A Closer Look at the Probabilistic Description Logic Prob-$\mathcal{EL}$

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Abstract

We study probabilistic variants of the description logic $\mathcal{EL}$. For the case where probabilities apply only to concepts, we provide a careful analysis of the borderline between tractability and EXPTIME-completeness. One outcome is that *any* probability value except zero and one leads to intractability in the presence of general TBoxes, while this is not the case for classical TBoxes. For the case where probabilities can also be applied to roles, we show PSPACE-completeness. This result is (positively) surprising as the best previously known upper bound was 2-EXPTIME and there were reasons to believe in completeness for this class.

Introduction

Classical description logics (DLs) are fragments of first-order logic (FOL) and thus do not provide any built-in means for representing uncertainty. This shortcoming has been addressed in a number of proposals for probabilistic DLs, see for example (Lukasiewicz and Straccia 2008; Jaeger 1994; da Costa and Laskey 2006; Lukasiewicz 2008) and references therein. Recently, a new family of probabilistic DLs was introduced in (Lutz and Schröder 2010), with the distinguishing feature that its members relate to the well-established probabilistic FOL of (Halpern 2003; Bacchus 1990) in the same way as classical DLs relate to traditional FOL. The main purpose of DLs from the new family, from now on called *Prob-DLs*, is to enable concept definitions that require reference to (degrees of) possibility, likelihood, and certainty. To this effect, Prob-DLs provide a probabilistic constructor $P_{\sim p}$ with $\sim \in \{<, \leq, =, \geq, >\}$ and $p \in [0, 1]$ that can be applied to concepts and sometimes also to roles. For example,

$$\mathsf{Patient} \sqcap \exists \mathsf{finding}.(\mathsf{Disease} \sqcap P_{>0.25}\mathsf{Infectious})$$

describes Patients having a disease that is infectious with probability at least .25.

As argued in (Lutz and Schröder 2010), Prob-DLs are well-suited to capture aspects of uncertainty that are present in almost all biomedical ontologies such as SNOMED CT. Such ontologies, which typically reach considerable size but still require efficient reasoning, are often formulated in lightweight DLs of the $\mathcal{EL}$ family for which the central reasoning problem of *subsumption* can be solved in polynomial time (Baader, Brandt, and Lutz 2005; Schulz, Suntisrivaraporn, and Baader 2007). Consequently, studying probabilistic extensions of $\mathcal{EL}$ in the style of the Prob-DL family is particularly relevant in this context. Some initial results in this direction have already been obtained in (Lutz and Schröder 2010).

The purpose of this paper is to establish a more complete picture of subsumption in probabilistic variants of $\mathcal{EL}$. In the first part of the paper, we consider *Prob-*$\mathcal{EL}$ in which probabilities can only be applied to concepts, but not to roles. It was known that some concrete combinations of probability constructors such as $P_{>0}$ and $P_{>0.4}$ lead to intractability (in fact, EXPTIME-completeness) of subsumption while a restriction to the probability values zero and one does not. We prove the much more general result that the extension of $\mathcal{EL}$ with *any* single concept constructor $P_{\sim p}$, where $\sim \in \{<, \leq, =, \geq, >\}$ and $p \in (0, 1)$, results in EXPTIME-completeness. More specifically, this result applies to *general TBoxes*, i.e., to sets of concept inclusions $C \sqsubseteq D$ when $\sim \in \{=, \geq, >\}$ and even to the empty TBox when $\sim \in \{<, \leq, \}$. Inspired by the observation that many biomedical ontologies such as SNOMED CT are *classical TBoxes*, i.e., sets of concept definitions $A \equiv D$ with A atomic, we then show that probabilities other than zero and one *can* be used without losing tractability in classical TBoxes for the cases $\sim \in \{>, \geq\}$. More precisely, subsumption in Prob-$\mathcal{EL}$ is tractable when only the constructors $P_{\sim p}$ and $P_{=1}$ are admitted, for any (single!) choice of $\sim \in \{\geq, >\}$ and $p \in (0, 1)$. The resulting logics actually coincide for all possible choices. We also show that when a second probability value from the range $(0, 1)$ sufficiently 'far away' from p is added, the complexity of subsumption snaps back to EXPTIME-completeness.

In the second part, we consider *Prob-*$\mathcal{EL}_r$, where probabilities can be applied to both concepts and roles, concentrating on general TBoxes. While decidability is an open problem for full Prob-$\mathcal{EL}_r$, it was known that subsumption is in 2-EXPTIME and PSPACE-hard in *Prob-*$\mathcal{EL}_r^{>0;=1}$, where probability values are restricted to zero and one. It is interesting to note that Prob-DLs are a special kind of two-dimensional DLs as studied for example in (Gabbay et al. 2003) and that, until now, any two-dimensional extension of $\mathcal{EL}$ turned out to have the same complexity as the corresponding extension of the expressive DL $\mathcal{ALC}$, see e.g. (Artale et al. 2007). Since subsumption in the $\mathcal{ALC}$-variant of Prob-$\mathcal{EL}_r^{>0;=1}$ is 2-EXPTIME-complete, it was thus tempting to conjecture that the same holds for Prob-$\mathcal{EL}_r^{>0;=1}$. We show that this is not the case by establishing a tight PSPACE upper bound for subsumption in Prob-$\mathcal{EL}_r^{>0;=1}$. This also implies PSPACE-completeness for the two-dimensional DL $\mathsf{S5}_{\mathcal{EL}}$, in sharp contrast with the 2-EXPTIME-completeness of $\mathsf{S5}_{\mathcal{ALC}}$.

Most proofs are deferred to the appendix of the long version, which is available from http://www.informatik.uni-bremen.de/∼clu/papers/index.html.

Preliminaries

Description logic concepts are built from a set of concept names $\mathsf{N_C}$ and a set of role names $\mathsf{N_R}$ (both countably infinite), using the available concept constructors. In the basic description logic $\mathcal{EL}$, these constructors are conjunction and existential restriction, which gives rise to the syntax rule

$$C, D ::= \top \mid A \mid C \sqcap D \mid \exists r.C$$

where $\top$ denotes the 'top-concept' (logical truth), A ranges over $\mathsf{N_C}$ and r over $\mathsf{N_R}$. To obtain a probabilistic version of $\mathcal{EL}$, we can apply probabilities to concepts or roles. Starting with the former, we consider the set of constructors

$$P_{\sim p}C \text{ with } \sim \in \{<, \leq, =, \geq, >\} \text{ and } p \in [0, 1],$$

denoting objects that are an instance of C with probability $\sim p$. The extension of $\mathcal{EL}$ with all these constructors is called *Prob-$\mathcal{EL}$*. For example, the SNOMED CT concept 'animal bite by potentially rabid animal' can be expressed as

$$\mathsf{Bite} \sqcap \exists \mathsf{by}.(\mathsf{Animal} \sqcap P_{>0.5}\exists \mathsf{has}.\mathsf{Rabies}).$$

When we admit only a few values for $\sim$ and n, we put them in superscript; for example, Prob-$\mathcal{EL}^{>0.4,<0,1}$ denotes the extension of $\mathcal{EL}$ with $P_{>0.4}C$ and $P_{<0.1}C$.

Probabilities can be applied to roles using the concept constructors $\exists P_{\sim p}r.C$ where $\sim$ and p range over the same values as above, expressing that there is an element satisfying C that is related to the current element by the role name r with probability $\sim p$. For example, the SNOMED CT concept 'disease of possible viral origin' can be modeled as

$$\mathsf{Disease} \sqcap \exists P_{>0}\mathsf{origin}.\mathsf{Viral}.$$

We denote the extension of Prob-$\mathcal{EL}$ with all the above (concept and role) constructors with *Prob-$\mathcal{EL}_r$*. We will also consider the restriction of Prob-$\mathcal{EL}_r$ to the constructors $P_{>0}$ and $P_{=1}$ both on concepts and roles, which is called *Prob-$\mathcal{EL}_r^{>0;=1}$*.

The semantics of classical DLs such as $\mathcal{EL}$ is based on interpretations $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$, where $\Delta^\mathcal{I}$ is a non-empty set called the *domain* and $\cdot^\mathcal{I}$ is an *interpretation function* that maps each $A \in \mathsf{N_C}$ to a subset $A^\mathcal{I} \subseteq \Delta^\mathcal{I}$ and each $r \in \mathsf{N_R}$ to a subset $r^\mathcal{I} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}$, see (Baader et al. 2003) for more details. The semantics of the probabilistic DLs considered here is given in terms of a *probabilistic interpretation* $\mathcal{I} = (\Delta^\mathcal{I}, W, (\mathcal{I}_w)_{w \in W}, \mu)$, where $\Delta^\mathcal{I}$ is the (non-empty) domain, W a non-empty set of *possible worlds*, μ a discrete probability distribution on W, and for each $w \in W$, $\mathcal{I}_w$ is a classical DL interpretation with domain $\Delta^\mathcal{I}$. We usually write $C^{\mathcal{I},w}$ for $C^{\mathcal{I}_w}$, and likewise for $r^{\mathcal{I},w}$. For concept names A and role names r, we define the probability

- $p_d^\mathcal{I}(A)$ that $d \in \Delta^\mathcal{I}$ is an A as $\mu(\{w \in W \mid d \in A^{\mathcal{I},w}\})$;
- $p_{d,e}^\mathcal{I}(r)$ that $d, e \in \Delta^\mathcal{I}$ are related by r as $\mu(\{w \in W \mid (d, e) \in r^{\mathcal{I},w}\})$.

Next, we extend $p_d^\mathcal{I}(A)$ to compound concepts C and define the extension $C^{\mathcal{I},w}$ of compound concepts by mutual recursion on C. The definition of $p_d^\mathcal{I}(C)$ is exactly as in the base case, with A replaced by C. The extension of compound concepts is defined as follows:

$$\begin{aligned}
\top^{\mathcal{I},w} &= \Delta^\mathcal{I} \\
(C \sqcap D)^{\mathcal{I},w} &= C^{\mathcal{I},w} \cap D^{\mathcal{I},w} \\
(\exists r.C)^{\mathcal{I},w} &= \{d \in \Delta^\mathcal{I} \mid \exists e \in C^{\mathcal{I},w}.(d, e) \in r^{\mathcal{I},w}\} \\
(P_{\sim p}C)^{\mathcal{I},w} &= \{d \in \Delta^\mathcal{I} \mid p_d^\mathcal{I}(C) \sim p\} \\
(\exists P_{\sim p}r.C)^{\mathcal{I},w} &= \{d \in \Delta^\mathcal{I} \mid \exists e \in C^{\mathcal{I},w} : p_{d,e}^\mathcal{I}(r) \sim p\}
\end{aligned}$$

In DLs, an ontology is formalized as a TBox. In this paper, we consider two kinds of TBoxes. A *general TBox* is a finite set of *concept inclusions* $C \sqsubseteq D$, where C, D are concepts. A *classical TBox* is a set of *concept definitions* $A \equiv C$, where A is a concept name and the left-hand sides of concept definitions are unique. Note that cyclic definitions are allowed.

A probabilistic interpretation $\mathcal{I}$ *satisfies* a concept inclusion $C \sqsubseteq D$ if $C^{\mathcal{I},w} \subseteq D^{\mathcal{I},w}$ and a concept definition $A \equiv C$ if $A^{\mathcal{I},w} = C^{\mathcal{I},w}$, for all $w \in W$. $\mathcal{I}$ is a *model* of a TBox $\mathcal{T}$ if it satisfies all inclusions/definitions in $\mathcal{T}$. A concept C is *subsumed by a concept D relative to a TBox $\mathcal{T}$* (written $\mathcal{T} \models C \sqsubseteq D$) if every model $\mathcal{I}$ of $\mathcal{T}$ satisfies the inclusion $C \sqsubseteq D$. Deciding subsumption is the most important reasoning task for DLs as it underlies the computation of the concept hierarchy, a central tool for structuring and accessing ontologies (Baader et al. 2003).

The above definition is the result of transferring the notion of subsumption from standard DLs to probabilistic DLs in a straightforward way. However, there is an alternative variant of subsumption that is natural for probabilistic DLs: a concept C is *positively subsumed by a concept D relative to a TBox $\mathcal{T}$* (written $\mathcal{T} \models^+ C \sqsubseteq D$) if $C^{\mathcal{I},w} \subseteq D^{\mathcal{I},w}$ for every probabilistic model $\mathcal{I} = (\Delta^\mathcal{I}, W, (\mathcal{I}_w)_{w \in W}, \mu)$ and every $w \in W$ with $\mu(w) > 0$. Intuitively, classical subsumption is about subsumptions that are *logically implied* whereas positive subsumption is about subsumptions that are *certain*. For example, when $\mathcal{T}_\emptyset$ is the empty TBox, then $\mathcal{T}_\emptyset \not\models P_{=1}A \sqsubseteq A$, but we can only have $d \in (P_{=1}A)^{\mathcal{I},v} \setminus A^{\mathcal{I},v}$ when $\mu(v) = 0$, thus non-subsumption is only witnessed by worlds that we are certain to not be the actual world. Consequently, $\mathcal{T}_\emptyset \models^+ P_{=1}A \sqsubseteq A$. In the extension Prob-$\mathcal{ALC}$ of Prob-$\mathcal{EL}$ with negation studied in (Lutz and Schröder 2010), positive subsumption can easily be reduced to subsumption. This does not seem easily possible in Prob-$\mathcal{EL}$. In fact, we will sometimes use (Turing) reductions in the opposite direction.

Probabilistic Concepts

In (Lutz and Schröder 2010), it was shown that subsumption in Prob-$\mathcal{EL}^{>0;=1}$ with general TBoxes is in PTIME, whereas the same problem is EXPTIME-complete in Prob-$\mathcal{EL}^{>0;>0.4}$ (both in the positive and in the unrestricted case). This raises the question whether *any* probability except 0,1 can be admitted in Prob-$\mathcal{EL}$ without losing tractability. The following theorem provides a strong negative result.

Theorem 1. *For all* $p \in (0,1)$*, (positive) subsumption in Prob-*$\mathcal{EL}^{\sim p}$ *relative to*

1. *general TBoxes is* EXPTIME*-hard when* $\sim \in \{=,>,\geq\}$
2. *the empty TBox is* EXPTIME*-hard when* $\sim \in \{\leq,<\}$

Matching upper bounds are an immediate consequence of the fact that each logic Prob-$\mathcal{EL}^{\sim p}$ is a fragment of the DL Prob-$\mathcal{ALC}_c$ for which subsumption was proved EXPTIME-complete in (Lutz and Schröder 2010). To prove the lower bounds, it suffices to show that each logic Prob-$\mathcal{EL}^{\sim p}$ is *non-convex* i.e., that there are a general TBox $\mathcal{T}$ and concepts $C, D_1, \ldots, D_n$, $n \geq 2$, such that $\mathcal{T} \models C \sqsubseteq D_1 \sqcup \cdots \sqcup D_n$, but $\mathcal{T} \not\models C \sqsubseteq D_i$ for all i (the semantics of disjunction is defined in the obvious way). Once that this is established, standard proof techniques from (Baader, Brandt, and Lutz 2005) can be used to reduce satisfiability in $\mathcal{ALC}$ relative to general TBoxes, which is EXPTIME-complete, to subsumption in Prob-$\mathcal{EL}^{\sim p}$. The following constructions work for standard subsumption and positive subsumption alike.

