such that for every model $\mathcal{J}$ of $(\mathcal{T},\mathcal{A})$ there exists a simulation S between $\mathcal{I}_{\mathcal{T},\mathcal{A}}$ and $\mathcal{J}$ with $(a,a^{\mathcal{J}}) \in S$ for all $a \in \mathsf{Ind}(\mathcal{A})$.

Proof The construction of $\mathcal{I}_{\mathcal{T},\mathcal{A}}$ will be given by induction on the number of nestings of $\exists^{sim}$ in $\mathcal{T}$. For a $\mathcal{EL}^{sa}$-concept C, denote by $\mathsf{sub}_0(C)$ the set of subconcepts of C that are not within the scope of any $\exists^{sim}$. For a TBox $\mathcal{T}$ we set

$$\mathsf{sub}_0(\mathcal{T}) = \bigcup_{C\sqsubseteq D\in\mathcal{T}} \mathsf{sub}_0(C)\cup\mathsf{sub}_0(D).$$

Suppose $(\mathcal{T},\mathcal{A})$ is given. By induction, we may assume that for any

$$F = \exists^{sim}\Sigma.(\mathcal{T}',\mathcal{A}',a') \in \mathsf{sub}_0(\mathcal{T})$$

we have decided already whether $(\mathcal{T}',\mathcal{A}')$ is satisfiable and, if so, have constructed an interpretation $\mathcal{I}_F = \mathcal{I}_{\mathcal{T}',\mathcal{A}'}$ satisfying (CAN). If F is not satisfiable, we replace F by $\bot$ everywhere in $\mathcal{T}$. Clearly the resulting TBox (denoted, for simplicity, by $\mathcal{T}$ as well) is equivalent to $\mathcal{T}$. If F is satisfiable, we construct an isomorphic copy $\mathcal{I}'_F$ of $\mathcal{I}_F$ with the following modifications:

- the domains $\Delta^{\mathcal{I}'_F}$ of $\mathcal{I}'_F$ are mutually disjoint and disjoint from $\mathsf{Ind}(\mathcal{A})$;
- individual names are not interpreted in $\mathcal{I}'_F$;
- all Σ-symbols are interpretated as empty sets;
- the point a' of $\mathcal{I}_F$ is renamed to $d_{\mathcal{I}_F}$ in $\mathcal{I}'_F$.

We can construct a model of $(\mathcal{T},\mathcal{A})$ only if $(\mathcal{T},\mathcal{A})$ is satisfiable. To enable us to construct a model that can be used to check satisfiability of $(\mathcal{T},\mathcal{A})$, we first replace every occurrence of $\bot$ in $\mathcal{T}$ that is not within the scope of any $\exists^{sim}$ by a fresh concept names $A_\bot$ and denote the resulting TBox by $\mathcal{T}^\bot$. We construct the model $\mathcal{I}_{\mathcal{T}^\bot,\mathcal{A}}$ satisfying (CAN) for $(\mathcal{T}^\bot,\mathcal{A})$ and then decide, depending on the interpretation of $A_\bot$, whether be obtain a model of $(\mathcal{T},\mathcal{A})$ or $(\mathcal{T},\mathcal{A})$ is unsatisfiable. Let

$$\Delta_0 = \mathsf{Ind}(\mathcal{A}) \cup \{d_C \mid \exists r.C \in \mathsf{sub}_0(\mathcal{T}^\bot)\},$$

where the d_C are pairwise distinct fresh objects. We define an interpretation $\mathcal{I}_0$ as follows. Let

$$\Delta^{\mathcal{I}_0} = \Delta_0 \cup \bigcup_{F\in\mathsf{sub}_0(\mathcal{T}^\bot),F=\exists^{sim}\Sigma.(\mathcal{T}',\mathcal{A}',a')} \Delta^{\mathcal{I}'_F}$$

and set $(d_1,d_2) \in r^{\mathcal{I}_0}$ iff

1.0 $d_1 = a_1, d_2 = a_2$ and $r(a_1,a_2) \in \mathcal{A}$, or
1.1 $d_1 = d_{C_1} \in \Delta_0, d_2 = d_{C_2} \in \Delta_0$ and $\exists r.C_2$ is a top-tevel conjunct of C_1, or
1.2 $d_1,d_2 \in \Delta^{\mathcal{I}'_F}$ for some F and $(d_1,d_2) \in r^{\mathcal{I}'_F}$, or
1.3 $d_1 = d_{C_1}$, C_1 has a top-level conjunct $F = \exists^{sim}\Sigma.(\mathcal{T}',\mathcal{A}',a')$ and $(d_{\mathcal{I}_F},d_2) \in r^{\mathcal{I}'_F}$.

Finally, set $d \in A^{\mathcal{I}_0}$ iff

2.0 $d = a_0$ and $A(a_0) \in \mathcal{A}$, or
2.1 $d = d_D \in \Delta_0$ and A is a top-level conjunct of D;
2.2 $d \in \Delta^{\mathcal{I}'_F}$ for some F and $d \in A^{\mathcal{I}'_F}$, or
2.3 $d = d_D$, D has a top-level conjunct $F = \exists^{sim}\Sigma.(\mathcal{T}',\mathcal{A}',a')$ and $d_{\mathcal{I}_F} \in A^{\mathcal{I}'_F}$.

