The Complexity of Monotone Hybrid Logics over Linear Frames and the Natural Numbers

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Abstract

Hybrid logic with binders is an expressive specification language. Its satisfiability problem is undecidable in general. If frames are restricted to $\mathbb{N}$ or general linear orders, then satisfiability is known to be decidable, but of non-elementary complexity. In this paper, we consider monotone hybrid logics (i.e., the Boolean connectives are conjunction and disjunction only) over $\mathbb{N}$ and general linear orders. We show that the satisfiability problem remains non-elementary over linear orders, but its complexity drops to PSPACE-completeness over $\mathbb{N}$. We categorize the strict fragments arising from different combinations of modal and hybrid operators into NP-complete and tractable (i.e. complete for NC^1 or LOGSPACE). Interestingly, NP-completeness depends only on the fragment and not on the frame. For the cases above NP, satisfiability over linear orders is harder than over $\mathbb{N}$, while below NP it is at most as hard. In addition we examine model-theoretic properties of the fragments in question.

Keywords: satisfiability, modal logic, complexity, hybrid logic

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1 Introduction

Hybrid logic is an extension of modal logic with nominals, satisfaction operators and binders. The downarrow binder $\downarrow$, which is related to the freeze operator in temporal logic [12], provides high expressivity. The price paid is the undecidability of the satisfiability problem for the hybrid language with the downarrow binder $\downarrow$ [4,11,1]. In contrast, modal logic, and its extension with nominals and the satisfaction operator, is PSPACE-complete [13,1].

In order to regain decidability, syntactic and semantic restrictions have been considered. It has been shown in [22] that the absence of certain combinations of universal operators ($\Box$, $\wedge$) with $\downarrow$ brings back decidability, and that the hybrid language with $\downarrow$ is decidable over frames of bounded width. Furthermore, this language is decidable over transitive and complete frames [17], and over frames with an equivalence relation (ER frames) [16]. Adding the at-operator @—which allows to jump to states named by nominals—leads to undecidability over transitive frames [17], but not over ER frames [16]. Over linear frames and transitive trees, $\downarrow$ on its own does not add expressivity, but combinations with @ or the global modality—an additional $\Diamond$ interpreted over the universal relation—do. These languages are decidable and of non-elementary complexity [9,17]; if the number of state variables is bounded, then they are of elementary complexity [19,24,5].

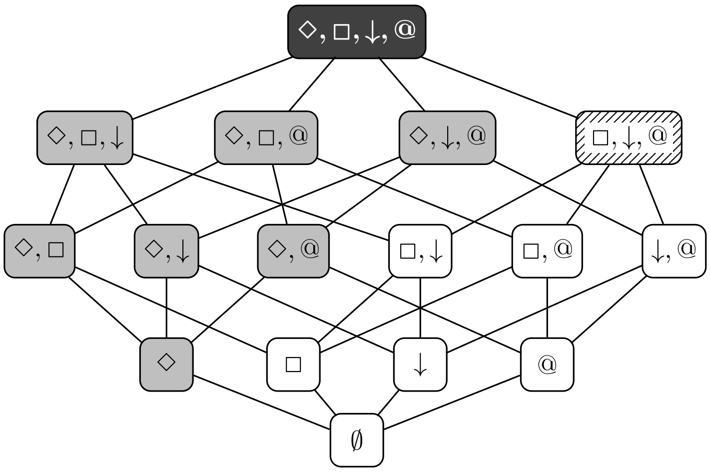
We aim for a more fine-grained distinction between fragments of different complexities by systematically restricting the set of Boolean connectives and combining this with restrictions to the modal/hybrid operators and to the underlying frames. In [15], we have focussed on four frame classes that allow cycles, and studied the complexity of satisfiability for fragments obtained by arbitrary combinations of Boolean connectives and four modal/hybrid operators. The main open question in [15] is the one for tight upper bounds for monotone fragments including the $\Box$-operator. Even though there are many logics for which the restriction to monotone Boolean connectives leads to a significant decrease in complexity, it is not straightforward, and therefore interesting to find out, where this happens for hybrid logics.

In this study, we classify the computational complexity of satisfiability for monotone fragments of hybrid logic with arbitrary combinations of the operators $\Diamond$, $\Box$, $\downarrow$ and @ over linear orders and the natural numbers. Whereas the full logic is non-elementary and decidable [17] for both frame classes, we show that in the monotone case this high complexity is gained only over linear orders and drops to PSPACE-completeness over the natural numbers. Informally speaking, the reason is that linearly ordered frames may consist of arbitrarily many dense parts that can be distinguished using the expressive power of all four operators. These dense parts and their distances are used to store information that cannot be stored in a frame without dense parts as, e.g., the natural numbers. For all other monotone fragments that contain the $\Diamond$-operator, we show NP-completeness independent on the frame class, for linear orders, all remaining fragments (i.e. the fragments without $\Diamond$) can be shown to be NC^1-complete. The reason is, informally speaking, that all (sub-)formulas of the form

$\Box\alpha$ are easily satisfied in a state without successor, which can essentially be used to reduce this problem to the satisfiability problem for monotone propositional formulae. This argument does not go through over the natural numbers, a total frame where every state has a successor. Over this frame class, we give a decision procedure that runs in logarithmic space for the fragment with all operators except $\Diamond$ (and prove a matching lower bound), and in NC^1 for all other fragments.

These results give rise to two interesting observations. First, the NP-completeness results are independent on the frame class. Second, for the fragment whose satisfiability problem is above NP, linear orders make the problem harder than the natural numbers, and for the richest fragment below NP, it is the opposite way round—the natural numbers make the problem harder than linear orders. Notice also that, in the case where Boolean operators are not restricted to monotone ones, all fragments are NP-hard.

Our results are shown in Figure 1.



lin: decidable, non-elementary
ℕ: PSPACE-complete

lin: NC^1-complete; ℕ: LOGSPACE-compl.
canonical model property

NP-complete
quasi-polysize model property

NC^1-complete
canonical model property

Fig. 1. Our complexity results for satisfiability over linear frames (lin) and the natural numbers (ℕ) for hybrid logic with monotone Boolean operators and different combinations of modal/hybrid operators

2 Preliminaries

Hybrid Logic. In the following, we introduce the notions and definitions of hybrid logic. The terminology is largely taken from [2].

Let PROP be a countable set of *atomic propositions*, NOM be a countable set of *nominals*, SVAR be a countable set of *variables* and ATOM = PROP ∪ NOM ∪ SVAR. We adhere to the common practice of denoting atomic propositions by $p, q, \ldots$, nominals by $i, j, \ldots$, and variables by $x, y, \ldots$ We define the language of *hybrid (modal) logic* $\mathcal{HL}$ as the set of well-formed formulae of the form

$$\varphi ::= a \mid \top \mid \bot \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Diamond\varphi \mid \Box\varphi \mid {\downarrow}x.\varphi \mid @_t\varphi$$

where $a \in$ ATOM, $x \in$ SVAR and $t \in$ NOM ∪ SVAR.

