Description Logic TBoxes: Model-Theoretic Characterizations and Rewritability

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Abstract

We characterize the expressive power of description logic (DL) TBoxes, both for expressive DLs such as $\mathcal{ALC}$ and $\mathcal{ALCQIO}$ and lightweight DLs such as DL-Lite and $\mathcal{EL}$. Our characterizations are relative to first-order logic, based on a wide range of semantic notions such as bisimulation, equisimulation, disjoint union, and direct product. We exemplify the use of the characterizations by a first study of the following novel family of decision problems: given a TBox $\mathcal{T}$ formulated in a DL $\mathcal{L}$, decide whether $\mathcal{T}$ can be equivalently rewritten as a TBox in the fragment $\mathcal{L}'$ of $\mathcal{L}$.

1 Introduction

Since the emergence of description logics (DLs) in the 1970s and 80s, research in the area has been driven by the fundamental trade-off between expressive power and computational complexity [Baader *et al.*, 2003]. Over the years, the idea of what complexity is 'acceptable' has varied tremendously, from insisting on tractability in the 1980s gradually up to NEXPTIME- or even 2NEXPTIME-hard DLs in the 2000s, soon intermixed with a revival of DLs for which reasoning is tractable or even in AC_0 (in a database context). Nowadays, it is widely accepted that there is no universal definition of acceptable computational complexity, but that a variety of DLs is needed to cater for the needs of different applications. For example, this is reflected in the recent OWL 2 standard by the W3C, which comprises one very expressive (and 2NEXPTIME-complete) DL and three tractable 'profiles' to be used in applications where the full expressive power is not needed and efficient reasoning is crucial.

While DLs have greatly benefited from this development, becoming much more varied and usable, there are also new challenges that arise: how to choose a DL for a given application? What to do when you have an ontology formulated in a DL $\mathcal{L}$, but would prefer to use a different DL $\mathcal{L}'$ in your application? How do the various DLs interrelate? The first aim of this paper is to lay ground for the study of these and similar questions by providing exact model-theoretic characterizations of the expressive power of TBoxes formulated in the most important DLs, including expressive ones such as $\mathcal{ALC}$ and $\mathcal{ALCQIO}$ (the core of the expressive DL formalized as OWL 2) and lightweight ones such as $\mathcal{EL}$ and DL-Lite (the cores of two of the OWL 2 profiles). We characterize the expressive power of DL TBoxes relative to first-order logic (FO) as a reference point, which (indirectly) also yields a characterization of the expressive power of a DL relative to other DLs. The second aim of this paper is to exemplify the use of the obtained characterizations by developing algorithms for the novel decision problem *$\mathcal{L}_1$-to-$\mathcal{L}_2$-TBox rewritability*: given an $\mathcal{L}_1$-TBox $\mathcal{T}$, decide whether there is an $\mathcal{L}_2$-TBox that is equivalent to $\mathcal{T}$. Note the connection to TBox approximation, studied e.g. in [Ren *et al.*, 2010; Botoeva *et al.*, 2010; Tserendorj *et al.*, 2008]: when $\mathcal{L}_1$ is computationally complex and the goal is to approximate $\mathcal{T}$ in a less expressive DL $\mathcal{L}_2$, the optimal result is of course an equivalent $\mathcal{L}_2$-TBox $\mathcal{T}'$, i.e., when $\mathcal{T}$ can be rewritten into $\mathcal{L}_2$ without any loss of information.

We prepare the study of TBox expressive power with a characterization of the expressive power of DL *concepts* in Section 3. These are in the spirit of the well-known van Benthem Theorem [Goranko and Otto, 2007], giving an exact condition for when an FO-formula with one free variable is equivalent to a DL concept. We use different versions of bisimulation for $\mathcal{ALC}$ and its extensions, and simulations and direct products for $\mathcal{EL}$ and DL-Lite. There is related work by de Rijke and Kurtonina [Kurtonina and de Rijke, 1999], which, however, does not cover those DLs that are considered central today. We then move on to our main topics, characterizing the expressive power of DL TBoxes and studying TBox rewritability in Sections 4 and 5. To characterize when a TBox is equivalent to an FO sentence, we use 'global' and symmetric versions of the model-theoretic constructions in Section 3, enriched with various versions of (disjoint and non-disjoint) unions and direct products. These results are loosely related to work by Borgida [Borgida, 1996], who focusses on DLs with complex role constructors, and by Baader [Baader, 1996], who uses a more liberal definition of expressive power. We use our characterizations to establish decidability of TBox rewritability for the $\mathcal{ALCI}$-to-$\mathcal{ALC}$ and $\mathcal{ALC}$-to-$\mathcal{EL}$ cases. The algorithms are highly non-trivial and a more detailed study of TBox rewritability has to remain as future work.

Most proofs in this paper are deferred to the appendix.

Name	Syntax	Semantics
inverse role	r^-	$(r^{\mathcal{I}})^{\smile} = \{(d,e) \mid (e,d) \in r^{\mathcal{I}}\}$
nominal	$\{a\}$	$\{a^{\mathcal{I}}\}$
negation	$\neg C$	$\Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$
conjunction	$C \sqcap D$	$C^{\mathcal{I}} \cap D^{\mathcal{I}}$
disjunction	$C \sqcup D$	$C^{\mathcal{I}} \cup D^{\mathcal{I}}$
at-least restriction	$(\geqslant n\ r\ C)$	$\{d \in \Delta^{\mathcal{I}} \mid \#(r^{\mathcal{I}}(d) \cap C^{\mathcal{I}}) \geq n\}$
at-most restriction	$(\leqslant n\ r\ C)$	$\{d \in \Delta^{\mathcal{I}} \mid \#(r^{\mathcal{I}}(d) \cap C^{\mathcal{I}}) \leq n\}$

Figure 1: Syntax and semantics of $\mathcal{ALCQIO}$.

2 Preliminaries

In DLs, *concepts* are defined inductively based on a set of *constructors*, starting with a set $\mathsf{N_C}$ of *concept names*, a set $\mathsf{N_R}$ of *role names*, and a set $\mathsf{N_I}$ of *individual names* (all countably infinite). The concepts of the expressive DL $\mathcal{ALCQIO}$ are formed using the constructors shown in Figure 1.

In Figure 1 and in general, we use $r^{\mathcal{I}}(d)$ to denote the set of all r-successors of d in $\mathcal{I}$, $\#S$ for the cardinality of a set S, a and b to denote individual names, r and s to denote roles (i.e., role names and inverses thereof), A, B to denote concept names, and C, D to denote (possibly compound) concepts. As usual, we use $\top$ as abbreviation for $A \sqcup \neg A$, $\bot$ for $\neg\top$, $\rightarrow$ and $\leftrightarrow$ for the usual Boolean abbreviations, $\exists r.C$ (*existential restriction*) for $(\geqslant 1\ r\ C)$, and $\forall r.C$ (*universal restriction*) for $(\leqslant 0\ r\ \neg C)$.

Throughout the paper, we consider the expressive DL $\mathcal{ALCQIO}$, which can be viewed as a core of the OWL 2 recommendation, and several relevant fragments; a basic such fragment underlying the OWL 2 EL profile of OWL 2 is the lightweight DL $\mathcal{EL}$, which allows only for $\top$, $\bot$, conjunction, and existential restrictions. By adding negation, one obtains the basic Boolean-closed DL $\mathcal{ALC}$. Additional constructors are indicated by concatenation of a corresponding letter: $\mathcal{Q}$ stands for number restrictions, $\mathcal{I}$ for inverse roles, and $\mathcal{O}$ for nominals. This explains the name $\mathcal{ALCQIO}$ and allows us to refer to fragments such as $\mathcal{ALCI}$ and $\mathcal{ALCQ}$. From the DL-Lite family of lightweight DLs [Calvanese *et al.*, 2005; Artale *et al.*, 2009], which underlies the OWL 2 QL profile of OWL 2, we consider *DL-Lite*$_{\mathsf{horn}}$ whose concepts are conjunctions of *basic concepts* of the form A, $\exists r.\top$, $\bot$, or $\top$, where $A \in \mathsf{N_C}$ and r is a role name or its inverse. We will also consider the DL-Lite$_{\mathsf{core}}$ variant, but defer a detailed definition to Section 4. We use DL to denote the set of DLs just introduced, and ExpDL to denote the set of *expressive DLs*, i.e., $\mathcal{ALC}$ and its extensions introduced above.

The semantics of DLs is defined in terms of an *interpretation* $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where $\Delta^{\mathcal{I}}$ is a non-empty set and $\cdot^{\mathcal{I}}$ maps each concept name $A \in \mathsf{N_C}$ to a subset $A^{\mathcal{I}}$ of $\Delta^{\mathcal{I}}$, each role name $r \in \mathsf{N_R}$ to a binary relation $r^{\mathcal{I}}$ on $\Delta^{\mathcal{I}}$, and each individual name $a \in \mathsf{N_I}$ to an $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$. The extension of $\cdot^{\mathcal{I}}$ to inverse roles and arbitrary concepts is inductively defined as shown in the third column of Figure 1.

For $\mathcal{L} \in \mathsf{DL}$, an *$\mathcal{L}$-TBox* is a finite set of *concept inclusions* (CIs) $C \sqsubseteq D$, where C and D are $\mathcal{L}$ concepts. An interpretation $\mathcal{I}$ *satisfies* a CI $C \sqsubseteq D$ if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ and is a *model* of a TBox $\mathcal{T}$ if it satisfies all inclusions in $\mathcal{T}$.

[Atom]	for all $(d_1, d_2) \in S$: $d_1 \in A^{\mathcal{I}_1}$ iff $d_2 \in A^{\mathcal{I}_2}$
[AtomR]	if $(d_1, d_2) \in S$ and $d_1 \in A^{\mathcal{I}_1}$, then $d_2 \in A^{\mathcal{I}_2}$
[Forth]	if $(d_1, d_2) \in S$ and $d_1' \in \mathsf{succ}_r^{\mathcal{I}_1}(d_1)$, $r \in \mathsf{N_R}$, then there is a $d_2' \in \mathsf{succ}_r^{\mathcal{I}_2}(d_2)$ with $(d_1', d_2') \in S$.
[Back]	dual of [Forth]
[QForth]	if $(d_1, d_2) \in S$ and $D_1 \subseteq \mathsf{succ}_r^{\mathcal{I}_1}(d_1)$ finite, $r \in \mathsf{N_R}$, then there is a $D_2 \subseteq \mathsf{succ}_r^{\mathcal{I}_2}(d_2)$ such that S contains a bijection between D_1 and D_2.
[QBack]	dual of [QForth]
[FSucc]	if $(d_1, d_2) \in S$, r a role, and $\mathsf{succ}_r^{\mathcal{I}_1}(d_1) \neq \emptyset$, then $\mathsf{succ}_r^{\mathcal{I}_2}(d_2) \neq \emptyset$.

Figure 2: Conditions on $S \subseteq \Delta^{\mathcal{I}_1} \times \Delta^{\mathcal{I}_2}$.

Concepts and TBoxes formulated in any $\mathcal{L} \in \mathsf{DL}$ can be regarded as formulas in first-order logic (FO) with equality using unary predicates from $\mathsf{N_C}$, binary predicates from $\mathsf{N_R}$, and constants from $\mathsf{N_I}$. More precisely, for every concept C there is an FO-formula $C^\sharp(x)$ such that $\mathcal{I} \models C^\sharp[d]$ iff $d \in C^{\mathcal{I}}$, for all interpretations $\mathcal{I}$ and $d \in \Delta^{\mathcal{I}}$ [Baader *et al.*, 2003]. For every TBox $\mathcal{T}$, the FO sentence

$$\mathcal{T}^\sharp = \bigwedge_{C \sqsubseteq D \in \mathcal{T}} \forall x.(C^\sharp(x) \rightarrow D^\sharp(x))$$

is logically equivalent to $\mathcal{T}$. We will often not explicitly distinguish between DL-concepts and TBoxes and their translation into FO. For example, we write $\mathcal{T} \equiv \varphi$ for a TBox $\mathcal{T}$ and an FO-sentence φ whenever $\mathcal{T}^\sharp$ is equivalent to φ.

3 Characterizing Concepts

We characterize DL-concepts relative to FO-formulas with one free variable, mainly to provide a foundation for subsequent characterizations on the TBox level. We use the notion of an *object* $(\mathcal{I}, d)$, which consists of an interpretation $\mathcal{I}$ and a $d \in \Delta^{\mathcal{I}}$ and, intuitively, represents an object from the real world. Two objects $(\mathcal{I}_1, d_1)$ and $(\mathcal{I}_2, d_2)$ are *$\mathcal{L}$-equivalent*, written $(\mathcal{I}_1, d_1) \equiv_{\mathcal{L}} (\mathcal{I}_2, d_2)$, if $d_1 \in C^{\mathcal{I}_1} \Leftrightarrow d_2 \in C^{\mathcal{I}_2}$ for all $\mathcal{L}$-concepts C. Our first aim is to provide, for each $\mathcal{L} \in \mathsf{DL}$, a relation $\sim_{\mathcal{L}}$ on objects such that $\equiv_{\mathcal{L}} \supseteq \sim_{\mathcal{L}}$ and the converse holds for a large class of interpretations. To ease notation, we use only d to denote the object $(\mathcal{I}, d)$ when $\mathcal{I}$ is understood.

We start by introducing the classical notion of a bisimulation, which corresponds to $\equiv_{\mathcal{ALC}}$ in the described sense. Two objects $(\mathcal{I}_1, d_1)$ and $(\mathcal{I}_2, d_2)$ are *bisimilar*, in symbols $(\mathcal{I}_1, d_1) \sim_{\mathcal{ALC}} (\mathcal{I}_2, d_2)$, if there exists a relation $S \subseteq \Delta^{\mathcal{I}_1} \times \Delta_2^{\mathcal{I}}$ such that the conditions [Atom] (for $A \in \mathsf{N_C}$), [Forth] and [Back] from Figure 2 hold, where $\mathsf{succ}_r^{\mathcal{I}}(d) = \{d' \in \Delta^{\mathcal{I}} \mid (d, d') \in r^{\mathcal{I}}\}$ and 'dual' refers to swapping the rôles of $\mathcal{I}_1, d_1, d_1'$ and $\mathcal{I}_2, d_2, d_2'$; we call such an S a *bisimulation* between $(\mathcal{I}_1, d_1)$ and $(\mathcal{I}_2, d_2)$. To address $\mathcal{ALCQ}$, we extend this to *counting bisimilarity* (cf. [Janin and Lenzi, 2004]), in symbols $\sim_{\mathcal{ALCQ}}$, and defined as bisimilarity, but with [Forth] and [Back] replaced by [QForth] and [QBack] from Figure 2. Given $\sim_{\mathcal{L}}$, the relation $\sim_{\mathcal{LO}}$ for the extension $\mathcal{LO}$ of $\mathcal{L}$ with nominals is defined by additionally requiring S to satisfy [Atom] for all concepts $A = \{a\}$ with $a \in \mathsf{N_I}$. Similarly, $\sim_{\mathcal{LI}}$ for the extension $\mathcal{LI}$ of $\mathcal{L}$ with inverse roles demands that in all conditions of $\sim_{\mathcal{L}}$, r additionally ranges over inverse roles.

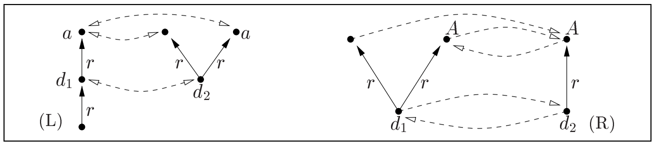

Figure 3: Examples for $d_1 \sim_{\mathcal{L}} d_2$

Example 1. In Figure 3 (L), $d_1 \sim_{\mathcal{ALC}} d_2$ and a bisimulation is indicated by dashed arrows. In contrast, $d_1 \not\sim_{\mathcal{L}} d_2$ for $\mathcal{L} \in \{\mathcal{ALCQ}, \mathcal{ALCO}, \mathcal{ALCI}\}$. It is instructive to construct $\mathcal{L}$-concepts C that show $d_1 \not\equiv_{\mathcal{L}} d_2$.

We have provided a relation $\sim_{\mathcal{L}}$ for each $\mathcal{L} \in \mathsf{ExpDL}$. For lightweight DLs with their restricted use of negation, it will be useful to consider *non-symmetric* relations between objects. A relation $S \subseteq \Delta^{\mathcal{I}_1} \times \Delta^{\mathcal{I}_2}$ is an *$\mathcal{EL}$-simulation* from $\mathcal{I}_1$ to $\mathcal{I}_2$ if it satisfies [AtomR] (for $A \in \mathsf{N_C}$) and [Forth] from Figure 2. S is a *DL-Lite$_{\mathsf{horn}}$-simulation* from $\mathcal{I}_1$ to $\mathcal{I}_2$ if it satisfies [AtomR] (for $A \in \mathsf{N_C}$) and [FSucc]. Let $\mathcal{L} \in \{\mathcal{EL}, \text{DL-Lite}_{\mathsf{horn}}\}$. Then $(\mathcal{I}_1, d_1)$ is *$\mathcal{L}$-simulated* by $(\mathcal{I}_2, d_2)$, in symbols $d_1 \leq_{\mathcal{L}} d_2$, if there exists an $\mathcal{L}$-simulation S with $(d_1, d_2) \in S$. The relation $\sim_{\mathcal{L}}$ that corresponds to (the inherently symmetric) $\equiv_{\mathcal{L}}$ is $\mathcal{L}$-equisimilarity: d_1 and d_2 are *$\mathcal{L}$-equisimilar*, written $d_1 \sim_{\mathcal{L}} d_2$, if $d_1 \leq_{\mathcal{L}} d_2$ and $d_2 \leq_{\mathcal{L}} d_1$.

Example 2. In Figure 3 (R), $d_1 \sim_{\mathcal{EL}} d_2$, the $\mathcal{EL}$-simulations are indicated by the dashed arrows. But $d_1 \not\sim_{\mathcal{ALC}} d_2$.

It is known from modal logic that $\equiv_{\mathcal{ALC}} \supseteq \sim_{\mathcal{ALC}}$ [Goranko and Otto, 2007], but that the converse holds only for certain classes of interpretations, called Hennessy-Milner classes, such as the class of all interpretations of finite out-degree. For our purposes, we need a class such that (i) $\equiv_{\mathcal{L}} \subseteq \sim_{\mathcal{L}}$ holds in this class, for all $\mathcal{L} \in \mathsf{DL}$ and (ii) every interpretation is elementary equivalent (indistinguishable by FO sentences) to an interpretation in the class. These conditions are satisfied by the class of all ω-saturated interpretations, as known from classical model theory [Chang and Keisler, 1990] and defined in full detail in the long version. For the reader, it is most important that this class satisfies the above Conditions (i) and (ii). It can be seen that every finite interpretation and modally saturated interpretation in the sense of [Goranko and Otto, 2007] is ω-saturated.

Theorem 3. *Let $\mathcal{L} \in \mathsf{DL}$ and $(\mathcal{I}_1, d_1)$ and $(\mathcal{I}_2, d_2)$ be objects.*

1. *If $d_1 \sim_{\mathcal{L}} d_2$, then $d_1 \equiv_{\mathcal{L}} d_2$;*
2. *If $d_1 \equiv_{\mathcal{L}} d_2$ and $\mathcal{I}_1, \mathcal{I}_2$ are ω-saturated, then $d_1 \sim_{\mathcal{L}} d_2$.*

We now characterize concepts formulated in expressive DLs relative to FO. An FO-formula $\varphi(x)$ is *invariant under* $\sim_{\mathcal{L}}$ if for any two objects $(\mathcal{I}_1, d_1)$ and $(\mathcal{I}_2, d_2)$, from $\mathcal{I}_1 \models \varphi[d_1]$ and $d_1 \sim_{\mathcal{L}} d_2$ it follows that $\mathcal{I}_2 \models \varphi[d_2]$.

Theorem 4. *Let $\mathcal{L} \in \mathsf{ExpDL}$ and $\varphi(x)$ an FO-formula. Then the following conditions are equivalent:*

1. *there exists an $\mathcal{L}$-concept C such that $C \equiv \varphi(x)$;*
2. *$\varphi(x)$ is invariant under $\sim_{\mathcal{L}}$.*

For $\mathcal{ALC}$, this result is exactly van Benthem's characterization of modal formulae as the bisimulation invariant fragment of FO [Goranko and Otto, 2007]. For the modal logic variant of $\mathcal{ALCQ}$, a similar, though more complex, characterization has been given in [de Rijke, 2000].

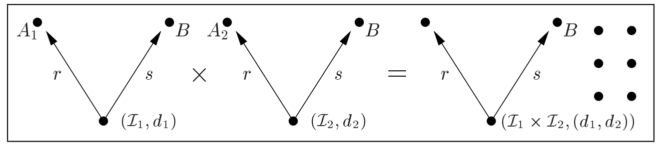

Figure 4: A product

Concept definability in the lightweight DLs $\mathcal{EL}$ and DL-Lite$_{\mathsf{horn}}$ cannot be characterized exactly as in Theorem 3. In fact, one can show that invariance under $\sim_{\mathcal{EL}}$ characterizes FO-formulae equivalent to *Boolean combinations* of $\mathcal{EL}$-concepts, and invariance under $\sim_{\text{DL-Lite}_{\mathsf{horn}}}$ characterizes FO-formulae equivalent to DL-Lite$_{bool}$-concepts, see [Artale *et al.*, 2009]. To fix this problem, we switch from $\sim_{\mathcal{L}}$ to $\leq_{\mathcal{L}}$ and additionally require the FO-formula $\varphi(x)$ to be preserved under direct products. Intuitively, the first modification addresses the restricted use of negation and the second one the lack of disjunction in $\mathcal{EL}$ and DL-Lite$_{\mathsf{horn}}$.

Let $\mathcal{I}_i$, $i \in I$, be a family of interpretations. The *(direct) product* $\prod_{i\in I} \mathcal{I}_i$ is the interpretation defined as follows:

$$\begin{aligned}
\Delta^{\prod \mathcal{I}_i} &= \{\bar{d} : I \to \textstyle\bigcup_{i\in I} \Delta^{\mathcal{I}_i} \mid \text{ for } i \in I : \bar{d}_i = \bar{d}(i) \in \Delta^{\mathcal{I}_i}\} \\
A^{\prod \mathcal{I}_i} &= \{\bar{d} \in \Delta^{\prod \mathcal{I}_i} \mid \text{for } i \in I : d_i \in A^{\mathcal{I}_i}\} \quad \text{ for } A \in \mathsf{N_C} \\
r^{\prod \mathcal{I}_i} &= \{(\bar{d}, \bar{e}) \mid \text{for } i \in I : (d_i, e_i) \in r^{\mathcal{I}_i}\} \quad \text{ for } r \in \mathsf{N_R}
\end{aligned}$$

Note that products are closely related to Horn logic, both in the case of full FO [Chang and Keisler, 1990] and modal logic [Sturm, 2000]. An FO-formula $\varphi(x)$ is *preserved under products* if for all families $(\mathcal{I}_i)_{i\in I}$ of interpretations and all $\bar{d} \in \Delta^{\prod \mathcal{I}_i}$ with $\mathcal{I}_i \models \varphi[\bar{d}_i]$ for all $i \in I$, we have $\prod_{i\in I} \mathcal{I}_i, \models \varphi[\bar{d}]$. This notion is adapted in the obvious way to FO sentences. For $\mathcal{L} \in \{\mathcal{EL}, \text{DL-Lite}_{\mathsf{horn}}\}$, an FO-formula $\varphi(x)$ is *preserved under* $\leq_{\mathcal{L}}$ if $(\mathcal{I}_1, d_1) \leq_{\mathcal{L}} (\mathcal{I}_2, d_2)$ and $\mathcal{I}_1 \models \varphi[d_1]$ imply $\mathcal{I}_2 \models \varphi[d_2]$.

Theorem 5. *Let $\mathcal{L} \in \{\mathcal{EL}, \text{DL-Lite}_{\mathsf{horn}}\}$ and $\varphi(x)$ an FO-formula. Then the following conditions are equivalent:*

1. *there exists an $\mathcal{L}$-concept C such that $C \equiv \varphi(x)$;*
2. *$\varphi(x)$ is preserved under $\leq_{\mathcal{L}}$ and under products.*

Example 6. In Figure 4, $d_i \in (\exists r.A_1 \sqcup \exists r.A_2)^{\mathcal{I}_i}$ for $i = 1, 2$, but $(d_1, d_2) \notin (\exists r.A_1 \sqcup \exists r.A_2)^{\mathcal{I}_1 \times \mathcal{I}_2}$. Thus, disjunctions of $\mathcal{EL}$-concepts are not preserved under products.

It is known that an FO-formula is preserved under products in the above sense iff it is preserved under binary products (where I has cardinality 2) [Chang and Keisler, 1990]. Likewise (and because of that), all results stated in this paper hold both for unrestricted produces and for binary ones.