It remains to satisfy the inclusions of $\mathcal{T}^\bot$ in $\mathcal{I}_0$. We may assume that, in each $C \sqsubseteq D \in \mathcal{T}^\bot$, D is a concept name, of the form $\exists r.D'$, or of the form $\exists^{sim}\Sigma.(\mathcal{T}',\mathcal{A}',a')$. Now we expand $\mathcal{I}_0$ by applying exhaustively the following rules:

3.1 Let $C \sqsubseteq A \in \mathcal{T}^\bot$ and assume that $d \in C^{\mathcal{I}_0}$ but $d \notin A^{\mathcal{I}_0}$. Then update $\mathcal{I}_0$ by setting $A^{\mathcal{I}_0} := \{d\} \cup A^{\mathcal{I}_0}$ (and leaving everything else unchanged).
3.2 Let $C \sqsubseteq \exists r.D \in \mathcal{T}^\bot$ and assume that $d \in C^{\mathcal{I}_0}$ but $d \notin (\exists r.D)^{\mathcal{I}_0}$. Then update $\mathcal{I}_0$ by setting $r^{\mathcal{I}_0} := \{(d,d_D)\} \cup r^{\mathcal{I}_0}$ (and leaving everything else unchanged).
3.3 Let $C \sqsubseteq F \in \mathcal{T}^\bot$ for $F = \exists^{sim}\Sigma.(\mathcal{T}',\mathcal{A}',a')$ and assume that $d \in C^{\mathcal{I}_0}$ but $d \notin F^{\mathcal{I}_0}$. Then update $\mathcal{I}_0$ by adding (d,d') to $r^{\mathcal{I}_0}$ whenever $(d_{\mathcal{I}_F},d') \in r^{\mathcal{I}'_F}$ and by adding d to $A^{\mathcal{I}_0}$ whenever $d_{\mathcal{I}_F} \in A^{\mathcal{I}'_F}$.

The resulting interpretation is denoted by $\mathcal{I}_{\mathcal{T}^\bot,\mathcal{A}}$. We show that $\mathcal{I}_{\mathcal{T}^\bot,\mathcal{A}}$ satisfies (CAN) for $(\mathcal{T}^\bot,\mathcal{A})$, that the construction above is in poly-time, and that the domain of $\mathcal{I}_{\mathcal{T}bot,\mathcal{A}}$ is of linear size. The latter is easily proved and left to the reader. We now prove (CAN).

Claim 1. For every model $\mathcal{J}$ of $(\mathcal{T}^\bot,\mathcal{A})$ there exists a simulation S between $\mathcal{I}_{\mathcal{T}^\bot,\mathcal{A}}$ and $\mathcal{J}$ with $(a,a^{\mathcal{J}}) \in S$ for all $a \in \mathsf{Ind}(\mathcal{A})$.

The proof is by induction on the number of nestings of $\exists^{sim}$ in $\mathcal{T}^\bot$. We consider the induction step from n to $n+1$. The case $n = 0$ is proved in the same way and left to the reader.

Assume Claim 1 has been proved for all $(\mathcal{T}',\mathcal{A}')$ with at most n nestings of $\exists^{sim}$ and let $\mathcal{T}^\bot$ have $n+1$ nestings of $\exists^{sim}$. Assume that $\mathcal{J}$ is a model of $(\mathcal{T}^\bot,\mathcal{A})$. We construct a simulation $S : \mathcal{I}_{\mathcal{T}^\bot,\mathcal{A}} \leq \mathcal{J}$ with $(a,a^{\mathcal{J}}) \in S$ for all $a \in \mathsf{Ind}(\mathcal{A})$. By induction hypothesis, for every $F = \exists^{sim}\Sigma.(\mathcal{T}',\mathcal{A}',a') \in \mathsf{sub}_0(\mathcal{T}^\bot)$, Claim 1 holds for $(\mathcal{T}',\mathcal{A}')$. So we have, in particular, $(\mathcal{I}'_F,d_{\mathcal{I}_F}) \leq (\mathcal{I},a'^{\mathcal{I}})$ for every model $\mathcal{I}$ of $(\mathcal{T}',\mathcal{A}')$.

To construct S, we first define a set S_0 as the union of

$$\{(d_C,e) \in \Delta_0 \times \Delta^{\mathcal{J}} \mid C \in \mathsf{sub}_0(\mathcal{T}^\bot), e \in C^{\mathcal{J}}\}$$

and

$$\{(a,a^{\mathcal{J}}) \mid a \in \mathsf{Ind}(\mathcal{A})\}.$$

By definition, for each $F = \exists^{sim}\Sigma.(\mathcal{T}',\mathcal{A}',a') \in \mathsf{sub}_0(\mathcal{T}^\bot)$ and $e \in F^{\mathcal{J}}$ there exists a model $\mathcal{J}'$ of $(\mathcal{T}',\mathcal{A}')$ with $f = a'^{\mathcal{J}'}$ such that

$$(\mathcal{J}',f) \leq_\Gamma (\mathcal{J},e),$$

where $\Gamma = (\mathsf{N_C} \cup \mathsf{N_R}) \setminus \Sigma$. By induction hypothesis, we have $(\mathcal{I}_{\mathcal{T}',\mathcal{A}'}, a') \leq (\mathcal{J}', f)$. Thus,

$$(\mathcal{I}'_F, d_{\mathcal{I}_F}) \leq (\mathcal{J}, e). \quad (1)$$

Let $S_{F,e}$ be a simulation that witnesses $(\mathcal{I}'_F, d_{\mathcal{I}_F}) \leq (\mathcal{J}, e)$ and let

$$S := S_0 \cup \bigcup \{S_{F,e} \mid e \in F^{\mathcal{J}}, F := \exists^{sim}\Sigma.(\mathcal{T}', \mathcal{A}', a') \in \mathsf{sub}_0(\mathcal{T}^\perp)\}.$$

We prove that S is a simulation between $\mathcal{I}_{\mathcal{T},C}$ and $\mathcal{J}$. Then, by definition, Claim 1 is proved. To this end, we first prove the following claim.