We define the usual Kripke semantics only to be able to refer to already existing results. We will then simplify the standard semantics for monotone formulae. Formulae of $\mathcal{HL}$ are interpreted on *(hybrid) Kripke structures* $K = (W, R, \eta)$, consisting of a set of *states* W, a *transition relation* $R\colon W \times W$, and a *labeling function* $\eta\colon$ PROP ∪ NOM $\to \wp(W)$ that maps PROP and NOM to subsets of W with $|\eta(i)| = 1$ for all $i \in$ NOM. The relational structure (W, R) is the *Kripke frame* underlying K. In order to evaluate ↓-formulae, an assignment $g\colon$ SVAR $\to W$ is necessary. Given an assignment g, a state variable x and a state w, an *x-variant g^x_w of g* is defined by $g^x_w(x) = w$ and $g^x_w(x') = g(x')$ for all $x \neq x'$. For any $a \in$ ATOM, let $[\eta, g](a) = \{g(a)\}$ if $a \in$ SVAR and $[\eta, g](a) = \eta(a)$, otherwise. The satisfaction relation of hybrid formulae is defined as follows.

$K, g, w \models \varphi \wedge \psi$	if and only if	$\exists w' \in W(wRw' \,\&\, K, g, w' \models \varphi)$
$K, g, w \models a$	if and only if	$w \in [\eta, g](a)$, $a \in$ ATOM,
$K, g, w \models \top$,	and $K, g, w \not\models \bot$,	
$K, g, w \models \neg\varphi$	if and only if	$K, g, w \not\models \varphi$,
$K, g, w \models \varphi \wedge \psi$	if and only if	$K, g, w \models \varphi$ and $K, g, w \models \psi$,
$K, g, w \models \varphi \vee \psi$	if and only if	$K, g, w \models \varphi$ or $K, g, w \models \psi$,
$K, g, w \models \Diamond\varphi$	if and only if	$\exists w' \in W(wRw' \,\&\, K, g, w' \models \varphi)$,
$K, g, w \models \Box\varphi$	if and only if	$\forall w' \in W(wRw' \Rightarrow K, g, w' \models \varphi)$,
$K, g, w \models @_t\varphi$	if and only if	$K, g, [\eta, g](t) \models \varphi$,
$K, g, w \models {\downarrow}x.\varphi$	if and only if	$K, g^x_w, w \models \varphi$.

A hybrid formula φ is said to be *satisfiable* if there exists a Kripke structure $K = (W, R, \eta)$, a $w \in W$ and an assignment $g\colon$ SVAR $\to W$ with $K, g, w \models \varphi$.

The *at* operator $@_t$ shifts evaluation to the state named by $t \in$ NOM ∪ SVAR. The *downarrow binder* ${\downarrow}x.$ binds the state variable x to the current state. The symbols $@_x$, ${\downarrow}x.$ are called *hybrid operators* whereas the symbols $\Diamond$ and $\Box$ are called *modal operators*.

The scope of an occurrence of the binder ↓ is defined as usual. For a state variable x, an occurrence of x or $@_x$ in a formula φ is called *bound* if this occurrence is in the scope of some ↓ in φ, *free* otherwise. φ is said to contain a free state variable if some x or $@_x$ occurs free in φ.

Given two formulae φ, α and a subformula ψ of φ, we use $\varphi[\psi/\alpha]$ to denote the result of replacing each occurrence of ψ in φ with α. For considering fragments of hybrid logics, we define subsets of the language $\mathcal{HL}$ as follows. Let

O be a set of hybrid and modal operators, i.e., a subset of $\{\Diamond, \Box, \downarrow, @\}$. We define $\mathcal{HL}(O)$ to denote the set of well-formed hybrid formulae using only the operators in O, and $\mathcal{MHL}(O)$ to be the set of all formulae in $\mathcal{HL}(O)$ that do not use $\neg$.

Properties of Frames. A *frame* F is a pair (W, R), where W is a set of states and $R \subseteq W \times W$ a transition relation. A frame $F = (W, R)$ is called

- transitive if R is transitive (for all $u, v, w \in W$: $uRv \wedge vRw \to uRw$),
- linear if R is transitive, irreflexive and trichotomous ($\forall u, v \in W$: uRv or $u = v$ or vRu),

In this paper we consider the class of all linear frames, denoted by lin, and the singleton frame class $\{(\mathbb{N}, <)\}$, denoted by $\mathbb{N}$. Obviously, $\mathbb{N} \subseteq \mathsf{lin}$.

Notational convenience. We can make some simplifying assumptions about syntax and semantics, of $\mathcal{HL}(O)$ and $\mathcal{MHL}(O)$, which do not restrict generality. (1) If $\downarrow \in O$, then formulae do not contain any nominals. Those can be simulated by free state variables. (2) Free state variables are never bound later in the formula, and every state variable is bound at most once. The latter is no significant restriction because variables bound multiple times can be named apart, which is a well-established and computationally easy procedure. (3) Monotone formulae do not contain any atomic propositions. This restriction is correct because every monotone formula φ is satisfiable if and only if φ with all atomic propositions replaced by $\top$ is satisfiable. This justifies the following restrictions. (4) For binder-free fragments, the domain of the labelling function η is restricted to nominals, and we re-define $\eta\colon \textsc{Nom} \to W$. Furthermore, the absence of $\downarrow$ makes assignments superfluous: we write $F, w \models \varphi$ instead of $F, g, w \models \varphi$. (5) For binder fragments, the satisfaction relation $\models$ is restricted to Kripke *frames* $F = (W, <)$, where $<$ is a linear order, and assignments $g : \textsc{SVar} \to W$, i.e., we write $F, g, w \models \varphi$. (6) Over $\mathbb{N}$, we omit the single Kripke frame, i.e., we write $\eta, i \models \varphi$ with $\eta : \textsc{Nom} \to \mathbb{N}$ and $i \in \mathbb{N}$ for binder-free fragments, and $g, i \models \varphi$ with $g : \textsc{SVar} \to \mathbb{N}$ for binder fragments.

Satisfiability Problems. The *satisfiability problem* for $\mathcal{HL}(O)$ over the frame class $\mathfrak{F}$ is defined as follows:

Problem: $\mathfrak{F}$-$\mathsf{SAT}(O)$
Input: an $\mathcal{HL}(O)$-formula φ (without nominals, see above)
Output: Is there a Kripke structure K based on a frame $(W, R) \in \mathfrak{F}$, an assignment $g\colon \textsc{SVar} \to W$ and a $w \in W$ such that $K, g, w \models \varphi$?

The *monotone satisfiability problem* for $\mathcal{MHL}(O)$ over the frame class $\mathfrak{F}$ is defined as follows:

Problem: $\mathfrak{F}$-$\mathsf{MSAT}(O)$
Input: an $\mathcal{MHL}(O)$-formula φ without nominals and atomic propositions
Output: Is there a Kripke frame $(W, R) \in \mathfrak{F}$, an assignment $g\colon \textsc{SVar} \to W$ and a $w \in W$ such that $F, g, w \models \varphi$?

If $\mathfrak{F}$ is the class of all frames, we simply write $\mathsf{SAT}(O)$ or $\mathsf{MSAT}(O)$. Furthermore, we often omit the set parentheses when giving O explicitly, e.g., $\mathsf{SAT}(\Diamond, \Box, \downarrow, @)$.

Complexity Theory. We assume familiarity with the standard notions of complexity theory as, e. g., defined in [18]. In particular, we make use of the classes LOGSPACE, NLOGSPACE, NP, PSPACE, and coRE. The complexity class NONELEMENTARY is the set of all languages A that are decidable and for which there exists no $k \in \mathbb{N}$ such that A can be decided using an algorithm whose running time is bounded by $\exp_k(n)$, where $\exp_k(n)$ is the k-th iteration of the exponential function (e.g., $\exp_3(n) = 2^{2^{2^n}}$).

