Generalized Satisfiability for the Description Logic $\mathcal{ALC}$ (Extended Abstract)*

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Abstract. The standard reasoning problem, concept satisfiability, in the basic description logic $\mathcal{ALC}$ is PSPACE-complete, and it is EXPTIME-complete in the presence of unrestricted axioms. Several fragments of $\mathcal{ALC}$, notably logics in the $\mathcal{FL}$, $\mathcal{EL}$, and DL-Lite families, have an easier satisfiability problem; sometimes it is even tractable. We classify the complexity of the standard satisfiability problems for all possible Boolean and quantifier fragments of $\mathcal{ALC}$ in the presence of general axioms.

1 Introduction

Standard reasoning problems of description logics, such as satisfiability or subsumption, have been studied extensively. Depending on the expressivity of the logic, the complexity of reasoning for DLs between fragments of the basic DL $\mathcal{ALC}$ and the OWL 2 standard $\mathcal{SROIQ}$ is between trivial and NEXPTIME.

For $\mathcal{ALC}$, concept satisfiability is PSPACE-complete [32]. In the presence of unrestricted axioms, it is EXPTIME-complete due to the correspondence with propositional dynamic logic [30, 34, 17]. Since the standard reasoning tasks are interreducible, subsumption has the same complexity.

Several fragments of $\mathcal{ALC}$, such as logics in the $\mathcal{FL}$, $\mathcal{EL}$ or DL-Lite families, are well-understood. They usually restrict the use of Boolean operators and of quantifiers, and it is known that their reasoning problems are often easier than for $\mathcal{ALC}$. We now need to distinguish between satisfiability and subsumption because they are no longer obviously interreducible if certain Boolean operators are missing. Concept subsumption with respect to acyclic and cyclic terminologies, and even with general axioms, is tractable in the logic $\mathcal{EL}$, which allows only conjunctions and existential restrictions, [4, 11], and it remains tractable under a variety of extensions such as nominals, concrete domains, role chain inclusions, and domain and range restrictions [5, 6]. Satisfiability for $\mathcal{EL}$, in contrast, is trivial, i.e., every $\mathcal{EL}$-ontology is satisfiable. However, the presence of universal quantifiers usually breaks tractability: Subsumption in $\mathcal{FL}_0$, which allows only conjunction and universal restrictions, is coNP-complete [27] and increases to PSPACE-complete

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with respect to cyclic terminologies [3, 21] and to EXPTIME-complete with general axioms [5, 20]. In [15, 16], concept satisfiability and subsumption for several logics below and above $\mathcal{ALC}$ that extend $\mathcal{FL}_0$ with disjunction, negation and existential restrictions and other features, is shown to be tractable, NP-complete, coNP-complete or PSPACE-complete. Subsumption in the presence of general axioms is EXPTIME-complete in logics containing both existential and universal restrictions plus conjunction or disjunction [18], as well as in $\mathcal{AL}$, where only conjunction, universal restrictions and unqualified existential restrictions are allowed [14]. In DL-Lite, where atomic negation, unqualified existential and universal restrictions, conjunctions and inverse roles are allowed, satisfiability of ontologies is tractable [13]. Several extensions of DL-Lite are shown to have tractable and NP-complete satisfiability problems in [1, 2]. The logics in the $\mathcal{EL}$ and DL-Lite families are so important for (medical and database) applications that OWL 2 has two profiles that correspond to logics in these families.

This paper revisits restrictions to the Boolean operators in $\mathcal{ALC}$. Instead of looking at one particular subset of $\{\sqcap, \sqcup, \neg\}$, we are considering all possible sets of Boolean operators, and therefore our analysis includes less commonly used operators such as the binary exclusive *or* $\oplus$. Our aim is to find for *every* possible combination of Boolean operators whether it makes satisfiability of the corresponding restriction of $\mathcal{ALC}$ hard or easy. Since each Boolean operator corresponds to a Boolean function—i.e., an n-ary function whose arguments and values are in $\{\mathtt{0}, \mathtt{1}\}$—there are infinitely many sets of Boolean operators that determine fragments of $\mathcal{ALC}$. The complexity of the corresponding concept satisfiability problems without theories has already been classified in [19] between being PSPACE-complete, coNP-complete, tractable and trivial for all combinations of Boolean operators and quantifiers.