Claim 1.a. S is a simulation between $\mathcal{I}_0$ and $\mathcal{J}$.

Let $(d_1, e_1) \in S$. First, we show that if $d_1 \in A^{\mathcal{I}_0}$, then $e_1 \in A^{\mathcal{J}}$. Let $d_1 \in A^{\mathcal{I}_0}$. If $d_1 \in \Delta_0$, then $(d_1, e_1) \in S_0$. Assume first $d_1 = d_C$ and $e_1 \in C^{\mathcal{J}}$ for some concept C. If $d_C \in A^{\mathcal{I}_0}$, then we have two cases:

2.1 A is a top-level conjunct of C. As $e_1 \in C^{\mathcal{J}}$, we have $e_1 \in A^{\mathcal{J}}$, as required.

2.3 C has the top-level conjunct $F := \exists^{sim}\Sigma.(\mathcal{T}', \mathcal{A}', a')$ and $d_{\mathcal{I}_F} \in A^{\mathcal{I}'_F}$. Then $e_1 \in F^{\mathcal{J}}$ and so, by (1), we obtain $e_1 \in A^{\mathcal{J}}$.

Assume now that $d_1 = a \in \mathsf{Ind}(\mathcal{A})$. Then $e_1 = a^{\mathcal{J}}$ and $A(a) \in \mathcal{A}$. We obtain $e_1 = a^{\mathcal{J}} \in A^{\mathcal{J}}$ from the condition that $\mathcal{J}$ is a model of $(\mathcal{T}, \mathcal{A})$.

In case $d_1 \in \Delta^{\mathcal{I}'_F}$ for some $F = \exists^{sim}\Sigma.(\mathcal{T}', \mathcal{A}', a') \in \mathsf{sub}_0(\mathcal{T}^\perp)$ Item 2.2 applies. As $(d_1, e_1) \in S$ and $d_1 \in \Delta^{\mathcal{I}'_F}$ the definition of S stipulates $(d_1, e_1) \in S_{F,e}$ for some $e \in \Delta^{\mathcal{J}}$. Then $e_1 \in A^{\mathcal{J}}$ because $S_{F,e}$ is a simulation.

Let now $(d_1, d_2) \in r^{\mathcal{I}_{\mathcal{T},\mathcal{A}}}$. We have to show that there is $e_2 \in \Delta^{\mathcal{J}}$ such that $(e_1, e_2) \in r^{\mathcal{J}}$ and $(d_2, e_2) \in S$. We distinguish the following cases:

1.0 $d_1 = a_1$, $d_2 = a_2$ and $r(a_1, a_2) \in \mathcal{A}$. By definition of S, $e_1 = a_1^{\mathcal{J}}$. As $\mathcal{J}$ is a model of $(\mathcal{T}, \mathcal{A})$, we obtain $(e_1, e_2) \in r^{\mathcal{J}}$ for $e_2 = a_2^{\mathcal{J}}$. Moreover, $(d_2, e_2) \in S$.

1.1 Both, $d_1 = d_{C_1}$ and $d_2 = d_{C_2}$ are in Δ_0 and $\exists r.C_2$ a top-level conjunct of C_1. As $(d_{C_1}, e_1) \in S_0$ we have $e_1 \in C_1^{\mathcal{J}}$ and so there is $e_2 \in C_2^{\mathcal{J}}$ with $(e_1, e_2) \in r^{\mathcal{J}}$. Hence $(d_1, e_2) \in S$.

1.3 $d_1 = d_{C_1} \in \Delta^{\mathcal{I}_0}$ where $F := \exists^{sim}\Sigma(\mathcal{T}', \mathcal{A}', a')$ is a top-level conjunct of C_1, and $(d_{\mathcal{I}_F}, d_2) \in r^{\mathcal{I}'_F}$. Since $e_1 \in C_1^{\mathcal{J}}$ we have $e_1 \in F^{\mathcal{J}}$. By (1), $(\mathcal{I}'_F, d_{\mathcal{I}_F}) \leq (\mathcal{J}, e_1)$. Thus, there exists $e_2 \in \Delta^{\mathcal{J}}$ with $(e_1, e_2) \in r^{\mathcal{J}}$ and $(d_2, e_2) \in S_{F,e_1}$. We obtain $(d_2, e_2) \in S$.

1.2 Both d_1 and d_2 are in $\Delta^{\mathcal{I}'_F}$. As we assume $(d_1, e_1) \in S$ there must be $S_{F,e'} \subseteq S$ containing (d_1, e_1). We have $(d_1, d_2) \in r^{\mathcal{I}'_F}$ and so, since $S_{F,e'}$ is a simulation, there exists $e_2 \in \Delta^{\mathcal{J}}$ with $(e_1, e_2) \in r^{\mathcal{J}}$ such that $(d_2, e_2) \in S_{F,e'}$. Hence $(d_2, e_2) \in S$.