4 Characterizing TBoxes, Expressive DLs

A natural first idea for lifting Theorem 4 from the concept level to the level of TBoxes is to replace the 'local' relations $\sim_{\mathcal{L}}$ with their 'global' counterpart $\sim^g_{\mathcal{L}}$, i.e., $\mathcal{I}_1 \sim^g_{\mathcal{L}} \mathcal{I}_2$ iff for all $d_1 \in \Delta^{\mathcal{I}_1}$ there exists $d_2 \in \Delta^{\mathcal{I}_2}$ with $(\mathcal{I}_1, d_1) \sim_{\mathcal{L}} (\mathcal{I}_2, d_2)$ and vice versa. It turns out that, in this way, we characterize Boolean $\mathcal{L}$-TBoxes rather than $\mathcal{L}$-TBoxes for all $\mathcal{L} \in \mathsf{ExpDL}$, where a *Boolean $\mathcal{L}$-TBox* is an expression built up from $\mathcal{L}$-concept inclusions and the Boolean operators $\neg$, $\wedge$, $\vee$. The proof exploits compactness and Theorem 3.

Theorem 7. *Let $\mathcal{L} \in \mathsf{ExpDL}$ and φ an FO-sentence. Then the following conditions are equivalent:*

1. *there exists a Boolean $\mathcal{L}$-TBox $\mathcal{T}$ such that $\mathcal{T} \equiv \varphi$;*
2. *φ is invariant under $\sim^g_{\mathcal{L}}$.*

To characterize TBoxes rather than Boolean TBoxes, we thus need to strengthen the conditions on φ. We first consider DLs without nominals. Let $(\mathcal{I}_i)_{i\in I}$ be a family of interpretations. The *union* $\sum_{i\in I} \mathcal{I}_i$ is defined by setting

- $\Delta^{\sum_{i\in I}\mathcal{I}_i} = \bigcup_{i\in I} \Delta^{\mathcal{I}_i}$;
- $X^{\sum_{i\in I}\mathcal{I}_i} = \bigcup_{i\in I} X^{\mathcal{I}_i}$ for $X \in \mathsf{N_C} \cup \mathsf{N_R}$.

If $\Delta^{\mathcal{I}_i} \cap \Delta^{\mathcal{I}_j} = \emptyset$ for all distinct $i, j \in I$, then $\sum_{i\in I} \mathcal{I}_i$ is a *disjoint union*. An FO-sentence φ is *invariant under disjoint unions* if for all families $(\mathcal{I}_i)_{i\in I}$ of interpretations with pairwise disjoint domains, we have $\sum_{i\in I} \mathcal{I}_i \models \varphi$ iff $\mathcal{I}_i \models \varphi$ for all $i \in I$. Similar to products, one can show that an FO-sentence is invariant under disjoint unions iff it is invariant under binary disjoint unions.

Example 8. Examples of Boolean TBoxes not invariant under disjoint unions are (i) $\varphi_1 = (\top \sqsubseteq A) \vee (\top \sqsubseteq B)$, since the disjoint union $\mathcal{I}$ of interpretations $\mathcal{I}_1, \mathcal{I}_2$ with $A^{\mathcal{I}_1} = \Delta^{\mathcal{I}_1}$, $B^{\mathcal{I}_1} = \emptyset$, and, respectively, $B^{\mathcal{I}_2} = \Delta^{\mathcal{I}_2}$, $A^{\mathcal{I}_2} = \emptyset$ is not a model of φ_1; and (ii) $\varphi_2 = \neg(\top \sqsubseteq A)$, since $\mathcal{I}$ is a model of φ_2, but $\mathcal{I}_1$ is not.

Theorem 9. *Let $\mathcal{L} \in \mathsf{ExpDL}$ not contain nominals and φ be an FO-sentence. The following conditions are equivalent:*

1. *there exists a $\mathcal{L}$-TBox $\mathcal{T}$ such that $\mathcal{T} \equiv \varphi$;*
2. *φ is invariant under $\sim^g_{\mathcal{L}}$ and disjoint unions.*

Proof. (sketch) The direction $1 \Rightarrow 2$ is straightforward based on Theorem 3, Point 1. For the converse, let φ be invariant under $\sim^g_{\mathcal{L}}$ and disjoint unions and consider the set $\mathsf{cons}(\varphi)$ of all $\mathcal{L}$-concept inclusions $C \sqsubseteq D$ such that $\varphi \models C \sqsubseteq D$. We are done if we can show that $\mathsf{cons}(\varphi) \models \varphi$: by compactness, one can find a finite $\mathcal{T} \subseteq \mathsf{cons}(\varphi)$ with $\mathcal{T} \models \varphi$, thus $\mathcal{T}$ is the desired $\mathcal{L}$-TBox. Assume to the contrary that $\mathsf{cons}(\varphi) \not\models \varphi$. Our aim is to construct ω-saturated interpretations $\mathcal{I}^-$ and $\mathcal{I}^+$ such that $\mathcal{I}^- \not\models \varphi$, $\mathcal{I}^+ \models \varphi$, and for all $d_1 \in \Delta^{\mathcal{I}_1}$ there exists $d_2 \in \Delta^{\mathcal{I}_2}$ with $(\mathcal{I}_1, d_1) \equiv_{\mathcal{L}} (\mathcal{I}_2, d_2)$ and vice versa. By Theorem 3, this implies $\mathcal{I}^- \sim^g_{\mathcal{L}} \mathcal{I}^+$, in contradiction to φ being invariant under $\sim^g_{\mathcal{L}}$. For each $\mathcal{L}$-concept inclusion $C \sqsubseteq D \notin \mathsf{cons}(\varphi)$, take a model $\mathcal{I}_{C \not\sqsubseteq D}$ of φ that refutes $C \sqsubseteq D$. Then $\mathcal{I}^+$ is defined as the disjoint union of all $\mathcal{I}_{C \not\sqsubseteq D}$ and $\mathcal{I}^-$ is defined as the disjoint union of $\mathcal{I}^+$ with a model of $\mathsf{cons}(\varphi) \cup \{\neg\varphi\}$. It follows from invariance of φ under disjoint unions that $\mathcal{I}^- \not\models \varphi$ and $\mathcal{I}^+ \models \varphi$. Moreover, $\mathcal{I}^-$ and $\mathcal{I}^+$ satisfy the same $\mathcal{L}$-concept inclusions. Using the condition that $\mathcal{L} \in \mathsf{ExpDL}$, one can now show that ω-saturated interpretations that are elementary equivalent to $\mathcal{I}^+$ and $\mathcal{I}^-$ are as required. ❑

In a modal logic context, disjoint unions have first been used to characterize global consequence in [de Rijke and Sturm, 2001]. We exploit the purely model-theoretic characterizations given in Theorems 7 and 9 to obtain an easy, worst-case optimal algorithm deciding whether a Boolean TBox is equivalent to a TBox.

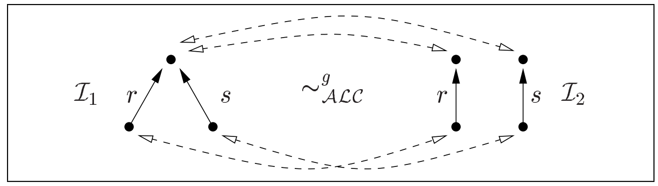

Figure 5: Globally bisimilar interpretations

Theorem 10. *Let $\mathcal{L} \in \mathsf{ExpDL}$ not contain nominals. Then it is* ExpTime*-complete to decide whether a Boolean $\mathcal{L}$-TBox is invariant under disjoint unions (equivalently, whether it is equivalent to an $\mathcal{L}$-TBox).*

Proof. (sketch) The proof is by mutual reduction with the unsatisfiability problem for Boolean $\mathcal{L}$-TBoxes, which is ExpTime-complete in all cases [Baader *et al.*, 2003]. We focus on the upper bound. Let φ be a Boolean $\mathcal{L}$-TBox. For a concept name A, denote by φ_A the relativization of φ to A, i.e., a Boolean TBox such that any interpretation $\mathcal{I}$ is a model of φ_A iff the restriction of $\mathcal{I}$ to the domain $A^{\mathcal{I}}$ is a model of φ. Take fresh concept names A_1, A_2 and let χ be the conjunction of

$A_1 \sqcap A_2 \sqsubseteq \bot$, $\top \sqsubseteq A_1 \sqcup A_2$, $A_i \sqsubseteq \forall r.A_i$, $\neg(A_i \sqsubseteq \bot)$,

for all role names r in φ and $i \in \{1, 2\}$, expressing that $\mathcal{I}$ is partitioned into two disjoint and unconnected parts, identified by A_1 and A_2. Then φ is invariant under binary disjoint unions iff the Boolean $\mathcal{L}$-TBox $\chi \rightarrow (\varphi_{A_1} \wedge \varphi_{A_2} \leftrightarrow \varphi)$ is a tautology. ❑

A further algorithmic application of Theorem 9 and of other characterizations that we will establish later is based on the following notion.

Definition 11 (TBox-rewritability)**.** Let $\mathcal{L}_1, \mathcal{L}_2 \in \mathsf{DL}$. A TBox $\mathcal{T}$ is $\mathcal{L}_1$-*rewritable* if it is equivalent to some $\mathcal{L}_1$-TBox. Then *$\mathcal{L}_1$-to-$\mathcal{L}_2$ TBox-rewritability* is the problem to decide whether a given $\mathcal{L}_1$-TBox is $\mathcal{L}_2$-rewritable.

If $\mathcal{L}_1, \mathcal{L}_2 \in \mathsf{ExpDL}$ do not contain nominals, then it follows from Theorem 9 that an $\mathcal{L}_1$-TBox $\mathcal{T}$ is $\mathcal{L}_2$-rewritable iff $\mathcal{T}$ it is invariant under $\sim^g_{\mathcal{L}_2}$. This provides a way to obtain decision procedures for TBox-rewritability, which we explore for the first few steps in this paper: we consider $\mathcal{ALCI}$-to-$\mathcal{ALC}$ rewritability in this section, and $\mathcal{ALC}$-to-$\mathcal{EL}$ and $\mathcal{ALCI}$-to-DL-Lite rewritability in the subsequent one. The basis of the algorithms is that a TBox $\mathcal{T}$ is *not* $\mathcal{L}_2$-rewritable iff there are two interpretations related by $\sim^g_{\mathcal{L}_2}$ such that one is a model of $\mathcal{T}$, but the other one is not.

Example 12. A typical rewriting between $\mathcal{ALCI}$ and $\mathcal{ALC}$ are range restrictions, which can be expressed by $\exists r^-.\top \sqsubseteq B$ in $\mathcal{ALCI}$ and rewritten as $\top \sqsubseteq \forall r.B$ in $\mathcal{ALC}$. Contrastingly, the $\mathcal{ALCI}$-TBox $\mathcal{T} = \{\exists r^-.\top \sqcap \exists s^-.\top \sqsubseteq B\}$ is not invariant under $\sim^g_{\mathcal{ALC}}$: in Figure 5, $\mathcal{T}$ is satisfied in $\mathcal{I}_2$, but not in $\mathcal{I}_1$ (where $B^{\mathcal{I}_1} = B^{\mathcal{I}_2} = \emptyset$). Thus, $\mathcal{T}$ is not equivalent to any $\mathcal{ALC}$-TBox.

The following result is proved by a non-trivial refinement of the method of type elimination known from complexity proofs in modal and description logic. We leave a matching lower complexity bound as an open problem for now.

Theorem 13. *$\mathcal{ALCI}$-to-$\mathcal{ALC}$ TBox rewritability is decidable in 2-*ExpTime*.*

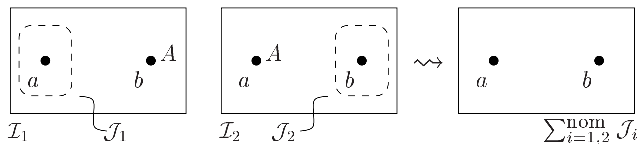

Figure 6: Nominal disjoint union

Theorem 9 excludes DLs with nominals since it is not clear how to interpret nominals in a disjoint union such that they are still singletons. In the following, we devise a relaxed variant of disjoint unions that respects nominals. For simplicity, we only consider DLs with nominals that have inverse roles as well (our approach can also be made to work otherwise, but becomes more technical).

A *component* of an interpretation $\mathcal{I}$ is a set $D \subseteq \Delta^{\mathcal{I}}$ that is closed under neighbors, i.e., if $d \in D$ and $(d, d') \in \bigcup_{r \in \mathsf{N_R}} r^{\mathcal{I}} \cup (r^-)^{\mathcal{I}}$, then $d' \in D$. A *component interpretation* of $\mathcal{I}$ is the restriction $\mathcal{J}$ of $\mathcal{I}$ to some domain $\Delta^{\mathcal{J}} \subseteq \Delta^{\mathcal{I}}$ that is a component of $\mathcal{I}$, i.e., $A^{\mathcal{J}} = A^{\mathcal{I}} \cap \Delta^{\mathcal{J}}$ for all $A \in \mathsf{N_C}$, $r^{\mathcal{J}} = r^{\mathcal{I}} \cap (\Delta^{\mathcal{J}} \times \Delta^{\mathcal{J}})$ for all $r \in \mathsf{N_R}$, and $a^{\mathcal{J}} = a^{\mathcal{I}}$ for $a \in \mathsf{N_I}$ if $a^{\mathcal{I}} \in \Delta^{\mathcal{J}}$; otherwise, $a^{\mathcal{J}}$ is simply undefined. We denote by $\mathsf{Nom}(\mathcal{J})$ the set of individual names interpreted by $\mathcal{J}$. Now let $(\mathcal{J}_i)_{i \in I}$ be a family of component interpretations such that

- $\bigcup_{i \in I} \mathsf{Nom}(\mathcal{J}_i) = \mathsf{N_I}$;
- $\mathsf{Nom}(\mathcal{J}_i) \cap \mathsf{Nom}(\mathcal{J}_j) = \emptyset$ for all $i \neq j$.

Then the *nominal disjoint union* of $(\mathcal{J}_i)_{i \in I}$, denoted $\sum_{i \in I}^{\mathsf{nom}} \mathcal{J}_i$, is the interpretation obtained by taking the disjoint union of $(\mathcal{J}_i)_{i \in I}$ and then interpreting each $a \in \mathsf{N_I}$ as $a^{\mathcal{J}_i}$ for the unique $i \in I$ with $a^{\mathcal{J}_i}$ defined.

An FO-sentence φ is *invariant under nominal disjoint unions* if the following conditions hold for all families $(\mathcal{I}_i, \mathcal{J}_i)_{i \in I}$ with $\mathcal{I}_i$ an interpretation and $\mathcal{J}_i$ a component interpretation of $\mathcal{I}_i$, for all $i \in I$:

(a) if $\mathcal{I}_i$ is a model of φ for all $i \in I$, then so is $\sum_{i \in I}^{\mathsf{nom}} \mathcal{J}_i$;

(b) if $\sum_{i \in I}^{\mathsf{nom}} \mathcal{J}_i$ is a model of φ and $\mathcal{I}_{i_0} = \mathcal{J}_{i_0}$ for some $i_0 \in I$, then $\mathcal{I}_{i_0}$ is a model of φ.

Note that, in Condition (b), $\mathcal{I}_{i_0} = \mathcal{J}_{i_0}$ implies that $\mathsf{Nom}(\mathcal{J}_{i_0})$ is the set of all individual names, but not necessarily that $\sum_{i \in I}^{\mathsf{nom}} \mathcal{J}_i = \mathcal{J}_{i_0}$. We can now characterize TBoxes formulated in expressive DLs with nominals.

Theorem 14. *Let $\mathcal{L} \in \{\mathcal{ALCIO}, \mathcal{ALCQIO}\}$ and φ be an FO-sentence. Then the following conditions are equivalent:*

1. *there exists an $\mathcal{L}$-TBox $\mathcal{T}$ such that $\mathcal{T} \equiv \varphi$;*
2. *φ is invariant under $\sim_{\mathcal{L}}^{g}$ and nominal disjoint unions.*

Example 15. Condition (a) of nominal disjoint unions can be used to show that $\varphi = A(a) \vee A(b)$ cannot be rewritten as an $\mathcal{ALCQIO}$-TBox. To see this, observe that $\mathcal{I}_1$ and $\mathcal{I}_2$ of Figure 6 satisfy φ and $\sum_{i=1,2}^{\mathsf{nom}} \mathcal{J}_i$ does not satisfy φ.

Similar to the proof of Theorem 10, one can use relativization to reduce the problem of checking invariance under nominal disjoint unions of Boolean $\mathcal{L}$-TBoxes to the unsatisfiability problem for Boolean $\mathcal{L}$-TBoxes (which is ExpTime-complete for $\mathcal{ALCIO}$ and coNExpTime-complete for $\mathcal{ALCQIO}$ [Baader *et al.*, 2003]):

Theorem 16. *It is* ExpTime*-complete to decide whether a Boolean $\mathcal{ALCIO}$-TBox is invariant under nominal disjoint unions (equivalently, whether it is equivalent to an $\mathcal{ALCIO}$-TBox). The problem is co*NExpTime*-complete for Boolean $\mathcal{ALCQIO}$-TBoxes.*

5 Characterizing TBoxes, Lightweight DLs

We characterize TBoxes formulated in $\mathcal{EL}$ and members of the DL-Lite families. We start with an analogue of Theorem 5: since the considered DLs are 'Horn' in nature, we add products to the closure properties identified in Section 4 and refine our proofs accordingly.

Theorem 17. *Let $\mathcal{L} \in \{\mathcal{EL}, \text{DL-Lite}_{\mathsf{horn}}\}$ and let φ be an FO-sentence. The following conditions are equivalent:*

1. *φ is equivalent to an $\mathcal{L}$-TBox;*
2. *φ is invariant under $\sim_{\mathcal{L}}^{g}$ and disjoint unions, and preserved under products.*

Proof. (sketch) In principle, we follow the strategy of the proof of Theorem 9. A problem is posed by the fact that, unlike in the case of expressive DLs, two ω-saturated interpretations $\mathcal{I}^-$ and $\mathcal{I}^+$ that satisfy the same $\mathcal{L}$-CIs need not satisfy $\mathcal{I}^- \equiv_{\mathcal{L}}^{g} \mathcal{I}^+$ (e.g. when $\mathcal{I}^-$ consists of three elements that satisfy $A \sqcap \neg B$, and $B \sqcap \neg A$, and $\neg A \sqcap \neg B$, respectively, and $\mathcal{I}^+$ consists of two elements that satisfy $A \sqcap \neg B$ and $B \sqcap \neg A$, respectively). To deal with this, we ensure that $\mathcal{I}^-$ and $\mathcal{I}^+$ satisfy the same *disjunctive $\mathcal{L}$-CIs*, i.e., CIs of the form $C \sqsubseteq D_1 \sqcup \cdots \sqcup D_n$ with $C, D_1, \ldots, D_n$ $\mathcal{L}$-concepts; this suffices to prove $\mathcal{I}^- \equiv_g \mathcal{I}^+$ as required. The construction of $\mathcal{I}^-$ is essentially as in the proof of Theorem 9 while the construction of $\mathcal{I}^+$ uses products to bridge the gap between $\mathcal{L}$-CIs and disjunctive $\mathcal{L}$-CIs. ❑

We apply Theorem 17 to TBox rewritability, starting with the $\mathcal{ALC}$-to-$\mathcal{EL}$ case. By Theorems 9 and 17, an $\mathcal{ALC}$-TBox is equivalent to some $\mathcal{EL}$-TBox iff it is invariant under $\sim_{\mathcal{EL}}^{g}$ and preserved under binary products. The following theorem, the proof of which is rather involved, establishes the complexity of both problems.

Theorem 18. *Invariance of $\mathcal{ALC}$-TBoxes under $\sim_{\mathcal{EL}}^{g}$ is* ExpTime*-complete. Preservation of $\mathcal{ALC}$-TBoxes under products is co*NExpTime*-complete.*

From Theorems 18 and 17 we obtain:

Theorem 19. *$\mathcal{ALC}$-to-$\mathcal{EL}$ TBox rewritability is in co*NExpTime.

One can easily show ExpTime-hardness of $\mathcal{ALC}$-to-$\mathcal{EL}$ TBox rewritability by reduction of satisfiability of $\mathcal{ALC}$-TBoxes. Namely, $\mathcal{T}$ is satisfiable iff $\mathcal{T} \cup \{A \sqsubseteq \forall r.B\}$ cannot be rewritten into an $\mathcal{EL}$-TBox, where A, B, r do not occur in $\mathcal{T}$. Finding a tight bound remains open.

We now consider $\mathcal{ALCI}$-to-DL-Lite$_{\mathsf{horn}}$ TBox rewritability and establish ExpTime-completeness. In contrast to $\mathcal{ALC}$-to-$\mathcal{EL}$ rewritability, where it is not clear whether or not the computationally expensive check for preservation under products can be avoided, here a rather direct approach is possible that relies only on deciding invariance under $\sim_{\text{DL-Lite}_{\mathsf{horn}}}$.

Theorem 20. *$\mathcal{ALCI}$-to-DL-Lite$_{\mathsf{horn}}$-TBox rewritability is* ExpTime*-complete.*

Proof. (sketch) First decide in ExpTime whether $\mathcal{T}$ is invariant under $\sim_{\text{DL-Lite}_{\text{horn}}}$. If not, then $\mathcal{T}$ is not equivalent to any DL-Lite$_{\text{horn}}$-TBox. If yes, check, in exponential time, whether for every $B_1 \sqcap \cdots \sqcap B_n \sqsubseteq B'_1 \sqcup \cdots \sqcup B'_m$ that follows from $\mathcal{T}$ with all B_i, B'_i basic concepts, there exists j such that $B_1 \sqcap \cdots \sqcap B_n \sqsubseteq B'_j$ follows from $\mathcal{T}$. $\mathcal{T}$ is equivalent to some DL-Lite$_{\text{horn}}$-TBox iff this is the case. ❑

The original DL-Lite dialects do not admit conjunction as a concept constructor, or only to express disjointness constraints. More precisely, a *DL-Lite$_{\text{core}}$-TBox* is a finite set of inclusions $B_1 \sqsubseteq B_2$, where B_1, B_2 are basic DL-Lite concepts as defined in Section 2. A *DL-Lite$^d_{\text{core}}$-TBox* admits, in addition, inclusions $B_1 \sqcap B_2 \sqsubseteq \bot$ expressing disjointness of B_1 and B_2. To characterize TBoxes formulated in DL-Lite$_{\text{core}}$ and DL-Lite$^d_{\text{core}}$, we additionally require preservation under (non-disjoint) unions and compatible unions, respectively. The latter are unions of interpretations $(\mathcal{I}_i)_{i \in I}$ that can be formed only if the family $(\mathcal{I}_i)_{i \in I}$ is *compatible*, i.e., for any $d \in \Delta^{\mathcal{I}_i} \cap \Delta^{\mathcal{I}_j}$ and basic DL-Lite concepts B_1, B_2 such that $d \in B_1^{\mathcal{I}_i} \cap B_2^{\mathcal{I}_j}$ there exists $\mathcal{I}_\ell$ with $(B_1 \sqcap B_2)^{\mathcal{I}_\ell} \neq \emptyset$. Preservation of FO-sentences under (compatible) unions is defined in the obvious way. The proof of the following theorem is similar to that of Theorem 17, except that the construction of $\mathcal{I}^+$ is yet a bit more intricate.

Theorem 21. *Let φ be an FO-sentence. Then the following conditions are equivalent:*

1. *φ is equivalent to a DL-Lite$_{\text{core}}$-TBox (DL-Lite$^d_{\text{core}}$-TBox);*
2. *φ is invariant under $\sim^g_{\text{DL-Lite}_{\text{horn}}}$ and disjoint unions, and preserved under products and unions (compatible unions).*

Note that it is not possible to strengthen Condition 2 of Theorem 21 by requiring φ to be *invariant* under unions as this results in failure of the implication $1 \Rightarrow 2$.

Because of the fact that there are only polynomially many concept inclusions over any finite signature, TBox rewritability into DL-Lite$_{\text{core}}$ and DL-Lite$^d_{\text{core}}$ is a comparably simple problem and semantic characterizations are less fundamental here than for more expressive DLs. In fact, for $\mathcal{L} \in \mathsf{ExpDL}$ that contains inverse roles, one can reduce $\mathcal{L}$-to-DL-Lite$_{\text{core}}$ rewritability to Boolean $\mathcal{L}$-TBox unsatisfiability. Conversely (and trivially), $\mathcal{L}$-TBox unsatisfiability can be reduced to $\mathcal{L}$-to-DL-Lite$_{\text{core}}$ TBox rewritability. As for all expressive DLs in this paper the complexity of TBox satisfiability and Boolean TBox satisfiability coincide, this yields tight complexity bounds. The same holds for DL-Lite$^d_{\text{core}}$. For a related study of approximation in DL-Lite, see [Botoeva *et al.*, 2010].