Furthermore, we need two non-standard complexity classes whose definition relies on circuit complexity and formal languages, see for instance [23,14]. The class NC^1 is defined as the set of languages recognizable by a logtime-uniform family of Boolean circuits of logarithmic depth and polynomial size over $\{\wedge, \vee, \neg\}$, where the fan-in of $\wedge$ and $\vee$ gates is fixed to 2. The class LOGDCFL is defined as the set of languages reducible in logarithmic space to some deterministic context-free language.

The following relations between the considered complexity classes are known.

$$\mathsf{NC}^1 \subseteq \mathsf{LOGSPACE} \subseteq \mathsf{LOGDCFL} \subseteq \mathsf{NP} \subseteq \mathsf{PSPACE} \subset \mathsf{coRE}.$$

It is unknown whether LOGDCFL contains NLOGSPACE or vice versa.

A language A is *constant-depth reducible* to D, $A \leqslant_{\mathrm{cd}} D$, if there is a logtime-uniform AC^0-circuit family with oracle gates for D that decides membership in A. Unless otherwise stated, all reductions in this paper are $\leqslant_{\mathrm{cd}}$-reductions.

Known results. The following theorem summarizes results for hybrid languages with Boolean operators $\wedge, \vee, \neg$ that are known from the literature. Since $\Box\varphi \equiv \neg\Diamond\neg\varphi$, the $\Box$-operator is implicitly present in all fragments containing $\Diamond$ and negation.

Theorem 2.1 ([1,2,3,9,17])

(1) $\mathsf{SAT}(\Diamond, \downarrow, @)$ *and* $\mathsf{SAT}(\Diamond, \downarrow)$ *are* coRE*-complete. [1]*

(2) $\mathsf{MSAT}(\Diamond, \Box)$ *is* PSPACE*-hard. [3]*

(3) $\mathfrak{F}$-$\mathsf{SAT}(\Diamond, \downarrow, @)$*, for* $\mathfrak{F} \in \{\mathsf{lin}, \mathbb{N}\}$*, are in* NONELEMENTARY*. [9,17]*

(4) $\mathfrak{F}$-$\mathsf{SAT}(\Diamond, \downarrow)$*,* $\mathfrak{F}$-$\mathsf{SAT}(\Diamond, @)$ *and* $\mathfrak{F}$-$\mathsf{SAT}(\Diamond)$*, with* $\mathfrak{F} \in \{\mathsf{lin}, \mathbb{N}\}$*, are* NP*-complete. [2,9]*

Our contribution. In this paper, we consider the monotone satisfiability problems $\mathfrak{F}$-$\mathsf{MSAT}(O)$ for $\mathfrak{F} \in \{\mathsf{lin}, \mathbb{N}\}$ and all $O \subseteq \{\Diamond, \Box, \downarrow, @\}$.

3 The hard cases: Non-elementary and PSPACE results

The hardest cases are those with the complete set of operators. In the non-monotone case, both satisfiability problems are non-elementary and decidable [17]. We show that in the monotone case even this hardness is reached, but only on linear frames, i.e. lin-MSAT($\Diamond, \Box, \downarrow, @$) is non-elementary and decidable. In contrast, on the natural numbers the complexity decreases, i.e. we show that $\mathbb{N}$-MSAT($\Diamond, \Box, \downarrow, @$) is PSPACE-complete.

Our proofs use reductions to and from fragments of first-order logic on the natural numbers. Let $\mathcal{FOL}(<, P)$ be the set of all first-order formulae that use $<$ as the unique binary relation symbol, and P as the unique unary relation symbol.[1] Let $\mathbb{N}$-SAT$_{\mathcal{FOL}}(<, P)$ denote the set of formulae from $\mathcal{FOL}(<, P)$ which are satisfied by a model that has $\mathbb{N}$ as its universe, interprets $<$ as the less-than relation on $\mathbb{N} \times \mathbb{N}$, and has an arbitrary interpretation for the predicate symbol P. It was shown by Stockmeyer [21] that $\mathbb{N}$-SAT$_{\mathcal{FOL}}(<, P)$ is non-elementary.

Let $\mathcal{FOL}(<)$ be the fragment of $\mathcal{FOL}(<, P)$ in which the predicate symbol P is not used. Accordingly, $\mathbb{N}$-SAT$_{\mathcal{FOL}}(<)$ denotes the set of formulae that are satisfiable over $\mathbb{N}$ and the natural interpretation of $<$. It was shown by Ferrante and Rackoff [8] that $\mathbb{N}$-SAT$_{\mathcal{FOL}}(<)$ is in PSPACE.

Notice that in both fragments $x = y$ can be expressed as $\neg(x < y \vee y < x)$. Moreover, every $n \in \mathbb{N}$ can be expressed by x_n in the formula $\exists x_0 \cdots \exists x_{n-1}[(\bigwedge_{i=0,1,\dots,n-1} x_i < x_{i+1}) \wedge \forall y(x_n < y \vee \bigvee_{i=0,1,\dots,n} y = x_i)]$.

Theorem 3.1 lin-MSAT($\Diamond, \Box, \downarrow, @$) *is non-elementary and decidable.*

Proof. Decidability follows from Theorem 2.1 (3). To establish non-elementary complexity, we give a reduction from $\mathbb{N}$-SAT$_{\mathcal{FOL}}(<, P)$.

We first show how to encode the intepretation of a predicate symbol, represented by a set $P \subseteq \mathbb{N}$, in a linear frame $F = (W, <)$ – without using atomic propositions and nominals as agreed in Section 2. Using free state variables, we can only distinguish linearly many states at any given time. We therefore use finite intervals (finite subchains of $(W, <)$) to encode whether $n \in P$. Such an interval—we call it a *marker*—has length 2 (resp. 3) if for the corresponding n holds $n \notin P$ (resp. $n \in P$). Accordingly, we call a marker of length 2 (resp. 3) *negative* (resp. *positive*). These finite intervals are separated by dense intervals—those are intervals wherein every two states have an intermediate state, e.g., $[0, 1]_{\mathbb{Q}} = \{q \in \mathbb{Q} \mid 0 \leqslant q \leqslant 1\}$. For example, the set P with $0, 2 \notin P$ and $1 \in P$ is represented by the chain in Figure 2. In our fragment, it is possible to distinguish between dense and finite intervals. We now show how to achieve this. In order to encode the alternating sequence of finite and dense intervals that represents a subset $P \subseteq \mathbb{N}$, we use the free state variable a to mark a state in a dense interval that is directly followed by the first marker. We furthermore use the following macros, where x and y are state variables that are already bound before the use of the macro, and r, s, t, u are fresh state variables.

- *The state named y is a* direct *successor of the state named x.* It suffices to

[1] I.e. $\mathcal{FOL}(<, P)$ is defined as set of all formulae φ as follows.

$$\varphi ::= \top \mid x < y \mid P(x) \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \exists x\, \varphi \mid \forall x\, \varphi$$

for variable symbols $x, y \in \textsc{SVar}$.