Claim 1.a. is proved. To prove Claim 1, it remains to show that iterated applications of the rules (3.1)-(3.3) to $\mathcal{I}_0$ preserve the condition that S is a simulation. So, assume that $\mathcal{I}_0 \subseteq \mathcal{I}_1 \subseteq \ldots$ is a sequence resulting from applications of the rules (3.1)-(3.3) and assume that S is a simulation between $\mathcal{I}_k$ and $\mathcal{J}$ such that $(a, a^{\mathcal{J}}) \in S$ for all $a \in \mathsf{Ind}(\mathcal{A})$. We show S is a simulation between $\mathcal{I}_{k+1}$ and $\mathcal{J}$.

3.1 Assume $d_1 \in C^{\mathcal{I}_k}$, $C \sqsubseteq A \in \mathcal{T}$, and $\mathcal{I}_{k+1}$ coincides with $\mathcal{I}_k$ except that $A^{\mathcal{I}_{k+1}} := A^{\mathcal{I}_k} \cup \{d_1\}$). Clearly it is sufficient to show that $e_1 \in A^{\mathcal{J}}$ for every $(d_1, e_1) \in S$. We have $S : (\mathcal{I}_k, d_1) \leq (\mathcal{J}, e_1)$ and so $e_1 \in C^{\mathcal{J}}$. Hence, since $\mathcal{J} \models \mathcal{T}$, $e_1 \in A^{\mathcal{J}}$, as required.

3.2 Assume $d_1 \in C^{\mathcal{I}_k}$, $C \sqsubseteq \exists r.D \in \mathcal{T}$, and $\mathcal{I}_{k+1}$ coincides $\mathcal{I}_k$ except that $r^{\mathcal{I}_k} := r^{\mathcal{I}_k} \cup \{(d_1, d_D)\}$). Assume that $(d_1, e_1) \in S$ for some $e_1 \in \Delta^{\mathcal{J}}$. Clearly it is sufficient to show that there exists e_2 with $(e_1, e_2) \in r^{\mathcal{J}}$ such that $(d_D, e_2) \in S$. We have $e_1 \in C^{\mathcal{J}}$ and as $\mathcal{J} \models \mathcal{T}$ by assumption there is an $e_2 \in D^{\mathcal{J}}$ with $(e_1, e_2) \in r^{\mathcal{J}}$. According to the definition of S we have $(d_D, e_2) \in S$, as required.

3.3 Assume $d_1 \in C^{\mathcal{I}_k}$, $C \sqsubseteq F \in \mathcal{T}$ with $F = \exists^{sim}\Sigma.(\mathcal{T}', \mathcal{A}', a')$ and $\mathcal{I}_{k+1}$ coincides with $\mathcal{I}_k$ except that

$$r^{\mathcal{I}_{k+1}} = r^{\mathcal{I}_k} \cup \{(d_1, d_2) \mid (d_{\mathcal{I}'_F}, d_2) \in r^{\mathcal{I}'_F}\}$$

and

$$A^{\mathcal{I}_{k+1}} = r^{\mathcal{I}_k} \cup \{d_1 \mid d_{\mathcal{I}'_F} \in A^{\mathcal{I}'_F}\},$$

for $r \in \mathsf{N_R}$ and $A \in \mathsf{N_C}$. We consider the condition on roles. Let $(d_{\mathcal{I}'_F}, d_2) \in r^{\mathcal{I}'_F}$ and let $(d_1, e_1) \in S$ for some $e_1 \in \Delta^{\mathcal{J}}$. Clearly it is enough to show that there exists e_2 with $(e_1, e_2) \in r^{\mathcal{J}}$ and $(d_2, e_2) \in S$. We have $e_1 \in C^{\mathcal{J}}$ and as $\mathcal{J} \models \mathcal{T}$ by assumption, we have $e_1 \in F^{\mathcal{J}}$. So $S_{F,e_1} \subseteq S$ and we have $S_{F,e_1} : (\mathcal{I}'_F, d_{\mathcal{I}_F}) \leq (\mathcal{J}, e_1)$. Hence there exists e_2 with $(e_1, e_2) \in r^{\mathcal{J}}$ and $(d_2, e_2) \in S_{F,e_1}$, as required.

This finishes the proof of Claim 1. We now show

Claim 2. The following conditions hold.

- $\mathcal{I}_{\mathcal{T}^\perp,\mathcal{A}}$ is a model of $(\mathcal{T}^\perp \mathcal{A})$;
- for all $a \in \mathsf{Ind}(\mathcal{A})$ and all D: $(\mathcal{T}^\perp, \mathcal{A}) \models D(a)$ iff $a \in D^{\mathcal{I}_{\mathcal{T}^\perp,\mathcal{A}}}$;
- for all $d_C \in \Delta_0$ and all D: $(\mathcal{T}^\perp, \mathcal{A}) \models C \sqsubseteq D$ iff $d_C \in D^{\mathcal{I}_{\mathcal{T}^\perp,\mathcal{A}}}$.

The proof is again by induction on the number of nestings of $\exists^{sim}$. Assume this has been proved for all $\mathcal{T}', \mathcal{A}'$ such that $\exists^{sim}\Sigma.(\mathcal{T}', \mathcal{A}', a')$ occurs in $\mathcal{T}^\perp$.