6 Discussion

We believe that the results established in this paper have many potential applications in areas where the expressive power of TBoxes plays a central role, such as TBox approximation and modularity. We also believe that the problem of TBox rewritability, studied here as an example application of our characterization results, is interesting in its own right. A more comprehensive study, including the actual computation of rewritten TBoxes, remains as future work.

The DLs standardized as OWL 2 and its profiles have additional expressive power compared to the 'core DLs' studied in this paper. While full OWL 2 is probably too complex to admit really succinct characterizations of the kind established here, some extensions are possible as follows: each of Theorems 9, 14, and 17 still holds when the admissible interpretations are restricted to some class that is definable by an FO-sentence preserved under the notion of (disjoint) union and product used in that theorem. This captures many features of OWL such as transitive roles, role hierarchy axioms, and even role inclusion axioms.

References

[Artale *et al.*, 2009] A. Artale, D. Calvanese, R. Kontchakov, and M. Zakharyaschev. The DL-Lite family and relations. *J. Artif. Intell. Res. (JAIR)*, 36:1–69, 2009.

[Baader *et al.*, 2003] F. Baader, D. Calvanese, D. McGuinness, D. Nardi, and P. Patel-Schneider. *The Description Logic Handbook*. Cambridge University Press, 2003.

[Baader, 1996] F. Baader. A formal definition for the expressive power of terminological knowledge representation languages. *J. Log. Comput.*, 6(1):33–54, 1996.

[Borgida, 1996] A. Borgida. On the relative expressiveness of description logics and predicate logics. *Artif. Intell.*, 82(1-2):353–367, 1996.

[Botoeva *et al.*, 2010] E. Botoeva, D. Calvanese, and M. Rodriguez-Muro. Expressive approximations in DL-Lite ontologies. In *Proc. of AIMSA*, volume 6304 of *LNCS*, pages 21–31. Springer, 2010.

[Calvanese *et al.*, 2005] D. Calvanese, G. De Giacomo, D. Lembo, M. Lenzerini, and R. Rosati. DL-Lite: Tractable description logics for ontologies. In *Proc. of AAAI*, pages 602–607. AAAI Press, 2005.

[Chang and Keisler, 1990] C. C. Chang and H. J. Keisler. *Model Theory*. Elsevier, 1990.

[de Rijke, 2000] M. de Rijke. A note on graded modal logic. *Studia Logica*, 64(2):271–283, 2000.

[de Rijke and Sturm, 2001] M. de Rijke and H. Sturm. Global definability in modal logic. In *Essays on Non-classical Logic*, pages 111–135. World Scientific Publishers, 2001.

[Goranko and Otto, 2007] V. Goranko and M. Otto. Model theory of modal logic. In *Handbook of Modal Logic*. Elsevier, 2007.

[Janin and Lenzi, 2004] D. Janin and G. Lenzi. On the Relationship Between Monadic and Weak Monadic Second Order Logic on Arbitrary Trees, with Applications to the mu-Calculus. *Fundam. Inform*, 61(3-4):247–265, 2004.

[Kurtonina and de Rijke, 1999] N. Kurtonina and M. de Rijke. Expressiveness of concept expressions in first-order description logics. *Artif. Intell.*, 107(2):303–333, 1999.

[Ren *et al.*, 2010] Y. Ren, J. Z. Pan, and Y. Zhao. Soundness preserving approximation for TBox reasoning. In *Proc. of AAAI*, AAAI Press, 2010.

[Sturm, 2000] H. Sturm. Modal Horn Classes. *Studia Logica*, 64(3):301–313, 2000.

[Tserendorj *et al.*, 2008] T. Tserendorj, S. Rudolph, M. Krötzsch, and P. Hitzler. Approximate OWL-reasoning with screech. In *Proc. of RR*, volume 5341 of *LNCS*, pages 165–180. Springer, 2010.

A Proofs for Section 3

To begin this section, we give a precise definition of ω-saturated interpretations. In what follows we assume that $\mathsf{N_C} \cup \mathsf{N_R} \cup \mathsf{N_I}$ and the domain $\Delta^\mathcal{I}$ of an interpretation $\mathcal{I}$ are disjoint sets. We can regard elements of $\Delta^\mathcal{I}$ as additional individual symbols that have a fixed interpretation in $\mathcal{I}$, defined by setting $a^\mathcal{I} = a$ for all $a \in \Delta^\mathcal{I}$.

Let $\mathcal{I}$ be an interpretation. A set Γ of FO-formulas with free variables among $x_1, \dots, x_n$, predicate symbols from $\mathsf{N_C} \cup \mathsf{N_R}$, and individual symbols from $\mathsf{N_I} \cup \Delta^\mathcal{I}$ is called

- *realizable* in $\mathcal{I}$ if there exists a variable assignment $a(x_i) \in \Delta^\mathcal{I}$, $1 \leq i \leq n$, such that $\mathcal{I} \models_a \varphi$ for all $\varphi \in \Gamma$.
- *finitely realizable* in $\mathcal{I}$ if for every finite subset Γ' of Γ there exists a variable assignment $a(x_i) \in \Delta^\mathcal{I}$, $1 \leq i \leq n$, such that $\mathcal{I} \models_a \varphi$ for all $\varphi \in \Gamma'$.

We call an interpretation $\mathcal{I}$ *ω-saturated* if the following holds for every such set Γ that uses only finitely many individual symbols from $\Delta^\mathcal{I}$: if Γ is finitely realizable in $\mathcal{I}$, then Γ is realizable in $\mathcal{I}$.

We apply the following existence theorem for ω-saturated interpretations (cf. [Chang and Keisler, 1990]).

Theorem 22. *For every interpretation $\mathcal{I}$ there exists an interpretation $\mathcal{I}^*$ that is ω-saturated and satisfies the same FO-sentences as $\mathcal{I}$ (is elementary equivalent to $\mathcal{I}$).*

In our proofs, we will often use the notion of a type. Formally, for a DL $\mathcal{L}$, an interpretation $\mathcal{I}$, and a $d \in \Delta^\mathcal{I}$, the *$\mathcal{L}$-type* of d in $\mathcal{I}$, denoted $t^\mathcal{I}_\mathcal{L}(d)$, is the set of $\mathcal{L}$-concepts C such that $d \in C^\mathcal{I}$.

We are in the position now to prove the results of Section 3.

Theorem 3 Let $\mathcal{L} \in \{\mathcal{EL}, \text{DL-Lite}_\mathsf{horn}\} \cup \mathsf{ExpDL}$ and let $(\mathcal{I}_1, d_1)$ and $(\mathcal{I}_2, d_2)$ objects.

- If $d_1 \sim_\mathcal{L} d_2$, then $d_1 \equiv_\mathcal{L} d_2$;
- If $d_1 \equiv_\mathcal{L} d_2$ and both objects are ω-saturated, then $d_1 \sim_\mathcal{L} d_2$.

For $\mathcal{ALC}$, various proofs of this result are known, mostly from the modal logic literature [Goranko and Otto, 2007]. Many of them are easily extended so as to cover $\mathcal{ALCO}$, $\mathcal{ALCI}$, and $\mathcal{ALCIO}$. Here we present proofs for $\mathcal{ALCQ}$, $\mathcal{EL}$, and DL-Lite$_\mathsf{horn}$. The extensions to the remaining members of ExpDL ($\mathcal{ALCQI}$, $\mathcal{ALCQIO}$) are straightforward and left to the reader.

Proof for $\mathcal{ALCQ}$. Assume first that $(\mathcal{I}_1, d_1) \sim_\mathcal{ALCQ} (\mathcal{I}_2, d_2)$ and let $S \subseteq \Delta^{\mathcal{I}_1} \times \Delta^{\mathcal{I}_2}$ satisfy [Atom] for all $A \in \mathsf{N_C}$, [QForth], and [QBack] such that $(d_1, d_2) \in S$. We show $e_1 \equiv_\mathcal{ALCQ} e_2$ for all $(e_1, e_2) \in S$; it follows that $d_1 \equiv_\mathcal{ALCQ} d_2$, as required. The proof is by induction over the construction of $\mathcal{ALCQ}$-concepts. Thus, we show by induction for all $\mathcal{ALCQ}$-concepts C:

Claim 1. $e_1 \in C^{\mathcal{I}_1}$ iff $e_2 \in C^{\mathcal{I}_2}$, for all $(e_1, e_2) \in S$.

If C is a concept name, then Claim 1 follows from [Atom]. The steps for the Boolean connectives are straightforward. Now assume $C = (\geqslant n\ r\ D)$ and let $e_1 \in (\geqslant n\ r\ D)^{\mathcal{I}_1}$. Let $X \subseteq \mathsf{succ}_r^{\mathcal{I}_1}(e_1)$ be of cardinality n such that $e \in D^{\mathcal{I}_1}$ for all $e \in X$. By [QForth], there exists $Y \subseteq \mathsf{succ}_r^{\mathcal{I}_2}(e_2)$ such that S contains a bijection between X and Y. By induction hypothesis $e' \in D^{\mathcal{I}_2}$ for all $e' \in Y$. Thus $e_2 \in (\geqslant n\ r\ D)^{\mathcal{I}_2}$, as required. The reverse condition can be proved in the same way using [QBack]. The case $(\leqslant\ n\ r\ D)$ can be proved similarly.

Conversely, assume that $(\mathcal{I}_1, d_1) \equiv_\mathcal{ALCQ} (\mathcal{I}_2, d_2)$ and $\mathcal{I}_1, \mathcal{I}_2$ are ω-saturated. Set

$$S := \{(e_1, e_2) \in \Delta^{\mathcal{I}_1} \times \Delta^{\mathcal{I}_2} \mid e_1 \equiv_\mathcal{ALCQ} e_2\}$$

We show that S satisfies [Atom], [QForth], and [QBack]. (Then $d_1 \sim_\mathcal{ALCQ} d_2$, as required.) As [Atom] follows directly from the definition of S and [QBack] can be proved in the same way as [QForth], we focus on [QForth]. Assume $(e_1, e_2) \in S$ and $D_1 \subseteq \mathsf{succ}_r^{\mathcal{I}_1}(e_1)$ is finite. Take an individual variable x_d for every $d \in D_1$ and consider the set of FO-formulas $\Gamma = \Gamma^{\neq} \cup \Gamma^r \cup \bigcup_{d \in D_1} \mathsf{type}(d)$, where

- $\Gamma^{\neq} = \{\neg(x_d = x_{d'}) \mid d \neq d', d, d' \in D_1\}$;
- $\mathsf{type}(d) = \{C^\sharp(x_d) \mid C \in t^{\mathcal{I}_1}_\mathcal{ALCQ}(d)\}$;
- $\Gamma^r = \{r(e_2, x_d) \mid d \in D_1\}$.

Note that Γ', the set Γ with e_2 replaced by e_1, is realizable in $\mathcal{I}_1$ by the assignment $a(x_d) = d$, for $d \in D_1$. Using ω-saturatedness of $\mathcal{I}_2$ and $e_1 \equiv_\mathcal{ALCQ} e_2$, it is readily check that Γ is realizable in $\mathcal{I}_2$. Assume Γ is realizable in $\mathcal{I}_2$ by the variable assignment $a(x_d)$, $d \in D_1$. Let

$$D_2 = \{a(x_d) \mid d \in D_1\}.$$

Then $d \equiv_\mathcal{ALCQ} a(x_d)$ for all $d \in D_1$ (by $\mathsf{type}(d)$), $D_2 \subseteq \mathsf{succ}_r^{\mathcal{I}_2}(e_2)$ (by Γ^r), and $d \mapsto a(x_d)$ is a bijection from D_1 to D_2 (by $\Gamma^{\neq}$). Thus [QForth] holds.

This finishes the proof for $\mathcal{ALCQ}$.

Proof for $\mathcal{EL}$. Assume first that $(\mathcal{I}_1, d_1) \sim_\mathcal{EL} (\mathcal{I}_2, d_2)$. Then $(\mathcal{I}_1, d_1) \leq_\mathcal{EL} (\mathcal{I}_2, d_2)$ and $(\mathcal{I}_2, d_2) \leq_\mathcal{EL} (\mathcal{I}_1, d_1)$ and so there exists an $\mathcal{EL}$-simulation S_1 between $\mathcal{I}_1$ and $\mathcal{I}_2$ with $(d_1, d_2) \in S_1$ and an $\mathcal{EL}$-simulation S_2 between $\mathcal{I}_2$ and $\mathcal{I}_1$ with $(d_2, d_1) \in S_2$. We show the following

- if $(e_1, e_2) \in S_1$ and $e_1 \in C^{\mathcal{I}_1}$, then $e_2 \in C^{\mathcal{I}_2}$, for all $\mathcal{EL}$-concepts C;
- if $(e_2, e_1) \in S_2$ and $e_2 \in C^{\mathcal{I}_2}$, then $e_1 \in C^{\mathcal{I}_1}$, for all $\mathcal{EL}$-concepts C.

Points 1 and 2 together and $(d_1, d_2) \in S_1$, $(d_2, d_1) \in S_2$ imply $d_1 \equiv_\mathcal{EL} d_2$, as required. We provide a proof of Point 1. The proof is by induction on the construction of C. For concept names, the claim follows from [AtomR]. For $\top$ and $\bot$ the claim is trivial. For conjunction the proof is trivial. Now assume $C = \exists r.D$, $(e_1, e_2) \in S_1$ and $e_1 \in C^{\mathcal{I}_1}$. There exists e_1' with $(e_1, e_1') \in r^{\mathcal{I}_1}$ such that $e_1' \in D^{\mathcal{I}_2}$. By [Forth], there exists e_2' with $(e_2, e_2') \in r^{\mathcal{I}_2}$ such that $(e_1', e_2') \in S_1$. By induction hypothesis, $e_2' \in D^{\mathcal{I}_2}$. Thus, $e_2 \in C^{\mathcal{I}_2}$, as required.

Conversely, let $(\mathcal{I}_1, d_1) \equiv_\mathcal{EL} (\mathcal{I}_2, d_2)$ and assume that $\mathcal{I}_1, \mathcal{I}_2$ are ω-saturated. Let

$$S_1 = \{(e_1, e_2) \in \Delta^{\mathcal{I}_1} \times \Delta^{\mathcal{I}_2} \mid t^{\mathcal{I}_1}_\mathcal{EL}(e_1) \subseteq t^{\mathcal{I}_2}_\mathcal{EL}(e_2)\}$$

and

$$S_2 = \{(e_2, e_1) \in \Delta^{\mathcal{I}_2} \times \Delta^{\mathcal{I}_1} \mid t^{\mathcal{I}_2}_{\mathcal{EL}}(e_2) \subseteq t^{\mathcal{I}_1}_{\mathcal{EL}}(e_1)\}.$$

We show that S_1 is a $\mathcal{EL}$-simulation between $\mathcal{I}_1$ and $\mathcal{I}_2$. The same argument shows that S_2 is a $\mathcal{EL}$-simulation between $\mathcal{I}_2$ and $\mathcal{I}_1$. Thus, from $(d_1, d_2) \in S_1$ and $(d_2, d_1) \in S_2$, we obtain $d_1 \sim_{\mathcal{EL}} d_2$, as required.

Property [AtomR] follows directly from the definition of S_1. We consider [Forth]. Let $(e_1, e_2) \in S_1$ and $(e_1, e_1') \in r^{\mathcal{I}_1}$. Take an individual variable x and consider the set of FO-formulas $\Gamma = \mathsf{type}(e_1') \cup \Gamma^r$, where

- $\mathsf{type}(e_1') = \{C^\sharp(x) \mid C \in t^{\mathcal{I}_1}_{\mathcal{EL}}(e_1')\}$;
- $\Gamma^r = \{r(e_2, x)\}$.

Note that Γ', the set Γ with e_2 replaced by e_1, is realizable in $\mathcal{I}_1$ by the assignment $a(x) = e_1'$. Using ω-saturatedness of $\mathcal{I}_2$ and $t^{\mathcal{I}_1}_{\mathcal{EL}}(e_1) \subseteq t^{\mathcal{I}_2}_{\mathcal{EL}}(e_2)$, it is readily check that Γ is realizable in $\mathcal{I}_2$. Assume Γ is realizable in $\mathcal{I}_2$ by the variable assignment $a(x)$. Then $(e_1', a(x)) \in S_1$ (by $\mathsf{type}(e_1')$) and $(e_2, a(x)) \in r^{\mathcal{I}_2}$ (by Γ^r). Thus [Forth] holds.

This finishes the proof for $\mathcal{EL}$.

Proof for DL-Lite$_{\mathsf{horn}}$. The proof for DL-Lite$_{\mathsf{horn}}$ is rather straightforward: no induction over concepts is required as there are no nestings of existential restrictions. Moreover, ω-saturatedness is not required for the implication from $\equiv_{\text{DL-Lite}_{\mathsf{horn}}}$ to $\sim_{\text{DL-Lite}_{\mathsf{horn}}}$.

Assume first that $(\mathcal{I}_1, d_1) \sim_{\text{DL-Lite}_{\mathsf{horn}}} (\mathcal{I}_2, d_2)$. Then $(\mathcal{I}_1, d_1) \leq_{\text{DL-Lite}_{\mathsf{horn}}} (\mathcal{I}_2, d_2)$ and $(\mathcal{I}_2, d_2) \leq_{\text{DL-Lite}_{\mathsf{horn}}} (\mathcal{I}_1, d_1)$ and so there exists a DL-Lite$_{\mathsf{horn}}$-simulation S_1 between $\mathcal{I}_1$ and $\mathcal{I}_2$ with $(d_1, d_2) \in S_1$ and a DL-Lite$_{\mathsf{horn}}$-simulation S_2 between $\mathcal{I}_2$ and $\mathcal{I}_1$ with $(d_2, d_1) \in S_2$. It is straightforward to show using the conditions on DL-Lite$_{\mathsf{horn}}$-simulations that

- if $(e_1, e_2) \in S_1$ and $e_1 \in C^{\mathcal{I}_1}$, then $e_2 \in C^{\mathcal{I}_2}$, for all DL-Lite$_{\mathsf{horn}}$-concepts C;
- if $(e_2, e_1) \in S_2$ and $e_2 \in C^{\mathcal{I}_2}$, then $e_1 \in C^{\mathcal{I}_1}$, for all DL-Lite$_{\mathsf{horn}}$-concepts C.

Points 1 and 2 together and $(d_1, d_2) \in S_1$, $(d_2, d_1) \in S_2$ imply $d_1 \equiv_{\text{DL-Lite}_{\mathsf{horn}}} d_2$, as required.

Conversely, assume $(\mathcal{I}_1, d_1) \equiv_{\text{DL-Lite}_{\mathsf{horn}}} (\mathcal{I}_2, d_2)$. Let

$$S = \{(e_1, e_2) \in \Delta^{\mathcal{I}_1} \times \Delta^{\mathcal{I}_2} \mid t^{\mathcal{I}_1}_{\text{DL-Lite}}(e_1) = t^{\mathcal{I}_2}_{\text{DL-Lite}}(e_2)\}.$$

It is reasily checked that S is a DL-Lite$_{\mathsf{horn}}$-simulation between $\mathcal{I}_1$ and $\mathcal{I}_2$ and that S^- is a DL-Lite$_{\mathsf{horn}}$-simulation between $\mathcal{I}_2$ and $\mathcal{I}_1$. We obtain $d_1 \sim_{\text{DL-Lite}_{\mathsf{horn}}} d_2$, as required. This finishes the proof for DL-Lite$_{\mathsf{horn}}$.

In the proof of Theorem 5 we will employ the following non-symmetric version of Theorem 3 for $\mathcal{EL}$ and DL-Lite$_{\mathsf{horn}}$ that follows directly from the proof of Theorem 3 above:

Lemma 23. *Let $\mathcal{L} \in \{\mathcal{EL}, \text{DL-Lite}_{\mathsf{horn}}\}$ and let $(\mathcal{I}_1, d_1)$ and $(\mathcal{I}_2, d_2)$ be objects.*

- *If $d_1 \leq_{\mathcal{L}} d_2$, then $t^{\mathcal{I}_1}_{\mathcal{L}}(d_1) \subseteq t^{\mathcal{I}_2}_{\mathcal{L}}(d_2)$;*
- *If $t^{\mathcal{I}_1}_{\mathcal{L}}(d_1) \subseteq t^{\mathcal{I}_2}_{\mathcal{L}}(d_2)$ and both objects are ω-saturated , then $d_1 \leq_{\mathcal{L}} d_2$.*

For a set Γ of FO-formulas and a FO-formula φ (all possibly containing free variables), we write $\Gamma \models \varphi$ if for every interpretation $\mathcal{I}$ with variable assigment a, we have $\mathcal{I} \models_a \varphi$ whenever $\mathcal{I} \models_a \psi$ for all $\psi \in \Gamma$. $\varphi \models \psi$ stands for $\{\varphi\} \models \psi$.

Theorem 4 Let $\mathcal{L} \in \mathsf{ExpDL}$ and $\varphi(x)$ a first-order formula with free variable x. Then the following conditions are equivalent:

1. there exists an $\mathcal{L}$-concept C such that $C^\sharp(x) \equiv \varphi(x)$;
2. $\varphi(x)$ is invariant under $\sim_{\mathcal{L}}$.

Proof. Let $\mathcal{L} \in \mathsf{ExpDL}$.

The direction $1 \Rightarrow 2$ follows from the fact $\mathcal{L}$-concepts are invariant under $\sim_{\mathcal{L}}$ (which has been shown in Theorem 3).

For the direction $2 \Rightarrow 1$ let $\varphi(x)$ be invariant under $\sim_{\mathcal{L}}$ but assume there is no $\mathcal{L}$-concept C such that $C^\sharp(x)$ is equivalent to $\varphi(x)$. Let

$$\mathrm{cons}(\varphi(x)) := \{C^\sharp(x) \mid C \text{ and } \mathcal{L}\text{-concept}, \varphi(x) \models C^\sharp(x))\}.$$

By compactness, $\mathrm{cons}(\varphi(x)) \cup \{\neg\varphi(x)\}$ is satisfiable. Let $\mathcal{I}^-$ be an interpretation satisfying $\mathrm{cons}(\varphi(x)) \cup \{\neg\varphi(x)\}$ under the assignment $a_2(x) = d_2$. We may assume that $\mathcal{I}^-$ is ω-saturated.

Claim 1. $\{\varphi(x)\} \cup \{C^\sharp(x) \mid C \in t^{\mathcal{I}}_{\mathcal{L}}(d_2)\}$ is satisfiable.

Assume that Claim 1 does not hold. Then, by compactness, there is a finite set $\Gamma \subseteq t^{\mathcal{I}}_{\mathcal{L}}(d_2)$ such that $\{\varphi(x)\} \cup \{C^\sharp(x) \mid C \in \Gamma\}$ is unsatisfiable. Thus,

$$\models \varphi(x) \rightarrow (\neg \bigsqcap_{C \in \Gamma} C)^\sharp(x)$$

which implies that $(\neg \bigsqcap_{C \in \Gamma} C)^\sharp(x) \in \mathrm{cons}(\varphi(x))$ (here we use the fact that $\mathcal{L}$-concepts are closed under forming negations and conjunctions) and so leads to a contradiction as $\mathrm{cons}(\varphi(x)) \subseteq \{C^\sharp(x) \mid C \in t^{\mathcal{I}}_{\mathcal{L}}(d_2)\}$.

Take an ω-saturated interpretation $\mathcal{I}^+$ satisfying $\{\varphi(x)\} \cup \{C^\sharp(x) \mid C \in t^{\mathcal{I}}_{\mathcal{L}}(d_2)\}$ under the assignment $a_1(x) = d_1$. By definition, $(\mathcal{I}_1, d_1) \equiv_{\mathcal{L}} (\mathcal{I}_2, d_2)$. By Theorem 3, $(\mathcal{I}_1, d_1) \sim_{\mathcal{L}} (\mathcal{I}_2, d_2)$. We have derived a contradiction as $\mathcal{I}_1 \models \varphi[d_1]$ but $\mathcal{I}_2 \not\models \varphi[d_2]$. ❑

Before proving Theorem 5, we determine the behaviour of $\mathcal{EL}$ and DL-Lite-concepts in direct products.

Lemma 24. *Let $\mathcal{L} \in \{\mathcal{EL}, \text{DL-Lite}_{\mathsf{horn}}\}$, C a $\mathcal{L}$-concept, and $(\mathcal{I}_i, d_i)$, $i \in I$, a family of objects. Then*

$$(d_i)_{i \in I} \in C^{\prod_{i \in I} \mathcal{I}_i} \quad \Leftrightarrow \quad \forall i \in I : d_i \in C^{\mathcal{I}_i}$$

Proof. Straightforward. ❑

Theorem 5 Let $\mathcal{L} \in \{\mathcal{EL}, \text{DL-Lite}_{\mathsf{horn}}\}$ and $\varphi(x)$ an FO-formula with free variable x. Then the following conditions are equivalent:

1. there exists an $\mathcal{L}$-concept C such that $C^\sharp(x) \equiv \varphi(x)$;
2. $\varphi(x)$ is preserved under $\mathcal{L}$-simulation and direct products.