First consider $\sim = \geq$ and assume $p \leq 0.5$. Fix a $k > 0$ such that $k \cdot p > 1$ and set

$$\begin{aligned} \mathcal{T} &= \{A_i \sqcap A_j \sqsubseteq P_{\geq p} B_{ij} \mid 1 \leq i < j \leq k\} \\ C &= P_{\geq p} A_1 \sqcap \cdots \sqcap P_{\geq p} A_k \\ D_{ij} &= P_{\geq p} B_{ij} \end{aligned}$$

Intuitively, the probabilities stipulated by C sum up to > 1, thus some of the A_i have to overlap, but there is a choice as to which ones these are. Formally, we can show non-convexity by proving that $\mathcal{T} \models C \sqsubseteq \bigsqcup_{1 \leq i < j \leq k} D_{ij}$, but $\mathcal{T} \not\models C \sqsubseteq D_{ij}$ for any i, j. The comparisons $\sim \in \{=,>\}$ can be handled similarly. For $\sim = >$ and $p > 0.5$, we use a variation of the above. The main idea is to use $P_{>p}C$ to simulate $P_{>q}C$, for some $q \leq 0.5$, which brings us back to a case already dealt with. More precisely, let $n > 0$ be smallest such that $n > \frac{1}{2(1-p)}$ and set $q = pn - n + 1$. An easy computation shows that $0 \leq q < 0.5$. Moreover, it can be shown that

$$P_{>p} X_1 \sqcap \cdots \sqcap P_{>p} X_n \sqsubseteq P_{>q}(X_1 \sqcap \cdots \sqcap X_n)$$

which allows us to redo the above reduction with probability $q < 0.5$. The comparisons $\sim \in \{=,\geq\}$ can be dealt with similarly.

For the remaining cases $\sim \in \{<,\leq\}$, there is a very simple argument for non-convexity even w.r.t. the empty TBox: we have $\top \sqsubseteq P_{<p}A \sqcup P_{<p}P_{<p}A$, but neither $\top \sqsubseteq P_{<p}A$ nor $\top \sqsubseteq P_{<p}P_{<p}A$, and likewise when $\sim$ is $\leq$.

When $\sim \in \{=,>,\geq\}$, the proof of Theorem 1 relies on general TBoxes in a crucial way. It turns out that when we restrict ourselves to classical TBoxes, tractability can be attained even with probabilities other than 0 and 1.

R1 If $\exists r.B \in C_A$, and $C_{B'} \subseteq C_B$
then replace $A \equiv C_A$ with $A \equiv C_A \cup \{\exists r.B'\}$

R2 If $P_{=1}B \in C_A$
then replace $A \equiv C_A$ with $A \equiv C_A \cup C_B$

R3 If $P_{=1}B \in C_A$
then replace $A \equiv C_A$ with $A \equiv C_A \cup \{P_{\sim p}B\}$

R4 If $P_{\sim p}B \in C_A$, and $D \in \mathsf{cert}(C_B)$
then replace $A \equiv C_A$ with $A \equiv C_A \cup \{D\}$

R5 If $C_B \subseteq \mathsf{cert}(C_A)$
then replace $A \equiv C_A$ with $A \equiv C_A \cup \{P_{=1}B\}$

R6 If $P_{\sim p}B \in C_A$ and $C_{B'} \subseteq \mathsf{cert}(C_A) \cup C_B$
then replace $A \equiv C_A$ with $A \equiv C_A \cup \{P_{\sim p}B'\}$

Figure 1: TBox completion rules for positive subsumption

Theorem 2. *For all* $\sim \in \{>,\geq\}$ *and* $p \in [0,1]$*, (positive) subsumption in Prob-*$\mathcal{EL}^{\sim p;=1}$ *relative to classical TBoxes is in* PTIME.

To prove Theorem 2, we start with positive subsumption. We can assume $p > 0$ since subsumption in Prob-$\mathcal{EL}^{>0;=1}$ is in PTIME even with general TBoxes. To prove a PTIME upper bound, we use a 'consequence-driven' procedure similar to the ones in (Baader, Brandt, and Lutz 2005; Kazakov 2009). A concept name A is *defined* in a classical TBox $\mathcal{T}$ if there is a concept definition $A \equiv C \in \mathcal{T}$, and *primitive* otherwise. We can w.l.o.g. restrict our attention to the subsumption of *defined concept names* relative to TBoxes. We also assume that the input TBox is normalized to a set of concept definitions of the form

$$A \equiv P_1 \sqcap \cdots \sqcap P_n \sqcap C_1 \sqcap \cdots \sqcap C_m$$

$n, m \geq 0$, and where $P_1, \ldots, P_n$ are primitive concept names and $C_1, \ldots, C_m$ are of the form $P_{\sim p}A$, $P_{=1}A$, and $\exists r.A$ with A a defined concept name (note that the top concept is completely normalized away). It is well-known that such a normalization can be achieved in polytime, see (Baader 2003) for details. For a given TBox $\mathcal{T}$ and a defined concept name A in $\mathcal{T}$, we write C_A to denote the *defining concept* for A in $\mathcal{T}$, i.e., $A \equiv C_A \in \mathcal{T}$. Moreover, we deliberately confuse the concept $C_A = D_1 \sqcap \cdots \sqcap D_k$ with the set $\{D_1, \ldots, D_k\}$. We define a set of concepts 'certain for C_A' as

$$\mathsf{cert}(C_A) = \{P_* B \mid P_* B \in C_A\} \cup \bigcup_{P_{=1}B \in C_A} \{C_B\}$$

where, here and in what follows, P_* ranges over $P_{=1}$ and $P_{>p}$. Intuitively $\mathsf{cert}(C_A)$ contains concepts that hold with probability 1 whenever A is satisfied in some world. The algorithm starts with the normalized input TBox and then exhaustively applies the completion rules displayed in Figure 1. As a general proviso, each rule can be applied only if it adds a concept that occurs in $\mathcal{T}$ and actually changes the TBox, e.g., **R1** can only be applied when $\exists r.B'$ occurs in $\mathcal{T}$ and $\exists r.B' \notin C_A$. Exemplarily, we explain rule **R5** in more detail. If all defining concepts C_B of B are certain for A, then $A \sqsubseteq P_{=1}B$, thus we can add $P_{=1}B$ to C_A. Let $\mathcal{T}^*$ be the result of exhaustive rule application and let C^*_A be the defining concept for A in $\mathcal{T}^*$, for all concept names A. The 'only if' direction requires a careful and surprisingly subtle model construction.

Lemma 3. *For all defined concept names A, B, we have $\mathcal{T} \models^+ A \sqsubseteq B$ iff $C^*_B \subseteq C^*_A$.*

It is easy to see that TBox completion requires only polytime: every rule application extends the TBox, but both the number of concept definitions and of conjuncts in each concept definition is bounded by the size of the original TBox.

To prove Theorem 2 for unrestricted subsumption, we provide a Turing reduction from unrestricted subsumption to positive subsumption. We again assume that the input TBox is in the described normal form and then exhaustively apply the rules shown in Figure 2, calling the result $\mathcal{T}^*$ with defining concept of the form C^*_A.

Lemma 4. *For all defined concept names A, B, we have $\mathcal{T} \models A \sqsubseteq B$ iff $C^*_B \subseteq C^*_A$.*

Clearly, the Turing reduction and thus the overall algorithm runs in polytime.

It is interesting to note that the proof of Theorem 2 is based on exactly the same algorithm, for all $\sim \in \{\geq, >\}$ and $p \in (0, 1]$. It follows that there is in fact only *a single logic* Prob-$\mathcal{EL}^{\sim p}$, for all such $\sim$ and p. Formally, given a Prob-$\mathcal{EL}^{\sim p}$-concept C, $\approx \in \{\geq, >\}$ and $q \in (0, 1]$, let $C_{\approx q}$ denote the result of replacing each subconcept $P_{\sim p}D$ in C with $P_{\approx q}D$ in C and similarly for Prob-$\mathcal{EL}^{\sim p}$-TBoxes $\mathcal{T}$.

Theorem 5. *For any $p, q > 0$, $\sim, \approx \in \{>, \geq\}$, $\mathcal{EL}^{\sim p}$-concepts C, D and -TBox $\mathcal{T}$, we have $\mathcal{T} \models^+ C \sqsubseteq D$ iff $\mathcal{T}_{\approx q} \models^+ C_{\approx q} \sqsubseteq D_{\approx q}$, and likewise for unrestricted subsumption.*

Consequently, the (potentially difficult!) choice of a concrete $\sim \in \{\geq, >\}$ and $p \in (0, 1]$ is moot. In fact, it might be more intuitive to replace the constructor $P_{\sim p}C$ with a constructor $\mathcal{L}\,C$ that describes elements which 'are likely to be a C', and to replace $P_{=1}C$ with the constructor $\mathcal{C}\,C$ to describe elements that 'are certain to be a C', see e.g. (Halpern and Rabin 1987; Herzig 2003) for other approaches to logics of likelihood. Note that the case $p = 0$ is different from the cases considered above: for example, we have $\mathcal{T}_\emptyset \models^+ \exists r.A \sqsubseteq \exists r.P_{>p}A$ iff $p = 0$, and likewise $\mathcal{T}_\emptyset \models P_{>p}\exists r.A \sqsubseteq P_{>p}\exists r.P_{>p}A$ iff $p = 0$. In the spirit of the constructors $\mathcal{C}$ and $\mathcal{L}$, $P_{>0}C$ can be replaced with a constructor $\mathcal{P}C$ that describes elements for which 'it is possible that they are a C'. For example, the SNOMED CT concepts 'definite thrombus' and 'possible thrombus' can then be written as $\mathcal{C}\,\mathsf{Thrombus}$ and $\mathcal{P}\,\mathsf{Thrombus}$ (although we speculate that the SNOMED CT designers mean 'likely' rather than 'possible').

It is a natural question whether the PTIME upper bound for classical TBoxes extends to the case of multiple probability values (except one, which is apparently always uncritical). The following result shows that many combinations of two probability values lead to (non-convexity, thus) intractability, even without any TBox.

S1	If $\exists r.B \in C_A$, and $C_{B'} \subseteq C_B$ then replace $A \equiv C_A$ with $A \equiv C_A \cup \{\exists r.B'\}$
S2	If $\mathcal{T} \models^+ \mathsf{cert}(C_A) \sqsubseteq P_*B$ then replace $A \equiv C_A$ with $A \equiv C_A \cup \{P_*B\}$

Figure 2: TBox completion rules for Turing reduction

Theorem 6. *Let $\sim \in \{>, \geq\}$, and $p, q \in [0, 1)$. Then (positive) subsumption in Prob-$\mathcal{EL}^{\sim p; \sim q}$ relative to the empty TBox is* EXPTIME*-hard if (i) $q = 0$, (ii) $p \leq 1/2$ and $q < p^2$, or more generally (iii) $2p - 1 < q < p^2$.*

In particular, we cannot combine the constructors $\mathcal{P}$ and $\mathcal{L}$ mentioned above without losing tractability. The above formulation of Theorem 6 is actually only a consequence of a more general (but also more complicated to state) result established in the appendix of the long version. We conjecture that (positive) subsumption in Prob-$\mathcal{EL}^{\sim p; \sim q}$ relative to the empty TBox is in PTIME relative to classical TBoxes whenever $p \geq q \geq p^2$ and that, otherwise, it is EXPTIME-hard.

Probabilistic Roles

Adding probabilistic roles to Prob-$\mathcal{EL}$ tends to increase the complexity of subsumption. While for full Prob-$\mathcal{EL}_r$ even decidability is open, it was shown in (Lutz and Schröder 2010) that subsumption is in 2-EXPTIME and PSPACE-hard in Prob-$\mathcal{EL}_r^{>0;=1}$. As discussed in the introduction, there were reasons to believe that this problem is actually 2-EXPTIME-complete. We show that this is not the case by proving a PSPACE upper bound, thus establishing PSPACE-completeness. This result holds both for positive and unrestricted subsumption, we start with the positive case.

We again concentrate on subsumption between concept *names* and assume that the input TBox is in a certain normal form, defined as follows. A *basic* concept is a concept of the form $\top$, A, $P_{>0}A$, $P_{=1}A$, or $\exists \alpha.A$, where A is a concept name and, here and in what follows, α is a *role*, i.e., of the form r, $P_{>0}r$, or $P_{=1}r$ with r a role name. Now, every concept inclusion in the input TBox is required to be of the form

$$X_1 \sqcap \cdots \sqcap X_n \sqsubseteq X$$

with $X_1, \ldots, X_n, X$ basic concepts. It is not hard to show that every TBox can be transformed into this normal form in polynomial time such that (non-)subsumption between the concept names that occur in the original TBox is preserved.

Let $\mathcal{T}$ be the input TBox in normal form, CN the set of concept names that occur in $\mathcal{T}$, BC the set of basic concepts in $\mathcal{T}$, and ROL the set of roles in $\mathcal{T}$. Call a role *probabilistic* if it is of the form $P_{=1}r$ or $P_{>0}r$. Our algorithm maintains the following data structures:

- a mapping Q that associates with each $A \in \mathsf{CN}$ a subset $Q(A) \subseteq \mathsf{BC}$ such that $\mathcal{T} \models A \sqsubseteq X$ for all $X \in Q(A)$;
- a mapping Q_{cert} that associates with each $A \in \mathsf{CN}$ a subset $Q_{\mathsf{cert}}(A) \subseteq \mathsf{BC}$ such that $\mathcal{T} \models A \sqsubseteq P_{=1}X$ for all $X \in Q_{\mathsf{cert}}(A)$;

R1	If $X_1 \sqcap \ldots \sqcap X_n \sqsubseteq X \in \mathcal{T}$ and $X_1, \ldots, X_n \in \Gamma$ then add X to Γ
R2	If $P_{=1}A \in \Gamma$ then add A to Γ
R3	If $\exists P_{=1}r.A \in \Gamma$ then add $\exists r.A$ to Γ
R4	If $A \in \Gamma$ then add $P_{>0}A$ to Γ
R5	If $\exists r.A \in \Gamma$ then add $\exists P_{>0}r.A$ to Γ
R6	If $\exists \alpha.A \in \Gamma$ and $B \in Q(A)$ then add $\exists \alpha.B$ to Γ

Figure 3: Saturation rules for $\mathsf{cl}(\Gamma)$

- a mapping R that associates with each probabilistic role $\alpha \in \mathsf{ROL}$ a binary relation $R(\alpha)$ on CN such that $\mathcal{T} \models A \sqsubseteq P_{>0}(\exists \alpha.B)$ for all $(A, B) \in R(\alpha)$.

Some intuition about the data structures is already provided above; e.g., $X \in Q(A)$ means that $\mathcal{T} \models A \sqsubseteq X$. However, there is also another view on these structures that will be important in what follows: they represent an abstract view of a model of $\mathcal{T}$, where each set $Q(A)$ describes the concept memberships of a domain element d in a world w with $d \in A^{\mathcal{I},w}$ and R describes role memberships, i.e., when $(A, B) \in R(\alpha)$, then $d \in A^{\mathcal{I},w}$ implies that in some world v with positive probability, d has an element described by $Q(B)$ as an α-successor. In this context, $Q_{\mathsf{cert}}(A)$ contains all concepts that must be true with probability 1 for any domain element that satisfies A in *some* world. Note that non-probabilistic roles r and probabilistic roles $P_{=1}r$ are not represented in the $R(\cdot)$ data structure; we will treat them in a more implicit way later on.

The data structures are initialized as follows, for all $A \in \mathsf{CN}$ and relevant roles α:

$$Q(A) = \{\top, A\} \qquad Q_{\mathsf{cert}}(A) = \{\top\} \qquad R(\alpha) = \emptyset.$$

The sets $Q(\cdot)$, $Q_{\mathsf{cert}}(\cdot)$, and $R(\cdot)$ are then repeatedly extended by the application of various rules. Before we can introduce these rules, we need some preliminaries. As the first step, Figure 3 presents a (different!) set of rules that serves the purpose of saturating a set of concepts Γ. We use $\mathsf{cl}(\Gamma)$ to denote the set of concepts that is the result of exhaustively applying the displayed rules to Γ, where any rule can only be applied if the added concept is in BC, but not yet in Γ. The rules access the data structure $Q(\cdot)$ introduced above and shall later be applied to the sets $Q(A)$ and $Q_{\mathsf{cert}}(A)$, but they will also serve other purposes as described below. It is not hard to see that rule application terminates after polynomially many steps.

The rules that are used for completing the data structures $Q(\cdot)$, $Q_{\mathsf{cert}}(\cdot)$, and $R(\cdot)$ are more complex and refer to 'traces' through these data structures, which we introduce next.