For Point 1 note that $\mathcal{I}_{\mathcal{T}^\perp,\mathcal{A}}$ is, by definition, a model of $\mathcal{A}$. $\mathcal{I}_{\mathcal{T}^\perp,\mathcal{A}} \models \mathcal{T}$ follows immediately from the fact that none of the rules (3.1)-(3.3) is applicable to $\mathcal{I}_{\mathcal{T}^\perp,\mathcal{A}}$ and that rule (3.3) constructs a model satisfying the required $F = \exists^{sim}\Sigma.(\mathcal{T}', \mathcal{A}', a')$ because $\mathcal{I}'_F$ is the reduct of the interpretation $\mathcal{I}_{\mathcal{T}',\mathcal{A}'}$ not interpreting Σ-symbols that is, by induction hypothesis, a model of $(\mathcal{T}', \mathcal{A}')$. Points 2 and 3 follow immediately from the induction hypothesis, Claim 1, and Point 1.

It follows that $\mathcal{I}_{\mathcal{T}^\perp,\mathcal{A}}$ satisfies (CAN) for $(\mathcal{T}^\perp, \mathcal{A})$. We now use this model to check satisfiability of $(\mathcal{T}, \mathcal{A})$ and construct $\mathcal{I}_{\mathcal{T},\mathcal{A}}$ in case it is satisfiable. Denote by $\mathcal{I}_{\mathcal{T},\mathcal{A}}$ the restriction of $\mathcal{I}_{\mathcal{T}^\perp,\mathcal{A}}$ to all $d \in \Delta^{\mathcal{I}_{\mathcal{T}^\perp,\mathcal{A}}}$ such that there exist $a \in \mathsf{Ind}(\mathcal{A})$, $d_0, \ldots, d_n = d$, and role names $r_1, \ldots, r_n$ with $d_0 = a$ and $(d_i, d_{i+1}) \in r_{i+1}^{\mathcal{I}_{\mathcal{T}^\perp,\mathcal{A}}}$ for $i < n$.

Claim 3. $(\mathcal{T}, \mathcal{A})$ is satisfiable iff $A_\perp^{\mathcal{I}_{\mathcal{T},\mathcal{A}}} = \emptyset$. Moreover, if $A_\perp^{\mathcal{I}_{\mathcal{T},\mathcal{A}}} = \emptyset$, then $\mathcal{I}_{\mathcal{T},\mathcal{A}}$ satisfies (CAN).

Clearly, $\mathcal{I}_{\mathcal{T},\mathcal{A}}$ still has the properties from Claim 1 and 2. Moreover, for $d \in \Delta^{\mathcal{I}_{\mathcal{T},\mathcal{A}}}$, we have $d \in A_\perp^{\mathcal{I}_{\mathcal{T},\mathcal{A}}}$ iff there exists $a \in \mathsf{Ind}(\mathcal{A})$ such that

$$(\mathcal{T}^\perp, \mathcal{A}) \models \exists r_1.\exists r_2.\cdots\exists r_n.A_\perp(a)$$

for some sequence $r_1, \ldots, r_n$ of role names. It thus follows from the construction of $\mathcal{T}^\perp$ from $\mathcal{T}$ that $(\mathcal{T}, \mathcal{A})$ is satisfiable iff $A_\perp^{\mathcal{I}_{\mathcal{T},\mathcal{A}}} = \emptyset$. The proof of the second claim is straightforward and left to the reader.

Finally, we show that $\mathcal{I}_{\mathcal{T},\mathcal{A}}$ can be constructed in polynomial time. As the domain of $\mathcal{I}_{\mathcal{T},\mathcal{A}}$ is clearly of linear size, it is sufficient to show that one can check the pre-condition "$d \in C^{\mathcal{I}}$" of the rules (3.1)-(3.3) in polynomial time. The only concepts C of interest are of the form $\exists^{sim}\Sigma.(\mathcal{T}, \mathcal{A}, a)$. But we know that checking $d \in (\exists^{sim}\Sigma.(\mathcal{T}, \mathcal{A}, a))^{\mathcal{I}}$ is equivalent to checking whether $(\mathcal{T}, \mathcal{A})$ is satisfiable and there exists a Γ-simulation between between $(\mathcal{I}_{\mathcal{T},\mathcal{A}}, a)$ and $(\mathcal{I}, d)$, for $\Gamma = (\mathsf{N_C} \cup \mathsf{N_I}) \setminus \Sigma$. Thus, checking this pre-condition is in polynomial time, by Lemma 31.

D Proofs for Section 6

To prove Theorem 15, we require two basic observations.

Observation 3 *Let $(\mathcal{I}, d)$, $(\mathcal{J}, e)$, and $(\mathcal{K}, f)$ be pointed interpretations. Then*

1. $(\mathcal{I} \times \mathcal{J}, (d, e)) \leq (\mathcal{I}, d)$ *and* $(\mathcal{I} \times \mathcal{J}, (d, e)) \leq (\mathcal{J}, e)$.
2. *If* $(\mathcal{K}, f) \leq (\mathcal{I}, d)$ *and* $(\mathcal{K}, f) \leq (\mathcal{J}, e)$, *then*

$$(\mathcal{K}, f) \leq (\mathcal{I} \times \mathcal{J}, (d, e)).$$

Proof For the second claim observe that if S_1 is a simulation between $(\mathcal{K}, f)$ and $(\mathcal{I}, d)$ and S_2 is a simulation between $(\mathcal{K}, f)$ and $(\mathcal{J}, e)$, then

$$S = \{(f', (d', e')) \mid (f', d') \in S_1 \text{ and } (f', e') \in S_2\}$$

is a simulation between $(\mathcal{K}, f)$ and $(\mathcal{I} \times \mathcal{J}, (d, e))$.