Proof. It follows from Theorem 3 and Lemma 24 that $\mathcal{EL}$ and DL-Lite$_{\mathsf{horn}}$-concepts are preserved under the corresponding simulations and under forming direct products. The direction $1 \Rightarrow 2$ follows.

For the direction $1 \Rightarrow 2$, assume that $\varphi(x)$ is preserved under $\mathcal{L}$-simulations and direct products but is not equivalent to any $\mathcal{L}$-concept. Let

$$\mathrm{cons}(\varphi(x)) = \{C^\sharp(x) \mid C \text{ an } \mathcal{L}\text{-concept}, \varphi(x) \models C^\sharp(x)\}.$$

By compactness, $\mathrm{cons}(\varphi(x)) \cup \{\neg\varphi(x)\}$ is satisfiable. Let $\mathcal{I}^-$ be an ω-saturated interpretation satisfying $\mathrm{cons}(\varphi(x)) \cup \{\neg\varphi(x)\}$ under an assignment $a_2(x) = d_2$.

Let I be the set of $\mathcal{L}$-concepts C with $d_2 \notin C^{\mathcal{I}^-}$. For any $C \in I$, the set $\{\varphi(x), \neg(C^\sharp(x))\}$ is satisfiable, because otherwise $\varphi(x) \models C^\sharp(x)$ and hence $C^\sharp(x) \in \mathrm{cons}(\varphi(x))$, a contradiction to $\mathcal{I}^- \models_{a_2} \mathrm{cons}(\varphi(x))$. Let $\mathcal{I}_C$ denote an interpretation such that for some $d_C \in \Delta^{\mathcal{I}_C}$ we have $\mathcal{I}_C \models \varphi[d_C] \wedge \neg C^\sharp[d_C]$.

Define

$$\mathcal{I} = \prod_{C \in I} \mathcal{I}_C, \quad \overline{d} = (d_C)_{C \in I}$$

As $\varphi(x)$ is preserved under products, $\mathcal{I} \models \varphi[\overline{d}]$. As $\mathcal{L}$ concepts are invariant under products (Lemma 24), we have $\overline{d} \notin C^{\mathcal{I}}$, for all $C \in I$. Thus $\overline{d} \in D^{\mathcal{I}}$ implies $d_2 \in D^{\mathcal{I}^-}$, for all $\mathcal{L}$-concepts D. Thus, we can take an ω-saturated interpretation $\mathcal{I}^+$ satisfying the same FO-sentences as $\mathcal{I}$ and a $d_1 \in \Delta^{\mathcal{I}^+}$ such that $\mathcal{I}^+ \models \varphi[d_1]$ and $d_1 \in D^{\mathcal{I}}$ implies $d_2 \in D^{\mathcal{I}^-}$, for all $\mathcal{L}$-concepts D. It follows from Lemma 23 that $(\mathcal{I}^+, d_1) \leq_{\mathcal{L}} (\mathcal{I}^-, d_2)$ and we have derived a contradiction to the condition that $\varphi(x)$ is preserved under $\mathcal{L}$-simulations. ❑

B Proofs for Section 4

Theorem 7. Let $\mathcal{L} \in \mathsf{ExpDL}$ and φ an FO-sentence. Then the following conditions are equivalent:

1. there exists a Boolean $\mathcal{L}$-TBox $\mathcal{T}$ such that $\mathcal{T} \equiv \varphi$;
2. φ is invariant under $\sim^g_{\mathcal{L}}$.

Proof. For the direction $1 \Rightarrow 2$, let $\mathcal{T}$ be Boolean $\mathcal{L}$-TBox $\mathcal{T}$ such that $\mathcal{T} \equiv \varphi$ and assume w.l.o.g. that $\mathcal{T} = \{\top \sqsubseteq C_{\mathcal{T}}\}$. Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be interpretations such that $\mathcal{I}_1 \models \varphi$ and $\mathcal{I}_1 \sim^g_{\mathcal{L}} \mathcal{I}_2$. Then $\mathcal{I}_1 \models \mathcal{T}$, thus $C_{\mathcal{T}}^{\mathcal{I}_1} = \Delta^{\mathcal{I}_1}$. Since $\mathcal{I}_1 \sim^g_{\mathcal{L}} \mathcal{I}_2$ and by Point 1 of Theorem 3, this yields $C_{\mathcal{T}}^{\mathcal{I}_2} = \Delta^{\mathcal{I}_2}$, thus $\mathcal{I}_2 \models \varphi$.

For $2 \Rightarrow 1$, let φ be invariant under $\sim^g_{\mathcal{L}}$ and consider the set $\mathsf{cons}(\varphi)$ of Boolean $\mathcal{L}$-TBoxes that are implied by φ. We are done if we can show that $\mathsf{cons}(\varphi) \models \varphi$, because by compactness there then is a finite $\Gamma \subseteq \mathsf{cons}(\varphi)$ with $\Gamma \models \varphi$, thus $\bigwedge \Gamma$ is the desired Boolean $\mathcal{L}$-TBox. Assume to the contrary that $\mathsf{cons}(\varphi) \not\models \varphi$. Our aim is to construct ω-saturated interpretations $\mathcal{I}^-$ and $\mathcal{I}^+$ such that $\mathcal{I}^- \not\models \varphi$, $\mathcal{I}^+ \models \varphi$, and $\mathcal{I}^- \equiv^g_{\mathcal{L}} \mathcal{I}^+$, i.e., for all $d_1 \in \Delta^{\mathcal{I}_1}$ there exists $d_2 \in \Delta^{\mathcal{I}_2}$ with $(\mathcal{I}_1, d_1) \equiv_{\mathcal{L}} (\mathcal{I}_2, d_2)$ and vice versa. By Theorem 3, this implies $\mathcal{I}^- \sim^g_{\mathcal{L}} \mathcal{I}^+$, in contradiction to φ being invariant under $\sim^g_{\mathcal{L}}$. We start with $\mathcal{I}^-$, which is any model of $\mathsf{cons}(\varphi) \cup \{\neg\varphi\}$. Let Γ be the set of all $\mathcal{L}$-concept literals true in $\mathcal{I}^-$, where a *concept literal* is a concept inlusion or the negation thereof. We have that $\Gamma \cup \{\varphi\}$ is satisfiable: if this is not the case, then by compactness there is a finite $\Gamma_f \subseteq \Gamma$ with $\Gamma_f \cup \{\varphi\}$ unsatisfiable, thus the Boolean TBox $\neg \bigwedge \Gamma_f$ is in $\mathsf{cons}(\varphi)$, in contradiction to the existence of $\mathcal{I}^-$. Let $\mathcal{I}^+$ be a model of $\Gamma \cup \{\varphi\}$. By Theorem 22, we can assume w.l.o.g. that $\mathcal{I}^-$ and $\mathcal{I}^+$ are ω-saturated.

It remains to show that $\mathcal{I}^- \equiv^g_{\mathcal{L}} \mathcal{I}^+$, based on the fact that $\mathcal{I}^-$ and $\mathcal{I}^+$ satisfy the same $\mathcal{L}$-concept inclusions (namely those that occur positively in Γ). Take a $d \in \Delta^{\mathcal{I}^-}$. We have to show that there is an $e \in \Delta^{\mathcal{I}^+}$ with $t_{\mathcal{L}}^{\mathcal{I}^-}(d) = t_{\mathcal{L}}^{\mathcal{I}^+}(e)$. For any finite $\Gamma_f \subseteq t_{\mathcal{L}}^{\mathcal{I}^-}(d)$, there is an $e_{\Gamma_f} \in \Delta^{\mathcal{I}^+}$ such that $e_{\Gamma_f} \in (\bigsqcap \Gamma_f)^{\mathcal{I}^+}$: since $\mathcal{I}^-$ does not satisfy $\top \models \neg \bigsqcap \Gamma_f$, neither does $\mathcal{I}^+$, which yields the desired e_{Γ_f}. As $\mathcal{I}^+$ is ω-saturated, the existence of the e_{Γ_f} for all finite $\Gamma_f \subseteq t_{\mathcal{L}}^{\mathcal{I}^-}(d)$ implies the existence of an $e \in \Delta^{\mathcal{I}^+}$ such that $e \in C^{\mathcal{I}^+}$ for all $C \in \Gamma$. It follows that $t_{\mathcal{L}}^{\mathcal{I}^-}(d) = t_{\mathcal{L}}^{\mathcal{I}^+}(e)$. The direction from $\mathcal{I}^+$ to $\mathcal{I}^-$ is analogous. ❑

Before we come to the proof of Theorem 13, we introduce some notation that will be used in other proofs as well.

We assume that $\mathcal{ALCI}$-concepts are defined using conjunction, negation, and existential restrictions. Other connectives such as disjunction and value restrictions will be used as abbreviations. Thus, in definitions and in inductive proofs, we only consider concepts constructed using those three constructors.

Define the *role depth* $\mathrm{rd}(C)$ of an $\mathcal{ALCI}$-concept C in the usual way as the number of nestings of existential restrictions in C. The role depth $\mathrm{rd}(\mathcal{T})$ of a TBox $\mathcal{T}$ is the maximum of all $\mathrm{rd}(C)$ such that C occurs in $\mathcal{T}$. By $\mathsf{sub}(\mathcal{T})$ we denote the closure under single negation of the set of subconcept of concepts that occur in $\mathcal{T}$. A $\mathcal{T}$*-type* t is a subset of $\mathsf{sub}(\mathcal{T})$ such that

- $C \in t$ or $\neg C \in t$ for all $\neg C \in \mathsf{sub}(\mathcal{T})$;
- $C \sqcap D \in t$ iff $C \in t$ and $D \in t$, for all $C \sqcap D \in \mathsf{sub}(\mathcal{T})$.

By tp we denote the set of all $\mathcal{T}$-types and by $\mathsf{tp}(\mathcal{T})$ the set of all $\mathcal{T}$-types that are satisfiable in a model of $\mathcal{T}$. A $t \in \mathsf{tp}$ is *realized* by an object $(\mathcal{I}, d)$ if $C \in d^{\mathcal{I}}$ for all $C \in t$. We also set

$$t^{\mathcal{I}}(d) = \{C \in \mathsf{sub}(\mathcal{T}) \mid d \in C^{\mathcal{I}}\}$$

For an inverse role r, we denote by r^- the role name s with $r = s^-$. We say that two $\mathcal{T}$-types t_1, t_2 are *coherent for a role* r, in symbols $t_1 \rightsquigarrow_r t_2$, if $\neg\exists r.C \in t'$ implies $C \notin t$ and $\neg\exists r^-.C \in t'$ implies $C \notin t$. Note that $t \rightsquigarrow_r t'$ iff $t' \rightsquigarrow_{r^-} t$.

Proof of Theorem 13 $\mathcal{ALCI}$-to-$\mathcal{ALC}$ TBox rewritability is decidable in 2-ExpTime.

The proof extends the type elemination method known from complexity proofs in modal logic. Let $\mathcal{T}$ be an $\mathcal{ALCI}$-TBox. The idea is to decide non-$\mathcal{ALC}$-rewritability of $\mathcal{T}$ by checking whether there is an interpretation $\mathcal{I}_1$ refuting $\mathcal{T}$ and an interpretation $\mathcal{I}_2$ satisfying $\mathcal{T}$ such that $\mathcal{I}_1 \sim^g_{\mathcal{ALC}} \mathcal{I}_2$. In the proof, we determine the set Z of all pairs (s, S) with $s \in \mathsf{tp}$ and $S \subseteq \mathsf{tp}$ such that there exist an object $(\mathcal{I}_1, d)$, an interpretation $\mathcal{I}_2$, and a bisimulation B between $\mathcal{I}_1$ and $\mathcal{I}_2$ such that $\mathsf{dom}(B) = \Delta^{\mathcal{I}_1}$ and (r1) If $(s,S) \in Y$ and $\exists r.C \in s$ with r a role name and there does not exist $(s',S') \in Y$ with $C \in s'$ and $(s,S) \rightsquigarrow_r (s',S')$, then set $Y := Y \setminus \{(s,S)\}$.

(r2) If $(s,S) \in Y$ and $\exists r.C \in s$ with r an inverse role and there does not exist $(s',S') \in Y$ with $C \in s'$ and $(s',S') \rightsquigarrow_{r^-} (s,S)$, then set $Y := Y \setminus \{(s,S)\}$.

(r3) If $(s,S) \in Y$ and $\exists r.C \in t$ for some $t \in S$ with r a role name, and there do not exist $(s',S') \in Y$ and $t' \in S'$ with $C \in t'$, $t \rightsquigarrow_r t'$, and $(s,S) \rightsquigarrow_r (s',S')$, then set $Y := Y \setminus \{(s,S)\}$.

Figure 7: Elimination Rules

- $\mathcal{I}_2$ is a model of $\mathcal{T}$;
- $s = t^{\mathcal{I}_1}(d)$;
- $S = \{t^{\mathcal{I}_2}(d') \mid (d,d') \in B\}$.

Clearly, $\mathcal{T}$ is not $\mathcal{ALC}$-rewritable iff there exists $(s,S) \in Z$ such that $s \in \mathsf{tp} \setminus \mathsf{tp}(\mathcal{T})$. Denote by Init the set of all pairs (s,S) such that

- $s \in \mathsf{tp}$;
- $S \subseteq \mathsf{tp}(\mathcal{T})$;
- for all $A \in \mathsf{N_C}$ and $t,t' \in S \cup \{s\}$: $A \in t$ iff $A \in t'$.

We have $\mathsf{Init} \subseteq Z$ and Init can be determined in double exponential time. Thus, a double exponential time algorithm computing Z from Init is sufficient to prove the desired result. To formulate the algorithm, we have to lift the coherence relation $\rightsquigarrow_r$ between types to a coherence relation between members of Init. For $r \in \mathsf{N_R}$, set

- $S \rightsquigarrow_r S'$ if for every $t \in S$ there exists $t' \in S'$ with $t \rightsquigarrow_r t'$.
- $(s,S) \rightsquigarrow_r (s',S')$ if $s \rightsquigarrow_r s'$ and $S \rightsquigarrow_r S'$;

Denote by Final the subset of Init that is the result of applying the rules (r1) to (r3) from Figure 7 exhaustively to $Y := \mathsf{Init}$. Clearly, Final is obtained from Init in at most double exponentially many steps. Thus, we are done if we can prove the following result.

Lemma 25. $\mathsf{Final} = Z$.

Proof. We start by proving $\mathsf{Final} \subseteq Z$. To this end, we construct $\mathcal{I}_1$, $\mathcal{I}_2$ and B that witness $(s,S) \in Z$ for all $(s,S) \in \mathsf{Final}$. We first construct $\mathcal{I}_1$. Set

- $\Delta^{\mathcal{I}_1} = \mathsf{Final}$;
- For $A \in \mathsf{N_C}$: $A^{\mathcal{I}_1} = \{(s,S) \in \Delta^{\mathcal{I}_1} \mid A \in s\}$;
- For $r \in \mathsf{N_R}$: $((s,S),(s',S')) \in r^{\mathcal{I}_1}$ iff $(s,S) \rightsquigarrow_r (s',S')$.

The proof of the following claim uses non-applicability of (r1) and (r2) to members of Final:

Claim 1. For all $C \in \mathsf{sub}(\mathcal{T})$ and $(s,S) \in \mathsf{Final}$: $C \in s$ iff $(s,S) \in C^{\mathcal{I}_1}$.

It follows that $t^{\mathcal{I}_1}(s,S) = s$ for all $(s,S) \in \Delta^{\mathcal{I}_1}$.
We now construct $\mathcal{I}_2$. First define $\mathcal{J}_2$ by

- $\Delta^{\mathcal{J}_2} = \{(s,S,t) \mid (s,S) \in \mathsf{Final}, t \in S\}$,
- For $A \in \mathsf{N_C}$: $A^{\mathcal{J}_2} = \{(s,S,t) \in \Delta^{\mathcal{J}_2} \mid A \in t\}$;
- For $r \in \mathsf{N_R}$: $((s,S,t),(s',S',t')) \in r^{\mathcal{J}_2}$ iff $t \rightsquigarrow_r t'$ and $(s,S) \rightsquigarrow_r (s',S')$.

For $e = (s,S,t) \in \Delta^{\mathcal{J}_2}$, take for every $\exists r.C \in t$ with r an inverse role, an object $(\mathcal{J}_{e,\exists r.C}, e_{\exists r.C})$ such that $\mathcal{J}_{e,\exists r.C}$ satisfies $\mathcal{T}$ and $e_{\exists r.C} \in C^{\mathcal{J}_{e,\exists r.C}}$. Assume those interpretations are disjoint and let $\mathcal{J}_e$ be defined by taking the union of the $\mathcal{J}_{e,\exists r.C}$ and adding e to its domain as well as $(e_{\exists r.C}, e) \in r^{\mathcal{J}_e}$. We may assume that $\Delta^{\mathcal{J}_2} \cap \Delta^{\mathcal{J}_e} = \{e\}$ for all $e \in \Delta^{\mathcal{J}_2}$.

Define $\mathcal{I}_2$ as the union of $\mathcal{J}_2$ and all $\mathcal{I}_e$, $e \in \Delta^{\mathcal{J}_2}$. The following claim is proved using non-applicability of (r3) to Final:

Claim 2. For all $C \in \mathsf{sub}(\mathcal{T})$ and $(s,S,t) \in \Delta^{\mathcal{I}_2}$: $C \in t$ iff $(s,S,t) \in C^{\mathcal{I}_2}$.

It follows that $t^{\mathcal{I}_2}(s,S,t) = t$ for all $(s,S,t) \in \Delta^{\mathcal{I}_2}$ and, since $t \in \mathsf{tp}(\mathcal{T})$ for all such t, that $\mathcal{I}_2$ is a model of $\mathcal{T}$.

Define B as the set of all pairs $((s,S),(s,S,t))$ with $(s,S,t) \in \Delta^{\mathcal{J}_2}$.

Claim 3. B is a bisimulation.

To prove the claim, first assume $(s,S) \in \Delta^{\mathcal{I}_1}$, $((s,S),(s',S')) \in r^{\mathcal{I}_1}$, and $((s,S),(s,S,t)) \in B$. We have $(s,S) \rightsquigarrow_r (s',S')$. Hence $S \rightsquigarrow_r S'$ and so there exists $t' \in S'$ with $t \rightsquigarrow_r t'$. We have $((s,S,t),(s,S',t')) \in r^{\mathcal{I}_2}$ and $((s',S'),(s',S',t')) \in B$, as required.

Now assume $(s,S,t) \in \Delta^{\mathcal{I}_2}$, $((s,S,t),(s',S',t')) \in r^{\mathcal{I}_2}$, and $((s,S),(s,S,t)) \in B$. Then $((s,S),(s',S')) \in r^{\mathcal{I}_2}$ and $((s',S'),(s',S',t')) \in B$, as required.

Using Claims 1 to 3 one can now use $\mathcal{I}_1, \mathcal{I}_2$, and B to show that $\mathsf{Final} \subseteq Z$.

We come to $Z \subseteq \mathsf{Final}$. Clearly, $\mathsf{Init} \supseteq Z$. Thus, to prove that $\mathsf{Final} \supseteq Z$ it is sufficient to show that if $Y \supseteq Z$ and Y' is the result of applying one of the rules (r1) to (r3) to Y, then $Y' \supseteq Z$. We show this for (r2), the other rules are considered similarly.

Consider an application of (r2) that eliminates $(s,S) \in Y$ triggered by $\exists r.C \in s$. Assume to the contrary of what has to be shown that $(s,S) \in Z$. Take interpretations $\mathcal{I}_1, \mathcal{I}_2, d \in \mathcal{I}_1$ and a bisimulation B between $\mathcal{I}_1$ and $\mathcal{I}_2$ with $\mathsf{dom}(B) = \Delta^{\mathcal{I}_1}$ that are a witness for this. As $s = t^{\mathcal{I}_1}(d)$, there exists d' with $(d,d') \in r^{\mathcal{I}_1}$ and $C \in t^{\mathcal{I}_1}(d')$. Let $s' = t^{\mathcal{I}_1}(d')$ and $S' = \{t^{\mathcal{I}_2}(e') \mid (d',e') \in B\}$. We have $(s',S') \in Z$, and so $(s',S') \in Y$. We show $(s',S') \rightsquigarrow_{r^-} (s,S)$ which is a contradiction to the applicability of (r2). $s' \rightsquigarrow_{r^-} s$ is clear from $(d,d') \in r^{\mathcal{I}_1}$. B is a bisimulation and $s := r^-$ a role name. Thus, for every $(d',e') \in B$ there exists e with $(e',e) \in s^{\mathcal{I}_2}$ such that $(d,e) \in B$. Thus, for every $t' \in S'$ there exists $t \in S$ with $t \rightsquigarrow_s t'$. We obtain $S' \rightsquigarrow_{r^-} S$, as required. ❏

Theorem 14. Let $\mathcal{L} \in \{\mathcal{ALCIO}, \mathcal{ALCQIO}\}$ and φ be an FO-sentence. Then the following conditions are equivalent:

1. there exists an $\mathcal{L}$-TBox $\mathcal{T}$ such that $\mathcal{T} \equiv \varphi$;
2. φ is invariant under $\sim^g_{\mathcal{L}}$ and nominal disjoint unions.

Proof. The proof of $1 \Rightarrow 2$ is straightforward and left to the reader. Conversely, assume φ is invariant under $\sim^g_{\mathcal{L}}$ and under nominal disjoint unions but not equivalent to any $\mathcal{L}$-TBox. Our proof strategy is similar to the previous proofs. Let

$$\mathsf{cons}(\varphi) = \{C \sqsubseteq D \mid \varphi \models C \sqsubseteq D \text{ and } C, D \text{ are } \mathcal{L}\text{-concepts}\}$$

As in previous proofs, by compactness, $\mathsf{cons}(\varphi) \not\models \varphi$. We now construct, using invariance under nominal disjoint unions, interpretations $\mathcal{I}^-$ not satisfying φ and $\mathcal{I}^+$ satisfying φ such that $\mathcal{I}_1^- \equiv^g_{\mathcal{L}} \mathcal{I}_2^+$. Assuming ω-saturatedness, we obtain $\mathcal{I}_1^- \sim^g_{\mathcal{L}} \mathcal{I}_2^+$, and have derived a contradiction. We start with the construction of $\mathcal{I}^-$.

For an interpretation $\mathcal{I}$ and $e, f \in \Delta^{\mathcal{I}}$, we set $e \sim^R_{\mathcal{I}} f$ iff there exists a (possibly empty) sequence $r_1, \dots, r_n$ of roles and $d_0, \dots, d_n$ such that $d_0 = e$, $d_n = f$, and $(d_i, d_{i+1}) \in r^{\mathcal{I}}$ for all $i < n$.

Let $\mathcal{I}$ be an interpretation satisfying $\mathsf{cons}(\varphi)$ and refuting φ. Assume, for simplicity, that $a^{\mathcal{I}} = b^{\mathcal{I}}$ for all a, b that do not occur in φ. Let N denote the set of concepts all of the form

$$\forall r_1. \cdots .\forall r_n.\neg\{a\},$$

where $r_1, \dots, r_n$ are roles, $n \geq 0$ (thus the sequence can be empty), and $a \in \mathsf{N_I}$. Let Γ denote the set of $\mathcal{L}$-concepts C such that $\mathsf{cons}(\varphi) \cup \{C^\sharp(x)\} \cup \{F^\sharp(x) \mid F \in N\}$ is satisfiable. Note that Γ consists of exactly those $\mathcal{L}$-concepts C for which there exists an interpretation $\mathcal{J}$ satisfying $\mathsf{cons}(\varphi)$ and a $d \in \Delta^{\mathcal{J}}$ such that $d \in C^{\mathcal{J}}$ and no nominal is interpreted in the connected component generated by d.

Take for any $C \in \Gamma$ an interpretation $\mathcal{I}_C$ satisfying $\mathsf{cons}(\varphi) \cup \{C^\sharp(x)\} \cup \{F^\sharp(x) \mid F \in N\}$. Let $\mathcal{J}_C$ denote the maximal component of $\mathcal{I}_C$ with $\mathsf{Nom}(\mathcal{J}_C) = \emptyset$. Observe that C is satisfied in $\mathcal{J}_C$. Let $I = \Gamma \cup \{0\}$ and $\mathcal{J}_0 = \mathcal{I}_0 = \mathcal{I}$. We can form the nominal disjoint union $\mathcal{I}^- = \sum^{\mathsf{nom}}_{i \in I} \mathcal{J}_i$. Then

- $\mathcal{I}^-$ refutes φ (by condition (b));
- $\mathcal{I}^-$ satisfies $\mathsf{cons}(\varphi)$;
- for all $C \in \Gamma$, $C^{\mathcal{I}^-} \neq \emptyset$.