Definition 7. Let $B \in \mathsf{CN}$. A *trace to B* is a finite sequence $S, A_1, \alpha_2, A_2, \ldots, \alpha_n, A_n$ where

1. $S = A$ for some $P_{>0}A \in Q(A_1)$ or $S = (r, B)$ for some $(A_1, B) \in R(P_{>0}r)$;
2. each $A_i \in \mathsf{CN}$ and each $\alpha_i \in \mathsf{ROL}$ is a probabilistic role, such that $A_n = B$;
3. $(A_i, A_{i-1}) \in R(\alpha_i)$ for $1 < i \leq n$.

If t is a trace of length n, we use t_k, $k \leq n$, to denote the shorter trace $S, A_1, \alpha_2, \ldots, \alpha_k, A_k$. Intuitively, the purpose of a trace is to deal with worlds that are generated by concepts $P_{>0}A$ and $\exists P_{>0}r.A$; there can be infinitely many such worlds as Prob-$\mathcal{EL}_r^{>0;=1}$ lacks the finite model property, see (Lutz and Schröder 2010). The trace starts at some domain element represented by a set $Q(A_1)$ in the world generated by the first element S of the trace, then repeatedly follows role edges represented by $R(\cdot)$ backwards until it reaches the final domain element represented by $Q(B)$. The importance of traces stems from the fact that information can be propagated along them, as captured by the following notion.

Definition 8. Let $t = S, A_1, \alpha_2, \ldots, \alpha_n, A_n$ be a trace of length n. Then the *type* $\Gamma(t) \subseteq \mathsf{BC}$ of t is

- $\mathsf{cl}(\{A\} \cup Q_{\mathsf{cert}}(A_1))$ if $n = 1$ and $S = A$;
- $\mathsf{cl}(Q_{\mathsf{cert}}(A_1) \cup \{\exists r.B' \in \mathsf{BC} \mid B' \in Q_{\mathsf{cert}}(B)\})$ if $n = 1$ and $S = (r, B)$;
- $\mathsf{cl}(Q_{\mathsf{cert}}(A_n) \cup \{\exists \alpha_n.B' \in \mathsf{BC} \mid B' \in \Gamma(t_{n-1})\})$ if $n > 1$.

Note that the rules $\mathbf{R1}$ to $\mathbf{R6}$ are used in every step of this inductive definition. The mentioned propagation of information along traces is now as follows: if there is a trace t to B, then any domain element that satisfies B in *some* world must satisfy the concepts in $\Gamma(t)$ in some other world. So if for example $P_{>0}A \in \Gamma(t)$, we need to add $P_{>0}A$ also to $Q_{\mathsf{cert}}(B)$ and to $Q(B)$.

Figure 4 shows the rules used for completing the data structures $Q(\cdot)$, $Q_{\mathsf{cert}}(\cdot)$, and $R(\cdot)$. Note that $\mathbf{S6}$ and $\mathbf{S7}$ implement the propagation of information along traces, as discussed above. Our algorithm for deciding (positive) subsumption starts with the initial data structures defined above and then exhaustively applies the rules shown in Figure 4. To decide whether $\mathcal{T} \models A \sqsubseteq B$, it then simply checks whether $B \in Q(A)$.

Lemma 9. *Let $\mathcal{T}$ be a Prob-$\mathcal{EL}_r^{>0;=1}$-TBox in normal form and A, B be concept names. Then $\mathcal{T} \models^+ A \sqsubseteq B$ iff, after exhaustive rule application, $B \in Q(A)$.*

We now argue that the algorithm can be implemented using only polynomial space. First, it is easy to see that there can be only polynomially many rule applications: every rule application extends the data structures $Q(\cdot)$, $Q_{\mathsf{cert}}(\cdot)$, and $R(\cdot)$, but these structures consist of polynomially many sets, each with at most polynomially many elements. It thus remains to verify that each rule application can be executed using only polyspace, which is obvious for all rules except those involving traces, i.e., $\mathbf{S6}$ and $\mathbf{S7}$. For these rules, we first note that it is not necessary to consider all (infinitely many!) traces. In fact, a straightforward 'pumping argument' can be used to show that there is a trace t to B with some relevant concept $C \in \Gamma(t)$ iff there is a *non-repeating* such trace, i.e., a trace t' of length n such that for all distinct $k, \ell \leq n$, we have $\Gamma(t'_k) \neq \Gamma(t'_\ell)$. Clearly, the length of non-repeating traces is bounded by 2^m, m the size of $\mathcal{T}$. To get to polyspace, we

S1 apply **R1-R6** to $Q(A)$ and $Q_{\mathsf{cert}}(A)$

S2 if $P_*B \in Q(A)$
then add P_*B to $Q_{\mathsf{cert}}(A)$

S3 if $C \in Q_{\mathsf{cert}}(A)$
then add $P_{=1}C$ and C to $Q(A)$

S4 If $\exists\alpha.B \in Q(A)$ with α a probabilistic role
then add (A, B) to $R(\alpha)$.

S5 If $P_{>0}B_1 \in Q(A)$, $(B_1, B_2) \in R(\alpha)$, $B_3 \in Q_{\mathsf{cert}}(B_2)$
then add $\exists\alpha.B_3$ to $Q_{\mathsf{cert}}(A)$

S6 if t is a trace to B and $P_*A \in \Gamma(t)$
then add P_*A to $Q_{\mathsf{cert}}(B)$

S7 if t is a trace to B and $\exists\alpha.A \in \Gamma(t)$
with α a probabilistic role
then add (B, A) to $R(\alpha)$

Figure 4: The rules for completing the data structures.

use a non-deterministic approach, enabled by Savitch's theorem: to check whether there is a trace t to B with $C \in \Gamma(t)$, we guess t step-by-step, at each time keeping only a single A_i, α_i and $\Gamma(t_i)$ in memory. When we reach a situation where $A_i = B$ and $C \in \Gamma(t_i)$, our guessing was successful and we apply the rule. We also maintain a binary counter of the number of steps that have been guessed so far. As soon as this counter exceeds 2^m, the maximum length of non-repeating traces, we stop the guessing and do not apply the rule. Clearly, this yields a polyspace algorithm.

Theorem 10. *Positive subsumption in Prob-$\mathcal{EL}_r^{>0;=1}$ relative to general TBoxes is* PSPACE*-complete.*

As a byproduct, the proof of Lemma 9 yields a unique least model (in the sense of Horn logic), thus proving convexity of Prob-$\mathcal{EL}_r^{>0;=1}$. Note that positive subsumption in Prob-$\mathcal{EL}_r^{>0;=1}$ is actually the same as subsumption in the two-dimensional description logic $\mathsf{S5}_{\mathcal{EL}}$, which is thus also PSPACE-complete. Using a Turing reduction similar to that shown in Figure 2, we can 'lift' the result from positive subsumption to unrestricted subsumption.

Theorem 11. *Subsumption in Prob-$\mathcal{EL}_r^{>0;=1}$ relative to general TBoxes is* PSPACE*-complete.*

Conclusion

We have established a fairly complete picture of the complexity of subsumption in Prob-$\mathcal{EL}$, although some questions remain open. We speculate that Theorem 2 can be proved also when $\sim$ is $=$ with only minor changes (e.g. rule **R3** becomes unsound). It would be interesting to verify the conjecture made below Theorem 6 that (positive) subsumption in Prob-$\mathcal{EL}^{\sim p;\sim q}$ relative to classical TBoxes is in PTIME whenever $p \geq q \geq p^2$ and that, otherwise, it is EXPTIME-hard relative to the empty TBox. It is even conceivable that the conjectured PTIME result can be further generalized to any set of probability values $\mathcal{P} \subseteq [0, 1]$ as long as $q \geq p^2$ whenever $p, q \in \mathcal{P}$ and $p \geq q$. Moreover, variants of Theorem 6 that involve, additionally or exclusively, the case where $\sim$ is $=$ would also be of interest.

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Proofs for Section *Probabilistic Concepts*

Theorem 1

We first consider Prob-$\mathcal{EL}^{\sim p}$ with $\sim \in \{=, \geq, >\}$, starting with $\sim = \geq$ and $p \leq 0.5$. Let $k > 0$ be such that $k \cdot p > 1$ and set

$$\begin{array}{rcl} \mathcal{T} & = & \{A_i \sqcap A_j \sqsubseteq P_{\geq p} B_{ij} \mid 1 \leq i < j \leq k\} \\ C & = & P_{\geq p} A_1 \sqcap \cdots \sqcap P_{\geq p} A_k \\ D_{ij} & = & P_{\geq p} B_{ij} \end{array}$$

We show that the above witnesses non-convexity.

Lemma 12. *$\mathcal{T} \models C \sqsubseteq \bigsqcup_{1 \leq i < j \leq k} D_{ij}$, but $\mathcal{T} \not\models C \sqsubseteq D_{ij}$ for $1 \leq i < j \leq k$.*

Proof. For the former, let $\mathcal{I}$ be a model of $\mathcal{T}$ and $d \in C^{\mathcal{I},w}$. Since $d \in P_{\geq p} A_i$ for $1 \leq i \leq k$ and $k \cdot p > 1$, there is a world v with $d \in (A_i \sqcap A_j)^{\mathcal{I},w}$ for some i, j with $1 \leq i < j \leq k$. It follows that $d \in D_{ij}^{\mathcal{I},v}$, thus $d \in D_{ij}^{\mathcal{I},w}$.

For the latter, fix i_0, j_0 with $1 \leq i_0 < j_0 \leq k$. We construct a model $\mathcal{I} = (\Delta^{\mathcal{I}}, W, (\mathcal{I}_w)_{w \in W}, \mu)$ of $\mathcal{T}$ with $\Delta^{\mathcal{I}} = \{d\}$ and $W = \{w_1, w_2\}$ such that $d \in C^{\mathcal{I},v}$ and $d \notin D_{i_0 j_0}^{\mathcal{I},v}$ for any $v \in W$ by setting, for all i, j with $1 \leq i < j \leq k$:

$$\begin{array}{rcl} A_i^{\mathcal{I}_{w_1}} & := & \begin{cases} \{d\} & \text{if } i \neq i_0 \\ \emptyset & \text{otherwise} \end{cases} \\ A_i^{\mathcal{I}_{w_2}} & := & \begin{cases} \{d\} & \text{if } i = i_0 \\ \emptyset & \text{otherwise} \end{cases} \\ B_{ij}^{\mathcal{I}_{w_1}} & := & B_{ij}^{\mathcal{I}_{w_2}} \;:= \\ & & \begin{cases} \emptyset & \text{if } i_0 \in \{i, j\} \\ \{d\} & \text{otherwise} \end{cases} \\ \mu(w_1) & := & \mu(w_2) \;:=\; 0.5 \end{array}$$

It is easy to check that $\mathcal{I}$ is a model of $\mathcal{T}$. Moreover, we clearly have $p_d^{\mathcal{I}}(A_i) = 0.5 \geq p$ and there is no world w where $d \in A_{i_0}^{\mathcal{I}_w} \cap A_{j_0}^{\mathcal{I}_w}$. ❑

When we replace $\geq$ with $=$ in $\mathcal{T}$, C, and the D_{ij}, the first part of the proof of Lemma 12 still goes through, without any modifications. For the model construction we need to slightly modify the above interpretation: we add a new world w_3 with $A_i^{\mathcal{I}_{w_3}} := B_{ij}^{\mathcal{I}_{w_3}} := \emptyset$ and update the probability distribution as follows:

$$\begin{array}{rcl} \mu(w_1) & := & \mu(w_2) \;:=\; p \\ \mu(w_3) & := & 1 - 2p \end{array}$$

When we replace $\geq$ with $>$ and assume $p < 0.5$, the proof of Lemma 12 goes through without any modifications. However, the case $\sim = >$ and $p = 0.5$ requires a slightly different construction. Set

$$\begin{array}{l} \mathcal{T} = \{A_i \sqcap A_j \sqcap A_k \sqsubseteq P_{\geq p} B_{ijk} \mid 1 \leq i < j < \ell \leq 4\} \\ C = P_{\geq p} A_1 \sqcap \cdots \sqcap P_{\geq p} A_4 \\ D_{ijk} = P_{\geq p} B_{ijk} \end{array}$$

It is not hard to show that the above witnesses non-convexity, similary to the proof of Lemma 12.

We now show non-convexity of Prob-$\mathcal{EL}^{\sim p}$ with $\sim \in \{=, \geq, >\}$ and $p > 0.5$. The main idea is to use the constructor $P_{\sim p} C$ to simulate $P_{\sim q} C$, for some $q < 0.5$. First let $\sim = >$ and fix a $p > 0.5$. Let $n > 0$ be smallest such that $n > \frac{1}{2(1-p)}$. Note that $n \geq 2$. Set $q = pn - n + 1$. An easy computation shows that $0 \leq q < 0.5$. Intuitively, we use the fact that

$$P_{>p} X_1 \sqcap \cdots \sqcap P_{>p} X_n \sqsubseteq P_{>q}(X_1 \sqcap \cdots \sqcap X_n)$$

which allows us to simply redo the above reduction with probability $q < 0.5$. Thus, let $k > 0$ be such that $k \cdot q > 1$ and define

$$\begin{array}{rcl} \mathcal{T} & = & \{A_{i1} \sqcap \cdots \sqcap A_{in} \sqsubseteq A_i \mid 1 \leq i \leq k\} \cup \\ & & \{A_i \sqcap A_j \sqsubseteq P_{>p} B_{ij} \mid 1 \leq i < j \leq k\} \\ C & = & \bigsqcap_{1 \leq i \leq k} \bigsqcap_{1 \leq \ell \leq n} P_{>p} A_{ij} \\ D_{ij} & = & P_{>p} B_{ij} \end{array}$$

Indeed, the above witnesses non-convexity.

Lemma 13. *$\mathcal{T} \models C \sqsubseteq \bigsqcup_{1 \leq i < j \leq k} D_{ij}$, but $\mathcal{T} \not\models C \sqsubseteq D_{ij}$ for $1 \leq i < j \leq k$.*

Proof. For the former, let $\mathcal{I}$ be a model of $\mathcal{T}$ and $d \in C^{\mathcal{I},w}$. We first show that $d \in (P_{>q} A_i)^{\mathcal{I},w}$ for $1 \leq i \leq k$. Assume that this is not the case for some i. Then $p_d^{\mathcal{I}}(A_{i1} \sqcap \cdots \sqcap A_{in}) \leq q$. Since $d \in C^{\mathcal{I},w}$, we must have

$$\sum_{1 \leq \ell \leq n} p_d^{\mathcal{I}}(A_{i\ell}) > pn.$$

Since $p_d^{\mathcal{I}}(A_{i1} \sqcap \cdots \sqcap A_{in}) \leq q$, the left side is bounded by $nq + (n-1)(1-q)$ (the worlds that make all n concept names $A_{i1}, \ldots, A_{in}$ true sum up to q, and the remaining worlds, which make at most $n-1$ concept names true, sum up to $1-q$). Using $q = pn - n + 1$, we obtain $pn > pn$, which is a contradiction. We thus have $d \in P_{\geq q} A_i$ for $1 \leq i \leq k$. We can thus continue as in the proof of Lemma 12: since $k \cdot q > 1$, there is a world v with $d \in (A_i \sqcap A_j)^{\mathcal{I},w}$ for some i, j with $1 \leq i < j \leq k$. It follows that $d \in D_{ij}^{\mathcal{I},v}$, thus $d \in D_{ij}^{\mathcal{I},w}$.