We also use the easily proved fact that for canonical models $(\mathcal{I}_{\mathcal{T},D}, d_D)$ and $(\mathcal{I}_{\mathcal{T},C}, d_C)$ we have

$$\mathcal{T} \models C \sqsubseteq D \iff (\mathcal{I}_{\mathcal{T},D}, d_D) \leq (\mathcal{I}_{\mathcal{T},C}, d_C).$$

Proof of Theorem 15 Let $\mathcal{T}$ be a general TBox in $\mathcal{EL}^{\nu+}$ and C_1, C_2 be two $\mathcal{EL}^{\nu+}$ concepts. We show that

$$\exists^{sim}(\mathcal{I}_{\mathcal{T},C_1} \times \mathcal{I}_{\mathcal{T},C_2}, (d_{C_1}, d_{C_2}))$$

is the LCS of C_1 and C_2, for the canonical models $\mathcal{I}_{\mathcal{T},C_i}$ of $\mathcal{T}, C_i$.

(a) We show

$$\mathcal{T} \models C_i \sqsubseteq \exists^{sim}(\mathcal{I}_{\mathcal{T},C_1} \times \mathcal{I}_{\mathcal{T},C_2}, (d_{C_1}, d_{C_2})).$$

for $i = 1, 2$. Let $i = 1$ ($i = 2$ is considered in the same way). Observation 3 shows

$$(\mathcal{I}_{\mathcal{T},C_1} \times \mathcal{I}_{\mathcal{T},C_2}, (d_{C_1}, d_{C_2})) \leq (\mathcal{I}_{\mathcal{T},C_1}, d_{C_1}).$$

Let now $\mathcal{J}$ be a model of $\mathcal{T}$ and $e \in C_1^{\mathcal{J}}$. Then $(\mathcal{I}_{\mathcal{T},C_1}, d_{C_1}) \leq (\mathcal{J}, e)$, which by transitivity of the simulation relation yields $e \in \exists^{sim}(\mathcal{I}_{\mathcal{T},C_1} \times \mathcal{I}_{\mathcal{T},C_2}, (d_{C_1}, d_{C_2}))^{\mathcal{J}}$. Hence $\mathcal{T} \models C_1 \sqsubseteq \exists^{sim}(\mathcal{I}_{\mathcal{T},C_1} \times \mathcal{I}_{\mathcal{T},C_2}, (d_{C_1}, d_{C_2}))$, as required.

(b) for all $\mathcal{EL}^{\nu+}$-concepts D we have to show: If $\mathcal{T} \models C_1 \sqsubseteq D$ and $\mathcal{T} \models C_2 \sqsubseteq D$ then

$$\mathcal{T} \models \exists^{sim}(\mathcal{I}_{\mathcal{T},C_1} \times \mathcal{I}_{\mathcal{T},C_2}, (d_{C_1}, d_{C_2})) \sqsubseteq D.$$

To this end, it is sufficient to show

$$(\mathcal{I}_{\mathcal{T},D}, d_D) \leq (\mathcal{I}_{\mathcal{T},C_1} \times \mathcal{I}_{\mathcal{T},C_2}, (d_{C_1}, d_{C_2})).$$

As $\mathcal{T} \models C_i \sqsubseteq D$ for $i = 1, 2$, we have

$$(\mathcal{I}_{\mathcal{T},D}, d_D) \leq (\mathcal{I}_{\mathcal{T},C_i}, d_{C_i}).$$

The second part of Observation 3 yields

$$(\mathcal{I}_{\mathcal{T},D}, d_D) \leq (\mathcal{I}_{\mathcal{T},C_1} \times \mathcal{I}_{\mathcal{T},C_2}, (d_{C_1}, d_{C_2})),$$

as required.

Finally, we show the interpolation property for $\mathcal{EL}^{\nu}$. We formulate the result to be proved again.

Theorem 20 Let $\mathcal{T}_1 \cup \mathcal{T}_2 \models C_0 \sqsubseteq D_0$ and assume that $\mathcal{T}_1$ and $\mathcal{T}_2$ are Γ-inseparable w.r.t. $\mathcal{EL}^{\nu}$ for $\Gamma = \mathsf{sig}(\mathcal{T}_1, C_0) \cap \mathsf{sig}(\mathcal{T}_2, D_0)$. Then $\mathcal{T}_2 \models \exists^{sim}\Sigma.(\mathcal{T}_1, C_0) \sqsubseteq D_)$, for $\Sigma = \mathsf{sig}(\mathcal{T}_1, C_0) \setminus \Gamma$.

Proof Assume that $\mathcal{T}_2 \not\models \exists^{sim}\Sigma.(\mathcal{T}_1, C_0) \sqsubseteq D_0$, for $\Sigma = \mathsf{sig}(\mathcal{T}_1, C_0) \setminus \Delta$. We show that $\mathcal{T}_1 \cup \mathcal{T}_2 \not\models C_0 \sqsubseteq D_0$.