We can assume that $\mathcal{I}^-$ is ω-saturated.

Claim 1. Γ coincides with the set of concepts C such that $\{\varphi, C^\sharp(x)\} \cup \{F^\sharp(x) \mid F \in N\}$ is satisfiable.

To prove the claim assume there exists C such that $\{\varphi, C^\sharp(x)\} \cup \{F^\sharp(x) \mid F \in N\}$ is not satisfiable, but $\mathsf{cons}(\varphi) \cup \{C^\sharp(c)\} \cup \{F^\sharp(x) \mid F \in N\}$ is satisfiable. By compactness,

$$\varphi \models \bigsqcap_{F \in N'} F \sqsubseteq \neg C,$$

for some finite subset N' of N. But then $(\bigsqcap_{F \in N'} F \sqsubseteq \neg C)^\sharp \in \mathsf{cons}(\varphi)$ and we obtain a contradiction.

Let $X \subseteq \mathsf{N_I}$ be a maximal set of individual names such that

- $a^{\mathcal{I}} \not\sim^R_{\mathcal{I}} b^{\mathcal{I}}$, for any two distinct $a, b \in X$;
- for every $a \in \mathsf{N_I}$ there is a $b \in X$ such that $a \sim^R_{\mathcal{I}} b$.

Note that X is finite since $a^{\mathcal{I}} = b^{\mathcal{I}}$ for all a, b that do not occur in φ.

Claim 2. For all $a \in X$, $\{\varphi\} \cup \{C^\sharp(x) \mid C \in t^{\mathcal{I}}_{\mathcal{L}}(a^{\mathcal{I}})\}$ is satisfiable.

Claim 2 follows from the fact that $\mathsf{cons}(\varphi) \not\models C \sqsubseteq \bot$ for any $C \in t^{\mathcal{I}}_{\mathcal{L}}(a^{\mathcal{I}})$, $a \in X$.

By Claim 1, we can take for every $C \in \Gamma$ an interpretation $\mathcal{I}'_C$ satisfying $\{\varphi, C^\sharp(x)\} \cup \{F^\sharp(x) \mid F \in N\}$. By Claim 2, we can take for every $a \in X$ an interpretation $\mathcal{I}_a$ satisfying $\{\varphi\} \cup \{C^\sharp(x) \mid C \in t^{\mathcal{I}}_{\mathcal{L}}(a^{\mathcal{I}})\}$.

For $C \in \Gamma$, let $\mathcal{J}'_C$ denote the maximal component of $\mathcal{I}'_C$ with $\mathsf{Nom}(\mathcal{J}'_C) = \emptyset$. Observe that C is satisfied in $\mathcal{J}'_C$.

For $a \in X$, let $\mathcal{J}_a$ denote the minimal component of $\mathcal{I}_a$ containing $a^{\mathcal{I}_a}$. Let $J = \Gamma \cup X$ and consider $\mathcal{I}^+ = \sum^{\mathsf{nom}}_{j \in J} \mathcal{J}_j$. As φ is preserved under nominal disjoint unions, $\mathcal{I}^+ \models \varphi$. We may assume that $\mathcal{I}^+$ is ω-saturated. By definition,

- $t^{\mathcal{I}^+}_{\mathcal{L}}(a^{\mathcal{I}^+}) = t^{\mathcal{I}^-}_{\mathcal{L}}(a^{\mathcal{I}^-})$, for all $a \in \mathsf{N_I}$;
- for all $C \in \Gamma$, $C^{\mathcal{I}^+} \neq \emptyset$.

It follows that $\mathcal{I}^- \equiv^g_{\mathcal{L}} \mathcal{I}^+$. Thus, $\mathcal{I}^- \sim^g_{\mathcal{L}} \mathcal{I}^+$, and we have obtained a contradiction. ❑

Theorem 16 For Boolean $\mathcal{ALCIO}$-TBoxes, it is ExpTime-complete to decide whether they are equivalent to $\mathcal{ALCIO}$-TBoxes. This problem is coNExpTime-complete for Boolean $\mathcal{ALCQIO}$-TBoxes.

Proof. The lower bounds can be proved by a straightforward reduction from the ExpTime-complete validity problem for Boolean $\mathcal{ALCIO}$-TBoxes and the co-NExpTime-complete validity problem for Boolean $\mathcal{ALCQIO}$-TBoxes, respectively.

Let $\mathcal{L} \in \{\mathcal{ALCIO}, \mathcal{ALCQIO}\}$. The upper bound for $\mathcal{L}$ is proved by a reduction to the validity problem for Boolean $\mathcal{L}$-TBoxes. Let φ be a Boolean $\mathcal{L}$-TBox and let X denote the set of nominals in φ. We may assume that $X \neq \emptyset$. We reduce the problem of checking invariance under nominal disjoint unions of φ. Note that one can show by induction that it is sufficient to consider condition (a) for nominal disjoint unions of families $(\mathcal{I}_i, \mathcal{J}_i)_{i \in I}$ in which $\mathsf{Nom}(\mathcal{J}_i) \cap X = \emptyset$ for at most one $i \in I$. Similarly, it is sufficient to consider condition (b) for nominal disjoint unions of families $(\mathcal{I}_i, \mathcal{J}_i)_{i \in I}$ with I of cardinality 2.

With any partition $\Xi = \{X_1, \dots, X_n\}$ of X (in which one X_i can be the empty set) we associate

- a Boolean $\mathcal{L}$-TBox φ^1_Ξ such that condition (a) for invariance under nominal disjoint unions holds for φ iff φ^1_Ξ is valid for all Ξ;
- a Boolean $\mathcal{L}$-TBox φ^2_Ξ such that condition (b) for invariance under nominal disjoint unions holds for φ iff φ^2_Ξ is valid for all Ξ.

Assume $\Xi = \{X_1, \dots, X_n\}$ is given.

To construct φ^1_Ξ, choose concepts names $A_1, \dots, A_n$ and $B_1, \dots, B_n$. Denote by φ_C the relativization of φ to C; i.e., the Boolean TBox such that any interpretation $\mathcal{I}$ is a model of φ_C iff the restriction of $\mathcal{I}$ to $C^{\mathcal{I}}$ is a model of φ. Now let

$$\varphi^1_{\Xi} = ((\chi \wedge (\bigwedge_{1\leq i\leq n} \varphi_{B_i}) \rightarrow \varphi_C),$$

where $C = \bigsqcup_{1\leq i\leq n} A_i$ and χ is the conjunction of

- $A_i \sqsubseteq B_i$, for $1 \leq i \leq n$;
- $A_i \sqsubseteq \forall r.A_i$ for all roles r in φ and $1 \leq i \leq n$;
- $\{a\} \sqsubseteq A_i$, for all $a \in X_i$ and $1 \leq i \leq n$;
- $B_i \sqcap B_j \sqsubseteq \bot$, for $1 \leq i < j \leq n$;
- $\neg(A_i \sqsubseteq \bot)$ for $1 \leq i \leq N$;
- $B_i \sqsubseteq \forall r.B_i$ for all roles r in φ and $1 \leq i \leq n$.

To prove our claim, observe that in any interpretation $\mathcal{I}$ satisfying χ, the interpretations $\mathcal{J}_i$, $1 \leq i \leq n$, induced by $A_i^{\mathcal{I}}$ and $\mathcal{I}_i$, $1 \leq i \leq n$, induced by $B_i^{\mathcal{I}_i}$ satisfy the conditions for nominal disjoint unions.

To construct φ^2_{Ξ}, choose concept names A_1, A_2, and B_1, B_2. Then let

$$\varphi^2_{\Xi} = ((\chi \wedge \varphi_{A_1 \sqcup A_2}) \rightarrow \varphi_{A_1}),$$

where χ is the conjunction of

- $A_1 \equiv B_1, A_2 \sqsubseteq B_2$;
- $A_i \sqsubseteq \forall r.A_i$ for all roles r in φ and $i = 1, 2$;
- $\{a\} \sqsubseteq A_1$, for all $a \in X$;
- $B_1 \sqcap B_2 \sqsubseteq \bot$;
- $\neg(A_i \sqsubseteq \bot)$ for $i = 1, 2$;
- $B_2 \sqsubseteq \forall r.B_2$ for all roles r in φ.

❏

C Proofs for Section 5

Theorem 17. Let $\mathcal{L} \in \{\mathcal{EL}, \text{DL-Lite}_{horn}\}$ and let φ be a first-order sentence. The following conditions are equivalent:

1. φ is equivalent to an $\mathcal{L}$-TBox;
2. φ is invariant under $\sim^g_{\mathcal{L}}$ and disjoint unions, and preserved under products.

Proof. The proof of 1 $\Rightarrow$ 2 is straightforward. For the converse direction, in principle we follow the strategy of the proof of Theorem 9. A problem is posed by the fact that, unlike in the case of expressive DLs, two ω-saturated interpretations $\mathcal{I}^-$ and $\mathcal{I}^+$ that satisfy the same $\mathcal{L}$-CIs need not satisfy $\mathcal{I}^- \equiv^g_{\mathcal{L}} \mathcal{I}^+$ (e.g. when $\mathcal{I}^-$ consists of two elements that satisfy A and B, respectively, and $\mathcal{I}^+$ consists of two elements that satisfy no concept name and A, B, respectively). To deal with this, we ensure that $\mathcal{I}^-$ and $\mathcal{I}^+$ satisfy the same *disjunctive $\mathcal{L}$-CIs*, i.e., CIs of the form $C \sqsubseteq D_1 \sqcup \cdots \sqcup D_n$ with $C, D_1, \ldots, D_n$ $\mathcal{L}$-concepts; this suffices to prove $\mathcal{I}^- \equiv_g \mathcal{I}^+$ as required.

Let $\mathsf{cons}(\varphi)$ be the set of all $\mathcal{L}$-CIs that are a consequence of φ and $\mathsf{cons}^{\sqcup}(\varphi)$ set of all disjunctive $\mathcal{L}$-CIs that are a consequence of $\mathsf{cons}(\varphi)$. As before, we are done when $\mathsf{cons}(\varphi) \models \varphi$, thus assume the opposite and derive a contradiction.

Our aim is to construct interpretations $\mathcal{I}^-$ and $\mathcal{I}^+$ such that $\mathcal{I}^- \not\models \varphi$, $\mathcal{I}^+ \models \varphi$, and both $\mathcal{I}^-$ and $\mathcal{I}^+$ satisfy precisely those disjunctive $\mathcal{L}$-CIs that are in $\mathsf{cons}^{\sqcup}(\varphi)$.

$\mathcal{I}^-$ is constructed as follows. For every disjunctive $\mathcal{L}$-CI $C \sqsubseteq D_1 \sqcup \cdots \sqcup D_n \notin \mathsf{cons}^{\sqcup}(\varphi)$, take a model $\mathcal{I}_{C \not\sqsubseteq D_1 \sqcup \cdots \sqcup D_n}$ of $\mathsf{cons}(\varphi)$ that violates $C \sqsubseteq D_1 \sqcup \cdots \sqcup D_n$. Then $\mathcal{I}^-$ is the disjoint union of all $\mathcal{I}_{C \not\sqsubseteq D_1 \sqcup \cdots \sqcup D_n}$ and a model of $\mathsf{cons}(\varphi) \cup \{\neg\varphi\}$. Clearly, $\mathcal{I}^-$ satisfies the desired properties.

To construct $\mathcal{I}^+$, first take for every $\mathcal{L}$-CI $C \sqsubseteq D \notin \mathsf{cons}(\varphi)$ a model $\mathcal{I}_{C \not\sqsubseteq D}$ of φ that violates $C \sqsubseteq D$. Second, take for every disjunctive $\mathcal{L}$-CI $C \sqsubseteq D_1 \sqcup \cdots \sqcup D_n \notin \mathsf{cons}^{\sqcup}(\varphi)$ the product

$$\mathcal{J}_{C \not\sqsubseteq D_1 \sqcup \cdots \sqcup D_n} = \prod_{1\leq i\leq n} \mathcal{I}_{C \not\sqsubseteq D_i}$$

Since φ is preserved under products and by Lemma 24, each $\mathcal{J}_{C \not\sqsubseteq (D_1 \sqcup \cdots \sqcup D_n)}$ is a model of φ that violates $C \sqsubseteq D_1 \sqcup \cdots \sqcup D_n$. By defining $\mathcal{I}^+$ as the disjoint union of all $\mathcal{J}_{C \not\sqsubseteq D_1 \sqcup \cdots \sqcup D_n}$, we clearly attain the properties desired for $\mathcal{I}^+$.

It remains to show that $\mathcal{I}^- \equiv^g_{\mathcal{L}} \mathcal{I}^+$, as then Theorem 3 implies $\mathcal{I}^- \sim^g_{\mathcal{L}} \mathcal{I}^+$, in contradiction to φ being invariant under $\sim^g_{\mathcal{L}}$. We can assume w.l.o.g. that $\mathcal{I}^-$ and $\mathcal{I}^+$ are ω-saturated. Take a $d \in \Delta^{\mathcal{I}^-}$. We have to show that there is an $e \in \Delta^{\mathcal{I}^+}$ with $t^{\mathcal{I}^-}_{\mathcal{L}}(d) = t^{\mathcal{I}^+}_{\mathcal{L}}(e)$. Let $\Gamma^+ = t^{\mathcal{I}^-}_{\mathcal{L}}(d)$ be the set of $\mathcal{L}$-concepts satisfied by d in $\mathcal{I}^-$ and Γ^- the set of $\mathcal{L}$-concepts not satisfied by d in $\mathcal{I}^-$. For any finite $\Gamma^-_f \subseteq \Gamma^-$ and $\Gamma^+_f \subseteq \Gamma^+$, there is an $e_{\Gamma^-_f, \Gamma^+_f} \in \Delta^{\mathcal{I}^+}$ such that $e_{\Gamma^-_f, \Gamma^+_f} \in (\bigsqcap \Gamma^+_f \sqcap \bigsqcap \Gamma^-_f)^{\mathcal{I}^+}$: since $\mathcal{I}^-$ does not satisfy $\bigsqcap \Gamma^+_f \sqsubseteq \bigsqcup \Gamma^-_f$ neither does $\mathcal{I}^+$, which yields the desired $e_{\Gamma^-_f, \Gamma^+_f}$. As $\mathcal{I}^+$ is ω-saturated, the existence of the $e_{\Gamma^-_f, \Gamma^+_f}$ implies the existence of an $e \in \Delta^{\mathcal{I}^+}$ such that $e \in C^{\mathcal{I}^+}$ for all $C \in \Gamma^+$ and $e \notin C^{\mathcal{I}^+}$ for all $C \in \Gamma^-$, i.e., $t^{\mathcal{I}^-}_{\mathcal{L}}(d) = t^{\mathcal{I}^+}_{\mathcal{L}}(e)$. The direction from $\mathcal{I}^+$ to $\mathcal{I}^-$ is analogous.

❏

We devide the proof of Theorem 18 into two parts and reserve a subsection for each part.

C.1 Proof of Theorem 18: Invariance under $\sim^g_{\mathcal{EL}}$

In this subsection, we prove the following result:

Theorem 26. *The problem of deciding whether an $\mathcal{ALC}$-TBox $\mathcal{T}$ is invariant under $\sim^g_{\mathcal{EL}}$ is* ExpTime*-complete.*

The lower bound proof is straightforward by a reduction of the ExpTime-hard satisfiability problem for $\mathcal{ALC}$-TBoxes:

Lemma 27. *Let $\mathcal{T}$ be an $\mathcal{ALC}$-TBox. The following conditions are equivalent*

1. *$\mathcal{T}$ is satisfiable;*
2. *$\mathcal{T}' = \mathcal{T} \cup \{A \sqsubseteq \forall r.B\}$ is not invariant under $\sim^g_{\mathcal{EL}}$ (where A, B, and r are fresh).*

Proof. The direction $2 \Rightarrow 1$ is trivial. For the direction $1 \Rightarrow 2$, assume that $\mathcal{T}$ is satisfiable. Let $\mathcal{I}$ be a model of $\mathcal{T}$ such that $\Delta^{\mathcal{I}}$ has at least four elements, $d_1, \dots, d_4$ (such a model exists by invariance of $\mathcal{ALC}$-TBoxes under disjoint unions). Expand $\mathcal{I}$ to $\mathcal{I}_1$ and $\mathcal{I}_2$ by setting

- $A^{\mathcal{I}_1} = A^{\mathcal{I}_2} = \{d_1\}$,
- $r^{\mathcal{I}_1} = r^{\mathcal{I}_2} = \{(d_1, d_2), (d_1, d_3)\}$;
- $B^{\mathcal{I}_1} = \{d_2, d_3\}$, $B^{\mathcal{I}_2} = \{d_2, d_4\}$,

Clearly $\mathcal{I}_1$ is a model of $\mathcal{T}'$, but $\mathcal{I}_2$ is not. On the other hand, $\mathcal{I}_1 \sim^g_{\mathcal{EL}} \mathcal{I}_2$. We show that $(\mathcal{I}_1, d_1) \sim_{\mathcal{EL}} (\mathcal{I}_2, d_1)$, equisimulations for the remaining domain elements are straightforward. Now,

$$S_1 = \{(d_1, d_1), (d_2, d_2), (d_3, d_2)\}$$

is a $\mathcal{EL}$-simulation between $(\mathcal{I}_1, d_1)$ and $(\mathcal{I}_2, d_2)$. Conversely,

$$S_2 = \{(d_1, d_1), (d_2, d_2), (d_3, d_3)\}$$

is a $\mathcal{EL}$-simulation between $(\mathcal{I}_2, d_2)$ and $(\mathcal{I}_1, d_1)$. ❑

The upper bound proof is more involved. Firstly, we require the following result about $\mathcal{EL}$-simulations:

Lemma 28. *Let $(\mathcal{I}_1, d_1) \sim_{\mathcal{EL}} (\mathcal{I}_2, d_2)$ and let $\mathcal{I}_1, \mathcal{I}_2$ be ω-saturated. Let $(d_1, d_1') \in r^{\mathcal{I}_1}$. Then there exist d_1'' and d_2'' with $(d_1, d_1'') \in r^{\mathcal{I}_1}$ and $(d_2, d_2'') \in r^{\mathcal{I}_2}$ such that*

$$d_1' \leq_{\mathcal{EL}} d_1'' \sim_{\mathcal{EL}} d_2''.$$

Proof. Let

$$X = \mathsf{succ}_r^{\mathcal{I}_1}(d_1) \cap \{d \mid (\mathcal{I}_1, d_1') \leq_{\mathcal{EL}} (\mathcal{I}_1, d)\}$$

We have $d_1' \in X$. X is ordered by the simulation relation $\leq_{\mathcal{EL}}$. Recall that, by Lemma 23, for all $d, d' \in X$, $d \leq_{\mathcal{EL}} d'$ iff $t^{\mathcal{I}_1}_{\mathcal{EL}}(d) \subseteq t^{\mathcal{I}_1}_{\mathcal{EL}}(d')$ since $\mathcal{I}_1$ is ω-saturated.

Claim 1. X contains a $\leq_{\mathcal{EL}}$-maximal element.

To prove Claim 1 it is sufficient to show that for every $\leq_{\mathcal{EL}}$-ascending chain $(e_i)_{i \in I}$ in X there exists $e \in X$ such that $e_i \leq_{\mathcal{EL}} e$ for all $i \in I$. Consider the set of FO-formulas

$$\Gamma = \{r(d_1, x)\} \cup \{C^\sharp(x) \mid C \in \bigcup_{i \in I} t^{\mathcal{I}_1}_{\mathcal{EL}}(e_i)\}.$$

Clearly Γ is finitely realizable in $\mathcal{I}_1$. By ω-saturatedness, Γ is realizable in $\mathcal{I}_1$ for an assigment $a(x) \in X$. Let $e = a(x)$. Then $e \in X$ and $e_i \leq_{\mathcal{EL}} e$ for all $i \in I$, as required.

Let d_1'' be a $\leq_{\mathcal{EL}}$-maximal element of X. Since $d_1 \leq_{\mathcal{EL}} d_2$, there exists $d_2'' \in \mathsf{succ}_r^{\mathcal{I}_2}(d_2)$ such that $d_1'' \leq_{\mathcal{EL}} d_2''$. Now $d_2'' \leq_{\mathcal{EL}} d_1''$ holds as well because there exists $e \in X$ such that $d_2'' \leq_{\mathcal{EL}} e$ and so $d_1'' \leq_{\mathcal{EL}} d_2''$ implies $d_2'' = e$ by $\leq_{\mathcal{EL}}$-maximality of d_1'' in X. We obtain $d_1'' \sim_{\mathcal{EL}} d_2''$, as required. ❑

We are now in the position to prove the ExpTime upper bound. It is proved by means of a generalization of the type elimination method to sequences of types rather than single types. Given an $\mathcal{ALC}$-TBox $\mathcal{T}$, by exponential time type elimination, we want to determine the set P of all pairs (t, s) of $\mathcal{T}$-types such that there exist $(\mathcal{I}, d)$ and $(\mathcal{J}, d')$ with t realized in d, s realized in d', and such that $\mathcal{J}$ is a model of $\mathcal{T}$, $d \sim_{\mathcal{EL}} d'$ and $\mathcal{I} \sim^g_{\mathcal{EL}} \mathcal{J}$. If P contains a pair (t, s) in which $t \in \mathsf{tp} \setminus \mathsf{tp}(\mathcal{T})$, then $\mathcal{T}$ is not preserved under $\sim^g_{\mathcal{EL}}$. If P does not contain any such pair, then $\mathcal{T}$ is preserved under $\sim^g_{\mathcal{EL}}$. The straightforward idea of a recursive procedure that computes P by eliminating pairs from the set of all pairs (t, s) with $t \in \mathsf{tp}$ and $s \in \mathsf{tp}(\mathcal{T})$ for which no appropriate witnesses for existential restrictions exist does not work: the length of the sequences of types required as witnesses for existential restrictions grows. However, as in the interpretation $\mathcal{I}$ we do not have to satisfy a fixed TBox, the role depth of the types to be realized in $\mathcal{I}$ decreases and, therefore, the length of the sequences of types one has to consider stabilizes after $\mathsf{rd}(\mathcal{T})$ man steps. We now give a detailed proof.

For $m \geq 0$, by tp^m we denote the set of all $t' \subseteq \mathrm{sub}(\mathcal{T})$ such that there exists $t \in \mathrm{tp}$ with

$$t' = \{C \in t \mid \mathrm{rd}(C) \leq m\}.$$

A $t \in \mathrm{tp}^m$ is *realized* by an object $(\mathcal{I}, d)$ if $C \in d^{\mathcal{I}}$ for all $C \in t$. Let k be the role depth of the $\mathcal{ALC}$-TBox $\mathcal{T}$. For $m = 0$ we set $m - 1 := 0$.

For $m, l \geq 0$ with $m + l \leq k$, we define X^m_l as the set of all tuples

$$(t, s, s_0, t_1, s_1, \dots, t_l, s_l),$$

such that

- $t, t_1, \dots, t_l \in \mathrm{tp}^m$,
- $s, s_0, s_1, \dots, s_l \in \mathrm{tp}(\mathcal{T})$,

and there exist objects $(\mathcal{I}, d)$, $(\mathcal{I}_1, d_1) \dots, (\mathcal{I}_l, d_l)$ and $(\mathcal{J}, d')$, $(\mathcal{J}_0, d_0'), \dots, (\mathcal{J}_l, d_l')$ such that

1. $(\mathcal{I}, d)$ realizes t and $(\mathcal{I}_i, d_i)$ realizes t_i for $1 \leq i \leq l$;
2. $(\mathcal{J}, d')$ realizes s and $(\mathcal{J}_i, d_i')$ realizes s_i for $0 \leq i \leq l$;
3. $\mathcal{J}$ and $\mathcal{J}_i$ satisfy $\mathcal{T}$, for $0 \leq i \leq l$;
4. $(\mathcal{I}, d) \leq_{\mathcal{EL}} (\mathcal{I}_i, d_i)$ for $1 \leq i \leq l$;
5. $(\mathcal{I}_i, d_i) \sim_{\mathcal{EL}} (\mathcal{J}_i, d_i')$ and $\mathcal{I}_i \sim^g_{\mathcal{EL}} \mathcal{J}_i$ for $1 \leq i \leq l$;
6. $(\mathcal{I}, d) \sim_{\mathcal{EL}} (\mathcal{J}, d')$ and $\mathcal{I} \sim^g_{\mathcal{EL}} \mathcal{J}$;
7. $(\mathcal{J}_0, d_0') \leq (\mathcal{J}, d')$.

Lemma 29. *$\mathcal{T}$ is not invariant under $\sim^g_{\mathcal{EL}}$ iff there exist $t \in \mathsf{tp} \setminus \mathsf{tp}(\mathcal{T})$, and $s = s_0 \in \mathsf{tp}(\mathcal{T})$ such that $(t, s, s_0) \in X^k_0$.*

Thus, the ExpTime upper bound follows if one can compute X^k_0 in exponential time. To this end, we will give an exponential time elimination algorithm that determines *all* sets X^m_l, $0 \leq m, l$ and $m + l \leq k$.