For the latter, fix i_0, j_0 with $1 \leq i_0 < j_0 \leq k$. We construct a model $\mathcal{I} = (\Delta^{\mathcal{I}}, W, (\mathcal{I}_w)_{w \in W}, \mu)$ of $\mathcal{T}$ with $\Delta^{\mathcal{I}} = \{d\}$ and $W = \{w_{1\ell}, w_{2\ell} \mid 1 \leq \ell \leq n\}$ such that $d \in C^{\mathcal{I},v}$ and $d \notin D_{i_0 j_0}^{\mathcal{I},v}$ for any $v \in W$ by setting, for all

i, j, ℓ with $1 \leq i < j \leq k$ and $1 \leq \ell, \ell' \leq n$:

$$
\begin{aligned}
A_i^{\mathcal{I}_{w_{1\ell}}} &:= \begin{cases} \{d\} & \text{if } i \neq i_0 \\ \emptyset & \text{otherwise} \end{cases} \\
A_i^{\mathcal{I}_{w_{2\ell}}} &:= \begin{cases} \{d\} & \text{if } i = i_0 \\ \emptyset & \text{otherwise} \end{cases} \\
A_{i\ell'}^{\mathcal{I}_{w_{1\ell}}} &:= \begin{cases} \{d\} & \text{if } i \neq i_0 \text{ or } \ell \neq \ell' \\ \emptyset & \text{otherwise} \end{cases} \\
A_{i\ell'}^{\mathcal{I}_{w_{2\ell}}} &:= \begin{cases} \{d\} & \text{if } i = i_0 \text{ or } \ell \neq \ell' \\ \emptyset & \text{otherwise} \end{cases} \\
B_{ij}^{\mathcal{I}_{w_{1\ell}}} &:= B_{ij}^{\mathcal{I}_{w_{2\ell}}} := \\
& \begin{cases} \emptyset & \text{if } i_0 \in \{i, j\} \\ \{d\} & \text{otherwise} \end{cases} \\
\mu(w_{1\ell}) &:= \mu(w_{2\ell}) := \tfrac{1}{2n}
\end{aligned}
$$

It is not hard to verify that $\mathcal{I}$ is a model of $\mathcal{T}$. Note that for $1 \leq i \leq k$ and $1 \leq \ell \leq n$, we have

$$p_d^{\mathcal{I}}(A_{i\ell}) = 0.5 + (n-1) \cdot \frac{1}{2n}$$

since for every $A_{i\ell}$, there is a $b \in \{1, 2\}$ such that $A_{i\ell}$ is true in $w_{b\ell'}$ for all ℓ' and in all $w_{\overline{b}\ell'}$, where $\overline{1} = 2$ and $\overline{2} = 1$, whenever $\ell \neq \ell'$. Using the fact that $n > \frac{1}{2(1-p)}$, an easy computation shows that $p_d^{\mathcal{I}}(A_{i\ell}) > p$. Thus, $d \in C^{\mathcal{I},v}$ for any $v \in W$. Finally, it is easy to check that $d \notin D_{i_0 j_0}^{\mathcal{I},v}$ for any $v \in W$. ❏

For the case $\sim \; = \; \geq$, we can use exactly the same construction and the proof goes through with only slight modifications. In case $\sim$ is equality $=$ again the first part of the proof of Lemma 13 goes through, but we have to change the model construction. We add a world w_3 such that in this world d is not in the extension of any concept, i.e., $A_{ij}^{\mathcal{I},w_3} := A_i^{\mathcal{I},w_3} := B_{ij}^{\mathcal{I},w_3} := \emptyset$. Moreover, we need to modify the probability distribution μ in the following way:

$$
\begin{aligned}
\mu(w_{1\ell}) &:= \mu(w_{2\ell}) := \tfrac{p}{2n-1} \\
\mu(w_3) &:= 1 - \tfrac{4np}{2n-1}
\end{aligned}
$$

It is not hard to verify that $\mu(w_3) \geq 0$ since $p > 0.5$ and $n > \frac{1}{2(p-1)}$, and $\sum_{w \in W} \mu(w) = 1$, thus μ is a valid probability distribution. Further, we can check with the same argumentation as in the proof above that $p_d^{\mathcal{I}}(A_{i\ell}) = p$, thus $d \in C^{\mathcal{I},v}$ for every v. Finally, it is easy to check that $d \notin D_{i_0 j_0}^{\mathcal{I},v}$ for any $v \in W$.

Summing up, Lemma 12 and 13 yield non-convexity of Prob-$\mathcal{EL}^{\sim p}$ for all $\sim \in \{>, \geq, =\}$ and $p \in (0, 1)$. It thus remains to deal with the cases $\sim \in \{<, \leq\}$. For showing non-convexity for Prob-$\mathcal{EL}^{<p,=1}$ we set

$$
\begin{aligned}
\mathcal{T} &= \{A_1 \sqsubseteq P_{=1}B, A_2 \sqsubseteq P_{<p}B\} \\
C &= \top \\
D_i &= P_{<p}A_i
\end{aligned}
$$

To show that the disjunction is implied, assume to the contrary that there is a model $\mathcal{I}$ of $\mathcal{T}$ that witnesses that $C \sqsubseteq D_1 \sqcup D_2$ is not implied, i.e. there is a domain element $d \in \Delta^{\mathcal{I}}$, and a world $w \in W$ such that $d \notin (P_{<p}A_1)^{\mathcal{I},w}$ and $d \notin (P_{<p}A_2)^{\mathcal{I},w}$, By the semantics, $d \in (P_{\geq p}A_1)^{\mathcal{I},w} \cap (P_{\geq p}A_2)^{\mathcal{I},w}$. Hence, there is are worlds w_1, w_2 such that $d \in A_1^{\mathcal{I},w_1}$ and $d \in A_2^{\mathcal{I},w_2}$. The TBox implies $d \in (P_{=1}B)^{\mathcal{I},w_1}$ and $d \in (P_{<p}B)^{\mathcal{I},w_2}$. Again by the semantics, $d \in (P_{=1}B)^{\mathcal{I},w}$ and $d \in (P_{<p}B)^{\mathcal{I},w}$ for all $w \in W$, which clearly is a contradiction. For showing that none of the D_i is implied, we give witnessing interpretations $\mathcal{I}_1, \mathcal{I}_2$:

- $\mathcal{I}_1 = (\Delta^{\mathcal{I}_1}, W, \mathcal{I}_w, \mu)$ with $\Delta^{\mathcal{I}_1} = \{d\}$, $W = \{w\}$, $\mu(w) = 1$, and $A_1^{\mathcal{I},w} = D_2^{\mathcal{I},w} = B^{\mathcal{I},w} = \{d\}$ and $A_2^{\mathcal{I},w} = D_1^{\mathcal{I},w} = \emptyset$.
- $\mathcal{I}_2 = (\Delta^{\mathcal{I}_2}, W, \mathcal{I}_w, \mu)$ with $\Delta^{\mathcal{I}_2} = \{d\}$, $W = \{w\}$, $\mu(w) = 1$, and $A_2^{\mathcal{I},w} = D_1^{\mathcal{I},w} = \{d\}$ and $A_2^{\mathcal{I},w} = D_1^{\mathcal{I},w} = B^{\mathcal{I},w} = \emptyset$.

We have thus shown non-convexity for all cases mentioned in Theorem 1. A standard proof technique can be used to show that this implies ExpTime-hardness. For the sake of completeness, we sketch this techinque in what follows. The lower bounds are shown by reduction of the satisfiability of a concept name w.r.t. a general $\mathcal{ALC}$-TBox (the extension of $\mathcal{EL}$ with negation and the bottom concept $\bot$), where a concept name A is *satisfiable* w.r.t. such a TBox $\mathcal{T}$ if there is a model $\mathcal{I}$ of $\mathcal{T}$ with $A^{\mathcal{I}} \neq \emptyset$. This problem is well-known to be ExpTime-complete (Baader et al. 2003). We only deal with the case $\sim \; = \; \geq$ and $p \leq 0.5$, the others are similar.

Suppose that an $\mathcal{ALC}$-TBox $\mathcal{T}$ and a concept name A_0 are given for which satisfiability is to be decided. First, we manipulate the TBox $\mathcal{T}$ as follows:

(a) Ensure that negation $\neg$ occurs in front of concept names, only: for every subconcept $\neg C$ in $\mathcal{T}$ with C complex, introduce a fresh concept name A, replace $\neg C$ with $\neg A$, and add $A \sqsubseteq C$ and $C \sqsubseteq A$ to $\mathcal{T}$.

(b) Eliminate negation: for every subconcept $\neg A$, introduce a fresh concept name $\overline{A}$, replace every occurrence of $\neg A$ with $\overline{A}$, and add $\top \sqsubseteq A \sqcup \overline{A}$ and $A \sqcap \overline{A} \sqsubseteq \bot$ to $\mathcal{T}$.

(c) Eliminate disjunction: modulo introduction of new concept names, we may assume that $\sqcup$ occurs in $\mathcal{T}$ only in the form (i) $A \sqcup B \sqsubseteq C$ and (ii) $C \sqsubseteq A \sqcup B$, where A and B are concept names and C is disjunction free. The former kind of inclusion is replaced with $A \sqsubseteq C$ and $B \sqsubseteq C$. The latter one is replaced with

$$
\begin{aligned}
A_i \sqcap A_j &\sqsubseteq P_{\geq p}B_{ij} \quad \text{for } 1 \leq i < j \leq k \\
C &\sqsubseteq P_{\geq p}A_1 \sqcap \cdots \sqcap P_{\geq p}A_k \\
P_{\geq p}(B_{12}) &\sqsubseteq A \\
P_{\geq p}(B_{ij}) &\sqsubseteq B \quad \text{for } 1 \leq i < j \leq k, ij \neq 12
\end{aligned}
$$

where $A_1, \ldots, A_k$ and the B_{ij} are fresh concept names.

Let $\mathcal{T}'$ be the TBox obtained by these manipulations. It is standard to prove that A_0 is satisfiable w.r.t. $\mathcal{T}$ iff A_0 is satisfiable w.r.t. $\mathcal{T}'$.

The TBox $\mathcal{T}'$ contains only the operators $\sqcap$, $\exists$, $\top$, $\bot$, and $P_{>n}$. We now reduce satisfiability of A_0 w.r.t. $\mathcal{T}'$ to (non-)subsumption in Prob-$\mathcal{EL}^{\geq p;=1}$. Introduce a fresh concept name L, replace every occurrence of $\bot$ with L and extend $\mathcal{T}'$ with $\exists r.L \sqsubseteq L$, for every role r from $\mathcal{T}'$. Then A_0 is satisfiable w.r.t. $\mathcal{T}'$ iff $\mathcal{T}'' \not\models A_0 \sqsubseteq L$.

Theorem 2

To ease notation, we use C_A to denote the defining concept for A in $\mathcal{T}^*$, the TBox obtained from the input TBox $\mathcal{T}$ by exhaustive application of the rules **R1** to **R6**. This is in contrast to the main text, where we use C_A^*.

Lemma 3. For all defined concept names A_0, B_0, we have $\mathcal{T} \models^+ A_0 \sqsubseteq B_0$ iff $C_{B_0} \subseteq C_{A_0}$.

Proof. We concentrate on the case $\sim = >$ and only sketch the modifications necessary for $\sim = \geq$. We assume w.l.o.g. that for each defined concept name A, the unique concept definition for A in $\mathcal{T}^*$ is $A \equiv C_A$. Using the semantics, it is straightforward to show that the rules are correct, i.e., if a TBox $\mathcal{T}_2$ is obtained from a TBox $\mathcal{T}_1$ by a single rule application, then every model of $\mathcal{T}_1$ that does not comprise worlds with probability zero is also a model of $\mathcal{T}_2$. This and the semantics yields the "if" direction of the lemma.

For "only if", let $C_{B_0} \not\subseteq C_{A_0}$. Our aim is to construct a model $\mathcal{I}$ of $\mathcal{T}$ that witnesses $\mathcal{T} \not\models^+ A_0 \sqsubseteq B_0$. Let Def denote the set of defined concept names in $\mathcal{T}$. We first fix some constants that will be used to define the probabilities of worlds in the model $\mathcal{I}$, in three steps:

- fix $\alpha, \alpha' \in (0,1)$ such that $\frac{\alpha}{2} < \alpha' < \alpha < p$;
- we can now find an $m \geq 2$ such that
$$(p - \alpha') + \frac{1 - (p - \alpha' + 3|\mathsf{Def}| \cdot \frac{\alpha}{2})}{m} < p,$$
simply by choosing m sufficiently large;
- finally, we can choose a $k \geq 2$ such that
$$(p - \alpha') \cdot \frac{k-1}{k} + \alpha > p,$$
since $(p - \alpha') + \alpha > p$ (again by choosing k sufficiently large).

Define the interpretation $\mathcal{I} = (\Delta^\mathcal{I}, W, (\mathcal{I}_w)_{w \in W}, \mu)$ as follows:

$$W = \{\delta_{Ai}, 1_j, p_\ell \mid A \in \mathsf{Def}, i \in \{1,2,3\}, j < m, \ell < k\}$$
$$\Delta^\mathcal{I} = \{(A, v) \mid A \in \mathsf{Def}, v \in W\}$$
$$\mu(p_\ell) = \tfrac{p-\alpha'}{k}$$
$$\mu(\delta_{Ai}) = \tfrac{\alpha}{2}$$
$$\mu(1_j) = \tfrac{1-(p-\alpha'+3|\mathsf{Def}|\cdot\frac{\alpha}{2})}{m}$$

The most important properties of μ are as follows:

(P1) for any set V of worlds that contains at least $k-1$ of the worlds in $\{p_\ell \mid \ell < k\}$ and at least two distinct δ_{Ai}, δ_{Bj} the probabilities sum up to more than p;

(P2) any set of worlds whose probabilities sum up to a value $> p$ includes at least two worlds from $W \setminus \{p_\ell \mid \ell < k\}$.

We use $\mathsf{sub}(\mathcal{T}^*)$ to denote all concepts of the form P (primitive concept name), $\exists r.A$, $P_{=1}A$, and $P_{>1}A$ that occur in $\mathcal{T}^*$. To define concept and role memberships, first define a map $\pi : (\Delta^\mathcal{I} \times W) \to 2^{\mathsf{sub}(\mathcal{T}^*)}$ such that each set $\pi(\cdot,\cdot)$ is minimal with the following conditions satisfied for all $A \in \mathsf{Def}$ and $w, v \in W$:

1. $C_A \subseteq \pi((A,w),w)$
2. if $P_*B \in C_A$, then $P_*B \in \pi((A,w),v)$;
3. if $P_{=1}B \in C_A$, then $C_B \subseteq \pi((A,w),v)$;
4. if $P_{>p}B \in C_A$, then $C_B \subseteq \pi((A,w),p_\ell)$ for all $\ell < k$ when $w \notin \{p_\ell \mid \ell < k\}$;
5. if $P_{>p}B \in C_A$, then $C_B \subseteq \pi((A,p_i),p_\ell)$; for all $\ell < k$ with $i \neq \ell$;
6. if $P_{>p}B \in C_A$, then $C_B \subseteq \pi((A,w),\delta_{B1})$ and $C_B \subseteq \pi((A,w),\delta_{B2})$ when $w \notin \{\delta_{B1}, \delta_{B2}, \delta_{B3}\}$;
7. if $P_{>p}B \in C_A$, then $C_B \subseteq \pi((A,\delta_{Bi}),\delta_{Bj})$ for all distinct $i,j \in \{1,2,3\}$;

Now define the interpretation of the defined concept names A, primitive concept names P, and role names r as

$$\begin{aligned} A^{\mathcal{I},w} &= \{d \in \Delta^\mathcal{I} \mid C_A \subseteq \pi(d,w)\} \\ P^{\mathcal{I},w} &= \{d \in \Delta^\mathcal{I} \mid P \in \pi(d,w)\} \\ r^{\mathcal{I},w} &= \{(d,(B,w)) \in \Delta^\mathcal{I} \times \Delta^\mathcal{I} \mid \exists r.B \in \pi(d,w)\} \end{aligned}$$

We establish the following, central claim.

Claim. For all $D \in \mathsf{sub}(\mathcal{T}^*)$, $(A,w) \in \Delta^\mathcal{I}$, and $v \in W$, we have $(A,w) \in D^{\mathcal{I},v}$ iff $D \in \pi((A,w),v)$.