Take the canonical model $(\mathcal{I}_{\mathcal{T}_1,C_0}, d_{C_0})$ defined above. We set $\mathcal{I}_0 = \mathcal{I}_{\mathcal{T}_1,C_0}$, $d_0 = d_{C_0}$, and $\Delta_0 = \Delta_{d_0} = \Delta^{\mathcal{I}_0}$. In the following, we construct an interpretation $\mathcal{I}^*$ of $\mathcal{T}_1 \cup \mathcal{T}_2$ refuting $C_0 \sqsubseteq D_0$. We define inductively an infinite sequence $\mathcal{I}_1, \mathcal{I}_2, \ldots$ of interpretations. The interpretation $\mathcal{I}^* = (\Delta^{\mathcal{I}^*}, \cdot^{\mathcal{I}^*})$ is then defined as the union of $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2, \ldots$ as follows:

$$\begin{aligned} \Delta^{\mathcal{I}^*} &:= \bigcup_{i \geq 0} \Delta^{\mathcal{I}_i}; \\ A^{\mathcal{I}^*} &:= \bigcup_{i \geq 0} A^{\mathcal{I}_i}, \text{ for all } A \in \mathsf{N_C}; \\ r^{\mathcal{I}^*} &:= \bigcup_{i \geq 0} r^{\mathcal{I}_i}, \text{ for all } r \in \mathsf{N_R}. \end{aligned}$$

Given an intepretation $\mathcal{I}$, we denote by $\mathcal{I}\restriction\Gamma$ the reduct of $\mathcal{I}$ interpreting the symbols in Γ only. For $d \in \Delta^{\mathcal{I}}$ and any TBox $\mathcal{T}$, we denote by $\mathcal{I}_{t_{\mathcal{I}}(d),\mathcal{T}}$ the canonical model $\mathcal{I}_{\mathcal{T},\exists^{sim}(\mathcal{I}\restriction\Gamma,d)}$ of the pair $\mathcal{T}, \exists^{sim}(\mathcal{I}\restriction\Gamma, d)$.

Let $n \geq 0$ and assume the interpretation $\mathcal{I}_n$ with domain Δ_n has been defined. If n is even, then take for every $d \in \Delta_n \setminus \Delta_{n-1}$ (we set $\Delta_{-1} = \emptyset$) the interpretation $\mathcal{I}_d = \mathcal{I}_{t_{\mathcal{I}_n}(d),\mathcal{T}_2}$ with domain Δ_d such that $\Delta_n \cap \Delta_d = \{d\}$ and the Δ_d, $d \in \Delta_n \setminus \Delta_{n-1}$, are mutually disjoint. If n is odd, then take for every $d \in \Delta_n \setminus \Delta_{n-1}$ the interpretation $\mathcal{I}_d = \mathcal{I}_{t_{\mathcal{I}_n}(d),\mathcal{T}_1}$ with domain Δ_d such that $\Delta_n \cap \Delta_d = \{d\}$ and the Δ_d, $d \in \Delta_n \setminus \Delta_{n-1}$, are mutually disjoint. Now set

$$\begin{aligned} \Delta_{n+1} &= \Delta_n \cup \textstyle\bigcup_{d \in \Delta_n \setminus \Delta_{n-1}} \Delta_d, \\ r^{\mathcal{I}_{n+1}} &= r^{\mathcal{I}_n} \cup \textstyle\bigcup_{d \in \Delta_n \setminus \Delta_{n-1}} r^{\mathcal{I}_d}, \\ A^{\mathcal{I}_{n+1}} &= A^{\mathcal{I}_n} \cup \textstyle\bigcup_{d \in \Delta_n \setminus \Delta_{n-1}} A^{\mathcal{I}_d}. \end{aligned}$$

For all $d \in \Delta^{\mathcal{I}^*}$ there exists a (uniquely) determined minimal natural number $n(d)$ with $d \in \Delta_{n(d)} \setminus \Delta_{n(d)-1}$. If $n(d) \neq 0$, then there exists a uniquely determined $d^* \in \Delta_{n(d)-1}$ with $d \in \Delta_{d^*}$. We set $d^* = d_0$ for $n(d) = 0$ and prove the following by induction on the construction of D. For all $d \in \Delta^{\mathcal{I}^*}$ and $\mathcal{EL}$-concepts D:

- if $n(d)$ is even then
  1. if $\mathsf{sig}(D) \cap \mathsf{sig}(\mathcal{T}_1, C_0) \subseteq \Gamma$, then $d \in D^{\mathcal{I}^*} \Leftrightarrow d \in D^{\mathcal{I}_d}$;
  2. if $\mathsf{sig}(D) \cap \mathsf{sig}(\mathcal{T}_2, D_0) \subseteq \Gamma$, then $d \in D^{\mathcal{I}^*} \Leftrightarrow d \in D^{\mathcal{I}_{d^*}}$;
- if $n(d)$ is odd then
  1. if $\mathsf{sig}(D) \cap \mathsf{sig}(\mathcal{T}_2, D_0) \subseteq \Gamma$, then $d \in D^{\mathcal{I}^*} \Leftrightarrow d \in D^{\mathcal{I}_d}$;
  2. if $\mathsf{sig}(D) \cap \mathsf{sig}(\mathcal{T}_1, C_0) \subseteq \Gamma$, then $d \in D^{\mathcal{I}^*} \Leftrightarrow d \in D^{\mathcal{I}_{d^*}}$.