First compute the sets Init^m_l, $0 \leq l, m$ and $l + m \leq k$, consisting of all

$$(t, s, s_0, t_1, s_1, \dots, t_l, s_l),$$

where $t, t_1, \dots, t_l \in \mathrm{tp}^m$, $s, s_0, s_1, \dots, s_l \in \mathrm{tp}(\mathcal{T})$ and for all $A \in \mathsf{N_C}$:

Let $(t, s, s_0, t_1, s_1, \ldots, t_l, s_l) \in Y_l^m$.

(r1) if $m > 0$ and there exists $\exists r.C \in t$ and such that there does not exist $(t', s', s'_0, t'_1, s'_1, \ldots, t'_l, s'_l, t'_{l+1}, s'_{l+1}) \in Y_{l+1}^{m-1}$ with $C \in t'$ and $t \rightsquigarrow_r t'$, $t \rightsquigarrow_r t'_{l+1}$, $s \rightsquigarrow_r s'_{l+1}$, and, for $1 \leq i \leq l$: $t_i \rightsquigarrow_r t'_i$, $s_i \rightsquigarrow_r s'_i$, then set

$$Y_l^m := Y_l^m \setminus \{(t, s, s_0, t_1, s_1, \ldots, t_l, s_l)\}$$

(r2) if there exists $\exists r.C \in s$ and there does not exist $(t', s', s'_0, t'_1, s'_1, \ldots, t'_l, s'_l) \in Y_l^{m-1}$ with $C \in s'_0$ and $s \rightsquigarrow_r s'_0$, $s \rightsquigarrow_r s'$, $t \rightsquigarrow_r t'$, and, for $1 \leq i \leq l$: $t_i \rightsquigarrow_r t'_i$, $s_i \rightsquigarrow_r s'_i$, then set

$$Y_l^m := Y_l^m \setminus \{(t, s, s_0, t_1, s_1, \ldots, t_l, s_l)\}$$

(r3) if there exists $\exists r.C \in s_0$ and there does not exist $(t', s', s'_0, t'_1, s'_1, \ldots, t'_l, s'_l) \in Y_l^{m-1}$ with $C \in s'_0$ and $s_0 \rightsquigarrow_r s'_0$, $s \rightsquigarrow_r s'$, $t \rightsquigarrow_r t'$, and, for $1 \leq i \leq l$: $t_i \rightsquigarrow_r t'_i$, $s_i \rightsquigarrow_r s'_i$, then set

$$Y_l^m := Y_l^m \setminus \{(t, s, s_0, t_1, s_1, \ldots, t_l, s_l)\}$$

(r4) if there exist $1 \leq i \leq l$ and $\exists r.C \in t_i$ such that there does not exist $(t', s', s'_0, t'_1, s'_1) \in Y_1^{m-1}$ with $C \in t'$ and $t_i \rightsquigarrow_r t'$, $t_i \rightsquigarrow_r t'_1$, $s_i \rightsquigarrow_r s'_1$, then set

$$Y_l^m := Y_l^m \setminus \{(t, s, s_0, t_1, s_1, \ldots, t_l, s_l)\}$$

(r5) if there exist $1 \leq i \leq l$ and $\exists r.C \in s_i$ such that there does not exist $(t', s', s'_0) \in Y_0^{m-1}$ with $C \in s'_0$ and $s_i \rightsquigarrow_r s'_0$, $s_i \rightsquigarrow_r s'$, $t_i \rightsquigarrow_r t'$, then set

$$Y_l^m := Y_l^m \setminus \{(t, s, s_0, t_1, s_1, \ldots, t_l, s_l)\}$$

Figure 8: Elimination Rules

- $A \in t$ implies $A \in t_i$ for $1 \leq i \leq l$;
- $A \in t_i$ iff $A \in s_i$ for $1 \leq i \leq l$;
- $A \in t$ iff $A \in s$;
- $A \in s_0$ implies $A \in s$.

Note that Init_l^m can be computed in exponential time since $\mathsf{tp}(\mathcal{T})$ can be computed in exponential time.

Now apply exhaustively the rules from Figure 8 to the sets $Y_l^m := \text{Init}_l^m$ and denote the resulting sets of tuples by Final_l^m.

It should be clear that the elimination algorithm terminates after at most exponentially many steps. Thus, the lower bound follows from the following lemma:

Lemma 30. *For all $m, l \geq 0$ with $m + l \leq k$, we have $X_l^m = \mathit{Final}_l^m$.*

Proof. We start with the proof of the inclusion $X_l^m \subseteq \text{Final}_l^m$. To this end, it is sufficient to observe that $X_l^m \subseteq \text{Init}_l^m$ and that the following holds for $1 \leq i \leq 5$:

Claim 1. If $X_l^m \subseteq Y_l^m$ for all $0 \leq m, l$ with $m + l \leq k$, and $\vec{x}$ is removed from $Y_{l_0}^{m_0}$ by an application of the rule (ri), then $\vec{x} \notin X_{l_0}^{m_0}$.

To prove the claim, first let $i = 1$. Assume that, in contrast to what has to be shown, there are $(t, s, s_0, t_1, s_1, \ldots, t_{l_0}, s_{l_0}) \in Y_{l_0}^{m_0}$ with $m_0 > 0$ and $\exists r.C \in t$ such that

- (r1) is applicable: there does not exist $(t', s', s'_0, t'_1, s'_1, \ldots, t'_{l_0}, s'_{l_0}, t'_{l_0+1}, s'_{l_0+1}) \in Y_{l_0+1}^{m_0-1}$ with $(*)$ $C \in t'$ and $t \rightsquigarrow_r t'$, $t \rightsquigarrow_r t'_{l_0+1}$, $s \rightsquigarrow_r s'_{l_0+1}$, and, for $1 \leq i \leq l_0$: $t_i \rightsquigarrow_r t'_i$, $s_i \rightsquigarrow_r s'_i$;
- $(t, s, s_0, t_1, s_1, \ldots, t_{l_0}, s_{l_0}) \in X_{l_0}^{m_0}$.

By Point 2, we can take objects $(\mathcal{I}, d), (\mathcal{I}_1, d_1), \ldots, (\mathcal{I}_{l_0}, d_{l_0})$ and $(\mathcal{J}, d'), (\mathcal{J}_1, d'_1), \ldots, (\mathcal{J}_{l_0}, d'_{l_0})$ with the properties 1–7. We may assume that those objects are ω-saturated. We find e with $(d, e) \in r^{\mathcal{I}}$ such that $e \in C^{\mathcal{I}}$. Let $1 \leq i \leq l_0$. We find $f_i \in \mathcal{I}_i$ with $(d_i, f_i) \in r^{\mathcal{I}_i}$ and $(\mathcal{I}, e) \leq_{\mathcal{EL}} (\mathcal{I}_i, f_i)$. By Lemma 28, we find e_i and e'_i with $(d_i, e_i) \in r^{\mathcal{I}_i}$ and $(d'_i, e'_i) \in r^{\mathcal{J}_i}$ such that

$$(\mathcal{I}, f_i) \leq_{\mathcal{EL}} (\mathcal{I}_i, e_i) \sim_{\mathcal{EL}} (\mathcal{J}_i, e'_i)$$

We also have $(\mathcal{I}, e) \leq_{\mathcal{EL}} (\mathcal{I}_i, e_i)$. Also, by Lemma 28, we find e_{l_0+1} and e'_{l_0+1} with $(d, e_{l_0+1}) \in r^{\mathcal{I}}$ and $(e, e'_{l_0+1}) \in r^{\mathcal{J}}$ such that

$$(\mathcal{I}, e) \leq_{\mathcal{EL}} (\mathcal{I}, e_{l_0+1}) \sim_{\mathcal{EL}} (\mathcal{J}, e'_{l_0+1})$$

Set $\mathcal{I}_{l_0+1} = \mathcal{I}$ and $\mathcal{J}_{l_0+1} = \mathcal{J}$. Now let

- t' be the type in tp^{m_0-1} realized by $(\mathcal{I}, e)$;
- t'_i, $1 \leq i \leq l_0 + 1$, be the type in tp^{m_0-1} realized by $(\mathcal{I}_i, e_i)$;
- s'_i, $1 \leq i \leq l_0 + 1$, be the $\mathcal{T}$-types realized by $(\mathcal{J}_i, e'_i)$;
- $s' = s'_0$ be the $\mathcal{T}$-type realized by some $(\mathcal{K}, f)$ such that $\mathcal{K}$ is a model of $\mathcal{T}$, $(\mathcal{I}, e) \sim_{\mathcal{EL}} (\mathcal{K}, f)$, and $\mathcal{I} \sim_{\mathcal{EL}}^g \mathcal{K}$.

Let $\vec{x} = (t', s', s'_0, t'_1, s'_1, \ldots, t'_{l_0}, s'_{l_0}, t'_{l_0+1}, s'_{l_0+1})$. Clearly $\vec{x} \in X_{l_0+1}^{m_0-1}$ and so $\vec{x} \in Y_{l_0+1}^{m_0-1}$. Moreover $\vec{x}$ satisfies $(*)$. Thus, we have derived a contradiction.

The rules (r2)–(r5) are considered similarly.

We now come to the inclusion $X_l^m \supseteq Z_l^m$. Denote the nth entry of $\vec{x} \in Z_l^m$ by $\vec{x}(n)$; $l(\vec{x})$ denotes the length on $\vec{x}$.

Now define an interpretation $\mathcal{I}$ by setting

$$\Delta^{\mathcal{I}} = \{(n, \vec{x}) \mid \vec{x} \in Z_l^m, l(\vec{x}) \geq n\}.$$

and, for $A \in \mathsf{N_C}$,

$$A^{\mathcal{I}} = \{(n, \vec{x}) \in \Delta^{\mathcal{I}} \mid A \in \vec{x}(n)\}.$$

Finally, for $r \in \mathsf{N_R}$,

$$\vec{x} = (t, s, s_0, t_1, s_1, \ldots, t_l, s_l) \in Z_l^m,$$

and $(n, \vec{y}) \in \Delta^{\mathcal{I}}$ we set $((n, \vec{x}), (m, \vec{y})) \in r^{\mathcal{I}}$ if

- $\vec{y} = (t', s', s'_0, t'_1, s'_1, \ldots, t'_l, s'_l, t'_{l+1}, s'_{l+1}) \in Z_{l+1}^{m-1}$, $(\vec{x}(n), \vec{y}(m))$ is one of the pairs (t, t'), (t, t'_{l+1}), (s, s'_{l+1}), (t_i, t'_i), (s_i, s'_i), for $1 \leq i \leq l$, and $t \rightsquigarrow_r t'$, $t \rightsquigarrow_r t'_{l+1}$, $s \rightsquigarrow_r s'_{l+1}$, and, for $1 \leq i \leq l$: $t_i \rightsquigarrow_r t'_i$, $s_i \rightsquigarrow_r s'_i$.

- $\vec{y} = (t', s', s'_0, t'_1, s'_1, \dots, t'_l, s'_l) \in Z_l^{m-1}$, $(\vec{x}(n), \vec{y}(m))$ is one of the pairs (s, s'_0), (s, s'), (t, t'), (t_i, t'_i), (s_i, s'_i), for $1 \leq i \leq l$, and $s \rightsquigarrow_r s'_0$, $s \rightsquigarrow_r s'$, $t \rightsquigarrow_r t'$, and, for $1 \leq i \leq l$: $t_i \rightsquigarrow_r t'_i$, $s_i \rightsquigarrow_r s'_i$.
- $\vec{y} = (t', s', s'_0, t'_1, s'_1, \dots, t'_l, s'_l) \in Z_l^{m-1}$, $(\vec{x}(n), \vec{y}(m))$ is one of the pairs (s_0, s'_0), (s, s'), (t, t') (t_i, t'_i), $(s_i, s_{i'})$, for $1 \leq i \leq l$, and $s_0 \rightsquigarrow_r s'_0$, $s \rightsquigarrow_r s'$, $t \rightsquigarrow_r t'$, and, for $1 \leq i \leq l$: $t_i \rightsquigarrow_r t'_i$, $s_i \rightsquigarrow_r s'_i$.
- there exists $1 \leq i \leq l$ such that such that $\vec{y} = (t', s', s'_0, t'_1, s'_1) \in Z_1^{m-1}$,
$$(\vec{x}(n), \vec{y}(m)) \in \{(t_i, t'), (t_i, t'_1), (s_i, s'_1)\}$$
and $t_i \rightsquigarrow_r t'$, $t_i \rightsquigarrow_r t'_1$, $s_i \rightsquigarrow_r s'_1$.
- there exists $1 \leq i \leq l$ such that $\vec{y} = (t', s', s'_0) \in Z_0^{m-1}$,
$$(\vec{x}(n), \vec{y}(m)) \in \{(s_i, s'_0), (s_i, s'), (t_i, t')\},$$
and $s_i \rightsquigarrow_r s'_0$, $s_i \rightsquigarrow_r s'$, $t_i \rightsquigarrow_r t'$.

The following can be proved by induction:

Claim 1. For all $(n, \vec{x}) \in \Delta^{\mathcal{I}}$, if $\vec{x}(n) \in \text{tp}^m$ for some $m \leq k$ and $C \in \text{sub}(\mathcal{T})$ has role depth $\leq m$, then
$$C \in \vec{x}(n) \quad \Leftrightarrow \quad (n, \vec{x}) \in C^{\mathcal{I}}.$$

Claim 2. If $\vec{x} = (t, s, s_0, t_1, s_1, \dots, t_l, s_l)$ and $(n, \vec{x}), (m, \vec{x}) \in \Delta^{\mathcal{I}}$. Then $(n, \vec{x}) \leq_{\mathcal{EL}} (m, \vec{x})$ whenever
$$(\vec{x}(n), \vec{x}(m)) \in \{(s_0, s)\} \cup \{(t, t_i) \mid 1 \leq i \leq l\}$$
and $(n, \vec{x}) \sim_{\mathcal{EL}} (m, \vec{x})$ whenever
$$(\vec{x}(n), \vec{x}(m)) \in \{(t, s)\} \cup \{(t_i, s_i) \mid 1 \leq i \leq l\}$$

Now let $\mathcal{I}_s$ be the interpretation induced by $\mathcal{I}$ on the set of all $(n, \vec{x}) \in \Delta^{\mathcal{I}}$ such that $\vec{x}(n) \in \{s, s_0, \dots, s_l\}$ for $\vec{x} = (t, s, s_0, t_1, s_1, \dots, t_l, s_l)$. Let $\mathcal{I}_t$ be the interpretation induced by $\mathcal{I}$ on $\Delta^{\mathcal{I}} \setminus \Delta^{\mathcal{I}_s}$. Observe that $\mathcal{I}$ is the *disjoint union* of $\mathcal{I}_s$ and $\mathcal{I}_t$.

Now assume that $\vec{x} = (t, s, s_0, t_1, s_1, \dots, t_l, s_l) \in Z_l^m$ is given. We set

- $\mathcal{I} = \mathcal{I}_1 = \cdots = \mathcal{I}_l := \mathcal{I}_t$;
- $d = (1, \vec{x})$ and, for $1 \leq i \leq l$, $d_i = (2 + 2i, \vec{x})$;
- $\mathcal{J} = \mathcal{J}_0 = \cdots = \mathcal{J}_l := \mathcal{I}_t$;
- $d' = (2, \vec{x})$ and, for $1 \leq i \leq l$, $d'_i = (3 + 2i, \vec{x})$.

It follows from Claims 1 and 2 that the defined objects satisfy the conditions 1–7. Thus, $\vec{x} \in X_l^m$, as required. ❑

C.2 Proof of Theorem 18: preservation under products

The aim of this subsection is to prove the following result:

Theorem 31. *It is co-*NExpTime*-complete to decide whether an $\mathcal{ALC}$-TBox is preserved under products.*

We start with the upper bound proof. An interpretation $\mathcal{I}$ is a *tree interpretetation* if the directed graph $(\Delta^{\mathcal{I}}, \bigcup_{r \in \mathsf{N_R}} r^{\mathcal{I}})$ is a tree and $r^{\mathcal{I}} \cap s^{\mathcal{I}} = \emptyset$ for any two distinct $r, s \in \mathsf{N_R}$.

Lemma 32. *If an $\mathcal{ALC}$-TBox $\mathcal{T}$ is not preserved under products, then there are tree-models $\mathcal{I}_1$ and $\mathcal{I}_2$ of $\mathcal{T}$ with outdegree at most 2^{n^2+n+1} such that $\mathcal{I}_1 \times \mathcal{I}_2$ is not a model of $\mathcal{T}$.*

Proof. Assume that $\mathcal{T} = \{\top \sqsubseteq C_{\mathcal{T}}\}$ is not preserved under products. Then there are models $\mathcal{I}_1$ and $\mathcal{I}_2$ of $\mathcal{T}$ such that $\mathcal{I}_1 \times \mathcal{I}_2$ is not a model of $\mathcal{T}$. Thus, there is a $(\hat{d}_1, \hat{d}_2) \in \Delta^{\mathcal{I}_1 \times \mathcal{I}_2}$ with $(\hat{d}_1, \hat{d}_2) \notin C_{\mathcal{T}}^{\mathcal{I}_1 \times \mathcal{I}_2}$. We proceed in two steps: first unravel $\mathcal{I}_1$ and $\mathcal{I}_2$ into tree-interpretations, then restrict their outdegree. An *i-path*, $i \in \{1, 2\}$, is a sequence $d_0 r_0 d_1 r_1 \cdots r_{k-1} d_k$, $k \geq 0$, alternating between elements of $\Delta^{\mathcal{I}_i}$ and role names that occur in $\mathcal{T}$ such that $d_0 = \hat{d}_i$ and for all $i < k$, we have $(d_i, d_{i+1}) \in r_i^{\mathcal{I}}$. Define new interpretations $\mathcal{I}'_1$ and $\mathcal{I}'_2$ as follows:
$$\begin{aligned}
\Delta^{\mathcal{I}'_i} &= \text{the set of } i\text{-paths} \\
A^{\mathcal{I}'_i} &= \{d_0 \cdots d_k \in \Delta^{\mathcal{I}'_i} \mid d_k \in A^{\mathcal{I}_i}\} \\
r^{\mathcal{I}'_i} &= \{(d_0 \cdots d_k, d_0 \cdots d_k r d_{k+1}) \mid d_0 \cdots d_k r d_{k+1} \in \Delta^{\mathcal{I}'_i}\}.
\end{aligned}$$
It can be proved by a straightforward induction that for all $C \in \mathsf{sub}(\mathcal{T})$ and $d_0 \cdots d_k \in \Delta^{\mathcal{I}'_i}$, $i \in \{1, 2\}$, we have $d_k \in C^{\mathcal{I}_i}$ iff $d_0 \cdots d_k \in C^{\mathcal{I}'_i}$. It follows that $\mathcal{I}'_1$ and $\mathcal{I}'_2$ are models of $\mathcal{T}$. To show that $\mathcal{I}'_1 \times \mathcal{I}'_2$ is not a model of $\mathcal{T}$, it suffices to establish the following claim, which yields $(\hat{d}_1, \hat{d}_2) \in (\neg C_{\mathcal{T}})^{\mathcal{I}'_1 \times \mathcal{I}'_2}$:

Claim. For all $C \in \mathsf{sub}(\mathcal{T})$, $p_1 = d_0^1 \cdots d_{k_1}^1 \in \Delta^{\mathcal{I}'_1}$, and $p_2 = d_0^2 \cdots d_{k_2}^2 \in \Delta^{\mathcal{I}'_2}$, we have $(d_{k_1}^1, d_{k_2}^2) \in C^{\mathcal{I}_1 \times \mathcal{I}_2}$ iff $(p_1, p_2) \in C^{\mathcal{I}'_1 \times \mathcal{I}'_2}$.

The proof is by induction on the structure of C, where the only interesting case is $C = \exists r.D$.

First let $(d_{k_1}^1, d_{k_2}^2) \in (\exists r.D)^{\mathcal{I}_1 \times \mathcal{I}_2}$. Then there is a $(d_1, d_2) \in D^{\mathcal{I}_1 \times \mathcal{I}_2}$ with $((d_{k_1}^1, d_{k_2}^2), (d_1, d_2)) \in r^{\mathcal{I}_1 \times \mathcal{I}_2}$. It follows that $(d_{k_1}^1, d_1) \in r^{\mathcal{I}_1}$ and $(d_{k_2}^2, d_2) \in r^{\mathcal{I}_2}$. Thus, $p_1 r d_1$ is a 1-path and $p_2 r d_2$ is a 2-path. Then $(p_1, p_1 r d_1) \in r^{\mathcal{I}_1}$ and $(p_2, p_2 r d_2) \in r^{\mathcal{I}_2}$ and $((p_1, p_2), (p_1 r d_1, p_2 r d)) \in r^{\mathcal{I}'_1 \times \mathcal{I}'_2}$. By IH, $(p_1 r d_1, p_2 r d_2) \in D^{\mathcal{I}'_1 \times \mathcal{I}'_2}$ and we are done.

Now let $(p_1, p_2) \in (\exists r.D)^{\mathcal{I}'_1 \times \mathcal{I}'_2}$. Then there are $(q_1, q_2) \in D^{\mathcal{I}'_1 \times \mathcal{I}'_2}$ such that $((p_1, p_2), (q_1, q_2)) \in r^{\mathcal{I}'_1 \times \mathcal{I}'_2}$. By definition of products and $\mathcal{I}'_1$ and $\mathcal{I}'_2$, we have $q_1 = p_1 r d_1$ and $p_2 = p_2 r d_2$ for some $d_1 \in \Delta^{\mathcal{I}_1}$ and $d_2 \in \Delta^{\mathcal{I}_2}$. Since q_1 is a 1-path and q_2 a 2-path, we have $(d_{k_1}^1, d_1) \in r^{\mathcal{I}_1}$ and $(d_{k_2}^2, d_2) \in r^{\mathcal{I}_2}$, thus $((d_{k_1}^1, d_{k_2}^2), (d_1, d_2)) \in r^{\mathcal{I}_1 \times \mathcal{I}_2}$. By IH, $(d_1, d_2) \in D^{\mathcal{I}_1 \times \mathcal{I}_2}$ and we are done.

We now define interpretations $\mathcal{I}''_1$ from $\mathcal{I}'_1$ and $\mathcal{I}''_2$ from $\mathcal{I}'_2$ by dropping 'unnecessary' subtrees, which results in a reduction of the maximum out-degree to 2^{n^2+n+1}. To select the subtrees in $\mathcal{I}'_1$ and $\mathcal{I}'_2$ that must not be dropped, we first need a notion of distance in the product interpretation $\mathcal{I}'_1 \times \mathcal{I}'_2$: for all $(p_1, p_2) \in \mathcal{I}'_1 \times \mathcal{I}'_2$, let $\delta_{12}(p_1, p_2)$ denote the length of the path from $(\hat{d}_1, \hat{d}_2)$ to (p_1, p_2) in $\mathcal{I}'_1 \times \mathcal{I}'_2$, if such a path exists (note that the path is unique if it exists); otherwise, $\delta_{12}(p_1, p_2)$ is undefined. Now choose for each $i \in \{1, 2\}$, a smallest set $\Gamma_i \subseteq \Delta^{\mathcal{I}'_i}$ such that the following conditions are satisfied:

(a) $\hat{d}_i \in \Gamma_i$;

(b) whenever $p \in \Gamma_i$ and $\exists r.C \in \mathsf{sub}(\mathcal{T})$ with $p \in (\exists r.C)^{\mathcal{I}'_i}$, then there is a $p' \in \Gamma_i$ such that $(p, p') \in r^{\mathcal{I}'_i}$ and $p' \in C^{\mathcal{I}'_i}$;

(c) whenever $p_1 \in \Gamma_1$ and $p_2 \in \Gamma_2$ and $\exists r.C \in \mathsf{sub}(\mathcal{T})$ with $(p_1, p_2) \in (\exists r.C)^{\mathcal{I}_1' \times \mathcal{I}_2'}$, $\delta_{12}(p_1, p_2)$ is defined, and $\mathsf{rd}(\exists r.C) \leq |\mathcal{T}| - \delta_{12}(p_1, p_2)$, then there is a $(p_1', p_2') \in C^{\mathcal{I}_1' \times \mathcal{I}_2'}$ such that $((p_1, p_2), (p_1', p_2')) \in r^{\mathcal{I}_1' \times \mathcal{I}_2'}$, $p_1 \in \Gamma_1$, and $p_2 \in \Gamma_2$.

Now let $\mathcal{I}_i''$ be obtained from $\mathcal{I}_i'$ by dropping all subtrees whose root is not in Γ_i, for $i \in \{1, 2\}$. The following can be proved by a straightforward structural induction.