Proof of claim. We start with the "if" direction. Let $D \in \pi((A,w),v)$. We distinguish the following cases:

- $D = P$ is a primitive concept name. Immediate by definition of $P^\mathcal{I}$.
- $D = P_{=1}B$. Since $D \in \pi((A,w),v)$, by definition of π one of the following cases applies:
  - $D \in C_A$. Then $B \in \pi((A,w),w')$ for all w'. By definition of $B^\mathcal{I}$, it follows that $(A,w) \in (P_{=1}B)^{\mathcal{I},w}$ as required.
  - $P_{=1}B' \in C_A$ and $D \in C_{B'}$. Due to non-applicability of rule **R2**, we then have $D \in C_A$ and can argue as in the previous case.
  - $P_{>p}B' \in C_A$, and $D \in C_{B'}$. Since rule **R4** is not applicable, we have $D \in C_A$ and can again argue as before.
- $D = P_{>p}B$. We distinguish the same cases as above, i.e.,
  - $D \in C_A$. Then $C_B \subseteq \pi((A,w),p)$ and $C_B \subseteq \pi((A,w),\delta_{Bi})$ and $C_B \subseteq \pi((A,w),\delta_{Bj})$ for distinct $i,j \in \{1,2,3\}$. By (P1) and definition of $B^\mathcal{I}$, it follows that $(A,w) \in (P_{>p}B)^{\mathcal{I},v}$, as required.
  - $P_{=1}B' \in C_A$ and $D \in C_{B'}$. Due to non-applicability of rule **R2**, we then have $D \in C_A$ and can argue as in the previous case.
  - $P_{>p}B' \in C_A$, and $D \in C_{B'}$. Since rule **R4** is not applicable, we have $D \in C_A$ and can again argue as before.

- $D = \exists r.B$. By definition of $r^{\mathcal{I}}$, we have $((A,w),(B,v)) \in r^{\mathcal{I},v}$. By Condition 1 of π and definition of $B^{\mathcal{I}}$, $(B,v) \in B^{\mathcal{I},v}$. By the semantics, $(A,w) \in (\exists r.B)^{\mathcal{I},v}$.

For the "only if" direction, assume that $(A,w) \in D^{\mathcal{I},v}$. Distinguish the following cases:

- $D = P$ is a primitive concept name. Immediate by definition of $P^{\mathcal{I}}$.
- $D = P_{=1}B$. Take a $1_j \in W$ such that $w \neq 1_j$ (exists since there are at least two worlds of the form 1_j). Since $(A,w) \in D^{\mathcal{I},v}$, we have $(A,w) \in B^{\mathcal{I},1_j}$. By definition of $B^{\mathcal{I}}$, we thus have $C_B \subseteq \pi((A,w),1_j)$. By definition of $\pi((A,w),1_1)$, it follows from Conditions 2 and 3 that for every $D' \in C_B$, we have (i) $D' \in C_A$ with D' of the form P_*B' or (ii) there is a $P_{=1}B' \in C_A$ with $D' \in C_{B'}$. Thus, $C_B \subseteq \mathsf{cert}(C_A)$ and non-applicability of the rule **R5** yields $P_{=1}B \in C_A$. By Condition 2 of π, we have $P_{=1}B \in \pi((A,w),v)$ as required.
- $D = P_{>p}B$. Since $(A,w) \in (P_{>p}B)^{\mathcal{I},v}$, (P2) yields the following cases:
  - $(A,w) \in B^{\mathcal{I},1_j}$ and $w \neq 1_j$. Then we can argue as above that $P_{=1}B \in C_A$. Thus rule **R3** yields $P_{>p}B \in C_A$ and by Condition 2 of π, we have $P_{>p}B \in \pi((A,w),v)$ as required.
  - $(A,w) \in B^{\mathcal{I},\delta_{B'j}}$ and $w \neq \delta_{B'j}$. By definition of $B^{\mathcal{I}}$, we thus have $C_B \subseteq \pi((A,w),\delta_{B'j})$. By definition of $\pi((A,w),\delta_{B'j})$, it follows that for every $D' \in C_B$, we have (i) $D' \in C_A$ with D' of the form P_*B', (ii) there is a $P_{=1}B'' \in C_A$ with $D' \in C_{B''}$, or (iii) $P_{>p}B' \in C_A$ and $D' \in C_{B'}$. Thus, $C_B \subseteq \mathsf{cert}(C_A)$ or $P_{>p}B' \in C_A$ and $C_B \subseteq \mathsf{cert}(C_A) \cup C_{B'}$. Now, non-applicability of the rules **R5** and **R6** yields $P_{>p}B \in C_A$. By Condition 2 of π, we have $P_{>p}B \in \pi((A,w),v)$ as required.
- $D = \exists r.B$. Then there is an (A',w') such that $((A,w),(A',v)) \in r^{\mathcal{I},v}$ and $(A',v) \in B^{\mathcal{I},v}$. By definition of $r^{\mathcal{I}}$, we have $\exists r.A' \in \pi((A,w),v)$. By definition of $B^{\mathcal{I}}$, we have $C_B \subseteq \pi((A',v),v)$. By definition of π, it follows that for every $D' \in C_B$ we have $D' \in C_{A'}$ or $P_{=1}B' \in C_{A'}$ with $D' \in C_{B'}$.[1] In the latter case, we also obtain $B \in C_A$ by non-applicability of **R2**. Thus $C_B \subseteq C_{A'}$. To continue, we make a case distinction as follows:
  - $v = w$. Then the definition of π yields that $\exists r.A' \in C_A$ or $P_{=1}B' \in C_A$ with $\exists r.A' \in C_{B'}$. In the latter case, we also obtain $\exists r.A' \in C_A$ by non-applicability of **R2**. This, $C_B \subseteq C_{A'}$, and non-applicability of **R1** yields $\exists r.B \in C_A$. By definition of π, we thus have $\exists r.B \in \pi((A,w),v)$.
  - $v = 1_j$, $v \neq w$. Since $\exists r.A' \in \pi((A,w),v)$, the definition of π yields a $P_{=1}B' \in C_A$ with $\exists r.A' \in C_{B'}$. By non-applicability of **R1**, we have $\exists r.B \in C_{B'}$. Thus, Condition 3 of π yields $\exists r.B \in \pi((A,w),v)$ as required.
  - $v = p$. Since $\exists r.A' \in \pi((A,w),v)$, the definition of π implies that there is a $P_{=1}B' \in C_A$ with $\exists r.A' \in C_{B'}$ or a $P_{>1}B' \in C_A$ with $\exists r.A' \in C_{B'}$. In the former case, we can continue as in the case $v = 1_j$ above. In the latter case, non-applicability of **R1** yields $\exists r.B \in C_{B'}$. Thus, Condition 4 of π yields $\exists r.B \in \pi((A,w),v)$ as required.
  - $v = \delta_{B'j}$. Identical to the previous case.

This finishes the proof of the claim.

It is an immediate consequence of Claim 1 and the interpretation of defined concept names that $\mathcal{I}$ is a model of $\mathcal{T}$. By Condition 1 on π, definition of $A_0^{\mathcal{I}}$, and Claim 1, we have $(A_0,1_1) \in A_0^{\mathcal{I},1_1}$. It thus remains to show that $(A_0,1_1) \notin B_0^{\mathcal{I},1_1}$. Assume that the contrary holds. By definition of $B_0^{\mathcal{I}}$, this means that $C_{B_0} \subseteq \pi((A_0,1_1),1_1)$. By definition of π, it follows that for every $D \in C_{B_0}$, we have $D \in C_{A_0}$ or $P_{=1B} \in C_{A_0}$ and $D \in C_B$. In the latter case, non-applicability of rule **R2** yields $D \in C_{A_0}$. In summary, we thus have $C_{B_0} \subseteq C_{A_0}$, which is a contradiction to our initial assumption that $C_{B_0} \not\subseteq C_{A_0}$.

This finishes the proof. When $\sim = \geq$, we can use exactly the same interpretation $\mathcal{I}$. The reason is that our construction also satisfies the following variations of (P1) and (P2):

(P1') for any set V of worlds that contains at least $k-1$ of the worlds in $\{p_\ell \mid \ell < k\}$ and at least two distinct δ_{Ai}, δ_{Bj} the probabilities sum up to at least p;

(P2') any set of worlds whose probabilities sum up to a value $\geq p$ includes at least two worlds from $W \setminus \{p_\ell \mid \ell < k\}$.

The rest of the proof is then identical. ❏

Lemma 4

To ease notation, we use C_A to denote the defining concept for A in $\mathcal{T}^*$, the TBox obtained from the input TBox $\mathcal{T}$ by exhaustive application of the roles **S1** and **S2**. This is in contrast to the main text, where we use C_A^*.

Lemma 4. For all defined concept names A_0, B_0, we have $\mathcal{T} \models A_0 \sqsubseteq B_0$ iff $C_{B_0} \subseteq C_{A_0}$.

Proof. Using the semantics, it is straightforward to show that the rules are correct, i.e., if a TBox $\mathcal{T}_2$ is obtained from a TBox $\mathcal{T}_1$ by a single rule application, then every model of $\mathcal{T}_1$ is also a model of $\mathcal{T}_2$. This and the semantics yields the "if" direction of the lemma. For "only if", let $C_{B_0} \not\subseteq C_{A_0}$. Our aim is to construct a model $\mathcal{I}$ of $\mathcal{T}$ that witnesses $\mathcal{T} \not\models A_0 \sqsubseteq B_0$.

Let Def be the set of defined concept names in $\mathcal{T}$. Since **R2** is not applicable, for every $A \in \mathsf{Def}$ and all $P_*B \notin C_A$ that occur in $\mathcal{T}$, we have

$$\mathcal{T}^* \not\models^+ \bigsqcap \mathsf{cert}(A) \sqsubseteq P_*B.$$

Thus, $\mathcal{T}_1 \not\models^+ \hat{A} \sqsubseteq \hat{B}$ where

$$\mathcal{T}_1 = \mathcal{T}^* \cup \{\hat{A} \equiv \bigsqcap \mathsf{cert}(A), \hat{B} \equiv P_*B\}$$

and $\hat{A}, \hat{B}$ are fresh concept names. The proof of Lemma 3 provides us with a model of $\mathcal{T}$ that witnesses the latter (and

[1] This is actually a bit subtle. Note that Conditions 4 to 7 cannot be responsible since we are concerned here with a set $\pi((A,v),v')$ with $v = v'$.

thus also the former) positive non-subsumption. An inspection of this model reveals that it is actually independent of the concept P_*B, i.e., for any P_*B exactly the same model $\mathcal{I}_A$ is generated. More precisely, this model has the set of worlds

$$W = \{p, \delta_{Ai}, 1_j \mid A \in \mathsf{Def}, i \in \{1,2,3\}, j \leq m\}$$

for some $m \geq 2$ and domain

$$\Delta^{\mathcal{I}_A} = \{(B, v) \mid B \in \mathsf{Def} \cup \{\hat{A}, \hat{B}\}, v \in W_A\}$$

and there is a domain element d_A with

$$d_A \in (\bigsqcap \mathsf{cert}(A))^{\mathcal{I}_A, v}$$

and

$$d_A \notin (P_*B)^{\mathcal{I}_A, v}$$

for any $P_*B \notin C_A$ that occurs in $\mathcal{T}$ and all worlds $v \in W_A$. We can thus assume that for all $A, A' \in \mathsf{Def}$, we have $W_A = W_{A'}$ and $\Delta^{\mathcal{I}_A} = \Delta^{\mathcal{I}_{A'}}$. Let us call these sets W^* and Δ^*. Moreover, all $\mathcal{I}_A, \mathcal{I}_{A'}$ agree on the assignment of probabilities to worlds.

We now combine the interpretations $\mathcal{I}_A$ into a model of $\mathcal{T}$ that witnesses $\mathcal{T} \not\models A_0 \sqsubseteq B_0$. Start with setting

$$\begin{aligned} W &= \{0\} \cup W^* \\ \Delta^{\mathcal{I}} &= \{(d, A) \mid d \in \Delta^*, A \in \mathsf{Def}\} \cup \mathsf{Def} \\ \mu(0) &= 0 \end{aligned}$$

For any world $v \in W^*$, $\mathcal{I}$ assigns the same probability $\mu(v)$ as the interpretations $\mathcal{I}_A$ do. Now define the interpretation of the primitive concept names P, defined concept names A, and role names r as follows:

$$\begin{aligned} P^{\mathcal{I},w} = &\{(d, B) \mid d \in P^{\mathcal{I}_B, w}\} \cup \\ &\{A \mid P \in C_A, w = 0\} \cup \\ &\{A \mid d_A \in P^{\mathcal{I}_A, w}, w \neq 0\} \\ A^{\mathcal{I},w} = &\{(d, B) \mid d \in A^{\mathcal{I}_B, w}\} \cup \\ &\{B \mid C_B \subseteq C_A, w = 0\} \cup \\ &\{B \mid d_B \in A^{\mathcal{I}_B, w}, w \neq 0\} \\ r^{\mathcal{I},w} = &\{((d, B), (d', B)) \mid (d, d') \in r^{\mathcal{I}_B, w}\} \cup \\ &\{(A, B) \mid \exists r.B \in C_A, w = 0\} \cup \\ &\{(A, (d, A)) \mid (d_A, d) \in r^{\mathcal{I}_A, w}, w \neq 0\} \end{aligned}$$

We use $\mathsf{sub}(\mathcal{T}^*)$ to denote all concepts of the form P (primitive concept name), $\exists r.A$, $P_{=1}A$, and $P_{>1}A$ that occur in $\mathcal{T}^*$. We prove the following central claim.

Claim. For all $D \in \mathsf{sub}(\mathcal{T}^*)$, $d \in \Delta^*$, $A \in \mathsf{Def}$, and $w \in W$, we have

1. $(d, A) \in D^{\mathcal{I},w}$ iff $d \in D^{\mathcal{I}_A, w}$;
2. $A \in D^{\mathcal{I},w}$ iff $d_A \in D^{\mathcal{I}_A, w}$ whenever $w \neq 0$;
3. $A \in D^{\mathcal{I},0}$ iff $D \in C_A$.