The tool used in [19] for classifying the infinitely many satisfiability problems was Post's lattice [29], which consists of all sets of Boolean functions closed under superposition. These sets directly correspond to all sets of Boolean operators closed under composition. Similar classifications have been achieved for satisfiability for classical propositional logic [22], Linear Temporal Logic [8], hybrid logic [24], and for constraint satisfaction problems [31, 33].

In this paper, we classify the concept satisfiability problems with respect to theories for $\mathcal{ALC}$ fragments obtained by arbitrary sets of Boolean operators and quantifiers. We separate these problems into EXPTIME-complete, NP-complete, P-complete and NL-complete, leaving only two single cases with non-matching upper and lower bound. We will also put these results into the context of the above listed results for $\mathcal{ALC}$ fragments.

This study extends our previous work in [25] by matching upper and lower bounds and considering restricted use of quantifiers.

2 Preliminaries

Description Logic. We use the standard syntax and semantics of $\mathcal{ALC}$ [7], with the Boolean operators $\sqcap$, $\sqcup$, $\neg$, $\top$, $\bot$ replaced by arbitrary operators $\circ_f$ that correspond to Boolean functions $f : \{0,1\}^n \to \{0,1\}$ of arbitrary arity n. Let $\mathsf{N_C}$, $\mathsf{N_R}$ and $\mathsf{N_I}$ be sets of atomic concepts, roles and individuals. Then the set of *concept descriptions*, for short *concepts*, is defined by

$$C := A \mid \circ_f(C, \dots, C) \mid \exists R.C \mid \forall R.C,$$

where $A \in \mathsf{N_C}$, $R \in \mathsf{N_R}$, and $\circ_f$ is a Boolean operator. For a given set B of Boolean operators, a *B-concept* is a concept that uses only operators from B. A *general concept inclusion (GCI)* is an axiom of the form $C \sqsubseteq D$ where C, D are concepts. We use "$C \equiv D$" as the usual syntactic sugar for "$C \sqsubseteq D$ and $D \sqsubseteq C$". A *TBox* is a finite set of GCIs without restrictions. An *ABox* is a finite set of axioms of the form $C(x)$ or $R(x,y)$, where C is a concept, $R \in \mathsf{N_R}$ and $x, y \in \mathsf{N_I}$. An *ontology* is the union of a TBox and an ABox. This simplified view suffices for our purposes.

An *interpretation* is a pair $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where $\Delta^{\mathcal{I}}$ is a nonempty set and $\cdot^{\mathcal{I}}$ is a mapping from $\mathsf{N_C}$ to $\mathfrak{P}(\Delta^{\mathcal{I}})$, from $\mathsf{N_R}$ to $\mathfrak{P}(\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}})$ and from $\mathsf{N_I}$ to $\Delta^{\mathcal{I}}$ that is extended to arbitrary concepts as follows:

$$\begin{aligned}
\circ_f(C_1, \dots, C_n)^{\mathcal{I}} &= \{x \in \Delta^{\mathcal{I}} \mid f(\|x \in C_1^{\mathcal{I}}\|, \dots, \|x \in C_n^{\mathcal{I}}\|) = 1\}, \\
&\quad \text{where } \|x \in C_1^{\mathcal{I}}\| = 1 \text{ if } x \in C_1^{\mathcal{I}} \text{ and } \|x \in C_1^{\mathcal{I}}\| = 0 \text{ if } x \notin C_1^{\mathcal{I}}, \\
\exists R.C^{\mathcal{I}} &= \{x \in \Delta^{\mathcal{I}} \mid \{y \in C^{\mathcal{I}} \mid (x,y) \in R^{\mathcal{I}}\} \neq \emptyset\}, \\
\forall R.C^{\mathcal{I}} &= \{x \in \Delta^{\mathcal{I}} \mid \{y \in C^{\mathcal{I}} \mid (x,y) \notin R^{\mathcal{I}}\} = \emptyset\}.
\end{aligned}$$

An interpretation $\mathcal{I}$ *satisfies* the axiom $C \sqsubseteq D$, written $\mathcal{I} \models C \sqsubseteq D$, if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$. Furthermore, $\mathcal{I}$ satisfies $C(x)$ or $R(x,y)$ if $x^{\mathcal{I}} \in C^{\mathcal{I}}$ or $(x^{\mathcal{I}}, y^{\mathcal{I}}) \in R^{\mathcal{I}}$. An interpretation $\mathcal{I}$ satisfies a TBox (ABox, ontology) if it satisfies every axiom therein. It is then called a *model* of this set of axioms.

