The Complexity of Decomposing Modal and First-Order Theories

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***Abstract*—We show that the satisfiability problem for the two-dimensional extension $\mathrm{K} \times \mathrm{K}$ of unimodal K is nonelementary, hereby confirming a conjecture of Marx and Mikulás from 2001. Our lower bound technique allows us to derive further lower bounds for many-dimensional modal logics for which only elementary lower bounds were previously known. We also derive nonelementary lower bounds on the sizes of Feferman-Vaught decompositions w.r.t. product for any decomposable logic that is at least as expressive as unimodal K. Finally, we study the sizes of Feferman-Vaught decompositions and formulas in Gaifman normal form for fixed-variable fragments of first-order logic.**

I. Introduction

A. Modal Logic and Many-Dimensional Modal Logic

Modal logic [1], [2] originated in philosophy and for a long time it was known as 'the logic of necessity and possibility'. Later, it has been discovered that modal logics are well-suited to talk about relational structures, so called *(Kripke) frames*. Relational structures appear in many branches of computer science, consider for example transition systems in verification, semantic networks in knowledge representation, or attribute value structures in linguistics. This has lead to various applications of modal logic in areas such as computer science, mathematics, and artificial intelligence.

Depending on the application, a lot of different modal operators have been introduced in the past, each of them tailored towards expressing different features of the domain. For instance there are modalities that talk about time, space, knowledge, beliefs, etc.

However, it turned out that recent application domains require to express properties that *combine* different modalities, e.g., talk about the evolution of knowledge over time. In order to reflect these requirements in theory, *many-dimensional* modal logics have been studied intensively [7], [8]. A particular way of combining two logics $\mathcal{L}_1$ and $\mathcal{L}_2$ is building their *product* $\mathcal{L}_1 \times \mathcal{L}_2$ [3]. For products, the semantics is given in terms of structures, whose frames are restricted to be asynchronous products of the (one-dimensional) component frames. The interpretation of the atomic propositions is done in an *uninterpreted* way, i.e., it is independent from the component frames.

An important and well-studied problem in this context is *satisfiability checking*, i.e., to decide whether a given formula admits a model. When considering products of modal logics, it has been shown that the computational complexity of satisfiability checking often increases drastically in comparison to the well-behaved component logics. As an example, consider the basic modal logic $\mathbf{K}$ and its variant $\mathbf{K4}$ for reasoning over the class of transitive frames. Satisfiability is PSPACE-complete for both $\mathbf{K}$ and $\mathbf{K4}$ [4], while for $\mathbf{K} \times \mathbf{K}$ and $\mathbf{K4} \times \mathbf{K}$ only elementary upper bounds were known [3]. Even worse, satisfiability becomes undecidable in $\mathbf{K} \times \mathbf{K} \times \mathbf{K}$ [5] and $\mathbf{K4} \times \mathbf{K4}$ [6]. To some extent, this can be explained by the grid-like shape of product structures.

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B. Logical Decomposition

Logical decomposition can concisely be summarized as follows: A logic $\mathcal{L}$ admits decomposition w.r.t. some operation op if all $\mathcal{L}$-properties that are interpreted on composed (with respect to the operation op) structures, are already determined by the $\mathcal{L}$-properties of the component structures. Logical decomposition dates back to the work of Mostowski [10] and Feferman and Vaught [11], where it is shown that first-order logic (FO) is decomposable w.r.t. a general product operation, which covers also disjoint union and product. Later, both for more expressive logics and for more sophisticated operations such decomposability results have been proven, see [12] for an excellent survey.

When proving decomposability for a logic $\mathcal{L}$, one often obtains an effective procedure for computing such decompositions: Given a formula φ from $\mathcal{L}$ evaluated on composed structures, one can effectively compute (i) a finite set of formulas $\{\varphi_1, \ldots, \varphi_n\}$, each being evaluated on some specific component, and (ii) a propositional formula β, whose propositions are tests of the form $\mathfrak{S}_i \models \varphi_j$, such that for all composed structures $\mathfrak{S} = \mathsf{op}(\mathfrak{S}_1, \ldots, \mathfrak{S}_k)$: $\mathfrak{S} \models \varphi$ if and only if β evaluates to true. The size of the resulting decomposition is typically nonelementary in the size of the original formula. Dawar et al. proved that this is unavoidable if $\mathcal{L} = \mathsf{FO}$ [9].