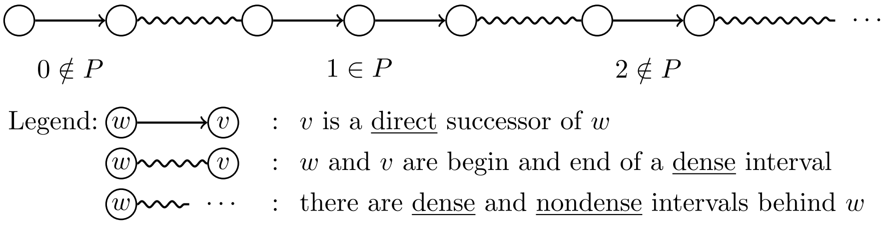


Fig. 2. An example with $0, 2 \notin P$ and $1 \in P$.

say that all successors of x are equal to, or occur after, y.

$$\mathsf{dirSuc}(x, y) := @_x \Box{\downarrow}z.(@_y z \vee @_y \Diamond z)$$

- *The state named x has no direct predecessor.* It suffices to say that, for all states r equal to, or after, the left bound a: if r is before x, then there is a state between r and x. We work around the implication by saying that one of the following three cases occurs: r is after x, or r equals x, or r is before x with a state in between.

$$\mathsf{noDirPred}(x) := @_a \Box{\downarrow}r.(@_x \Diamond r \vee @_x r \vee @_r \Diamond\Diamond x)$$

- *The state named x has a direct predecessor.* It suffices to say that there is a state r after a of which x is a direct successor.

$$\mathsf{dirPred}(x) := @_a \Diamond{\downarrow}r.\mathsf{dirSuc}(r, x)$$

- *The interval between states x, y is dense.* We say that, for all r with $x < r$: r is after y, or r has no direct predecessor.

$$\mathsf{dense}(x, y) := @_x \Box{\downarrow}r.(@_y \Diamond r \vee \mathsf{noDirPred}(r))$$

- *The state x is in a separator.* This macro says that, for some successor r of x, the interval between x and r is dense.

$$\mathsf{sep}(x) := @_x \Diamond{\downarrow}r.\mathsf{dense}(x, r)$$

- *The state x is the begin of a negative marker.* This macro says that x has a direct successor that is the begin of a separator, and x has no direct predecessor. The latter is necessary to avoid that, in the above example, the middle state of a positive marker is mistaken for the begin of a negative marker.

$$\mathsf{neg}(x) := @_x \Diamond{\downarrow}r.(\mathsf{dirSuc}(x, r) \wedge \mathsf{sep}(r)) \wedge \mathsf{noDirPred}(x)$$

- *The state x is the begin of a positive marker.* Similarly to the above macro, we express that x has a direct-successor sequence r, s with s being the begin of a separator, and x has no direct predecessor.

$$\mathsf{pos}(x) := @_x \Diamond{\downarrow}r.(\mathsf{dirSuc}(x, r) \wedge \Diamond{\downarrow}s.(\mathsf{dirSuc}(r, s) \wedge \mathsf{sep}(s))) \wedge \mathsf{noDirPred}(x)$$

- *The state x is in a separator whose end is a marker.* This macro says that, for some successor r of x, the interval between x and r is dense and r is the begin of a marker.

$$\mathsf{sepM}(x) := @_x \Diamond{\downarrow}r.(\mathsf{dense}(x, r) \wedge (\mathsf{neg}(r) \vee \mathsf{pos}(r)))$$

We now need the following two conjuncts to express that the part of the model starting at a represents a sequence of infinitely many markers.

- *a is in a separator that ends with a marker.* $\psi_1 := \mathsf{sepM}(a)$
- *Every marker has a direct successor marker.* We say that every state r after a satisfies one of the following conditions.
  - r is in a separator—this also includes that r is the end of a marker—that is followed by a marker.
  - r is the begin of a negative marker and its direct successor is the begin of a separator whose end is a marker.
  - r is the begin of a positive marker and its direct 2-step successor is the begin of a separator whose end is a marker.
  - r in the middle of a positive marker, i.e., r has a direct predecessor which is the begin of a positive marker, and r's direct successor is in a separator whose end is a marker.

$$
\begin{aligned}
\psi_2 := @_a\Box{\downarrow}r.\Big(&\mathsf{sepM}(r)\\
&\vee\Big(\mathsf{neg}(r)\wedge\Diamond{\downarrow}s.(\mathsf{dirSuc}(r,s)\wedge\mathsf{sepM}(s))\Big)\\
&\vee\Big(\mathsf{pos}(r)\wedge\Diamond{\downarrow}s.(\mathsf{dirSuc}(r,s)\wedge\Diamond{\downarrow}t.(\mathsf{dirSuc}(s,t)\wedge\mathsf{sepM}(t)))\Big)\\
&\vee\Big((@_a\Diamond{\downarrow}s.\mathsf{dirSuc}(s,r)\wedge\mathsf{pos}(s))\wedge\Diamond{\downarrow}t.(\mathsf{dirSuc}(r,t)\wedge\mathsf{sepM}(t))\Big)
\end{aligned}
$$

Finally, we encode formulae φ from $\mathcal{FOL}(<,P)$. We assume w.l.o.g. that such formulae have the shape $\varphi := Q_1x_1\dots Q_nx_n.\beta(x_1,\dots,x_n)$, where $Q_i\in\{\exists,\forall\}$ and β is quantifier-free with atoms $P(x)$ and $x<y$ for variables x,y, such that negations appear only directly before atoms. The transformation of φ reuses the x_i as state variables and proceeds inductively as follows.

$$
\begin{aligned}
f(P(x_i)) &:= \mathsf{pos}(x_i)\\
f(\neg P(x_i)) &:= \mathsf{neg}(x_i)\\
f(x_i<x_j) &:= @_{x_i}\Diamond x_j\\
f(\neg(x_i<x_j)) &:= @_{x_i}x_j\vee @_{x_j}\Diamond x_i\\
f(\alpha\wedge\beta) &:= f(\alpha)\wedge f(\beta)\\
f(\alpha\vee\beta) &:= f(\alpha)\vee f(\beta)\\
f(\exists x_i.\alpha) &:= @_a\Diamond{\downarrow}x_i.\Big((\mathsf{neg}(x_i)\vee\mathsf{pos}(x_i))\wedge f(\alpha)\Big)\\
f(\forall x_i.\alpha) &:= @_a\Box{\downarrow}x_i.\Big(\mathsf{sep}(x_i)\vee\mathsf{dirPred}(x_i)\vee f(\alpha)\Big)
\end{aligned}
$$

The transformation of φ into $\mathcal{MHL}(\Diamond,\Box,\downarrow,@)$ is now achieved by the function g defined as follows.

$$g(\varphi) := \psi_1\wedge\psi_2\wedge f(\varphi)$$

It is clear that the reduction function g can be computed in polynomial time. The correctness of the reduction is expressed by the following claim.

Claim 3.2 *For every formula* φ *from* $\mathcal{FOL}(<, P)$ *holds:*
$\varphi \in \mathbb{N}\text{-}\mathsf{SAT}_{\mathcal{FOL}}(<, P)$ *if and only if* $g(\varphi) \in \mathsf{lin\text{-}MSAT}(\Diamond, \Box, \downarrow, @)$.