Let B be a finite set of Boolean operators and $\mathcal{Q} \subseteq \{\exists, \forall\}$. We use $\mathsf{Con}_{\mathcal{Q}}(B)$, $\mathfrak{T}_{\mathcal{Q}}(B)$ and $\mathfrak{O}_{\mathcal{Q}}(B)$ to denote the set of all concepts, TBoxes and ontologies that use operators in B only and quantifiers from $\mathcal{Q}$ only. The following decision problems are of interest for this paper.

Concept satisfiability $\mathrm{CSAT}_{\mathcal{Q}}(B)$:
Given a concept $C \in \mathsf{Con}_{\mathcal{Q}}(B)$, is there an interpretation $\mathcal{I}$ s.t. $C^{\mathcal{I}} \neq \emptyset$?

TBox satisfiability $\mathrm{TSAT}_{\mathcal{Q}}(B)$:
Given a TBox $\mathcal{T} \subseteq \mathfrak{T}_{\mathcal{Q}}(B)$, is there an interpretation $\mathcal{I}$ s.t. $\mathcal{I} \models \mathcal{T}$?

TBox-concept satisfiability $\mathrm{TCSAT}_{\mathcal{Q}}(B)$:
Given $\mathcal{T} \subseteq \mathfrak{T}_{\mathcal{Q}}(B)$ and $C \in \mathsf{Con}_{\mathcal{Q}}(B)$, is there an $\mathcal{I}$ s.t. $\mathcal{I} \models \mathcal{T}$ and $C^{\mathcal{I}} \neq \emptyset$?

Ontology satisfiability $\mathrm{OSAT}_{\mathcal{Q}}(B)$:
Given an ontology $\mathcal{O} \subseteq \mathfrak{O}_{\mathcal{Q}}(B)$, is there an interpretation $\mathcal{I}$ s.t. $\mathcal{I} \models \mathcal{O}$?

Ontology-concept satisfiability $\mathrm{OCSAT}_{\mathcal{Q}}(B)$:
Given $\mathcal{O} \subseteq \mathfrak{O}_{\mathcal{Q}}(B)$ and $C \in \mathsf{Con}_{\mathcal{Q}}(B)$, is there an $\mathcal{I}$ s.t. $\mathcal{I} \models \mathcal{O}$ and $C^{\mathcal{I}} \neq \emptyset$?

By abuse of notation, we will omit set parentheses and commas when stating $\mathcal{Q}$ explicitly, as in $\mathrm{TSAT}_{\exists\forall}(B)$. The above listed decision problems are interreducible independently of B and $\mathcal{Q}$ in the following way:

$$\mathrm{CSAT}_{\mathcal{Q}}(B) \leq_{\mathrm{m}}^{\log} \mathrm{OSAT}_{\mathcal{Q}}(B)$$
$$\mathrm{TSAT}_{\mathcal{Q}}(B) \leq_{\mathrm{m}}^{\log} \mathrm{TCSAT}_{\mathcal{Q}}(B) \leq_{\mathrm{m}}^{\log} \mathrm{OSAT}_{\mathcal{Q}}(B) \equiv_{\mathrm{m}}^{\log} \mathrm{OCSAT}_{\mathcal{Q}}(B)$$

Some reductions in the main part of the paper consider another decision problem which is called *subsumption* (SUBS) and is defined as follows: Given a TBox $\mathcal{T}$ and two atomic concepts A, B, does every model of $\mathcal{T}$ satisfy $A \sqsubseteq B$?

Complexity Theory. We assume familiarity with the standard notions of complexity theory as, e.g., defined in [28]. In particular, we will make use of the classes NL, P, NP, coNP, and EXPTIME, as well as logspace reductions $\leq_{\mathrm{m}}^{\log}$.

Boolean operators. This study is complete with respect to Boolean operators, which correspond to Boolean functions. The table below lists all Boolean functions that we will mention, together with the associated DL operator where applicable.