Proof of the Claim. The claim is proved by making a case distinction according to the form of D. The claim holds trivially in the case that D is a primitive or defined concept name. In the following we list the remaining cases:

1. $(d, A) \in D^{\mathcal{I},w}$ iff $d \in D^{\mathcal{I}_A, w}$.
   - $D = \exists r.A$. "if": Let $d \in (\exists r.A)^{\mathcal{I}_B, w}$. By the semantics, there is a $d' \in A^{\mathcal{I}_B, w}$ such that $(d, d') \in r^{\mathcal{I}_B, w}$. This implies $(d', B) \in A^{\mathcal{I},w}$ and, by definition of the interpretation of role names $((d, B), (d', B)) \in r^{\mathcal{I},w}$. Hence $(d, B) \in (\exists r.A)^{\mathcal{I},w}$. "only if": Let $(d, B) \in (\exists r.A)^{\mathcal{I},w}$. By the semantics, there is ab $e \in A^{\mathcal{I},w}$ such that $((d, B), e) \in r^{\mathcal{I},w}$. By definition of the interpretation of role names, it is clear that e has to be of the form (d', B) and $(d, d') \in r^{\mathcal{I}_B, w}$. Since $(d', B) \in A^{\mathcal{I},w}$ we have $d' \in A^{\mathcal{I}_B, w}$. Thus, $d \in (\exists r.A)^{\mathcal{I}_B, w}$.
   - $D = P_{>p}A$ or $D = P_{=1}A$. We show only one case here, since the other is equivalent. "if": Let $d \in (P_{>p}A)^{\mathcal{I}_B, w}$. By the semantics, there is a set of worlds $W' \subseteq W_B$ with $\mu(W') > p$ and $d \in A^{\mathcal{I}_B, v}$ for all $v \in W'$. Thus, $(d, B) \in A^{\mathcal{I},v}$ for all $v \in W'$. Since $\mathcal{I}$ has the same set of worlds (except for 0 which does not account for μ) this implies $(d, B) \in (P_{>p}A)^{\mathcal{I},w}$. "only if": Let $(d, B) \in (P_{>p}A)^{\mathcal{I},w}$. By the semantics, there is a set of worlds $W' \subseteq W$ with $\mu(W') > p$ and $(d, B) \in A^{\mathcal{I},v}$ for all $v \in W'$. It follows that $d \in A^{\mathcal{I}_B, v}$ for all $v \in W'$. Thus, $d \in (P_{>p}A)^{\mathcal{I}_B, w}$.
2. $A \in D^{\mathcal{I},w}$ iff $d_A \in D^{\mathcal{I}_A, w}$ whenever $w \neq 0$.
   - $D = \exists r.A$. "if": Let $d_B \in (\exists r.A)^{\mathcal{I}_B, w}$. By the semantics, there is a $d' \in A^{\mathcal{I}_B, w}$ such that $(d_B, d') \in r^{\mathcal{I}_B, w}$. Thus, $(d', B) \in A^{\mathcal{I},w}$. By definition of the interpretation of role names, $(B, (d', B)) \in r^{\mathcal{I},w}$. Hence, it follows also $B \in (\exists r.A)^{\mathcal{I},w}$. "only if": Let $B \in (\exists r.A)^{\mathcal{I},w}$. By the semantics, there is a $d \in A^{\mathcal{I},w}$ with $(B, d) \in r^{\mathcal{I},w}$. By definition of the interpretation of role names, d is of the form (d', B) and $(d_B, d') \in r^{\mathcal{I}_B, w}$. From $(d', B) \in A^{\mathcal{I},w}$ we obtain $d' \in A^{\mathcal{I}_B, w}$. By the semantics, we have also $d_B \in (\exists r.A)^{\mathcal{I}_B, w}$.
   - $D = P_{>p}A$ or $D = P_{=1}A$. The two cases are very similar and we show only one of them. "if": Let $d_B \in (P_{>p}A)^{\mathcal{I}_B, w}$. By the semantics, there is a set of worlds $W' \in W_B$ with $\mu(W') > p$ such that $d_B \in A^{\mathcal{I}_B, v}$ for all $v \in W'$. Thus, $B \in A^{\mathcal{I},v}$ for all $v \in W'$ and $B \in (P_{>p}A)^{\mathcal{I},w}$. "only if": Let $B \in (P_{>p}A)^{\mathcal{I},w}$. By the semantics, there is a set of worlds $W' \subseteq W$ with $\mu(W') > p$ and $B \in A^{\mathcal{I},v}$ for all $v \in W'$. Thus, $d_B \in A^{\mathcal{I}_B, v}$ for all $v \in W'$. Therefore, $d_B \in (P_{>p}A)^{\mathcal{I},B,w}$.
3. $A \in D^{\mathcal{I},0}$ iff $D \in C_A$.
   - $D = \exists r.A$. "if": Let $\exists r.A \in C_B$. Then, by definition of the interpretation of role names, $(B, A) \in r^{\mathcal{I},0}$ and, by construction, $A \in A^{\mathcal{I},0}$. Thus $B \in (\exists r.A)^{\mathcal{I},0}$. "only if": By the semantics, there is $A' \in A^{\mathcal{I},0}$ and $(B, A') \in r^{\mathcal{I},0}$. The former implies that $C_{A'} \subseteq C_A$, and by the latter we have $\exists r.A' \in C_B$. Thus, by rule **S1**, we obtain $\exists r.A \in C_B$.
   - Again the cases $D = P_{>p}A$ and $D = P_{=1}A$ are equivalent, so we show only one. "if": Let $P_{>p}A \in C_B$. By assumption, $d_B \in (\bigsqcap \mathsf{cert}(B))^{\mathcal{I}_B, w}$ for all $w \in W_B$. Since $P_{>p}A \in \mathsf{cert}(B)$, we have in particular $d_B \in (P_{>p}A)^{\mathcal{I}_B, w}$ for all $w \in W_B$. By the semantics, there is a set of worlds $W' \subseteq W_B$ with $\mu(W') > p$

and $d_B \in A^{\mathcal{I}_B,v}$ for all $v \in W'$. Thus, by Point 2 of the claim, $B \in A^{\mathcal{I},v}$ for all $v \in W'$. But this implies $B \in (P_{>p}A)^{\mathcal{I},0}$. "only if": Let $B \in (P_{>p}A)^{\mathcal{I},0}$. Then, there exists a set of worlds $W' \subseteq W$ with $\mu(W') > p$ and $B \in A^{\mathcal{I},v}$ for all $v \in W'$. By the second point of the claim, $d_B \in A^{\mathcal{I}_B,v}$ for all $v \in W'$. Hence, $d_B \in (P_{>p}A)^{\mathcal{I}_B,v}$. Since, by assumption, $d_B \notin (P_{>p}B')^{\mathcal{I}_B,v}$ for all $P_{>p}B' \in \mathsf{sub}(\mathcal{T}^*) \setminus C_B$, we obtain that $P_{>p}A \in C_B$.

This finishes the proof of the claim.

By Point 3 of the claim and since $C_{B_0} \not\subseteq C_{A_0}$, we have $A_0 \in A_0^{\mathcal{I},0} \setminus B_0^{\mathcal{I},0}$. It thus remains to argue that $\mathcal{I}$ is a model of $\mathcal{T}$. This, however, is a consequence of the fact that the $\mathcal{I}_A$ are models of $\mathcal{T}$ together with the claim, the interpretation of defined concept names, and non-applicability of the rule **S2**. ❑

Theorem 6

Theorem 6. Let $\sim \in \{>, \geq\}$, and let $p, q \in [0, 1)$. Put $A = \{1 - m(1-p) \mid m \in \mathbb{N}, m \geq 1\}$. If there exists a non-negative $u \in A$, such that $u^2 > q > 2u - 1$, then (positive) subsumption in Prob-$\mathcal{EL}^{\sim p;\sim q}$ relative to the empty TBox is ExpTime-hard.

Proof. We first do the case $u = p$. We prove that Prob-$\mathcal{EL}^{\sim p;\sim q}$ is non-convex relative to the empty TBox. Let C_i be atomic concepts for $i = 1, \ldots, n$, and for $i, j \in \{1, \ldots, n\}, i \neq j$, let C_{ij} abbreviate the conjunction $C_i \sqcap C_j$. Then consider the subsumption

$$\prod_{i=1}^{n} P_{\sim p} C_i \sqsubseteq \bigsqcup_{i \neq j} P_{\sim q} C_{ij}. \tag{1}$$

To begin, note that none of the subsumptions

$$\prod_{i=1}^{n} P_{\sim p} C_i \sqsubseteq P_{\sim q} C_{ij} \qquad (i \neq j) \tag{2}$$

is valid. To see this for $\sim$ being $>$, fix $i \neq j$ and construct a countermodel where $C_i \cap C_j$ has measure exactly q (where we abuse concepts to denote sets in the model), and the disjoint sets $C_i \setminus C_j$ and $C_j \setminus C_i$ both receive half of the remaining space, i.e. have measure $(1-q)/2$. Then C_i has measure $q + (1-q)/2 = (q+1)/2 > p$. The other C_k are interpreted as holding universally. Thus, the countermodel does satisfy $P_{>p}C_k$ for all k, but not $P_{>q}C_{ij}$. For $\sim$ being $\geq$, we proceed similarly: let $C_i \cap C_j$ have measure $q - \varepsilon$, with $\varepsilon > 0$ to be determined later, and allocated half of the remaining space, i.e. $(1 - q + \varepsilon)/2$, to each of $C_i \setminus C_j$ and $C_j \setminus C_i$. Then C_i and C_j each have measure $(q-\varepsilon) + (1-q+\varepsilon)/2 = (q+1-\varepsilon)/2 = (q+1)/2 - \varepsilon/2$. The first summand in the last expression is, by assumption, strictly greater than p, so the sum can be made greater than p by taking ε sufficiently small.

Next we show that the subsumption (1) is valid for sufficiently large n. To this end, we use a sound and complete rule taken from (Schröder and Pattinson 2009) (and transformed into an obviously equivalent form): for n at least two (so that both sides of (1) are non-trivial), we can conclude (1) (irrespective of the choice of $\sim$) from the propositional formula

$$\sum r_{ij} C_{ij} - \sum r_i C_i \geq q \sum r_{ij} - p \sum r_i$$

where the r_{ij} and the r_i are positive natural numbers. Here, the sum notation refers to arithmetic of characteristic functions, i.e. the truth value $\top$ is counted as 1, and $\bot$ is counted as 0. We claim that we can find a solution where all r_{ij} are 1, and all r_i are equal to the single value r. Then the above becomes

$$\sum C_{ij} - \binom{n}{2} q \geq r \sum_{i=1}^{n} C_i - rnp. \tag{3}$$

We now note that since $C_{ij} = C_i \sqcap C_j$,

$$\sum_{i \neq j} C_{ij} = \binom{\sum_{i=1}^{n} C_i}{2}.$$

Thus, the propositional formula (3) is valid iff for all $l = 0, \ldots, n$,

$$\binom{l}{2} - \binom{n}{2} q \geq r(l - np). \tag{4}$$

Assuming w.l.o.g. that np is non-integer, such an r exists iff for all $l^+ > np$ and all $l^- < np$,

$$\frac{\binom{l^-}{2} - \binom{n}{2} q}{l^- - np} \leq \frac{\binom{l^+}{2} - \binom{n}{2} q}{l^+ - np}. \tag{5}$$

and moreover for all such l^+,

$$\frac{\binom{l^+}{2} - \binom{n}{2} q}{l^+ - np} > 0 \tag{6}$$

(recalling that r must be positive). We first deal with (6): this just means that $\binom{l^+}{2} - \binom{n}{2} q > 0$. Multiplying by 2 and estimating factorials, this holds if $(np-1)^2 > n^2 q$; the latter holds for sufficiently large n as $p^2 > q$.

To establish (5) for large n, we analyse

$$f(x) = \frac{x(x-1) - n(n-1)q}{x - np} \tag{7}$$

as a function in a real variable x. We have

$$f'(x) = \frac{x^2 - 2npx + n(p + (n-1)q)}{(x - np)^2}, \tag{8}$$

which has roots

$$x^+ = np + \sqrt{np - n(p + (n-1)q)}$$
$$x^- = np - \sqrt{np - n(p + (n-1)q)}$$

corresponding to local extrema; since $f'(x) \to 1$ from below for $x \to \infty$ and $x \to -\infty$ for large n and f' is continuous (also at 0), x^- is a maximum and x^+ is a minimum, so that we are done once we show that $f(x^-) \leq f(x^+)$ for large n. Abbreviating $u = \sqrt{np - n(p + (n-1)q)}$ and already multiplying out the denominators, which coincide up to the sign, we have to show

$$-((n-u)^2 - n + u - n(n-1)q) \leq (n+u)^2 - n - u - n(n-1)q, \tag{9}$$

$$-((n-u)^2-n+u-n(n-1)q) \leq (n+u)^2-n-u-n(n-1)q, \quad (10)$$

equivalently (cancelling identical summands, moving the rest to the right and dividing by 2)

$$0 \leq n^2p^2 + u^2 - np - n(n-1)q. \quad (11)$$

Since we can make n large, we can discard all summands not containing the dominant n^2, arriving at

$$0 \leq 2n^2p - n^2q - n^2q = 2n^2(p-q)$$

which is immediate from our assumption $p^2 > q$.

It remains to prove the general case, i.e. for $u \in A$ arbitrary. Since we can emulate $P_{\sim u}C$ using $P_{\sim p}$ for atomic concepts C using acyclic TBoxes (and hence no TBox at all) for all $u \in A$ as already used in the proof of Theorem 1, this case immediately reduces to the case $u = p$, in which we needed to apply $P_{\sim p}$ only to atomic concepts. ❑

Proofs for Section *Probabilistic Roles*

Lemma 9. Let $\mathcal{T}$ be a Prob-$\mathcal{EL}_r^{>0;=1}$-TBox in normal form and A, B be concept names. Then $\mathcal{T} \models^+ A \sqsubseteq B$ iff, after exhaustive rule application, $B \in Q(A)$.

Proof. For the "if" direction we show that the following invariants of the algorithm hold, i.e.,

$$C \in Q(A) \text{ implies } A \sqsubseteq C \quad \text{(I1)}$$

$$C \in Q_{\mathsf{cert}}(A) \text{ implies } A \sqsubseteq P_{=1}C \quad \text{(I2)}$$

$$(A, B) \in R(\alpha) \text{ implies } A \sqsubseteq P_{>0}(\exists\alpha.B) \quad \text{(I3)}$$

The proof is by induction on the number of applications of the rules in Figure 4. The induction base is trivial since $A \sqsubseteq A$ and $A \sqsubseteq \top$. For the induction step we start with showing soundness of the rules **R1-R6**, i.e., for every set of concepts Γ it holds

$$\bigsqcap \Gamma \sqsubseteq \bigsqcap \mathsf{cl}(\Gamma) \quad (*)$$

For the rules **R1-R5** it follows directly by the semantics. For **R6** assume $\exists\alpha.A \in \Gamma$ and $B \in Q(A)$. Invariant (I1) implies $A \sqsubseteq B$, which means that we can certainly add $\exists\alpha.B$ to Γ.

Next, we analyze traces a little closer and prove the following claim.

Claim. If t is a trace to B, then $B \sqsubseteq P_{>0}(\bigsqcap \Gamma(t))$.

Proof of the Claim. Let $t = S, A_1, \alpha_2, \dots, \alpha_n, A_n$. The proof is by induction on the length n of t. For the induction base, let $n = 1$ and consider first the case that the trace starts with $S = A$, i.e., $P_{>0}A \in Q(A_1)$. From the invariants (I1) and (I2) follows that $A_1 \sqsubseteq P_{>0}(A \sqcap \bigsqcap Q_{\mathsf{cert}}(A_1))$. Since $\Gamma(t) = \mathsf{cl}(\{A\} \cup Q_{\mathsf{cert}}(A_1))$, by (*), we obtain $A_1 \sqsubseteq P_{>0}(\bigsqcap \Gamma(t))$.

Assume now that the trace starts with $S = (r, B)$, i.e., $(A_1, B) \in R(P_{>0}r)$. From the invariant (I2), we get that $A_1 \sqsubseteq P_{=1}(\bigsqcap Q_{\mathsf{cert}}(A_1))$ and $B \sqsubseteq P_{=1}(\bigsqcap Q_{\mathsf{cert}}(B))$. Further, (I3) implies that $A_1 \sqsubseteq P_{>0}(\exists P_{>0}r.B)$. Thus, $A_1 \sqsubseteq P_{>0}(\exists r.P_{>0}B)$. Overall, we obtain that

$$A_1 \sqsubseteq P_{>0}\left(\bigsqcap Q_{\mathsf{cert}}(A_1) \sqcap \exists r.\bigsqcap Q_{\mathsf{cert}}(B)\right)$$

Since $\Gamma(t) = \mathsf{cl}(Q_{\mathsf{cert}}(A_1) \cup \{\exists r.B' \mid B' \in Q_{\mathsf{cert}}(B)\})$ then, by (*), it follows

$$A_1 \sqsubseteq P_{>0}(\bigsqcap \Gamma(t))$$

this finishes the proof of the induction base.