The proof of the claim should be clear. Since $\mathbb{N}\text{-}\mathsf{SAT}_{\mathcal{FOL}}(<, P)$ is non-elementary [21], it follows that $\mathsf{lin\text{-}MSAT}(\Diamond, \Box, \downarrow, @)$ is non-elementary, too.

Finally, we note that our reduction uses a single free state variable a, which could as well be bound to the first state of evaluation. □

The high complexity of $\mathsf{lin\text{-}MSAT}(\Diamond, \Box, \downarrow, @)$ relies on the possibility that the linear frame alternatingly has dense and non-dense parts. If we have the natural numbers as frame for a hybrid language, we lose this possibility. As a consequence, the satisfiability problem for monotone hybrid logics over the natural numbers has a lower complexity than that over linear frames.

Theorem 3.3 $\mathbb{N}\text{-}\mathsf{MSAT}(\Diamond, \Box, \downarrow, @)$ *is* PSPACE*-complete.*

Proof sketch. Let QBFSAT be the problem to decide whether a given quantified Boolean formula is valid. We show PSPACE-hardness by a polynomial-time reduction from the PSPACE-complete QBFSAT to $\mathbb{N}\text{-}\mathsf{MSAT}(\Diamond, \Box, \downarrow, @)$. Let φ be an instance of QBFSAT and assume w.l.o.g. that negations occur only directly in front of atomic propositions. We define the transformation as $f\colon \varphi \mapsto {\downarrow} r.\Diamond{\downarrow} s.\Diamond h(\varphi)$ where h is given as follows: let ψ, χ be quantified Boolean formulae and let x_k be a variable in φ, then

$$\begin{aligned} h(\exists x_k \psi) &:= @_r \Diamond {\downarrow} x_k . h(\psi), & h(\forall x_k \psi) &:= @_r \Box {\downarrow} x_k . h(\psi), \\ h(\psi \wedge \chi) &:= h(\psi) \wedge h(\chi), & h(\psi \vee \chi) &:= h(\psi) \vee h(\chi), \\ h(\neg x_k) &:= @_s \Diamond x_k, & h(x_k) &:= @_s x_k. \end{aligned}$$

For example, the QBF $\psi = \forall x \exists y (x \wedge y) \vee (\neg x \wedge \neg y)$ is mapped to
$f(\varphi) = {\downarrow} r.\Diamond{\downarrow} s.\Diamond @_r \Box {\downarrow} x_0 . @_r \Diamond {\downarrow} x_1 . (@_s x_0 \wedge @_s x_1) \vee (@_s \Diamond x_0 \wedge @_s \Diamond x_1)$.

Intuitively, this construction requires the existence of an initial state named r, a successor state s that represents the truth value $\top$, and one or more successor states of s which together represent $\bot$. The quantifiers $\exists, \forall$ are replaced by the modal operators $\Diamond, \Box$ which range over s and its successor states. Finally, positive literals are enforced to be true at s, negative literals strictly after s.

For every model of $f(\varphi)$, it holds that r is situated at the first state of the model and that state has a successor labelled by s. By virtue of the function h, positive literals have to be mapped to s, whereas negative literals have to be mapped to some state other than s. An easy induction on the structure of formulae shows that $\varphi \in$ QBFSAT iff $f(\varphi) \in \mathbb{N}\text{-}\mathsf{MSAT}(\Diamond, \Box, \downarrow, @)$.

We obtain PSPACE-membership via a polynomial-time reduction from $\mathbb{N}\text{-}\mathsf{MSAT}(\Diamond, \Box, \downarrow, @)$ to the satisfiability problem $\mathbb{N}\text{-}\mathsf{SAT}_{\mathcal{FOL}}(<)$ for the fragment of first-order logic with the relation "$<$" interpreted over the natural numbers. Let the first order language contain all members of SVar as variables and all members of Nom as constants. Based on the standard translation from hybrid to first-order logic [22], we devise a reduction H that maps hybrid formulae φ

and variables or constants z to first-order formulae.

$$H(p,z) := \top \text{ for } p \in \textsc{Prop} \qquad H(v,z) := v = z \quad \text{for } v \in \textsc{SVar} \cup \textsc{Nom}$$

$$H(\alpha \wedge \beta, z) := H(\alpha, z) \wedge H(\beta, z) \qquad H(\alpha \vee \beta, z) := H(\alpha, z) \vee H(\beta, z)$$

$$H(\Diamond\alpha, z) := \exists t(z < t \wedge H(\alpha, t)) \qquad H(\Box\alpha, z) := \forall t(z < t \rightarrow H(\alpha, t))$$

$$H(\downarrow x.\alpha, z) := \exists x(x = z \wedge H(\alpha, z)) \qquad H(@_x\alpha, z) := H(\alpha, x)$$

In the $\Diamond$, $\Box$ and @-cases we deviate from the usual definition of the standard translation because we do not insist on using only two variables in addition to SVar—therefore it suffices to require that t is a fresh variable—and we allow constants in the second argument.

For a first-order formula ψ with variables in SVar and an assignment $g : \textsc{SVar} \rightarrow \mathbb{N}$, let $\psi[g]$ denote the first-order formula that is obtained from ψ by substituting every free occurrence of $x \in \textsc{SVar}$ by the first-order term that describes $g(x)$.

Claim 3.4 *For every instance φ of $\mathbb{N}$-MSAT$(\Diamond, \Box, \downarrow, @)$, every assignment $g : \textsc{SVar} \rightarrow \mathbb{N}$ and every $n \in \mathbb{N}$, it holds that: $g, n \models \varphi$ if and only if $(\mathbb{N}, <) \models H(\varphi, z)[g^z_n]$, where z is a new variable that does not occur in φ.*

Now, $\varphi \in \mathbb{N}$-MSAT$(\Diamond, \Box, \downarrow, @)$ if and only if $g, 0 \models \varphi \vee \Diamond\varphi$ for some assignment g. By the above claim, this is equivalent to $(\mathbb{N}, <) \models H(\varphi \vee \Diamond\varphi, z)[g^z_0]$ for some g and a new variable z, which can also be expressed as $(\mathbb{N}, <) \models \forall x(\neg(x < z) \wedge H(\varphi \vee \Diamond\varphi, z))$. This shows that $\mathbb{N}$-MSAT$(\Diamond, \Box, \downarrow, @)$ is polynomial-time reducible to $\mathbb{N}$-SAT$_{\mathcal{FOL}}(<)$, which was shown to be in PSPACE in [8]. Therefore, $\mathbb{N}$-MSAT$(\Diamond, \Box, \downarrow, @)$ is in PSPACE. □

4 The easy cases: NC^1 and LOGSPACE results

In this section, we show that the fragments without the $\Diamond$-operator have an easy satisfiability problem. Our results can be structured into four groups. First, we consider fragments without modal operators. For these fragments we obtain NC^1-completeness. Simply said, without negation and $\Diamond$ we cannot express that two nominals or state variables are not bound to the same state. Therefore, the model that binds all variables to the first state satisfies every satisfiable formula in this fragment.