Claim. For all $C \in \mathsf{sub}(\mathcal{T})$, $p_1 \in \Delta^{\mathcal{I}_1''}$, $p_2 \in \Delta^{\mathcal{I}_2''}$, and $i \in \{1, 2\}$, we have

1. $p_i \in C^{\mathcal{I}_i'}$ iff $p_i \in C^{\mathcal{I}_i''}$;
2. $(p_1, p_2) \in C^{\mathcal{I}_1' \times \mathcal{I}_2'}$ iff $(p_1, p_2) \in C^{\mathcal{I}_1'' \times \mathcal{I}_2''}$ whenever $\delta_{12}(p_1, p_2)$ is defined and $\mathsf{rd}(C) \leq |\mathcal{T}| - \delta_{12}(p_1, p_2)$.

It follows that $\mathcal{I}_1''$ and $\mathcal{I}_2''$ are still models of $\mathcal{T}$, and that $(\hat{d}_1, \hat{d}_2) \notin C_{\mathcal{T}}^{\mathcal{I}_1'' \times \mathcal{I}_2''}$, thus $\mathcal{I}_1'' \times \mathcal{I}_2''$ is not a model of $\mathcal{T}$. It remains to verify that the out-degree of $\mathcal{I}_1''$ and $\mathcal{I}_2''$ is bounded by 2^{n^2+n+1}. First define distance functions δ_1 and δ_2 in $\mathcal{I}_1'$ and $\mathcal{I}_2'$, analogously to the definition of δ_{12}. Let $|\mathcal{T}| = n$, $f(0) = 2n$ and $f(i) = n + n \cdot f(i-1)$ for all $i > 0$. We establish the following

Claim. For all $i \in \{1, 2\}$ and $p \in \Delta^{\mathcal{I}''}$,

1. p has at most $f(\delta_i(p))$ successors;
2. p has at most n successors if $\delta_i(p) \geq n$.

Since Point 2 is obvious by our use of δ_{12} in Condition (c) of the definition of Γ_1 and Γ_2, we concentrate on Point 1 of the claim. It is proved by induction on $\delta_i(p)$. For the induction start, let $\delta_i(p) = 0$, i.e., $p = \hat{d}_i$. Then p has at most n successors selected due to Condition (b) of the definition of Γ_1 and Γ_2 and at most n successors selected due to Condition (c). It remains to remind that $f(0) = 2n$. For the induction step, we concentrate on the case $i = 1$; the case $i = 2$ is symmetric. Thus, let $\delta_1(p) > 0$. Again, at most n successors are selected due to Condition (b) of the definition of Γ_1 and Γ_2. In $\mathcal{I}_1' \times \mathcal{I}_2'$, the number of elements (p, q) for which $\delta_{12}(p, q)$ is defined is bounded by the maximal number of successors of elements $q' \in \Delta^{\mathcal{I}_2'}$ with $\delta_2(p') = \delta_1(p) - 1$; the reason is that $\delta_{12}(p, q)$ is defined only if the predecessor (p', q') of (p, q) satisfies the following properties:

1. p' is the unique predecessor of p in $\mathcal{I}_1$;
2. q is a successor of q' in $\mathcal{I}_2$.

By IH, there are thus at most $f(\delta_1(p) - 1)$ such elements (p, q). For each such (p, q), at most n successors of p are selected in Condition (c) of the definition of Γ_1 and Γ_2. Thus, the maximum number of successors of p is

$$n + n * f(\delta_1(p) - 1) = f(\delta_1(p)).$$

This finishes the proof of the claim. Now, an easy analysis of the recurrence in Point 1 of the above claim yields a maximum outdegree of 2^{n^2+n+1}. ❑

For an interpretation $\mathcal{I}$ and a $d \in \Delta^{\mathcal{I}}$, we use $\mathsf{tp}_{\mathcal{T}}^{\mathcal{I}}$ to denote the *semantic $\mathcal{T}$-type* of d in $\mathcal{I}$, i.e., $\mathsf{tp}_{\mathcal{T}}^{\mathcal{I}}(d) = \{C \in \mathsf{sub}(\mathcal{T}) \mid d \in C^{\mathcal{I}}\}$. The set of all semantic $\mathcal{T}$-types is

$$\mathfrak{T} = \{\mathsf{tp}^{\mathcal{I}}(d) \mid \mathcal{I} \text{ a model of } \mathcal{T},\ d \in \Delta^{\mathcal{I}}\}.$$

For $t_1, t_2 \in \mathfrak{T}$, set $t_1 \rightsquigarrow_r t_2$ if we have $\exists r.C \in t_1$ iff $C \in t_2$, for all $\exists r.C \in \mathsf{sub}(\mathcal{T})$. For $k \geq 0$, a *k-initial interpretation tree* is a triple $(\mathcal{I}, \rho^{\mathcal{I}}, t^{\mathcal{I}})$, where $\mathcal{I}$ is a tree-shaped interpretation of depth at most k and with root $\rho^{\mathcal{I}}$ and $t^{\mathcal{I}} : \Delta^{\mathcal{I}} \rightarrow \mathfrak{T}$. For $d \in \Delta^{\mathcal{I}}$, we use $\delta_{\mathcal{I}}(d)$ to denote the distance of d from the root of $\mathcal{I}$. We require that the following conditions are satisfied, for all $d, e \in \Delta^{\mathcal{I}}$:

1. $d \in A^{\mathcal{I}}$ iff $A \in t^{\mathcal{I}}(d)$ for all $A \in \mathsf{N_C}$;
2. if $\exists r.C \in t^{\mathcal{I}}(d)$ and $\delta_{\mathcal{I}}(d) < k$, then there is an $e \in \Delta^{\mathcal{I}}$ with $(d, e) \in r^{\mathcal{I}}$ and $C \in t^{\mathcal{I}}(e)$;
3. if $(d, e) \in r^{\mathcal{I}}$, then $t^{\mathcal{I}}(d) \rightsquigarrow t^{\mathcal{I}}(e)$.

When we speak about the *product* $\mathcal{I}_1 \times \mathcal{I}_2$ of two k-initial interpretation trees $\mathcal{I}_1$ and $\mathcal{I}_2$, we simply mean the product of the interpretations $(\Delta^{\mathcal{I}_1}, \cdot^{\mathcal{I}_1})$ and $(\Delta^{\mathcal{I}_2}, \cdot^{\mathcal{I}_2})$, i.e., the annotating components $\rho^{\mathcal{I}_i}$ and $t^{\mathcal{I}_i}$ are dropped before forming the product.

Lemma 33. *An $\mathcal{ALC}$-TBox $\mathcal{T} = \{\top \sqsubseteq C_{\mathcal{T}}\}$ is not preserved under products iff there are n-initial interpretation trees $\mathcal{I}_1$ and $\mathcal{I}_2$ of maximum outdegree 2^{n^2+n+1} such that $(\rho_1, \rho_2) \notin C_{\mathcal{T}}^{\mathcal{I}_1 \times \mathcal{I}_2}$, where $n = |\mathcal{T}|$.*

Proof. First assume that $\mathcal{T}$ is not preserved under products. By Lemma 32, there are tree-shaped models $\mathcal{J}_1$ and $\mathcal{J}_2$ of $\mathcal{T}$ of maximum outdegree 2^{n^2+2} such that $\mathcal{J}_1 \times \mathcal{J}_2$ is not a model of $\mathcal{T}$. Let ρ_i be the root of $\mathcal{J}_i$, for $i \in \{1, 2\}$. W.l.o.g., we can assume that $(\rho_1, \rho_2) \notin C_{\mathcal{T}}^{\mathcal{J}_1 \times \mathcal{J}_2}$ (if this is not the case, replace $\mathcal{J}_1$ and $\mathcal{J}_2$ by suitable subtrees of these models). Define n-initial interpretation trees $\mathcal{I}_1$ and $\mathcal{I}_2$ by starting with $\mathcal{J}_1$ and $\mathcal{J}_2$, removing all nodes of depth exceeding n, and adding the annotations $\rho^{\mathcal{I}_i} = \rho_i$ and $t^{\mathcal{I}_i}$, where the latter is defined by setting $t^{\mathcal{I}_i}(d) = \mathsf{tp}_{\mathcal{T}}^{\mathcal{J}_i}(d)$ for all $d \in \Delta^{\mathcal{I}_i}$. It remains to show that $(\rho^{\mathcal{I}_1}, \rho^{\mathcal{I}_2}) \notin C_{\mathcal{T}}^{\mathcal{I}_1 \times \mathcal{I}_2}$, which is an immediate consequence of $(\rho_1, \rho_2) \notin C_{\mathcal{T}}^{\mathcal{J}_1 \times \mathcal{J}_2}$ and the following claim, whose proof is left to the reader. For all $(d_1, d_2) \in \Delta^{\mathcal{I}_1 \times \mathcal{I}_2}$, we use $\delta_{12}(d_1, d_2)$ to denote the length of the unique path from $(\rho^{\mathcal{I}_1}, \rho^{\mathcal{I}_2})$ to (d_1, d_2) in $\mathcal{I}_1 \times \mathcal{I}_2$ if such a path exists; otherwise, $\delta_{12}(d_1, d_2)$ is undefined.

Claim. For all $(d_1, d_2) \in \Delta^{\mathcal{I}_1 \times \mathcal{I}_2}$ with $\delta_{12}(d_1, d_2)$ defined and $C \in \mathsf{sub}(\mathcal{T})$ with $\mathsf{rd}(C) \leq n - \delta_{12}(d_1, d_2)$, we have $(d_1, d_2) \in C^{\mathcal{J}_1 \times \mathcal{J}_2}$ iff $(d_1, d_2) \in C^{\mathcal{I}_1 \times \mathcal{I}_2}$.

Conversely, assume that there are n-initial interpretation trees $\mathcal{I}_1$ and $\mathcal{I}_2$ as stated in the lemma. W.l.o.g., we assume that $\Delta^{\mathcal{I}_1} \cap \Delta^{\mathcal{I}_2} = \emptyset$. For $i \in \{1, 2\}$, let $F_i = \{d \in \Delta^{\mathcal{I}_i} \mid \delta_{\mathcal{I}_i}(d) = n\}$. For each $d \in F_i$, choose a model $\mathcal{I}_d$ of $\mathcal{T}$ and a $\rho_d \in \Delta^{\mathcal{I}_d}$ such that $\mathsf{tp}_{\mathcal{T}}^{\mathcal{I}_d}(\rho_d) = t^{\mathcal{I}_i}(d)$ (which exists by definition of $\mathfrak{T}$ and initial interpretation trees). W.l.o.g., assume that $\Delta^{\mathcal{I}_d} \cap \Delta^{\mathcal{I}_e} = \emptyset$ whenever $d \neq e$ and $\Delta^{\mathcal{I}_i} \cap \Delta^{\mathcal{I}_d} = d$ for all $d \in F_i$, $i \in \{1, 2\}$. Now let $\mathcal{J}_i$ be the interpretation obtained by taking the union of $\mathcal{I}_i$ and all $\mathcal{I}^d$, $d \in F_i$. In detail:

$$\begin{array}{rcl} \Delta^{\mathcal{J}_i} & = & \Delta^{\mathcal{I}_i} \cup \bigcup_{d \in F_i} \Delta^{\mathcal{I}_d} \\ A^{\mathcal{J}_i} & = & A^{\mathcal{I}_i} \cup \bigcup_{d \in F_i} A^{\mathcal{I}_d} \\ r^{\mathcal{J}_i} & = & r^{\mathcal{I}_i} \cup \bigcup_{d \in F_i} r^{\mathcal{I}_d} \end{array}$$

The following claim can be proved by a straightforward induction. It implies that $\mathcal{J}_1$ and $\mathcal{J}_2$ are models of $\mathcal{T}$, but $\mathcal{J}_1 \times \mathcal{J}_2$ is not, whence $\mathcal{T}$ is not closed under products. Details are left to the reader. For $(d_1, d_2) \in \Delta^{\mathcal{I}_1 \times \mathcal{I}_2}$, we use $\delta_{1,2}(d_1, d_2)$ to denote the length of the path (d_1, d_2) from $(\rho^{\mathcal{I}_1}, \rho^{\mathcal{I}_2})$; such a path need not exist (then $\delta_{1,2}(d_1, d_2)$ is undefined), but it is unique if it exists.

Claim. For all $C \in \mathsf{sub}(\mathcal{T})$, $d_1 \in \Delta^{\mathcal{I}_1}$, $d_2 \in \Delta^{\mathcal{I}_2}$, $e_1 \in \Delta^{\mathcal{I}_{d_1}}$, and $e_2 \in \Delta^{\mathcal{I}_{d_2}}$, we have

1. $C^{\mathcal{I}_i} \in t^{\mathcal{I}_i}(d_i)$ iff $d_i \in C^{\mathcal{J}_i}$;
2. $e_i \in C^{\mathcal{I}_{d_i}}$ iff $e_i \in C^{\mathcal{J}_i}$;
3. $(d_1, d_2) \in C^{\mathcal{I}_1 \times \mathcal{I}_2}$ iff $(d_1, d_2) \in C^{\mathcal{J}_1 \times \mathcal{J}_2}$ whenever $\delta_{12}(d_1, d_2)$ is defined and $\mathsf{rd}(C) \leq n - \delta_{12}(d_1, d_2)$.

❏

By Lemma 33, to decide whether a given $\mathcal{ALC}$-TBox $\mathcal{T}$ is not preserved under products, it suffices to guess two initial interpretation trees $\mathcal{I}_1$ and $\mathcal{I}_2$ whose size is bounded exponentially in that of $|\mathcal{T}|$, and then verifying that $(\rho_1, \rho_2) \notin C_{\mathcal{T}}^{\mathcal{I}_1 \times \mathcal{I}_2}$. It is not hard to see that the latter can be done in time polynomial in the size of $\mathcal{I}_1$ and $\mathcal{I}_2$, by explicitly forming the product and then applying model checking. We have proved the upper bound stated in Theorem 31.

The lower bound stated in Theorem 31 is proved by reduction of the $2^{n+1} \times 2^{n+1}$-tiling problem.

Definition 34 (Tiling System)**.** A *tiling system* $\mathfrak{T}$ is a triple (T, H, V), where $T = \{0, \ldots, k-1\}$, $k \geq 0$, is a finite set of *tile types* and $H, V \subseteq T \times T$ represent the *horizontal and vertical matching conditions*. Let $\mathfrak{T}$ be a tiling system and $c = c_0, \ldots, c_{n-1}$ an *initial condition*, i.e. an n-tuple of tile types. A mapping $\tau : \{0, \ldots, 2^{n+1}-1\} \times \{0, \ldots, 2^{n+1}-1\} \to T$ is a *solution* for $\mathfrak{T}$ and c iff for all $x, y < 2^{n+1}$, the following holds (where $\oplus_i$ denotes addition modulo i):

- if $\tau(x, y) = t$ and $\tau(x \oplus_{2^{n+1}} 1, y) = t'$, then $(t, t') \in H$
- if $\tau(x, y) = t$ and $\tau(x, y \oplus_{2^{n+1}} 1) = t'$, then $(t, t') \in V$
- $\tau(i, 0) = c_i$ for $i < n$.

To represent grid positions, we use a binary counter that is implemented through concept names $X_0, \ldots, X_{2(n+1)}$ and $\overline{X}_0, \ldots, \overline{X}_{2(n+1)}$ where truth of X_i indicates that bit i is set, truth of $\overline{X}_i$ indicates that bit i is not set, the first $n+1$ bits represent the horizontal value of the grid position, and the remaining $n+1$ bits the vertical value. To reflect the latter, we will sometimes write $Y_0, \ldots, Y_n$ instead of $X_{n+1}, \ldots, X_{2(n+1)}$, and likewise for $\overline{Y}_0, \ldots, \overline{Y}_n$. Define

$$\begin{array}{rl} \mathsf{tree}_n = & \bigsqcap_{i \leq 2(n+1)} \forall r^i.(\exists r.X_i \sqcap \exists r.\overline{X}_i) \sqcap \\ & \bigsqcap_{i<j<2(n+1)} \forall r^j.\big((X_i \to \forall r.X_i) \sqcap (\overline{X}_i \to \forall r.\overline{X}_i) \big) \sqcap \\ & \forall r^{2(n+1)}.(\exists r.P \sqcap \exists r.R \sqcap \exists r.U) \sqcap \\ & \forall r^{2(n+1)}.(\forall r.(P \to X{=}{=}) \sqcap \forall r.(P \to Y{=}{=})) \sqcap \\ & \forall r^{2(n+1)}.(\forall r.(R \to X{+}{+}) \sqcap \forall r.(R \to Y{=}{=})) \sqcap \\ & \forall r^{2(n+1)}.(\forall r.(U \to X{=}{=}) \sqcap \forall r.(U \to Y{+}{+})) \sqcap \\ & \bigsqcap_{\substack{j \leq 2(n+1)+1 \\ i \leq 2(n+1)}} \forall r^j.\neg(X_i \sqcap \overline{X}_i) \end{array}$$

where $\forall r.(P \to X{=}{=})$ is a concept which expresses that the horizontal value of all r-successors that satisfy P is identical to the horizontal value at the current node, $\forall r.(U \to Y{+}{+})$ expresses that the vertical value of all r-successors that satisfy P can be obtained from the vertical value at the current node by incrementation, and so on. It is left to the reader to work out the details of these concepts, we only give $\forall r.(R \to Y{=}{=})$ as an example:

$$\bigsqcap_{i \leq n} (Y_i \to \forall r.(R \to Y_i)) \sqcap \bigsqcap_{i \leq n} (\overline{Y}_i \to \forall r.(R \to \overline{Y}_i)).$$

Intuitively, the concept tree_n generates a tree that contains all the grid positions, where each subtree rooted at level $2(n+1)$ represents a small fragment of the grid. More specifically, such a subtree has depth 1 and represents a grid node (the P-leaf, where P stands for 'current position'), its right neighbor (the R-leaf), and its upper neighbor (the U-leaf). To achieve that each such fragment has a proper tiling, define

$$\begin{array}{l} \mathsf{tiling}_{\mathfrak{T},c} = \\ \quad \forall r^{2(n+1)}.\big(\bigsqcup_{(t,t') \in H} (\forall r.(P \to T_t) \sqcap \forall r.(R \to T_{t'}) \big) \sqcap \\ \quad \forall r^{2(n+1)}.\big(\bigsqcup_{(t,t') \in V} (\forall r.(P \to T_t) \sqcap \forall r.(U \to T_{t'}) \big) \sqcap \\ \quad \forall r^{2(n+1)+1}.\big(\neg \bigsqcap_{t,t' \in T} (T_t \sqcap T_{t'}) \big) \sqcap \\ \quad \forall r^{2(n+1)+1}.\big(\bigsqcap_{i<n} ((P \sqcap (X{=}{=}i) \sqcap (Y{=}{=}0)) \to T_{c_i}) \big) \end{array}$$

where c_i is the i-th bit of the initial condition c, $(X{=}{=}i)$ is a concept expressing that the horizontal value at the current node is identical to the constant i, and similarly for $(Y{=}{=}0)$.

Note that each position (except those on the fringes of the grid) occurs at least three times in the tree: as a P-node, as an R-node, and as a U-node. To represent a proper solution to the tiling system, it remains to ensure that multiple occurrences of the same grid position are labeled with the same tile type. To achieve this, we use products. Assume there are *two* tree interpretations of the above form. The following concept is true in the root of their product interpretation iff the two component interpretations disagree on the tiling of some position:

$$\mathsf{defect}_n \quad = \quad \exists r^{2(n+1)+1}.(\bigsqcap_{i \leq 2(n+1)} (X_i \sqcup \overline{X}_i) \sqcap \bigsqcap_{t \in T} \neg T_t)$$

To assemble all the pieces into a single concept, set $C_{\mathfrak{T},c} = D_1 \sqcup D_2 \sqcup D_3$ where

$$\begin{aligned} D_1 &= (\mathsf{tree}_n \sqcap \mathsf{tiling}_{\mathfrak{T},c} \sqcap M) \\ D_2 &= (\mathsf{tree}_n \sqcap \mathsf{tiling}_{\mathfrak{T},c} \sqcap M') \\ D_3 &= (\mathsf{tree}_n \sqcap \mathsf{defect}_n) \end{aligned}$$

The above encoding of solutions of tiling systems works purely on the level of concepts, and does not necessarily need TBoxes. We believe that this is interesting and start with proving a strong form of correctness: the following lemma shows that given a concept C, it is co-NEXPTIME-hard to decide whether C is preserved under products. In a subsequent step, we will raise this result to the level of TBoxes.

Lemma 35. *There is a solution for $\mathfrak{T}$ and c iff $C_{\mathfrak{T},c}$ is not preserved under products.*

Proof. First assume that $\mathfrak{T}$ and c have a solution τ. We define tree interpretations $\mathcal{I}_1$ and $\mathcal{I}_2$ such that $\mathcal{I}_i$ is a model of D_i for $i \in \{1, 2\}$ (thus both $\mathcal{I}_1$ and $\mathcal{I}_2$ are models of $C_{\mathfrak{T},c}$), but their product is not a model of $C_{\mathfrak{T},c}$. For $i \in \{1, 2\}$, define

$$\Delta^{\mathcal{I}_i} = \bigcup_{i \leq 2(n+1)} \{0,1\}^i \cup \{0,1\}^{2(n+1)} \cdot \{P, R, U\},$$

i.e., $\Delta^{\mathcal{I}_i}$ is the set of all words over the alphabet $\{0, 1\}$ of length at most $2(n + 1)$ plus all words over $\{0, 1\}$ of length exactly $2(n+1)$ concatenated with a symbol from $\{P, R, U\}$. We will not distinguish between words of the former kind and numbers represented in binary, lowest bit first. We now define a function $\mu : \Delta^{\mathcal{I}_i} \to \mathbb{N}$ and extend τ to elements of $\Delta^{\mathcal{I}_i} \cap (\{0,1\}^{2(n+1)} \cdot \{P, R, U\})$ as follows:

- for each $w \in \Delta^{\mathcal{I}_i} \cap \{0, 1\}^*$, $\mu(w) = w$;
- for each $w \cdot P \in \Delta^{\mathcal{I}_i}$ with $w = w_x \cdot w_y$, where $w_x, w_y \in \{0, 1\}^{n+1}$, set $\mu(w \cdot P) = \mu(w)$ and $\tau(w \cdot P) = \tau(w_x, w_y)$;
- for each $w \cdot R \in \Delta^{\mathcal{I}_i}$ with $w = w_x \cdot w_y$, where $w_x, w_y \in \{0, 1\}^{n+1}$, set $\mu(w \cdot R) = \mu((w_x+1) \cdot w_y)$ and $\tau(w \cdot R) = \tau(w_x + 1, w_y)$;
- for each $w \cdot U \in \Delta^{\mathcal{I}_i}$ with $w = w_x \cdot w_y$, where $w_x, w_y \in \{0, 1\}^{n+1}$, set $\mu(w \cdot U) = \mu(w_x \cdot (w_y + 1))$ and $\tau(w \cdot U) = \tau(w_x, w_y + 1)$.

For $n \geq 0$, we use $\mathsf{bit}_j(n)$ to denote the j-th bit of the binary representation of the number n. To complete the definition of $\mathcal{I}_1$ and $\mathcal{I}_2$ set for $i \in \{1, 2\}$, $j \leq 2(n + 1)$, $t \in T$, and $G \in \{P, R, U\}$:

$$\begin{aligned} r^{\mathcal{I}_i} &= \{(w, w \cdot c) \mid w \cdot c \in \Delta^{\mathcal{I}_i}, c \in \{0, 1, P, R, U\}\} \\ X_j^{\mathcal{I}_i} &= \{w \mid w \in \Delta^{\mathcal{I}_i}, \mathsf{bit}_{j+1}(\mu(w)) = 1\} \\ \overline{X}_j &= \{w \mid w \in \Delta^{\mathcal{I}_i}, \mathsf{bit}_{j+1}(\mu(w)) = 0\} \\ T_t^{\mathcal{I}_i} &= \{w \cdot c \mid w \cdot c \in \Delta^{\mathcal{I}_i}, c \in \{P, R, U\}, \tau(w) = t\} \\ G^{\mathcal{I}_i} &= \{w \cdot G \mid w \cdot G \in \Delta^{\mathcal{I}_i}\} \\ M^{\mathcal{I}_1} &= M'^{\mathcal{I}_2} = \{\varepsilon\} \\ M'^{\mathcal{I}_1} &= M^{\mathcal{I}_2} = \emptyset \end{aligned}$$

Now consider the element $(\varepsilon, \varepsilon) \in \mathcal{I}_1 \times \mathcal{I}_2$. It is neither an instance of M nor of M', thus does neither satisfy D_1 nor D_2. Finally, it is not hard to show that $(\varepsilon, \varepsilon) \notin \mathsf{defect}_n^{\mathcal{I}_1 \times \mathcal{I}_2}$, essentially because all nodes in $\mathcal{I}_1$ and $\mathcal{I}_2$ that are on the same level and satisfy the same combination of X_i and $\overline{X}_i$ concepts are labeled with the same concept name T_t. Thus, $(\varepsilon, \varepsilon)$ does not satisfy D_3, and thus also not $C_{\mathfrak{T},c}$.