For the induction step, let $n > 1$. By Definition 7, $(A_n, A_{n-1}) \in R(\alpha_n)$. By (I3), we have $A_n \sqsubseteq P_{>0}(\exists\alpha_n.A_{n-1})$. Applying the induction hypothesis, we get

$$A_n \sqsubseteq P_{>0}\left(\exists\alpha_n.P_{>0}(\bigsqcap \Gamma(t_{n-1}))\right)$$

Since $\exists\alpha_n.P_{>0}C \sqsubseteq P_{>0}\exists\alpha_n.C$, then

$$A_n \sqsubseteq P_{>0}\left(\exists\alpha_n.\bigsqcap \Gamma(t_{n-1})\right)$$

On the other hand, (I2) implies $A_n \sqsubseteq P_{=1}\bigsqcap Q_{\mathsf{cert}}(A_n)$. Hence, we obtain the following:

$$A_n \sqsubseteq P_{>0}\left(\bigsqcap Q_{\mathsf{cert}}(A_n) \sqcap \exists\alpha_n.\bigsqcap \Gamma(t_{n-1})\right)$$

Since $\Gamma(t) = \mathsf{cl}(Q_{\mathsf{cert}}(A_n) \cup \{\exists\alpha_n.B \mid B \in \Gamma(t_{n-1})\})$ then, by (*), we get:

$$A_n \sqsubseteq P_{>0} \bigsqcap_{C \in \Gamma(t)} C$$

This finishes the proof of the claim.

It remains to show that the rules in Figure 4 preserve the invariants:

S1 Direct consequence of (*).

S2 Direct by the semantics: $P_{>0}B \sqsubseteq P_{=1}(P_{>0}B)$ and $P_{=1}B \sqsubseteq P_{=1}(P_{=1}B)$.

S3 $C \in Q_{\mathsf{cert}}(A)$ implies $A \sqsubseteq P_{=1}C$ by invariant (I2), hence also $A \sqsubseteq C$.

S4 $\exists\alpha.B \in Q(A)$ implies $A \sqsubseteq \exists\alpha.B$ by invariant (I1), thus also $A \sqsubseteq P_{>0}(\exists\alpha.B)$.

S5 By (I1), we get $A \sqsubseteq P_{>0}B_1$. Then, by invariant (I3), $B_1 \sqsubseteq P_{>0}\exists\alpha.B_2$ and, by invariant (I2), $B_2 \sqsubseteq P_{=1}B_3$. Combining these inclusions yields $A \sqsubseteq P_{>0}(\exists\alpha.P_{=1}B_3)$. The semantics then implies $A \sqsubseteq P_{=1}(\exists\alpha.B_3)$.

S6 Let t be a trace to B and $\Gamma = \Gamma(t)$ its type. By the above claim $B \sqsubseteq P_{>0}C$ for every $C \in \Gamma$. Thus in particular $B \sqsubseteq P_*A$, if $P_*A \in \Gamma$. Hence $B \sqsubseteq P_{=1}(P_*A)$, so P_*A can be added to $Q_{\mathsf{cert}}(B)$.

S7 Analogously to **S6**

Assume now that $B \in Q(A)$. Invariant (I1) implies $A \sqsubseteq B$ which finishes the proof of the "if"-direction.

For showing the "only if" direction, we provide a probabilistic model $\mathcal{I} = (\Delta^\mathcal{I}, W, (\mathcal{I}_w)_{w \in W})$ of $\mathcal{T}$ such that there is a world $w \in W$ and a domain element $d \in \Delta^\mathcal{I}$ with $d \in A^{\mathcal{I},w}$ but $d \notin B^{\mathcal{I},w}$.

We define sequences $\Delta^{\mathcal{I}_0}, \Delta^{\mathcal{I}_1}, \dots, W_0, W_1, \dots,$ and partial maps $\pi_1, \pi_2, \dots$ with $\pi_i : \Delta^{\mathcal{I}_i} \times W_i \to 2^{\mathsf{BC}}$. Our desired sets $\Delta^\mathcal{I}$ and W are then obtained in the limit. The elements of the sets $\Delta^{\mathcal{I}_i}$ are sequences of triples (α, w, A) where α is a role, $w \in W_i$, and A is a concept name. For such a sequence σ, we use $\sigma|_j$ to denote the prefix of σ that consists of the first j triples.

It is possible to view the sequences in $\Delta^{\mathcal{I}}$ as traces, in analogy with the traces from Definition 7. Assume $\sigma = (\alpha_1, w_1, A_1) \cdots (\alpha_n, w_n, A_n) \in \Delta^{\mathcal{I}_i}$ for some $i \geq 0$ (this $\mathcal{I}_i$ is yet to be defined), and that S is either A for some $P_{>0}A \in Q(A_n)$ or (r, B) for some $(A_n, B) \in R(P_{>0}r)$. For $j \leq n$, the *type* $\Gamma_j(S, \sigma) \in 2^{\mathsf{BC}}$ is defined as follows:

- $\mathsf{cl}(\{A\} \cup Q_{\mathsf{cert}}(A_n))$ if $j = n$ and $S = A$;
- $\mathsf{cl}(Q_{\mathsf{cert}}(A_n) \cup \{\exists r.B' \in \mathsf{BC} \mid B' \in Q_{\mathsf{cert}}(B)\})$ if $j = n$ and $S = (r, B)$;
- $\mathsf{cl}(Q_{\mathsf{cert}}(A_j) \cup \{\exists \hat{\alpha}_{j+1}.B' \in \mathsf{BC} \mid B' \in \Gamma_{j+1}(S, \sigma)\})$ if $j < n$, where $\hat{\alpha}_{j+1} = \alpha_{j+1}$ if α_{j+1} is a probabilistic role and $\hat{\alpha}_{j+1} = P_{>0}r$ if α_{j+1} is the role name r.

To start the construction of $\mathcal{I}$, set

- $\Delta^{\mathcal{I}_0} = \{(\alpha, \varepsilon, A_0)\}$ where α is any role (not important) and A_0 is the concept name from the left-hand side of the subsumption;
- $W_0 = \{\varepsilon, 0\}$,
- $\pi((\alpha, \varepsilon, A_0), \varepsilon) = Q(A_0)$ and $\pi((\alpha, \varepsilon, A_0), 0) = Q_{\mathsf{cert}}(A_0)$.

For the induction step, we start with setting $\Delta^{\mathcal{I}_i} = \Delta^{\mathcal{I}_{i-1}}$, $W_i = W_{i-1}$, and $\pi_i = \pi_{i-1}$, and then proceed as follows:

1. if $\exists \alpha.A \in \pi(\sigma, w)$, then add $\sigma \cdot (\alpha, w, A)$ to $\Delta^{\mathcal{I}_i}$ (if it does not yet exist) and set $\pi_i(\sigma \cdot (\alpha, w, A), w) = Q(A)$ and $\pi_i(\sigma \cdot (\alpha, w, A), v) = Q_{\mathsf{cert}}(A)$ for all $v \in W \setminus \{w\}$.
2. if $\sigma = (\alpha_1, w_1, A_1) \cdots (\alpha_n, w_n, A_n) \in \Delta^{\mathcal{I}_i}$ and $P_{>0}B \in Q(A_n)$, then add (σ, B) to W_i (if it does not yet exist) and set $\pi_i(\sigma|_j, (\sigma, B)) = \Gamma_j(B, \sigma)$ for all $j \leq n$; for all $\sigma' \cdot (\alpha, w, A) \in \Delta^{\mathcal{I}}$ that are not a prefix of σ, set $\pi_i(\sigma' \cdot (\alpha, w, A), (\sigma, B)) = Q_{\mathsf{cert}}(A)$.
3. if $\sigma = (\alpha_1, w_1, A_1) \cdots (\alpha_n, w_n, A_n) \in \Delta^{\mathcal{I}_i}$ and $(A_n, B) \in R(P_{>0}r)$, then add (σ, r, B) to W_i (if it does not yet exist) and set $\pi_i(\sigma|_j, (\sigma, r, B)) = \Gamma_j((r, B), \sigma)$ for all $j \leq n$; for all $\sigma' \cdot (\alpha, w, A) \in \Delta^{\mathcal{I}}$ that are not a prefix of σ, set $\pi_i(\sigma' \cdot (\alpha, w, A), (\sigma, r, B)) = Q_{\mathsf{cert}}(A)$.

Finally, set $\Delta^{\mathcal{I}} = \bigcup_{i \geq 0} \Delta^{\mathcal{I}_i}$ and $W = \bigcup_{i \geq 0} W_i$. Define μ such that $\mu(w) > 0$ for all $w \in W$ and $\sum_{w \in W} \mu(w) = 1$. It remains to define the interpretation of concept and role names:

$$
\begin{aligned}
A^{\mathcal{I},w} &= \{\sigma \in \Delta^{\mathcal{I}} \mid A \in \pi(\sigma, w)\} \\
r^{\mathcal{I},w} &= \{(\sigma, \sigma \cdot (P_{>0}r, v, A)) \mid \sigma \cdot (P_{>0}r, v, A) \in \Delta^{\mathcal{I}}, \\
&\qquad\qquad w = (\sigma \cdot (P_{>0}r, v, A), r, A)\} \cup \\
&\quad \{(\sigma, \sigma \cdot (r, w, A)) \mid \sigma \cdot (r, w, A) \in \Delta^{\mathcal{I}}\} \cup \\
&\quad \{(\sigma, \sigma \cdot (P_{=1}r, v, A)) \mid \sigma \cdot (P_{=1}r, v, A) \in \Delta^{\mathcal{I}}\}
\end{aligned}
$$

First, we show a correspondence between types in the above construction and types of a trace. For doing this, we define a function tr that maps $S, (\alpha_1, w_1, A_1) \cdots (\alpha_n, w_n, A_n)$ to the sequence $S, A_n, \alpha_n, \ldots, \alpha_2, A_1$. We prove the following claim:

Claim. For all $\sigma = (\alpha_1, w_1, A_1) \cdots (\alpha_n, w_n, A_n) \in \Delta^{\mathcal{I}}$, we have

- $(A_j, A_{j+1}) \in R(\hat{\alpha}_{j+1})$ for $1 \leq j < n$, and
- if S is either B for some $P_{>0}B \in Q(A_n)$ or (r, B) for some $(A_n, B) \in R(P_{>0}r)$, then $tr(S, \sigma)$ is a trace and $\Gamma_j(S, \sigma) = \Gamma(tr(S, \sigma)_{n-j+1})$ for all $1 \leq j \leq n$.

Proof of the Claim. We show the claim by induction on n. For the induction base, let $n = 1$. Then, $\sigma = (\alpha_1, w_1, A_1)$ and $tr(S, \sigma) = S, A_1$ is obviously a trace. The definition of the types in both cases yields $\Gamma_1(S, \sigma) = \Gamma(tr(S, \sigma)_1)$. Let now $n > 1$. By construction (rule 1), we have for $\sigma' = (\alpha_1, w_1, A_1) \cdots (\alpha_{n-1}, w_{n-1}, A_{n-1})$ that $\exists \alpha_n.A_n \in \pi(\sigma', w)$ for some world $w \in W$. Rule **R5** implies $\exists \hat{\alpha}_n.A_n \in \pi(\sigma', w)$. By construction, $\pi(\sigma', w)$ is either $Q(A_{n-1})$, $Q_{\mathsf{cert}}(A_{n-1})$, or $\Gamma_{n-1}(S', \sigma')$ for some S'. In the first case **S4** yields $(A_{n-1}, A_n) \in R(\hat{\alpha}_n)$. In the second case **S3** implies $\exists \hat{\alpha}_n.A_n \in Q(A_{n-1})$ and we argue as in the previous case. In the third case, induction hypothesis yields $\exists \hat{\alpha}_n.A_n \in \Gamma(tr(S', \sigma')_1)$, then **S7** implies $(A_{n-1}, A_n) \in R(\hat{\alpha}_n)$. By induction hypothesis, we also know that $(A_i, A_{i+1}) \in R(\hat{\alpha}_{i+1})$ for all $1 \leq i < n-1$, thus $tr(S, \sigma)$ is indeed a trace. Another induction on j yields the desired result: $\Gamma_j(S, \sigma) = \Gamma(tr(S, \sigma)_{n-j+1})$ for $1 \leq j \leq n$. This finishes the proof of the claim.

One consequence of this claim is that

$$P_*A \in \pi(\sigma, w) \text{ if and only if } P_*A \in \pi(\sigma, v) \qquad \text{(A1)}$$

for all $\sigma \in \Delta^{\mathcal{I}}$ and $w, v \in W$: if $P_*A \in Q(B)$, then by rule **S2** it will be in $Q_{\mathsf{cert}}(B)$. If $P_*A \in \Gamma_j(S, \sigma)$ (for some j) then by rule **S6** it will be in $Q_{\mathsf{cert}}(B)$. On the other hand, $Q_{\mathsf{cert}}(B)$ is a subset of both $Q(B)$ and $\Gamma_j(S, \sigma)$.

Another property that we will need later is that for all probabilistic roles α, it holds that

$$\sigma' = \sigma \cdot (\alpha, v, B) \wedge A \in \pi(\sigma', w) \Rightarrow \exists \alpha.A \in \pi(\sigma, w) \quad \text{(A2)}$$

This can be shown by a case distinction:

- If $v = w$ then construction rule 1 yields $\exists \alpha.B \in \pi(\sigma, w)$ and $A \in \pi(\sigma', w) = Q(B)$. Now by rule **R6** we obtain $\exists \alpha.A \in \pi(\sigma, w)$.
- $\pi(\sigma', w) = Q_{\mathsf{cert}}(B)$. Thus $A \in Q_{\mathsf{cert}}(B) \subseteq Q(B)$. Further, $\sigma' \in \Delta^{\mathcal{I}}$ implies that $\exists \alpha.B \in \pi(\sigma, v)$ by rule 1 of the construction. But this implies that $(A_n, B) \in R(\alpha)$, because of **S4** if $\pi(\sigma, v)$ equals $Q(A_n)$ or $Q_{\mathsf{cert}}(A_n)$, or because of **S7** if $\pi(\sigma, v)$ is the type of some trace. Further note that $P_{>0}A_n \in Q_{\mathsf{cert}}(A_n)$, since by definition and **R4**, $P_{>0}A_n \in Q(A_n)$. Now, we can apply **S5** in order to obtain $\exists \alpha.B \in Q_{\mathsf{cert}}(A_n)$. Finally, rule **R6** yields $\exists \alpha.A \in Q_{\mathsf{cert}}(A_n) \subseteq \pi(\sigma, w)$.
- $\pi(\sigma', w) = \Gamma_j(S, w)$ for some $j \leq n$ and S, and $A \in \Gamma_j(S, w)$. By definition, $\Gamma_{j-1}(S, w)$ contains $\exists \alpha.A$. Now by definition it is $\Gamma_{j-1}(S, w) = \pi(\sigma, w)$.

We are now ready to show the central property of our model construction.

Claim. For all $\sigma \in \Delta^{\mathcal{I}}$, $w \in W$, and $C \in \mathsf{BC}$, we have $\sigma \in C^{\mathcal{I},w}$ iff $C \in \pi(\sigma, w)$.

Proof of the Claim. We prove the claim by a case distinction on the form of C. Throughout the following we assume $\sigma = (\alpha_1, w_1, A_1) \cdots (\alpha_n, w_n, A_n)$.

- $C = \top$. Then both $\sigma \in \top^{\mathcal{I},w}$ and $\top \in \pi(\sigma, w)$ for all $\sigma \in \Delta^{\mathcal{I}}$ and $w \in W$.
- $C = A \in \mathsf{CN}$. For this case, the lemma holds trivially by definition of the interpretation of concept names.
- $C = P_{>0}A$. "if": Let $\sigma \in (P_{>0}A)^{\mathcal{I},w}$. Then, by the semantics, $\sigma \in A^{\mathcal{I},v}$ for some $v \in W$. This implies $A \in \pi(\sigma, v)$. By **R4**, also $P_{>0}A \in \pi(\sigma, v)$, and by (A1) $P_{>0}A \in \pi(\sigma, w)$. "only if": Let $P_{>0}A \in \pi(\sigma, w)$. By rule 2 of the construction, $A \in \pi(\sigma, (\sigma, A)) = \Gamma_n(A, \sigma)$. This yields $\sigma \in A^{\mathcal{I},(\sigma,A)}$, thus $\sigma \in (P_{>0}A)^{\mathcal{I},w}$.
- $C = P_{=1}A$. "if": Let $\sigma \in (P_{=1}A)^{\mathcal{I},w}$, thus $\sigma \in A^{\mathcal{I},v}$ for all $v \in W$. Hence $A \in \pi(\sigma, v)$ for all $v \in W$, and in particular both $A \in \pi(\sigma, 0)$ and $A \in \pi(\sigma, \varepsilon)$. By construction, both sets are not the type of a trace, since they are realized in worlds different from $0, \varepsilon$. Furthermore, by rule 1 the newly added domain elements σ will have $\pi(\sigma, w) = Q(B)$ for some B in exactly one world w. Thus, at least one of $\pi(\sigma, 0)$ and $\pi(\sigma, \varepsilon)$ will be $Q_{\mathsf{cert}}(B)$, where $\sigma = \sigma' \cdot (\alpha, w', B)$. W.l.o.g. let $\pi(\sigma, 0) = Q_{\mathsf{cert}}(B)$. By **S3** $P_{=1}A \in Q(B)$, and by **S2** also $P_{=1}A \in Q_{\mathsf{cert}}(B) = \pi(\sigma, 0)$. Then, by (A1), $P_{=1}A \in \pi(\sigma, w)$.