Lemma 4.1 *Let $F_0 = (\{0\}, \emptyset)$ and $g_0(y) = 0$ for every $y \in \textsc{SVar}$. Then $\varphi \in$ lin-MSAT$(\downarrow, @)$ (resp. $\varphi \in \mathbb{N}$-MSAT$(\downarrow, @)$) if and only if $F_0, g_0, 0 \models \varphi$.*

Proof. The implication direction from left to right follows from the monotonicity of the considered formulas. For the other direction, notice that $F_0 \in$ lin. For frame class $\mathbb{N}$, note that if $F_0, g_0, 0 \models \varphi$ and φ has no modal operators, then $g_0, 0 \models \varphi$. □

Theorem 4.2 *Let $O \subseteq \{\downarrow, @\}$. Then* lin-MSAT$(O)$ *and* $\mathbb{N}$-MSAT(O) *are* NC^1-*complete.*

Proof. NC^1-hardness of $\mathfrak{F}$-MSAT$(\emptyset)$ follows immediately from the NC^1-completeness of the Formula Value Problem for propositional formulae [6]. It remains to show that $\mathsf{lin\text{-}MSAT}(\downarrow, @)$ and $\mathbb{N}\text{-}\mathsf{MSAT}(\downarrow, @)$ are in NC^1. In order to decide whether φ is in $\mathsf{lin\text{-}MSAT}(\downarrow, @)$, according to Lemma 4.1 it suffices to check whether the propositional formula obtained from φ deleting all occurrences of $\downarrow x.$ and $@_x$, is satisfied by the assignment that sets all atoms to *true*. According to [6] this can be done in NC^1. Since $\mathsf{lin\text{-}MSAT}(\downarrow, @) = \mathbb{N}\text{-}\mathsf{MSAT}(\downarrow, @)$ by Lemma 4.1, we obtain the same for $\mathbb{N}\text{-}\mathsf{MSAT}(\downarrow, @)$. □

Second, we consider fragments with the □-operator over linear frames. We can show NC^1-completeness here, too. The main reason is that (sub-)formulas that begin with a □ are satisfied in a state that has no successor. Therefore similar as above, every formula of this fragment that is satisfiable over linear frames is satisfied by a model with only one state.

Theorem 4.3 $\mathsf{lin\text{-}MSAT}(\Box, \downarrow, @)$ *is* NC^1*-complete.*

Proof. NC^1-hardness follows from Theorem 4.2. It remains to show that $\mathsf{lin\text{-}MSAT}(\Box, \downarrow, @) \in \mathsf{NC}^1$. We show that essentially the □-operators can be ignored.

Claim 4.4 $\mathsf{lin\text{-}MSAT}(\Box, \downarrow, @) \leqslant_{\mathrm{cd}} \mathsf{lin\text{-}MSAT}(\downarrow, @)$.

Proof of Claim. For an instance φ of $\mathsf{lin\text{-}MSAT}(\Box, \downarrow, @)$, let φ'' be the formula obtained from φ by replacing every subformula $\Box\psi$ of φ with the constant $\top$. Then φ'' is an instance of $\mathsf{lin\text{-}MSAT}(\downarrow, @)$. If $\varphi \in \mathsf{lin\text{-}MSAT}(\Box, \downarrow, @)$, then $\varphi'' \in \mathsf{lin\text{-}MSAT}(\downarrow, @)$ due to the monotonicity of φ. On the other hand, if $\varphi'' \in \mathsf{lin\text{-}MSAT}(\downarrow, @)$, then $K_0, g, 0 \models \varphi''$ (Lemma 4.1). Since $K_0, g, 0 \models \Box\alpha$ for every α, we obtain $K_0, g, 0 \models \varphi$, hence $\varphi \in \mathsf{lin\text{-}MSAT}(\Box, \downarrow, @)$. As such simple substitutions can be realized using an AC^0-circuit, the stated reduction is indeed a valid $\leqslant_{\mathrm{cd}}$-reduction from $\mathsf{lin\text{-}MSAT}(\Box, \downarrow, @)$ to $\mathsf{lin\text{-}MSAT}(\downarrow, @)$. ◇

Since $\mathsf{lin\text{-}MSAT}(\downarrow, @) \in \mathsf{NC}^1$ (Theorem 4.2) and NC^1 is closed downwards under $\leqslant_{\mathrm{cd}}$, it follows from the Claim that $\mathsf{lin\text{-}MSAT}(\Box, \downarrow, @) \in \mathsf{NC}^1$. □

It is clear that this argument does not apply to the natural numbers.

Third, we show NC^1-completeness for the fragments with □ and one of $\downarrow$ and @ over $\mathbb{N}$. They receive separate treatment because, in $(\mathbb{N}, <)$, every state has a successor, and therefore □-subformulas cannot be satisfied as easily as above. It turns out that the complexity of the satisfiability problem increases only if both hybrid operators can be used.

Theorem 4.5 $\mathbb{N}\text{-}\mathsf{MSAT}(\Box, @)$ *is* NC^1*-complete.*

Proof sketch. NC^1-hardness follows from Theorem 4.2.

For the upper bound, we distinguish occurrences of nominals that are either *free*, or that are *bound* by a □, or that are bound by an @. Simply said, a free occurrence of i in α is bound by □ in $\Box\alpha$ and bound by @ in $@_x\alpha$ (even if $x \neq i$). Since the assignment g is not relevant for the considered fragment, we write $K, w \models \alpha$ for short instead of $K, g, w \models \alpha$.

Claim 4.6 *Let α' be the formula obtained from α by replacing every occurrence* *of a nominal that is bound by* $\Box$ *with* $\bot$*, and let* η *be a valuation. If* $\eta, k \models \alpha$*, then* $\eta, k \models \alpha'$*.*

Moreover, it turns out that binding every nominal to the initial state suffices to obtain a satisfying model.

Claim 4.7 $\varphi \in \mathbb{N}\text{-}\mathsf{MSAT}(\Box, @)$ *if and only if* $\eta_0, 0 \models \varphi$ *with* $\eta_0(x) = \{0\}$ *for every* $x \in \textsc{Nom}$*.*

Both claims together yield that, in order to decide $\varphi \in \mathbb{N}\text{-}\mathsf{MSAT}(\Box, @)$, it suffices to check whether $\eta_0, 0 \models \varphi'$. No nominal in φ' occurs bound by a $\Box$-operator. Therefore for every subformula $\Box\alpha$ of φ' and for every k holds: $\eta_0, k \models \alpha$ if and only if $\eta_0, 0 \models \alpha$. All nominals that occur free or bound by an @ evaluate to *true* in state 0 via η_0. Therefore, in order to decide $\eta_0, 0 \models \varphi'$, it suffices to ignore all $\Box$ and @-operators of φ' and evaluate it as a propositional formula under assignment η_0 that sets all atoms of φ' to *true*. This can be done in NC^1 [6]. The complete proof can be found in the Technical Report version of this paper [10]. $\Box$

Next, we consider $\mathbb{N}\text{-}\mathsf{MSAT}(\Box, \downarrow)$. According to our remarks in Section 2 about notational convenience, we assume that there are no nominals in $\mathcal{MHL}(\Box, \downarrow)$.

Theorem 4.8 $\mathbb{N}\text{-}\mathsf{MSAT}(\Box, \downarrow)$ *is* NC^1*-complete.*

Proof sketch. Now, we distinguish occurrences of state variables as the occurrences in the proof sketch above. They are either *free*, or they are *bound* by a $\Box$, or they are *bound* by $\downarrow$. Note that this phrasing differs from the standard usage of the terms 'free' and 'bound' in the context of state variables. A free occurrence of i in α is bound by $\Box$ in $\Box\alpha$, as above. It is bound by $\downarrow$ in $\downarrow i.\alpha$ only. Notice that y occurs free in $\downarrow x.y$ (for $x \neq y$).