The implications from right to left are trivial, so we consider the implications from left to right only. We concentrate on the case $n(d)$ even (the case $n(d)$ odd is proved in the same way) and prove the induction step for $D = \exists r.C$. First consider Point 1. So let $\mathsf{sig}(D) \cap \mathsf{sig}(\mathcal{T}_1, C_0) \subseteq \Gamma$ and assume $d \in D^{\mathcal{I}^*}$ with $n(d)$ even. There exists $c \in \Delta^{\mathcal{I}^*}$ such that $c \in C^{\mathcal{I}^*}$ and $(d, c) \in r^{\mathcal{I}^*}$. Assume first that $c \in \Delta_{n(d)}$. Then, by construction, $c \notin \Delta_{n(d)-1}$. Then $r \in \Gamma$ because for any $r \notin \mathsf{sig}(\mathcal{T}_1, C_0)$, $r^{\mathcal{I}^*} \cap (\Delta_{n(d)} \setminus \Delta_{n(d)-1})^2 = \emptyset$. We obtain

$n(c) = n(d)$ and, by IH, $c \in C^{\mathcal{I}_c}$. We obtain, by the construction of $\mathcal{I}_c$,

$$\mathcal{T}_2 \models \exists^{sim}(\mathcal{I}_{n(d)}\restriction\Gamma, c) \sqsubseteq C$$

But then

$$\mathcal{T}_2 \models \exists r.\exists^{sim}(\mathcal{I}_{n(d)}\restriction\Gamma, c) \sqsubseteq \exists r.C$$

and so from the validity of

$$\exists^{sim}(\mathcal{I}_{n(d)}\restriction\Gamma, d) \sqsubseteq \exists r.\exists^{sim}(\mathcal{I}_{n(d)}\restriction\Gamma, c)$$

we obtain

$$\mathcal{T}_2 \models \exists^{sim}(\mathcal{I}_{n(d)}\restriction\Gamma, d) \sqsubseteq D$$

which implies $d \in D^{\mathcal{I}_d}$, as required.

Now assume $c \notin \Delta_{n(d)}$. Then $c \in \Delta_d$, $c^* = d$, and $n(c) = n(d) + 1$. By IH (for $n(c)$ odd), $c \in C^{\mathcal{I}^*}$ iff $c \in C^{\mathcal{I}_{c^*}} = C^{\mathcal{I}_d}$. Hence $d \in (\exists r.C)^{\mathcal{I}_d}$.

Consider now Point 2. Let $\mathsf{sig}(D) \cap \mathsf{sig}(\mathcal{T}_2, D_0) \subseteq \Gamma$ and $d \in D^{\mathcal{I}^*}$. There exists $c \in \Delta^{\mathcal{I}^*}$ such that $c \in C^{\mathcal{I}^*}$ and $(d, c) \in r^{\mathcal{I}^*}$. Assume first that $c \in \Delta_{d^*}$. Then $c^* = d^*$ and, by IH, $c \in C^{\mathcal{I}_{d^*}}$. As we also have $(d, c) \in r^{\mathcal{I}_{d^*}}$, we obtain $d \in D^{\mathcal{I}_{d^*}}$.

Now assume $c \notin \Delta_{d^*}$. Then $c \in \Delta_d$. Then $r \in \Gamma$ because for any $r \notin \mathsf{sig}(\mathcal{T}_2, D_0)$, $r^{\mathcal{I}^*} \cap \Delta_d \times \Delta_d = \emptyset$. By IH, $c \in C^{\mathcal{I}_c}$. Hence

$$\mathcal{T}_1 \models \exists^{sim}(\mathcal{I}_{n(d)+1}\restriction\Gamma, c) \sqsubseteq C.$$

Then

$$\mathcal{T}_1 \models \exists r.\exists^{sim}(\mathcal{I}_{n(d)+1}\restriction\Gamma, c) \sqsubseteq \exists r.C.$$

We have $d \in (\exists r.\exists^{sim}(\mathcal{I}_{n(d)+1} \restriction \Gamma, c))^{\mathcal{I}_d}$ and by Γ-inseparability of $\mathcal{T}_1$ and $\mathcal{T}_2$, $d \in (\exists r.\exists^{sim}(\mathcal{I}_{n(d)+1} \restriction \Gamma, c))^{\mathcal{I}_{n(d)}}$. So, $d \in (\exists r.\exists^{sim}(\mathcal{I}_{n(d)+1} \restriction \Gamma, c))^{\mathcal{I}_{d^*}}$. $\mathcal{I}_{d^*}$ is a model of $\mathcal{T}_1$. Hence $d \in (\exists r.C)^{\mathcal{I}_{d^*}}$, as required.

Since $\mathcal{I}^*$ has finite outdegree, it follows that the claim above holds for all $\mathcal{EL}^\nu$-concepts D (not just all $\mathcal{EL}$-concepts).

It follows immediately that $\mathcal{I}^*$ is a model of $\mathcal{T}_1 \cup \mathcal{T}_2$: let $C \sqsubseteq D \in \mathcal{T}_i$. If $C_0^{\mathcal{I}^*} \setminus D_0^{\mathcal{I}^*} \neq \emptyset$, then there exists a an interpretation $\mathcal{I}_d$ of $\mathcal{T}_i$ with $C_0^{\mathcal{I}_d} \setminus D_0^{\mathcal{I}_d} \neq \emptyset$ which contradicts the claim above.

It remains to show that $d_0 \in C_0^{\mathcal{I}^*} \setminus D_0^{\mathcal{I}^*}$. $d_0 \in C_0^{\mathcal{I}^*}$ is clear by definition. Now assume $d_0 \in D_0^{\mathcal{I}^*}$. By the claim above (for $\mathcal{EL}^\nu$-concepts), we obtain $d_0 \in D_0^{\mathcal{I}_d}$. By construction of $\mathcal{I}_{d_0}$, this implies

$$\mathcal{T}_2 \models \exists^{sim}(\mathcal{I}_{\mathcal{T}_1, C_0}\restriction\Gamma, d_0) \sqsubseteq D_0.$$

But the latter statement is equivalent to $\mathcal{T}_2 \models \exists^{sim}\Sigma.(\mathcal{T}_1, C_0) \sqsubseteq D_0$ and we have derived a contradiction.