Function symbol	Description	DL operator symbol
0, 1	constant 0, 1	$\bot$, $\top$
and, or	binary conjunction/disjunction $\wedge$, $\vee$	$\sqcap$, $\sqcup$
neg	unary negation $\overline{\cdot}$	$\neg$
xor	binary exclusive *or* $\oplus$	$\boxplus$
andor	$x \wedge (y \vee z)$	
sd	$(x \wedge \overline{y}) \vee (x \wedge \overline{z}) \vee (\overline{y} \wedge \overline{z})$	
equiv	binary equivalence function	

Table 1. Boolean functions with description and corresponding DL operator symbol.

A set of Boolean functions is called a *clone* if it contains all projections (also known as identity functions, the eponym of the I-clones below) and is closed under composition (also referred to as superposition). The lattice of all clones has been established in [29], see [10] for a more succinct but complete presentation. Via the inclusion structure, lower and upper complexity bounds can be carried over to higher and lower clones under certain conditions. We will therefore state our results for minimal and maximal clones only, together with those conditions.

Given a finite set B of functions, the smallest clone containing B is denoted by $[B]$. The set B is called a *base* of $[B]$, but $[B]$ often has other bases as well. For example, nesting of binary conjunction yields conjunctions of arbitrary arity. The table below lists all clones that we will refer to, using the following definitions. A Boolean function f is called *self-dual* if $f(\overline{x_1}, \ldots, \overline{x_n}) = \overline{f(x_1, \ldots, x_n)}$, *c-reproducing* if $f(c, \ldots, c) = c$ for $c \in \{0, 1\}$, and *c-separating* if there is an $1 \leq i \leq n$ s.t. for each $(b_1, \ldots, b_n) \in f^{-1}(c)$, it holds that $b_i = c$.

From now on, we will use B to denote a finite set of Boolean operators. Hence, $[B]$ consists of all operators obtained by nesting operators from B. By abuse of notation, we will denote operator sets with the above clone names when this

Clone	Description	Base
BF	all Boolean functions	$\{\mathsf{and}, \mathsf{neg}\}$
R_0, R_1	0-, 1-reproducing functions	$\{\mathsf{and}, \mathsf{xor}\}$, $\{\mathsf{or}, \mathsf{equiv}\}$
M	all monotone functions	$\{\mathsf{and}, \mathsf{or}, 0, 1\}$
S_1	1-separating functions	$\{x \wedge \overline{y}\}$
S_{11}	1-separating, monotone functions	$\{\mathsf{andor}, 0\}$
D	self-dual functions	$\{\mathsf{sd}\}$
L	affine functions	$\{\mathsf{xor}, 1\}$
L_0	affine, 0-reproducing functions	$\{\mathsf{xor}\}$
L_3	affine, 0- and 1-reproducing functions	$\{x \ \mathsf{xor}\ y \ \mathsf{xor}\ z \ \mathsf{xor}\ 1\}$
E_0, E	conjunctions and 0 (and 1)	$\{\mathsf{and}, 0\}$, $\{\mathsf{and}, 0, 1\}$
V_0, V	disjunctions and 0 (and 1)	$\{\mathsf{or}, 0\}$, $\{\mathsf{or}, 0, 1\}$
N_2, N	negation (and 1)	$\{\mathsf{neg}\}$, $\{\mathsf{neg}, 1\}$
I_0, I	0 (and 1)	$\{0\}$, $\{0, 1\}$

Table 2. List of all relevant clones in this paper with their standard bases.

is not ambiguous. Furthermore, we call a Boolean operator corresponding to a monotone (self-dual, 0-reproducing, 1-reproducing, 1-separating) function a monotone (self-dual, $\bot$-reproducing, $\top$-reproducing, $\top$-separating) operator.

Known complexity results for CSAT. In [19], the complexity of concept satisfiability has been classified for modal logics corresponding to all fragments of $\mathcal{ALC}$ with arbitrary combinations of Boolean operators and quantifiers: $\mathrm{CSAT}_{\mathcal{Q}}(B)$ with $\mathcal{Q} \subseteq \{\exists, \forall\}$ is either PSPACE-complete, coNP-complete, or in P. Some of the latter cases are trivial, i.e., every concept in such a fragment is satisfiable. These results generalize known complexity results for $\mathcal{ALE}$ and the $\mathcal{EL}$ and $\mathcal{FL}$ families. On the other hand, results for $\mathcal{ALU}$ and the DL-Lite family cannot be put into this context because they only allow unqualified existential restrictions. See [25] for a more detailed discussion.