Claim 4.9 *Let* α' *be the formula obtained from* α *by replacing every occurrence of a state variable that is bound by* $\Box$ *with* $\bot$*, and let* g *be an assignment. If* $g, k \models \alpha$*, then* $g, k \models \alpha'$*.*

Claim 4.10 $\varphi \in \mathbb{N}\text{-}\mathsf{MSAT}(\Box, \downarrow)$ *if and only if* $g_0, 0 \models \varphi$*, for* $g_0(x) = 0$ *for every* $x \in \textsc{SVar}$*.*

Both claims together yield that, in order to decide $\varphi \in \mathbb{N}\text{-}\mathsf{MSAT}(\Box, \downarrow)$, it suffices to check whether $g_0, 0 \models \varphi'$. No state variable in φ' occurs bound by a $\Box$-operator. Therefore for every subformula $\Box\alpha$ of φ' and for every k holds: $g_0, k \models \alpha$ if and only if $g_0, 0 \models \alpha$. All occurrences of state variables in φ' that are bound by $\downarrow$ evaluate to *true*, because no $\Box$ occurs "between" the binding $\downarrow i$ and the occurrence of i, which means that the state where the variable is bound is the same as where the variable is used. All free occurrences of state variables evaluate to *true* in state 0 due to g_0. Therefore, in order to decide $g_0, 0 \models \varphi'$, it suffices to ignore all $\Box$ and $\downarrow$-operators of φ' and evaluate it as a propositional formula under an assignment that sets all atoms to *true*. This can be done in NC^1 [6]. The complete proof can be found in the Technical Report version of this paper [10]. $\Box$

The fourth part deals with the fragment with $\Box$ and both $\downarrow$ and @ over the natural numbers.

Lemma 4.11 $\mathbb{N}$-MSAT$(\Box, \downarrow, @)$ *is* LOGSPACE*-hard.*

Proof. This proof is very similar to the proof of Theorem 3.3. in [15]. We give a reduction from the problem *Order between Vertices* (ORD) which is known to be LOGSPACE-complete [7] and defined as follows.

Problem:	ORD
Input:	A finite set of vertices V, a successor-relation S on V, and two vertices $s, t \in V$.
Output:	Is $s \leqslant_S t$, where $\leqslant_S$ denotes the unique total order induced by S on V?

Notice that (V, S) is a directed line-graph. Let (V, S, s, t) be an instance of ORD. We construct an $\mathcal{MHL}(\Box, \downarrow, @)$-formula φ that is satisfiable if and only if $s \leqslant_S t$. We use $V = \{v_0, v_1, \ldots, v_n\}$ as state variables. The formula φ consists of three parts. The first part binds all variables except s to one state and the variable s to a successor of this state. The second part of φ binds a state variable v_l to the state labeled by s iff $s \leqslant_S v_l$. Let α denote the concatenation of all $@_{v_k}\downarrow v_l$ with $(v_k, v_l) \in S$ and $v_l \neq s$, and α^n denotes the n-fold concatenation of α. Essentially, α^n uses the assignment to collect eventually all v_i with $s \leqslant_S v_i$ in the state labeled s. The last part of φ checks whether s and t are bound to the same state after this procedure. That is, $\varphi = \downarrow v_0.\downarrow v_1.\downarrow v_2.\cdots\downarrow v_n.\Box\downarrow s.\ \alpha^n\ @_s t$. To prove the correctness of our reduction, we show that φ is satisfiable if and only if $s \leqslant_S t$.

Assume $s \leqslant_S t$. For an arbitrary assignment g, one can show inductively that $g, 0 \models \downarrow v_0.\downarrow v_1.\cdots\downarrow v_n.\Box\downarrow s.\ \alpha^i\ @_s r$ for $i = 0, 1, \ldots, n$ and for all r that have distance i from s. Therefore it eventually holds that $g, 0 \models \varphi$. For $s \not\leqslant_S t$ we show that $g, n \not\models \varphi$ for any assignment g and natural number n. Let g_0 be the assignment obtained from g after the bindings in the prefix $\downarrow v_0.\downarrow v_1.\cdots\downarrow v_n.\Box\downarrow s$ of φ, and let g_i be the assignment obtained from g_0 after evaluating the prefix of φ up to and including α^i. It holds that $g_i(s) \neq g_i(t) = 0$ for all $i = 0, 1, \ldots, n$. This leads to $g_n, 0 \not\models @_s t$ and therefore $g, 0 \not\models \varphi$. $\Box$

For the upper bound, we establish a characterisation of the satisfaction relation that assigns a *unique* assignment and state of evaluation to every subformula of a given formula φ. Using this new characterisation, we devise a decision procedure that runs in logarithmic space and consists of two steps: it replaces every occurrence of any state variable x in φ with 1 if its state of evaluation agrees with that of its $\downarrow x$-superformula, and with 0 otherwise; it then removes all $\Box$-, $\downarrow$- and @-operators from the formula and tests whether the resulting Boolean formula is valid.

Theorem 4.12 $\mathbb{N}$-MSAT$(\Box, \downarrow, @)$ *is in* LOGSPACE .

The proof is technically involved and can be found in the Technical Report version of this paper [10].

5 The intermediate cases: NP results

After we have seen that all fragments without $\Diamond$ have an easy satisfiability problem, we show that $\Diamond$ together with the use of nominals makes the satisfiability problem NP-hard. Recall that, owing to the presence of nominals, $\mathcal{MHL}(\Diamond)$ is not just modal logic with the $\Diamond$-operator. The absence of $\downarrow$ makes assignments superfluous: we write $K, w \models \varphi$ instead of $K, g, w \models \varphi$.

Lemma 5.1 lin-MSAT($\Diamond$) *and* $\mathbb{N}$-MSAT($\Diamond$) *both are* NP*-hard.*

Proof sketch. We reduce from 3SAT. Let $\varphi = c_1 \wedge \ldots \wedge c_n$ be an instance of 3SAT with clauses $c_1, \ldots, c_n$ (where $c_i = (l_1^i \vee l_2^i \vee l_3^i)$ for literals l_j^i) and variables $x_1, \ldots, x_m$. We define the transformation as

$$f\colon \varphi \mapsto \Diamond(i_0 \wedge \Diamond i_1) \ \wedge \ \left(\bigwedge_{\ell=1}^{m} \Diamond(i_0 \wedge x_\ell) \vee \Diamond(i_1 \wedge x_\ell)\right) \wedge h(\varphi),$$

where i_0, i_1 and all x_ℓ are nominals, and the function h is defined as follows: let l_k^j be a literal in clause c_j, then

$$\begin{aligned} h(l_k^j) &:= \begin{cases} (i_1 \wedge x), & \text{if } l_k^j = x \\ (i_0 \wedge x), & \text{if } l_k^j = \neg x \end{cases} \\ h(c_j) &:= \Diamond(h(l_1^j) \vee h(l_2^j) \vee h(l_3^j)), \quad \text{where } c_j = (l_1^j \vee l_2^j \vee l_3^j); \\ h(c_1 \wedge \cdots \wedge c_n) &:= h(c_1) \wedge \cdots \wedge h(c_n). \end{aligned}$$

Notice that f turns variables in the 3SAT instance into *nominals* in the lin-MSAT($\Diamond$) instance. The part $\Diamond(i_0 \wedge \Diamond i_1)$ enforces the existence of two successors w_1 and w_2 of the state satisfying $f(\varphi)$. The part $\bigwedge_{\ell=1}^{m} \Diamond(i_0 \wedge x_\ell) \vee \Diamond(i_1 \wedge x_\ell)$ simulates the assignment of the variables in φ, enforcing that each x_ℓ is true in either w_1 or w_2. The part $h(\varphi)$ then simulates the evaluation of φ on the assignment determined by the previous parts. With the following claim NP-hardness of lin-MSAT($\Diamond$) follows.