  "only if": Let $P_{=1}A \in \pi(\sigma, w)$. By (A1), $P_{=1}A \in \pi(\sigma, v)$ for all $v \in W$. Since all $\pi(\sigma, v)$ are closed under **R2**, $A \in \pi(\sigma, v)$ for all v. Hence $\sigma \in A^{\mathcal{I},v}$ for all $v \in W$, which implies $\sigma \in (P_{=1}A)^{\mathcal{I},w}$.
- $C = \exists r.A$. "if": Assume $\sigma \in (\exists r.A)^{\mathcal{I},w}$. By the semantics, there is a $\sigma' \in \Delta^{\mathcal{I}}$ such that $\sigma' \in A^{\mathcal{I},w}$ and $(\sigma, \sigma') \in r^{\mathcal{I},w}$. Due to the model construction there are three possibilities for (σ, σ') being in $r^{\mathcal{I},w}$:
  - $\sigma' = \sigma \cdot (P_{>0}r, v, B)$ and $w = (\sigma', r, B)$: By construction (rule 3), $\pi(\sigma', w) = Q_{\mathsf{cert}}(B)$, thus $A \in Q_{\mathsf{cert}}(B)$. Now, the definition of a type yields $\exists r.A \in \Gamma_n((r, B), \sigma) = \pi(\sigma, w)$.
  - $\sigma' = \sigma \cdot (r, w, B)$: By rule 1 of the construction, $\pi(\sigma', w) = Q(B)$ and thus $A \in Q(B)$. Also by rule 1, $\exists r.B \in \pi(\sigma, w)$. Hence by **R6**, $\exists r.A \in \pi(\sigma, w)$.
  - $\sigma' = \sigma \cdot (P_{=1}r, v, B)$: We can apply (A2) in order to obtain $\exists P_{=1}r.A \in \pi(\sigma, w)$. Now rule **R3** yields $\exists r.A \in \pi(\sigma, w)$.

  "only if": Let $\exists r.A \in \pi(\sigma, w)$. By rule 1 of the construction, there is a domain element $\sigma' = \sigma \cdot (r, w, A)$ with $\pi(\sigma', w) = Q(A)$, thus $A \in \pi(\sigma', w)$ and $\sigma' \in A^{\mathcal{I},w}$. By definition of the interpretation of role names, $(\sigma, \sigma') \in r^{\mathcal{I},w}$. Hence, $\sigma \in (\exists r.A)^{\mathcal{I},w}$.
- $C = \exists P_{=1}r.A$. "if": Let $\sigma \in (\exists P_{=1}r.A)^{\mathcal{I},w}$, thus there is a domain element σ' with $\sigma' \in A^{\mathcal{I},w}$ and $(\sigma, \sigma') \in r^{\mathcal{I},v}$ for all $v \in W$. Consider $v = 0$ and $v = \varepsilon$: From $(\sigma, \sigma') \in r^{\mathcal{I},0} \cap r^{\mathcal{I},\varepsilon}$ follows that $\sigma' = \sigma \cdot (P_{=1}r, v, B)$ for some world $v \in W$ and concept name B. Applying (A2) yields $\exists P_{=1}r.A \in \pi(\sigma, w)$.

  "only if": Let $\exists P_{=1}r.A \in \pi(\sigma, w)$. By rule 1 of the construction, there is a domain element $\sigma' = \sigma \cdot (P_{=1}r, w, A)$ with $\pi(\sigma', w) = Q(A)$, thus $A \in \pi(\sigma', w)$ and $\sigma' \in A^{\mathcal{I},w}$. By definition of the interpretation of role names, $(\sigma, \sigma') \in r^{\mathcal{I},v}$ for all $v \in W$. Hence, $\sigma \in (\exists P_{=1}r.A)^{\mathcal{I},w}$.
- $C = \exists P_{>0}r.A$. "if": Let $\sigma \in (\exists P_{>0}r.A)^{\mathcal{I},w}$, thus there is a domain element σ' with $\sigma' \in A^{\mathcal{I},w}$ and $(\sigma, \sigma') \in r^{\mathcal{I},v}$ for some $v \in W$. Again we distinguish the three cases of the interpretation of the roles.
  - $\sigma' = \sigma \cdot (P_{>0}r, v', B)$ and $w = (\sigma', r, B)$. By construction rule 3, $\pi(\sigma', w) = Q_{\mathsf{cert}}(B)$, thus $A \in Q_{\mathsf{cert}}(B)$. Also by construction rule 3, $\pi(\sigma, w) = \Gamma_n((r, B), \sigma)$, and by the definition of Γ_n this yields $\exists r.A \in \Gamma_n(r, B), \sigma)$. So by rule **R5**, $\exists P_{>0}r.A \in \pi(\sigma, w)$.
  - $\sigma' = \sigma \cdot (r, w, B)$. By rule 1, $\pi(\sigma', w) = Q(B)$ and $\exists r.B \in \pi(\sigma, w)$. This implies $A \in Q(B)$, and since $\pi(\sigma, w)$ is closed under **R6**, $\exists r.A \in \pi(\sigma, w)$. Thus, by **R5**, $\exists P_{>0}r.A \in \pi(\sigma, w)$.
  - $\sigma' = \sigma \cdot (P_{=1}r, v, B)$. Applying (A2) yields $\exists P_{=1}r.A \in \pi(\sigma, w)$. Using rules **R3** and **R5** we obtain $\exists P_{>0}r.A \in \pi(\sigma, w)$.

  "only if". Let $\exists P_{>0}r.A \in \pi(\sigma, w)$. By rule 1 of the construction there is a domain element $\sigma' = \sigma \cdot (P_{>0}r, w, A)$ $\pi(\sigma', w) = Q(A)$, thus $\sigma' \in A^{\mathcal{I},w}$. By definition of the interpretation of role names $(\sigma, \sigma') \in r^{\mathcal{I},v}$ for $v = (\sigma', A, r)$. Hence $\sigma \in (\exists P_{>0}r.A)^{\mathcal{I},w}$.

It remains to show that for $\sigma_0 = (\alpha, \varepsilon, A_0)$ we have $\sigma_0 \in A_0^{\mathcal{I},\varepsilon}$, but not $\sigma_0 \in B_0^{\mathcal{I},\varepsilon}$. However, both are obviously true: first we note that, by construction, $\pi(\sigma_0, \varepsilon) = Q(A_0)$. By definition, $A_0 \in Q(A_0)$, hence $\sigma_0 \in A_0^{\mathcal{I},\varepsilon}$ by the above claim. On the other hand, by assumption we have $B_0 \notin Q(A_0)$, thus by the above claim $\sigma_0 \notin B_0^{\mathcal{I},\varepsilon}$. ❑

Theorem 11.

Theorem 11. Subsumption in Prob-$\mathcal{EL}_r^{>0;=1}$ relative to general TBoxes is PSPACE-complete.

We only have to show the upper bound. The proof is based on a Turing reduction to positive subsumption, similar to that shown in Figure 2. We concentrate on subsumption between concept *names* and assume the input TBox $\mathcal{T}$ to be in the same normal form as in the proof of Theorem 10. The reduction consists starting with the input TBox $\mathcal{T}$, adding $A \sqsubseteq A$ and $A \sqsubseteq \top$ for all concept names that occur in $\mathcal{T}$, then exhaustively applying the rules shown in Figure 5, and finally checking whether the subsumption in question is syntactically contained in the resulting TBox $\mathcal{T}^*$. In rule **R3**, $\mathsf{prob}(X)$ denotes the set of all concepts Y where $X \sqsubseteq Y \in \mathcal{T}^*$ and Y is a *probabilistic concept*, i.e., of the form P_*A or $\exists P_*r.A$. Rule **R3** can only add $X \sqsubseteq P_*A$ if P_*A occurs in the input TBox. The following lemma asserts correctness.

Lemma 14. *For all concept names A, B, we have $\mathcal{T} \models A \sqsubseteq B$ iff $A \sqsubseteq B \in \mathcal{T}^*$.*

Proof. Using the semantics, it is straightforward to show that the rules are correct, i.e., if a TBox $\mathcal{T}_2$ is obtained from a TBox $\mathcal{T}_1$ by a single rule application, then every model of $\mathcal{T}_1$ is also a model of $\mathcal{T}_2$. This yields the "if" direction of the lemma. For "only if", let $A_0 \sqsubseteq B_0 \notin \mathcal{T}$. We describe the

R1 If $X \sqsubseteq X_1, \ldots, X \sqsubseteq X_n, X_1 \sqcap \cdots \sqcap X_n \sqsubseteq Y \in \mathcal{T}$ then add $X \sqsubseteq Y$ to $\mathcal{T}$

R2 If $X \sqsubseteq \exists r.A, A \sqsubseteq A', \exists r.A' \sqsubseteq X' \in \mathcal{T}$ then add $X \sqsubseteq Y$ to $\mathcal{T}$

R3 If $\mathcal{T} \models^+ \mathsf{prob}(X) \sqsubseteq P_* A$ then add $X \sqsubseteq P_* A$ to $\mathcal{T}$

Figure 5: TBox completion rules for Turing reduction

construction of a model $\mathcal{I}$ of $\mathcal{T}$ that witnesses $\mathcal{T} \not\models A_0 \sqsubseteq B_0$.

Let CN be the set of all concept names that occur in $\mathcal{T}$ and BC the set of all basic concepts in $\mathcal{T}$, and PC the set of all probabilistic concepts in $\mathcal{T}$ (thus $\mathsf{PC} \subseteq \mathsf{BC}$). For all $A \in \mathsf{CN}$ and $D \in \mathsf{PC}$ with $A \sqsubseteq D \notin \mathcal{T}^*$, non-applicability of **R3** yields

$$\mathcal{T}^* \not\models^+ \bigsqcap \mathsf{prob}(A) \sqsubseteq D.$$

Thus, $\mathcal{T}_1 \not\models^+ \hat{A} \sqsubseteq \hat{B}$ where

$$\mathcal{T}_1 = \mathcal{T}^* \cup \{\hat{A} \sqsubseteq \bigsqcap \mathsf{prob}(A), \hat{B} \sqsubseteq D\}$$

and $\hat{A}, \hat{B}$ are fresh concept names. The proof of Lemma 9 provides us with a model of $\mathcal{T}$ that witnesses the latter (and thus also the former) positive non-subsumption. An inspection of this model reveals that it is actually independent of the concept D, i.e., for any $D \in \mathsf{PC}$ with $A \sqsubseteq D \notin \mathcal{T}^*$, exactly the same model $\mathcal{I}_A$ is generated. By construction of this model, we can assume that for all concept names $A, A' \in \mathsf{CN}$, we have $W_A = W_{A'}$ and $\Delta^{\mathcal{I}_A} = \Delta^{\mathcal{I}_{A'}}$. Let us call these sets W^* and Δ^*. Moreover, all $\mathcal{I}_A, \mathcal{I}_{A'}$ agree on the assignments of probabilities to worlds and for all A, we have

$$(\alpha_0, \varepsilon, \hat{A}) \in (\bigsqcap \mathsf{cert}(A))^{\mathcal{I}_A, \varepsilon}$$

and

$$(\alpha_0, \varepsilon, \hat{A}) \notin D^{\mathcal{I}_A, \varepsilon}$$

for any $D \in \mathsf{PC}$ with $A \sqsubseteq D \notin \mathcal{T}$.

We now combine the interpretations $\mathcal{I}_A$ into a model of $\mathcal{T}$ that witnesses $\mathcal{T} \not\models A_0 \sqsubseteq B_0$, very similarly to what was done in the proof of Lemma 4. Start with setting

$$\begin{aligned} W &= \{0\} \cup W^* \\ \Delta^{\mathcal{I}} &= \{(d, B) \mid d \in \Delta^*, B \in \mathsf{CN}\} \cup \mathsf{CN} \\ \mu(0) &= 0 \end{aligned}$$

For any world $v \in W^*$, $\mathcal{I}$ assigns the same probability $\mu(v)$ as the interpretations $\mathcal{I}_A$ do. Now define the interpretation of concept and role names as follows:

$$\begin{aligned} A^{\mathcal{I},w} = &\{(d, B) \mid d \in A^{\mathcal{I}_B, w}\} \cup \\ &\{B \mid B \sqsubseteq A \in \mathcal{T}^*, w = 0\} \cup \\ &\{B \mid (\alpha_0, \varepsilon, \hat{B}) \in A^{\mathcal{I}_B, w}, w \neq 0\} \\ r^{\mathcal{I},w} = &\{((d, B), (d', B)) \mid (d, d') \in P^{\mathcal{I}_B, w}\} \cup \\ &\{(A, B) \mid A \sqsubseteq \exists r.B \in \mathcal{T}^*, w = 0\} \cup \\ &\{((A, (d, A)) \mid ((\alpha_0, \varepsilon, \hat{A}), d) \in r^{\mathcal{I}_A, w}, w \neq 0\} \end{aligned}$$

The following claim can be proved by making a case distinction according to the form of D. We leave details to the reader.

Claim. For all $D \in \mathsf{BC}$, $d \in \Delta^{\mathcal{I}}$, $A, B \in \mathsf{CN}$, roles α, and $w \in W$, we have

1. $((d, A), (d', A)) \in \alpha^{\mathcal{I},w}$ iff $(d, d') \in \alpha^{\mathcal{I}_A, w}$;
2. $(A, B) \in \alpha^{\mathcal{I},w}$ iff $A \sqsubseteq \exists \alpha.B \in \mathcal{T}^*$;
3. $(A, (d, A)) \in \alpha^{\mathcal{I},w}$ iff $((\alpha_0, \varepsilon, \hat{A}), d) \in \alpha^{\mathcal{I}_A, w}$;
4. $(d, A) \in D^{\mathcal{I},w}$ iff $d \in D^{\mathcal{I}_A, w}$;
5. $A \in D^{\mathcal{I},w}$ iff $(\alpha_0, \varepsilon, \hat{A}) \in D^{\mathcal{I}_A, w}$ whenever $w \neq 0$;
6. $A \in D^{\mathcal{I},0}$ iff $A \sqsubseteq D \in \mathcal{T}^*$.

By Point 6 of the claim and since $A_0 \sqsubseteq B_0 \notin \mathcal{T}^*$, we have $A_0 \in A_0^{\mathcal{I},0} \setminus B_0^{\mathcal{I},0}$. Using the above claim and non-applicability of the rules **R1** to **R3** to $\mathcal{T}^*$, it can be proved that $\mathcal{I}$ is a model of $\mathcal{T}^*$ (again, details are left to the reader), thus it is also a model of $\mathcal{T}$ which finishes the proof. ❑

It is easy to see that at most polynomially many rule applications are possible. Since the calls to positive subsumption can be decided using only polyspace by Theorem 10, we obtain the desired PSPACE upper bound.

Additional References for Appendix

Schröder, L., and Pattinson, D. 2009. PSPACE bounds for rank-1 modal logics. *ACM Trans. Comput. Log.* 10:13:1–13:33.