3 Complexity Results for TSAT, TCSAT, OSAT, OCSAT

In this section we will almost completely classify the above mentioned satisfiability problems for their tractability with respect to sub-Boolean fragments and put them into context with existing results for fragments of $\mathcal{ALC}$. Full proofs of every theorem and auxiliary lemmata are given in [26].

We use $\star\mathrm{SAT}_{\mathcal{Q}}(B)$ to speak about any of the four satisfiability problems $\mathrm{TSAT}_{\mathcal{Q}}(B), \mathrm{TCSAT}_{\mathcal{Q}}(B), \mathrm{OSAT}_{\mathcal{Q}}(B)$ and $\mathrm{OCSAT}_{\mathcal{Q}}(B)$ introduced above; for the three problems having the power to speak about a single individual, we abuse this notion and write $\star\mathrm{SAT}^{\sim}_{\mathcal{Q}}(B)$ for the problems $\star\mathrm{SAT}_{\mathcal{Q}}(B)$ without $\mathrm{TSAT}_{\mathcal{Q}}(B)$.

3.1 Both quantifiers

Theorem 1 ([30, 34, 17]). $\mathrm{OCSAT}_{\exists\forall}(\mathsf{BF}) \in \mathrm{EXPTIME}$.

Due to the interreducibilities stated in Section 2, it suffices to show lower bounds for TSAT and upper bounds for OCSAT. Moreover one can show that a base independence result holds which enables us to restrict the proofs to the standard basis of each clone for stating general results. Several proof sketches involve the ability to express the constant $\top$ through a fresh concept. This technique goes back to Lewis 1979 [22] and often will be referred to as $\top$-knack.

The following theorem improves [25] by stating completeness results.

Theorem 2. *Let B be a finite set of Boolean operators.*

1. *If $\mathsf{I} \subseteq [B]$ or $\mathsf{N}_2 \subseteq [B]$, then $\mathrm{TSAT}_{\exists\forall}(B)$ is EXPTIME-complete.*
2. *If $\mathsf{I}_0 \subseteq [B]$ or $\mathsf{N}_2 \subseteq [B]$, then $\star\mathrm{SAT}^{\sim}_{\exists\forall}(B)$ is EXPTIME-complete.*
3. *If $[B] \subseteq \mathsf{R}_0$, then $\mathrm{TSAT}_{\exists\forall}(B)$ is trivial.*
4. *If $[B] \subseteq \mathsf{R}_1$, then $\star\mathrm{SAT}_{\exists\forall}(B)$ is trivial.*

Proof sketch. 1. Membership is immediate from Theorem 1. Hardness can be shown in two steps: via a reduction from the positive entailment problem for Tarskian set constraints (cf. [18]) to $\mathrm{TSAT}_{\exists\forall}(\mathsf{E})$ and eliminating in this reduction the conjunction operator. The latter is achieved by an extended version of the normalization algorithm in [12]. The case $\mathsf{N}_2 \subseteq [B]$ then follows directly from Lemma 1 in [25]. *2.* follows from *1.* by simulating the constant $\top$ with a fresh concept. *3.* and *4.* follow from [25]. □

Part (2) for I_0 generalizes the EXPTIME-hardness of subsumption for $\mathcal{FL}_0$ and $\mathcal{AL}$ with respect to GCIs [18, 14, 5, 20]. The contrast to the tractability of subsumption with respect to GCIs in $\mathcal{EL}$, which uses only existential quantifiers, undermines the observation that, for negation-free fragments, the choice of the quantifier affects tractability and not the choice between conjunction and disjunction. DL-Lite and $\mathcal{ALU}$ cannot be put into this context because they use unqualified restrictions.

Parts (1) and (2) show that satisfiability with respect to theories is already intractable for even smaller sets of Boolean operators. One reason is that sets of axioms already contain limited forms of implication and conjunction. This also causes the results of this analysis to differ from similar analyses for sub-Boolean modal logics in that hardness already holds for bases of clones that are comparatively low in Post's lattice.

Part (3) reflects the fact that TSAT is less expressive than the other three decision problems: it cannot speak about one single individual.