Claim 5.2 $\varphi \in$ 3SAT *if and only if* $h(\varphi) \in$ lin-MSAT($\Diamond$).

Using this claim, NP-hardness of lin-MSAT($\Diamond$) follows. It is straightforward to show that 3SAT reduces to $\mathbb{N}$-MSAT($\Diamond$) using the same reduction. □

We will now establish NP-membership of the problems $\mathfrak{F}$-MSAT($\Diamond, \Box, \downarrow$), $\mathfrak{F}$-MSAT($\Diamond, \Box, @$), and $\mathfrak{F}$-MSAT($\Diamond, \downarrow, @$) for $\mathfrak{F} \in \{\mathsf{lin}, \mathbb{N}\}$. For the first two, this follows from the literature, see Theorem 2.1 (4). For the third, we observe that all modal and hybrid operators in a formula φ from the fragment $\mathcal{MHL}(\Diamond, \downarrow, @)$ are translatable into FOL by the standard translation using no universal quantifiers. The existential quantifiers introduced by the binder can be skolemised away, which corresponds to removing all binding from φ and replacing each state variable with a fresh nominal. The correctness of this translation is proven in [22]. Hence, $\mathfrak{F}$-MSAT($\Diamond, \downarrow, @$) polynomial-time reduces to $\mathfrak{F}$-MSAT($\Diamond, @$).

Lemma 5.3 lin-MSAT($\Diamond, \downarrow, @$) *and* $\mathbb{N}$-MSAT($\Diamond, \downarrow, @$) *are in* NP.

From the lower bounds in Lemma 5.1 and the upper bounds in Theorem 2.1 (4) and Lemma 5.3, we obtain the following theorem.

Theorem 5.4 *Let* $\{\Diamond\} \subseteq O$*, and* $O \subsetneq \{\Diamond, \Box, \downarrow, @\}$*. Then* lin-MSAT($O$) *and* $\mathbb{N}$-MSAT(O) *are* NP-complete*.*

In addition to the NP-membership of the fragments captured by Theorem 5.4, we are interested in their model-theoretic properties. We show that these logics enjoy a kind of linear-size model property, precisely a quasi-quadratic size model property: over the natural numbers, every satisfiable formula has a model where two successive nominal states have at most linearly many intermediary states, and the states behind the last such state are indistinguishable. This property allows for an alternative worst-case decision procedure for satisfiability that consists of guessing a linear representation of a model of the described form and symbolically model-checking the input formula on that model. Over general linear frames, which may have dense intervals, we formulate the model property in a more general way and prove it using additional technical machinery to deal with density. However, the result then carries over to the rationals, where we are not aware of any upper complexity bound in the literature.

In [20], Sistla and Clarke showed a variation of the linear-size model property for LTL(F), which corresponds to $\mathcal{HL}(\Diamond, \Box)$ over $\mathbb{N}$: whenever $\varphi \in \mathcal{HL}(\Diamond, \Box)$ is satisfiable over $\mathbb{N}$, then it is satisfiable in the initial state of a model over $\mathbb{N}$ which has a linear-sized prefix init and a remainder final such that final is maximal with respect to the property that every type (set of all atomic propositions true in a state) occurs infinitely often, and final contains only linearly many types. Such a structure can be guessed in polynomial time, represented in polynomial space and model-checked in polynomial time. While it is straightforward to extend Sistla and Clarke's proof to cover nominals and the @ operator, it will not go through if density is allowed (frame class lin).

We establish that $\mathcal{MHL}(\Diamond, \Box, @)$ over lin has a quadratic size model property, and we subsequently show how to extend the result to the other fragments from Theorem 5.4 and how to restrict them to $\mathbb{N}$.

Theorem 5.5 $\mathcal{MHL}(\Diamond, \Box, @)$ *has the quasi-quadratic size model property with respect to* lin *and* $\mathbb{N}$*.*

The proof can be found in the Technical Report version of this paper [10].

As an immediate consequence, the model property in Theorem 5.5 carries over to the subfragments $\mathcal{MHL}(\Diamond, \Box)$, $\mathcal{MHL}(\Diamond, @)$, $\mathcal{MHL}(\Box, @)$, $\mathcal{MHL}(\Diamond)$, $\mathcal{MHL}(\Box)$, $\mathcal{MHL}(@)$, and $\mathcal{MHL}(\emptyset)$. Moreover, our arguments in the proofs of Theorems 4.3 and 4.12 can be used to transfer it to $\mathcal{MHL}(\Box, \downarrow, @)$. Together with the observations that

- $\mathcal{MHL}(\Diamond, \downarrow, @)$ is no more expressive than $\mathcal{MHL}(\Diamond, @)$ (see the explanation before Lemma 5.3), and
- $\mathcal{MHL}(\Diamond, \Box, \downarrow)$ is no more expressive than $\mathcal{MHL}(\Diamond, \Box)$ (because, without @, one cannot jump to named states),

we obtain the following generalisation of Theorem 5.5.

Corollary 5.6 *Let* $O \subsetneq \{\Diamond, \Box, \downarrow, @\}$*. Then* $\mathcal{MHL}(O)$ *has the quasi-quadratic size model property with respect to* lin *and* $\mathbb{N}$*.*

6 Conclusion

We have completely classified the complexity of all fragments of hybrid logic with monotone Boolean operators obtained from arbitrary combinations of four modal and hybrid operators, over linear frames and the natural numbers. Except for the largest such fragment over linear frames, all fragments are of elementary complexity. We have classified their complexity into PSPACE-complete, NP-complete and tractable and shown that the tractable cases are complete for either NC^1 or LOGSPACE . Surprisingly, while the largest fragment is harder over linear frames than over $(\mathbb{N}, <)$, the largest $\Diamond$-free fragment is easier over linear frames than over $(\mathbb{N}, <)$.

The question remains whether the PSPACE-complete largest fragment over $(\mathbb{N}, <)$ admits some quasi-polynomial size model property. Furthermore, this study can be extended in several possible ways: by allowing negation on atomic propositions, by considering frame classes that consist only of dense frames, such as $(\mathbb{Q}, <)$, or by considering arbitrary sets of Boolean operators in the same spirit as in [15]. For atomic negation, it follows quite easily that the largest fragment is of non-elementary complexity over $(\mathbb{N}, <)$, too, and that all fragments except $O = (\Box, \downarrow, @)$ are NP-complete. However, our proof of the quasi-quadratic model property does not immediately go through in the presence of atomic propositions. Over $(\mathbb{Q}, <)$, we conjecture that all fragments, except possibly for the largest one, have the same complexity and model properties as over $(\mathbb{N}, <)$.

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