First assume that $\mathfrak{T}$ and c have a solution. Then take tree models $\mathcal{I}$ and $\mathcal{I}'$ encoding that solution such that $\mathcal{I}$ and $\mathcal{I}'$ are identical except that $\mathcal{I}$ satisfies $M \sqcap \neg M'$ at the root while $\mathcal{I}'$ satisfies $M' \sqcap \neg M$ there. In the product, $C_{\mathfrak{T},n}$ is false: D_1 is false as M is not satisfied at the root, D_2 is false as M' is not satisfied at the root, and D_3 is false as there is no defect.

Conversely, assume that there is no solution for $\mathfrak{T}$ and c and take two objects $(\mathcal{I}_1, d_1)$ and $(\mathcal{I}_2, d_2)$ such that $d_1 \in C_{\mathfrak{T},c}^{\mathcal{I}_1}$ and $d_2 \in C_{\mathfrak{T},c}^{\mathcal{I}_2}$. We have to show that $(d_1, d_2) \in C_{\mathfrak{T},c}^{\mathcal{I}_1 \times \mathcal{I}_2}$. As a preliminary, we state the following claim, which is easily proved by induction on the structure of the concept C.

Claim. All concepts C built from concept names, conjunction, existential restriction, universal restriction, and implication $A \to D$ with A a concept name, are preserved under products.

Using this claim, it is easy to show that the concept tree_n is preserved under products: it suffices to consider each conjunct separately, all conjuncts except the last one are captured by the claim, and the last conjunct is clearly also preserved under products.

We have $d_1 \in \mathsf{tree}_n^{\mathcal{I}_1}$ and $d_2 \in \mathsf{tree}_n^{\mathcal{I}_2}$, thus $(d_1, d_2) \in \mathsf{tree}_n^{\mathcal{I}_1 \times \mathcal{I}_2}$. It thus suffices to show $(d_1, d_2) \in \mathsf{defect}_n^{\mathcal{I}_1 \times \mathcal{I}_2}$: then, $(d_1, d_2) \in D_3^{\mathcal{I}_1 \times \mathcal{I}_2}$, thus $(d_1, d_2) \in C_{\mathfrak{T},c}^{\mathcal{I}_1 \times \mathcal{I}_2}$. Distinguish the following cases.

(i) $d_1 \in D_3^{\mathcal{I}_1}$ or $d_2 \in D_3^{\mathcal{I}_2}$.

We only treat the case $d_1 \in D_3^{\mathcal{I}_1}$, as $d_2 \in D_3^{\mathcal{I}_2}$ is symmetric. Since $d_1 \in \mathsf{defect}_n^{\mathcal{I}_1}$, there is an $e_1 \in \Delta^{\mathcal{I}_1}$ and a sequence $Z_0, \ldots, Z_{2(n+1)}$, $Z_i \in \{X_i, \overline{X}_i\}$, such that e_1 is reachable from d_1 in $2(n + 1) + 1$ steps along r-edges, $e_1 \in Z_i^{\mathcal{I}_1}$ for all $i \leq 2(n+1)$, and $e_1 \notin T_t^{\mathcal{I}_1}$ for any $t \in T$. Since $d_2 \in \mathsf{tree}_n^{\mathcal{I}_2}$, there is a node $e_2 \in \Delta^{\mathcal{I}_2}$ such that e_2 is reachable from d_2 in $2(n + 1) + 1$ steps along r-edges and $e_2 \in Z_i^{\mathcal{I}_2}$ for all $i \leq 2(n + 1)$. Then (e_1, e_2) is reachable from (d_1, d_2) in $2(n + 1) + 1$ steps along r-edges in $\mathcal{I}_1 \times \mathcal{I}_2$, witnessing that $(d_1, d_2) \in \mathsf{defect}_n^{\mathcal{I}_1 \times \mathcal{I}_2}$ as desired.

(ii) $d_1 \in (D_1 \sqcup D_2)^{\mathcal{I}_1}$ and $d_1 \in (D_1 \sqcup D_2)^{\mathcal{I}_2}$.

Then $d_1 \in \mathsf{tree}_n^{\mathcal{I}_1}$ and $d_1 \in \mathsf{tiling}_{\mathfrak{T},c}^{\mathcal{I}_1}$. Since there is no solution for $\mathfrak{T}$ and c, at least one position of the grid must have non-unique tile types, i.e., there is a sequence $Z_0, \ldots, Z_{2(n+1)}$, $Z_i \in \{X_i, \overline{X}_i\}$ and distinct $t, t' \in T$ such that

$$d_1 \in (\exists r^{2(n+1)+1}.(\bigsqcap_{i \leq 2(n+1)} Z_i \sqcap T_t))^{\mathcal{I}_1}$$

and

$$d_1 \in (\exists r^{2(n+1)+1}.(\bigsqcap_{i \leq 2(n+1)} Z_i \sqcap T_{t'}))^{\mathcal{I}_1}.$$

Take witnesses d_t and $d_{t'}$ for this, i.e., d_t is reachable from d_1 in $2(n + 1) + 1$ steps along r-edges and satisfies the concept inside the upper existential restriction, and analogously for $d_{t'}$. Since $d_2 \in \mathsf{tree}_n^{\mathcal{I}_2}$, there is a node $e_2 \in \Delta^{\mathcal{I}_2}$ such that e_2 is reachable from d_2 in $2(n+1)+1$ steps along r-edges and $e_2 \in Z_i^{\mathcal{I}_2}$ for all $i \leq 2(n+1)$. Since $d_2 \in \mathsf{tiling}_{\mathfrak{T},c}^{\mathcal{I}_2}$, we do not have both $d_2 \in T_t^{\mathcal{I}_2}$ and $d_2 \in T_{t'}^{\mathcal{I}_2}$. It follows that (d_t, e_2) or $(d_{t'}, e_2)$ is a witness for $(d_1, d_2) \in \mathsf{defect}_n^{\mathcal{I}_1 \times \mathcal{I}_2}$ as desired. ❑

It is now easy to reproduce this on the level of TBoxes.

Lemma 36. *There is a solution for $\mathfrak{T}$ and c iff the TBox $\{\top \sqsubseteq \exists s.C_{\mathfrak{T},c}\}$ is not preserved under products.*

Proof.(sketch) By Lemma 35, it suffices to show that $C_{\mathfrak{T}_c}$ is preserved under products iff $\exists s.C_{\mathfrak{T},c}$ is. This is straightforward. ❑

From Lemma 36, we obtain the desired lower bound stated in Theorem 31.

C.3 Proofs for DL-Lite

Theorem 37. *It is decidable in* ExpTime *whether an $\mathcal{ALCI}$-TBox is invariant under $\sim_{\text{DL-Lite}_{\text{horn}}}$.*

Proof. Assume $\mathcal{T}$ is given. Let $\mathsf{sig}(\mathcal{T})$ be the set of concept and role names that occur in $\mathcal{T}$ and denote by $B(\mathcal{T})$ the closure under single negation of the set of basic concepts over $\mathsf{sig}(\mathcal{T})$. In this proof, the set tp denotes the set of types over $\mathsf{sub}(\mathcal{T}) \cup B(\mathcal{T})$; i.e., all subsets t of $\mathsf{sub}(\mathcal{T}) \cup B(\mathcal{T})$ such that

- $C_1 \sqcap C_2 \in t$ iff $C_1 \in t$ and $C_2 \in t$, for all $C_1 \sqcap C_2 \in \mathsf{sub}(\mathcal{T}) \cup B(\mathcal{T})$;
- $\neg C \in t$ iff $C \notin t$ iff $\neg C \in \mathsf{sub}(\mathcal{T}) \cup B(\mathcal{T})$.

For $t \in \mathsf{tp}$, we set $t^B = t \cap B(\mathcal{T})$. Let $\mathsf{tp}^B = \{t^B \mid t \in \mathsf{tp}\}$ and call elements of tp^B *b-types*. The following is readily checked:

Claim 1. If $\mathcal{I}_1, \mathcal{I}_2$ only interpret symbols in $\mathsf{sig}(\mathcal{T})$, then $\mathcal{I}_1 \sim^g_{\text{DL-Lite}_{\text{horn}}} \mathcal{I}_2$ iff the sets of b-types realized in $\mathcal{I}_1$ and $\mathcal{I}_2$ coincide.

Denote by $\mathsf{tp}(\mathcal{T})$ the set of $t \in \mathsf{tp}$ that are realizable in models of $\mathcal{T}$ and set $\mathsf{tp}^B(\mathcal{T}) = \{t^B \mid t \in \mathsf{tp}(\mathcal{T})\}$. Apply the following rule exhaustively to $Q = \{t \in \mathsf{tp} \mid t^B \in \mathsf{tp}^B(\mathcal{T})\}$:

- If $\exists r.C \in t \in Q$ and there does no exists $s \in Q$ such that $t \rightsquigarrow_r s$ and $s \in Q$, then remove t of Q.

Denote the resulting set by P. The following is readily checked.

Claim 2. P consists of the set of all types that are realizable in interpretations realizing b-types from $\mathsf{tp}^B(\mathcal{T})$ only.

Observe that $P \supseteq \mathsf{tp}(\mathcal{T})$.

Claim 3. $P \not\subseteq \mathsf{tp}(\mathcal{T})$ iff $\mathcal{T}$ is not invariant under $\sim_{\text{DL-Lite}_{\text{horn}}}$.

Assume $P \not\subseteq \mathsf{tp}(\mathcal{T})$ and take the disjoint union $\mathcal{I}_1$ of interpretations $\mathcal{I}_t$, $t \in P$, that realize t and realize b-types from $\mathsf{tp}^B(\mathcal{T})$ only. On the other, take a model $\mathcal{I}_2$ of $\mathcal{T}$ that realizes all b-types in $\mathsf{tp}^B(\mathcal{T})$. Now, $\mathcal{I}_1$ and $\mathcal{I}_2$ realize exactly the b-types in $\mathsf{tp}^B(\mathcal{T})$, and since we may assume that $\mathcal{I}_1$ and $\mathcal{I}_2$ only interpret symbols from $\mathsf{sig}(\mathcal{T})$, we obtain from Claim 1 that $\mathcal{I}_1 \sim_{\text{DL-Lite}_{\text{horn}}} \mathcal{I}_2$. On the other hand, $\mathcal{I}_2$ is a model of $\mathcal{T}_1$ but $\mathcal{I}_1$ is not, since it realizes a type from $\mathsf{tp} \setminus \mathsf{tp}(\mathcal{T})$.

The converse direction is clear.

As $\mathsf{tp}(\mathcal{T})$ can be computed in exponential time (since satisfiability of $\mathcal{ALCI}$-concepts w.r.t. $\mathcal{ALCI}$-TBoxes is decidable in ExpTime) and since P is computed in exponential time, we have proved the ExpTime-upper bound. ❑

Theorem 21. Let φ be a first-order sentence. Then the following conditions are equivalent:

1. φ is equivalent to a DL-Lite$_{\mathsf{core}}$ TBox (resp. DL-Lite$^d_{\mathsf{core}}$ TBox);
2. φ is invariant under $\sim^g_{\text{DL-Lite}_{\text{horn}}}$ and disjoint unions, and is preserved under products and unions (resp. compatible unions).

Proof. The proof is a variation of the proof of Theorem 17. We again concentrate on $2 \Rightarrow 1$, in particular on showing that $\mathsf{cons}(\varphi) \models \varphi$, where $\mathsf{cons}(\varphi)$ is the set of all DL-Lite concept inclusions that are a consequence of φ. Assume to the contrary that $\mathsf{cons}(\varphi) \not\models \varphi$.

Let $\mathsf{cons}^{\sqcap,\sqcup}(\varphi)$ denote the set of extended $\mathcal{L}$-CIs that are a consequence of $\mathsf{cons}(\varphi)$, where an *extended $\mathcal{L}$-CI* has the form

$$B_1 \sqcap \cdots \sqcap B_m \sqsubseteq D_1 \sqcup \cdots \sqcup D_n,$$

with both the B_i and the D_i basic DL-Lite concepts. Our aim is to construct interpretations $\mathcal{I}^-$ and $\mathcal{I}^+$ such that $\mathcal{I}^- \not\models \varphi$, $\mathcal{I}^+ \models \varphi$, and both $\mathcal{I}^-$ and $\mathcal{I}^+$ satisfy precisely those extended $\mathcal{L}$-CIs that are in $\mathsf{cons}^{\sqcap,\sqcup}(\varphi)$.

$\mathcal{I}^-$ is constructed as in the proof of Theorem 17. For every extended $\mathcal{L}$-CI $C \sqsubseteq D \notin \mathsf{cons}^{\sqcap,\sqcup}(\varphi)$, take a model $\mathcal{I}_{C \not\sqsubseteq D}$ of $\mathsf{cons}(\varphi)$ that violates $C \sqsubseteq D$. Then $\mathcal{I}^-$ is the disjoint union of all $\mathcal{I}_{C \not\sqsubseteq D}$ and a model of $\mathsf{cons}(\varphi) \cup \{\neg\varphi\}$. Clearly, $\mathcal{I}^-$ satisfies the desired properties.

The main step in constructing $\mathcal{I}^+$ is to build a model $\mathcal{I}'_{C \not\sqsubseteq D}$ of φ that violates $C \sqsubseteq D$, for every extended $\mathcal{L}$-CI $C \sqsubseteq D \notin \mathsf{cons}^{\sqcap,\sqcup}(\varphi)$. When this is done, $\mathcal{I}^+$ will simply by the disjoint union of all $\mathcal{I}'_{C \not\sqsubseteq D}$. Let

$$C \sqsubseteq D = B_1 \sqcap \cdots \sqcap B_m \sqsubseteq D_1 \sqcup \cdots \sqcup D_n.$$

Let $i \leq m$ and $j \leq n$. Since $\varphi \not\models C \sqsubseteq D$, we also have $\varphi \not\models B_i \sqsubseteq D_j$ and thus there is a model $\mathcal{I}_{i,j}$ of φ that violates $B_i \sqsubseteq D_j$. Assume that $d_{i,j} \in (B_i \setminus D_j)^{\mathcal{I}_{i,j}}$. For $1 \leq i \leq m$, take the product

$$\mathcal{I}_i = \prod_{1 \leq j \leq n} \mathcal{I}_{i,j}.$$

Since φ is preserved under products, $\mathcal{I}_i$ is a model of φ. Moreover, the element $\overline{d}_i : j \mapsto d_{i,j} \in \Delta^{\mathcal{I}_i}$ satisfies $\overline{d}_i \in B_i^{\mathcal{I}_i}$ and $\overline{d}_i \notin (D_1 \sqcup \cdots \sqcup D_n)^{\mathcal{I}_i}$. By renaming domain elements, we can achieve that there is a $d \in \bigcap_{1 \leq i \leq m} \Delta^{\mathcal{I}_i}$ such that $d \in B_i^{\mathcal{I}_i} \setminus (D_1 \sqcup \cdots \sqcup D_n)^{\mathcal{I}_i}$ for $1 \leq i \leq m$. Now, $\mathcal{I}'_{C \not\sqsubseteq D}$ is the union of the interpretations $(\mathcal{I}_i)_{1 \leq i \leq m}$. Then $\mathcal{I}'_{C \not\sqsubseteq D}$ is a model of φ since φ is preserved under unions and we have $d \in (B_1 \sqcap \cdots \sqcap B_m)^{\mathcal{I}'_{C \not\sqsubseteq D}}$ and $d \notin (D_1 \sqcup \cdots \sqcup D_n)^{\mathcal{I}'_{C \not\sqsubseteq D}}$, thus $\mathcal{I}'_{C \not\sqsubseteq D}$ is as required.

We can again assume w.l.o.g. that $\mathcal{I}^-$ and $\mathcal{I}^+$ are ω-saturated. It remains to show $\mathcal{I}^- \equiv^g_{\mathcal{L}} \mathcal{I}^+$, which can be done as in the proof of Theorem 17. ❑

Theorem 38. *Let $\mathcal{L}_1 \in$ ExpDL contain inverse roles and $\mathcal{L}_2 \in \{$DL-Lite$_{\mathsf{core}}$, DL-Lite$^d_{\mathsf{core}}\}$. Then the complexity of $\mathcal{L}_1$-to-$\mathcal{L}_2$-TBox rewritability coincides with the complexity of TBox satisfiability in $\mathcal{L}_1$.*

Proof. It it common knowledge that for all $\mathcal{L}_1 \in$ ExpDL, TBox satisfiability and Boolean TBox satisfiability have the same complexity. It thus suffices to give a reduction from $\mathcal{L}_1$-TBox unsatisfiability to $\mathcal{L}_1$-to-$\mathcal{L}_2$-TBox rewritability and from $\mathcal{L}_1$-to-$\mathcal{L}_2$-TBox rewritability to the unsatisfiability of Boolean $\mathcal{L}_1$-TBoxes. The former is easy since an $\mathcal{L}_1$-TBox $\mathcal{T}$ is satisfiable iff $\mathcal{T} \cup \mathcal{T}'$ is not $\mathcal{L}_2$-rewritable, where $\mathcal{T}'$ is any fixed TBox that is not $\mathcal{L}_2$-rewritable.

For the reduction of $\mathcal{L}_1$-to-$\mathcal{L}_2$-TBox rewritability to the unsatisfiability of Boolean $\mathcal{L}_1$-TBoxes, fix an $\mathcal{L}_1$-TBox $\mathcal{T}$. Let Σ be the signature of $\mathcal{T}$, i.e., the set of all concept names, role names, and nominals that occur in $\mathcal{T}$. Moreover, let Γ be the set of all $\mathcal{L}_2$-concept inclusions over Σ, and

$$\Gamma_{\mathcal{T}} = \{\alpha \in \Gamma \mid \mathcal{T} \models \alpha\}.$$

Note that Γ is finite, and that its cardinality is polynomial in the size of $\mathcal{T}$.

Claim 1. $\mathcal{T}$ is $\mathcal{L}_2$-rewritable iff $\mathcal{T} \equiv \Gamma_{\mathcal{T}}$.

The "if" direction is trivial. For the "only if" direction, assume that $\mathcal{T}$ is $\mathcal{L}_2$-rewritable and that $\mathcal{T}'$ is an $\mathcal{L}_2$-TBox that is equivalent to $\mathcal{T}$. Clearly, every concept inclusion in $\Gamma_{\mathcal{T}}$ is a consequence of $\mathcal{T}'$. Conversely, every concept inclusion in $\mathcal{T}'$ must also be in $\Gamma_{\mathcal{T}}$. Thus, $\Gamma_{\mathcal{T}} \equiv \mathcal{T}_1 \equiv \mathcal{T}$.

By Claim 1, it suffices to reformulate the question 'is $\mathcal{T}$ equivalent to $\Gamma_{\mathcal{T}}$?' in terms of unsatisfiability of Boolean $\mathcal{L}_1$-TBoxes. This is what we do in the following. First, we may assume w.l.o.g. that $\mathcal{T}$ is of the form $\{\top \sqsubseteq C_{\mathcal{T}}\}$ with $C_{\mathcal{T}}$ and $\mathcal{L}_1$-concept in *negation normal form (NNF)*, i.e., negation is only applied to concept names and nominals. For each concept name $A \in \Sigma$ (role name $r \in \Sigma$, nominal $a \in \Sigma$) and $\alpha \in \Gamma$, reserve a fresh concept name A_α (role name r_α, nominal a_α). Moreover, for each $\alpha \in \Gamma \uplus \{\bullet\}$ reserve an additional concept name R_α. The new symbols give rise to signature-disjoint and relativized copies $\mathcal{T}_\alpha$ of $\mathcal{T}$, for each $\alpha \in \Gamma \uplus \{\bullet\}$, defined as follows:

1. replace in $C_{\mathcal{T}}$ each concept name A with A_α, each role name r with r_α, and each nominal a with a_α; call the result $C_{\mathcal{T},\alpha}$;
2. replace $\top \sqsubseteq C_{\mathcal{T},\alpha}$ with $R_\alpha \sqsubseteq C_{\mathcal{T},\alpha}$;
3. replace each subconcept $\exists r.C$ in $C_{\mathcal{T},\alpha}$ with $\exists r.(R_\alpha \sqcap C)$ and each subconcept $\forall r.C$ in $C_{\mathcal{T},\alpha}$ with $\forall r.(R_\alpha \rightarrow C)$.

Note that R_α is used for relativization, i.e., the TBox $\mathcal{T}_\alpha$ does not 'speak' about the entire domain, but only about the part identified by R_α.

Analogously, we introduce a renaming and relativization α_α for each $\alpha \in \Gamma \uplus \{\bullet\}$: first rename symbols as in Step 1 above, then replace $\alpha = B_1 \sqsubseteq B_2$ with $R_\alpha \sqcap B_1 \sqsubseteq B_2$. Note that the modified α is not in $\mathcal{L}_1$, but in $\mathcal{L}_2$ (since the latter contains inverse roles).

Define a Boolean TBox

$$\varphi = \bigwedge_{\alpha \in \Gamma} (\mathcal{T}_\alpha \wedge (\alpha_\alpha \rightarrow \alpha_\bullet)) \wedge \neg \mathcal{T}_\bullet$$

It suffices to prove the following

Claim 2. $\mathcal{T}$ is $\mathcal{L}_2$-rewritable iff φ is unsatisfiable.

"if". Let $\mathcal{T}$ not be $\mathcal{L}_2$-rewritable. By Claim 1, we then have $\Gamma_{\mathcal{T}} \not\models \mathcal{T}$. For each $\alpha \in \Gamma$, take a model $\mathcal{I}_\alpha$ of $\mathcal{T}_\alpha$ such that $\mathcal{I}_\alpha \models \alpha_\alpha$ iff $\alpha \in \Gamma_{\mathcal{T}}$. Additionally, take a model $\mathcal{I}_\bullet$ of $\{\alpha_\bullet \mid \alpha \in \Gamma_{\mathcal{T}}\}$ with $\mathcal{I}_\bullet \not\models \mathcal{T}_\bullet$. We can assume w.l.o.g. that $\mathcal{I}_\bullet$ and each $\mathcal{I}_\alpha$ have the same domain Δ: if they don't, then the relativization to R_α allows us to extend each domain $\Delta^{\mathcal{I}_\alpha}$ to $\bigcup_{\alpha \in \Gamma \uplus \{\bullet\}} \Delta^{\mathcal{I}_\alpha}$. Define a new interpretation $\mathcal{I}$ as follows:

- $\Delta^{\mathcal{I}} = \Delta$;
- $A^{\mathcal{I}}_\alpha = A^{\mathcal{I}_\alpha}_\alpha$, $r^{\mathcal{I}}_\alpha = r^{\mathcal{I}_\alpha}_\alpha$, and $a^{\mathcal{I}}_\alpha = a^{\mathcal{I}_\alpha}_\alpha$ for all concept names A including the relativization names A_α, role names r, nominals a, and $\alpha \in \Gamma \uplus \{\bullet\}$;

It is not hard to prove that $\mathcal{I}$ is a model of $\mathcal{T}_\alpha$ for all $\alpha \in \Gamma$, but not for $\mathcal{T}_\bullet$. Moreover, it is a model of α_α iff $\alpha \in \Gamma_{\mathcal{T}}$ and a model of $\alpha_\bullet$ if $\alpha \in \Gamma_{\mathcal{T}}$, for all $\alpha \in \Gamma$. Therefore, $\mathcal{I}$ satisfies φ.

"only if". Let $\mathcal{T}$ be $\mathcal{L}_2$-rewritable. Then $\mathcal{T} \equiv \Gamma_{\mathcal{T}}$ by Claim 1. Assume to the contrary of what is to be shown that there is a model $\mathcal{I}$ of φ. Since $\mathcal{T} \models \Gamma_{\mathcal{T}}$, it is easy to see that $\mathcal{T}_\alpha \models \alpha_\alpha$ for all $\alpha \in \Gamma_{\mathcal{T}}$. For this reason, the first conjunct of φ yields $\mathcal{I} \models \alpha_\bullet$ for all $\alpha \in \Gamma_{\mathcal{T}}$. Since $\mathcal{T} \equiv \Gamma_{\mathcal{T}}$, this yields $\mathcal{I} \models \mathcal{T}_\bullet$, in contradiction to $\mathcal{I}$ satisfying the last conjunct of φ. ❏

Note that, when inverse roles are not contained in $\mathcal{L}_1$, the above proof yields a reduction of $\mathcal{L}_1$-to-$\mathcal{L}_2$-TBox rewritability to satisfiability of Boolean $\mathcal{L}_1\mathcal{I}$-TBoxes, where $\mathcal{L}_1\mathcal{I}$ is the extension of $\mathcal{L}_1$ with inverse roles.