3.2 Restricted quantifiers

In this section we investigate the complexity of the problems $\mathrm{OCSAT}_{\mathcal{Q}}$, $\mathrm{OSAT}_{\mathcal{Q}}$, $\mathrm{TCSAT}_{\mathcal{Q}}$, and $\mathrm{TSAT}_{\mathcal{Q}}$, where $\mathcal{Q}$ contains at most one of the quantifiers $\exists$ or $\forall$. Even the case $\mathcal{Q} = \emptyset$ is nontrivial: for example, $\mathrm{TSAT}_{\mathcal{Q}}(B)$ does not reduce to propositional satisfiability for B because restricted use of implication and conjunction is implicit in sets of axioms.

TSAT-Results

Theorem 3. *Let B be a finite set of Boolean operators.*

1. *If* $\mathsf{L}_3 \subseteq [B]$ *or* $\mathsf{M} \subseteq [B]$, *then* $\mathrm{TSAT}_\emptyset(B)$ *is* NP-*complete.*
2. *If* $\mathsf{E} = [B]$ *or* $\mathsf{V} = [B]$, *then* $\mathrm{TSAT}_\emptyset(B)$ *is* P-*complete.*
3. *If* $[B] \in \{\mathsf{I}, \mathsf{N}_2, \mathsf{N}\}$, *then* $\mathrm{TSAT}_\emptyset(B)$ *is* NL-*complete.*
4. *Otherwise (if* $[B] \subseteq \mathsf{R}_1$ *or* $[B] \subseteq \mathsf{R}_0$*), then* $\mathrm{TSAT}_\emptyset(B)$ *is trivial.*

Proof sketch. For the monotone case in *1.* it holds that $\overline{\mathrm{IMP}(\mathsf{M})} \leq_{\mathrm{m}}^{\log} \mathrm{TSAT}_\emptyset(\mathsf{M})$ where $\mathrm{IMP}(\mathsf{M})$ being coNP-complete is shown in [9]. For L_3 using a knack from Theorem 2 lets us easily state a reduction from the NP-complete problem 1-in-3-SAT involving the binary exclusive-or. Hardness in the E-case of *2.* is achieved via the hypergraph accessibility problem HGAP; for $\mathsf{V} = [B]$ we argue via contraposition. Membership comes from containment in $\mathrm{OCSAT}_\exists(\mathsf{E})$. In case *3.* hardness is shown for $\mathsf{I} = [B]$ by reducing from the graph inaccessibility problem $\overline{\mathrm{GAP}}$. Membership is entailed by $\mathrm{TCSAT}_\emptyset(\mathsf{N})$. *4.* follows from Theorem 2. □

Theorem 4. *Let B be a finite set of Boolean operators and $\mathcal{Q} \in \{\forall, \exists\}$.*

1. *If* $\mathsf{M} \subseteq [B]$ *or* $\mathsf{N}_2 \subseteq [B]$, *then* $\mathrm{TSAT}_\mathcal{Q}(B)$ *is* EXPTIME-*complete.*
2. *If* $\mathsf{E} = [B]$, $\mathsf{V} = [B]$, *or* $\mathsf{I} = [B]$, *then* $\mathrm{TSAT}_\mathcal{Q}(B)$ *is* P-*complete.*
3. *Otherwise (if* $[B] \subseteq \mathsf{R}_1$ *or* $[B] \subseteq \mathsf{R}_0$*), then* $\mathrm{TSAT}_\mathcal{Q}(B)$ *is trivial.*

Proof sketch. In case *1.* with $\mathsf{M} \subseteq [B]$ and $\mathcal{Q} = \exists$ we reduce from $\overline{\mathcal{ELU}\text{-SUBS}}$, the subsumption problem of the logic $\mathcal{ELU}$, whose EXPTIME-completeness has been proven in [5]. For $\mathcal{Q} = \forall$ we reduce from $\mathrm{TSAT}_\exists(B)$ in combination with a contraposition argument. For N_2, negation can simulate both constants which leads to a simple reduction from $\mathrm{TSAT}_{\exists\forall}(\mathsf{I})$. The hardness results of item *3.* follow from $\mathrm{TSAT}_\exists(\mathsf{I})$ whose hardness is via a reduction from the word problem for a particular Turing machine model of the class P. The P-algorithm for the case $\forall$ and $\exists$ extends the algorithm in [11]. The upper bound for $\mathrm{TSAT}_\exists(\mathsf{E})$ results from $\mathrm{OCSAT}_\exists(\mathsf{E})$; for the remainder we use a contraposition argument. □

Part (3) generalizes the fact that every $\mathcal{EL}$- and $\mathcal{FL}_0$-TBox is satisfiable, and the whole theorem shows that separating either conjunction and disjunction, or the constants is the only way to achieve tractability for TSAT.

TCSAT-, OSAT-, OCSAT-Results.

Theorem 5. *Let B be a finite set of Boolean operators.*

1. *If* $\mathsf{S}_{11} \subseteq [B]$ *or* $\mathsf{L}_3 \subseteq [B]$ *or* $\mathsf{L}_0 \subseteq [B]$, *then* $\star\mathrm{SAT}_\emptyset^\sim(B)$ *is* NP-*complete.*
2. *If* $[B] \in \{\mathsf{E}_0, \mathsf{E}, \mathsf{V}_0, \mathsf{V}\}$, *then* $\star\mathrm{SAT}_\emptyset^\sim(B)$ *is* P-*complete.*
3. *If* $[B] \in \{\mathsf{I}_0, \mathsf{I}, \mathsf{N}_2, \mathsf{N}\}$, *then* $\star\mathrm{SAT}_\emptyset^\sim(B)$ *is* NL-*complete.*
4. *Otherwise (if* $[B] \subseteq \mathsf{R}_1$*), then* $\star\mathrm{SAT}_\emptyset^\sim(B)$ *is trivial.*

Proof sketch. Hardness in *1.* follows from the respective $\mathrm{TSAT}_\emptyset(B)$ case together with the fact that these fragments can simulate $\top$. Membership is via a reduction to SAT involving a construction using implication for the terminology. In case *2.* we can adjust the lower bounds from $\mathrm{TSAT}_\emptyset(B)$ and $[B] \in \{\mathsf{V}, \mathsf{E}\}$ by simulating $\top$ again. Membership is due to $\mathrm{OCSAT}_\exists(\mathsf{E})$. In *3.*, membership for N is via an algorithm that searches for cycles containing a concept and its negation in the directed graph induced by the terminology. Hardness results from the case $\mathrm{TSAT}_\emptyset(\mathsf{I})$ plus the $\top$-knack. The last item is due to Theorem 1. □

Theorem 6. *Let B be a finite set of Boolean operators, and $\mathcal{Q} \in \{\forall, \exists\}$.*

1. *If $\mathsf{S}_{11} \subseteq [B]$, $\mathsf{N}_2 \subseteq [B]$, or $\mathsf{L}_0 \subseteq [B]$ then $\star\mathrm{SAT}^{\sim}_{\mathcal{Q}}(B)$ is* EXPTIME-*complete.*
2. *If $\mathsf{I}_0 \subseteq [B] \subseteq \mathsf{V}$, then $\mathrm{TCSAT}_\exists(B)$ and $\star\mathrm{SAT}^{\sim}_{\forall}(B)$ are* P-*complete*[3].
3. *If $[B] \in \{\mathsf{E}_0, \mathsf{E}\}$, then $\star\mathrm{SAT}^{\sim}_{\forall}(B)$ is* EXPTIME-*complete,* *and $\star\mathrm{SAT}^{\sim}_{\exists}(B)$ is* P-*complete.*
4. *If $[B] \subseteq \mathsf{R}_1$, then $\star\mathrm{SAT}^{\sim}_{\mathcal{Q}}(B)$ is trivial.*

Proof sketch. For *1.*, combine EXPTIME-completeness of $\mathrm{TSAT}_{\mathcal{Q}}(B)$ with the $\top$-knack. EXPTIME-hardness of $\mathrm{TCSAT}_\forall(B)$ in case *3.* is achieved via a reduction from $\overline{\mathcal{FL}_0\text{-SUBS}}$ which is EXPTIME-complete [5, 20]. P-hardness in *2.* and *3.* result from $\mathrm{TSAT}_{\mathcal{Q}}(\mathsf{I})$ again with the help of the $\top$-knack. P-membership of $\mathrm{OCSAT}_\exists(\mathsf{E})$ is accomplished through a reduction to the subsumption problem for the logic $\mathcal{EL}^{++}$ [6], and a contraposition argument is used to reduce $\mathrm{OCSAT}_\forall(\mathsf{V})$ to $\mathrm{OCSAT}_\exists(\mathsf{E})$. *4.* is due to Theorem 2. □

Theorem 6 shows one reason why the logics in the $\mathcal{EL}$ family have been much more successful as "small" logics with efficient reasoning methods than the $\mathcal{FL}$ family: the combination of the $\forall$ with conjunction is intractable, while $\exists$ and conjunction are still in polynomial time. Again, separating either conjunction and disjunction, or the constants is crucial for tractability.

Table 3 gives an overview of our results. [26] contains figures showing how the results arrange in Post's lattice.

4 Conclusion

With Theorems 2 to 6, we have completely classified the satisfiability problems connected to arbitrary terminologies and concepts for $\mathcal{ALC}$ fragments obtained by arbitrary sets of Boolean operators and quantifiers—only the fragments emerging around ontologies with existential quantifier and disjunction as only allowed connective resisted a full classification. In particular we improved and finished the study of [25]. In more detail we achieved a dichotomy for all problems using both quantifiers (EXPTIME-complete vs. trivial fragments), a trichotomy when only one quantifier is allowed (trivial, EXPTIME-, and P-complete fragments), and a quartering for no allowed quantifiers ranging from trivial, NL-complete, P-complete, and NP-complete fragments.

[3] $\mathrm{OSAT}_\exists(B)$ and $\mathrm{OCSAT}_\exists(B)$ are P-hard for $[B] \in \{\mathsf{V}_0, \mathsf{V}\}$ and in EXPTIME.

$\mathrm{TSAT}_{\mathcal{Q}}(B)$	I	V	E	N/N_2	M	L_3 to BF	otherwise
$\mathcal{Q} = \emptyset$	NL	P		NL	NP		trivial
$\lvert\mathcal{Q}\rvert = 1$	P			EXPTIME			trivial
$\mathcal{Q} = \{\exists, \forall\}$	EXPTIME						trivial

$\star\mathrm{SAT}^{\sim}_{\mathcal{Q}}(B)$	I/I_0	V/V_0	E/E_0	N/N_2	S_{11} to M	$\mathsf{L}_3/\mathsf{L}_0$ to BF	otherwise
$\mathcal{Q} = \emptyset$	NL	P		NL	NP		trivial
$\mathcal{Q} = \{\exists\}$	P	$\mathrm{P}^{\S}$	P	EXPTIME			trivial
$\mathcal{Q} = \{\forall\}$	P		EXPTIME				trivial
$\mathcal{Q} = \{\exists, \forall\}$	EXPTIME						trivial

Table 3. Complexity overview for all Boolean function and quantifier fragments. All results are completeness results for the given complexity class, except for the case marked §: here, OCSAT and OSAT are in EXPTIME and P-hard.

Furthermore the connection to well-known logic fragments of $\mathcal{ALC}$, e.g., $\mathcal{FL}$ and $\mathcal{EL}$ now enriches the landscape of complexity by a generalization of these results. These improve the overall understanding of where the tractability border lies. The most important lesson learnt is that the separation of quantifiers together with the separation of either conjunction and disjunction, or the constants, is the only way to achieve tractability in our setting.

Especially in contrast to similar analyses of logics using *Post's lattice*, this study shows intractable fragments quite at the bottom of the lattice. This illustrates how expressive the concept of terminologies and assertional boxes is: restricted to only the Boolean function *false* besides both quantifiers we are still able to encode EXPTIME-hard problems into the decision problems that have a TBox and a concept as input. Thus perhaps the strongest source of intractability can be found in the fact that unrestricted theories already express limited implication and disjunction, and not in the set of allowed Boolean functions alone.

For future work, it would be interesting to see whether the picture changes if the use of general axioms is restricted, for example to cyclic terminologies—theories where axioms are cycle-free definitions $A \equiv C$ with A being atomic. Theories so restricted are sufficient for establishing taxonomies. Concept satisfiability for $\mathcal{ALC}$ w.r.t acyclic terminologies is still PSPACE-complete [23]. Is the tractability border the same under this restriction? One could also look at fragments with unqualified quantifiers, e.g., $\mathcal{ALU}$ or the DL-lite family, which are not covered by the current analysis. Furthermore, since the standard reasoning tasks are not always interreducible under restricted Boolean operators, a similar classification for other decision problems such as concept subsumption is pending.

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