Decomposition theorems have powerful implications in computer science logic. Let us mention only four of them.

Firstly, assume some decomposable logic $\mathcal{L}$: Then *decidability of the $\mathcal{L}$-theory* of some composed structure, for instance a product structure, can be derived from the decidability of the $\mathcal{L}$-theories of its component structures.

Secondly, let us mention that *model checking* a fixed $\mathcal{L}$-formula (i.e. the *data complexity*) in a composed structure is not harder than model checking fixed $\mathcal{L}$-formulas on the component structures: If the formula is fixed, also the decomposition is fixed (although possibly large).

Moreover, decompositional methods can be applied for showing decidability of *satisfiability checking*: Instead of asking whether a given formula φ is satisfiable in a composed model, one computes a decomposition for φ, translates the decomposition into disjunctive normal form, and finally checks satisfiability of a conjunction of formulas in their corresponding components. Rabinovich proved that basic modal logic $\mathbf{K}$ is decomposable w.r.t. *interpreted* products [13], where "interpreted" means that the interpretation of the propositions is inherited from the component structures. It is worth noting that this, however, does not lead to decidability of $\mathbf{K}\times\mathbf{K}$ w.r.t. the classical (uninterpreted) products mentioned above. To the contrary, satisfiability w.r.t. interpreted products is easily reducible to the uninterpreted version.

Finally, an important application of logical decomposition à la Feferman and Vaught is the (original) proof of *Gaifman's locality theorem* [14] stating that every first-order sentence is equivalent to a boolean combination of basic local sentences, where a basic local sentence admits quantification only relativized to finite neighbourhoods of elements. Gaifman's locality theorem has important applications such as inexpressibility results for first-order logic. For a further and more recent application of Gaifman's locality theorem we mention algorithmic meta-theorems for first-order logic [15], stating that first-order properties can be efficiently solved on numerous classes of structures.

C. Our Contributions and Related Work

As our first main result we show that (even the interpreted variant of) the satisfiability problem of two-dimensional modal logic $\mathbf{K}^2 = \mathbf{K}\times\mathbf{K}$ has nonelementary complexity, hereby solving a fundamental problem that has been open for more than 10 years. Gabbay and Shetman proved in 1998 that satisfiability in $\mathbf{K}^2$ is decidable in a tower of exponentials [3]. To the best of the authors' knowledge, the best known lower bound has been NEXP-hardness shown by Marx and Mikulás in 2001 [16]. In fact, we prove that satisfiability in $\mathbf{K}^2$ restricted to formulas of switching depth k (the minimal modal rank among the two dimensions) is k-NEXP-complete, hereby confirming a conjecture of Marx and Mikulás [16]. We derive nonelementary lower bounds for the two-dimensional modal logics $\mathbf{K4}\times\mathbf{K}$ and $\mathbf{S5}_2\times\mathbf{K}$ for which only elementary lower bounds were known [7].

Our lower bound technique allows us to derive a nonelementary lower bound for the size of Feferman-Vaught decompositions w.r.t. product for $\mathbf{K}$. Such a result was already shown in [17]. However, in contrast to [17], our proof technique implies that the nonelementary lower bound carries over to all decomposable logics that are at least as expressive as $\mathbf{K}$. An instance of such a logic is the two-variable fragment FO^2 of first-order logic. Moreover, we prove that the same lower bound holds when relativized to the class of finite trees, answering an open problem formulated in [17].

In the same fashion, for the three-variable fragment FO^3 of first-order logic, we can derive the following new results: (i) the size of Feferman-Vaught decompositions w.r.t. disjoint sum are inherently nonelementary and (ii) equivalent formulas in Gaifman normal form are inherently nonelementary. It is worth mentioning that (i) and (ii) were shown in [9] for full FO. By inspecting the formulas in [9] it turns out that they are in fact FO^4-formulas. However, it seems to be unclear whether the construction from [9] can be adapted so that it yields FO^3-formulas.

Finally, we provide effective doubly exponential (and hence elementary) upper bounds for the two-variable fragment FO^2 of first-order logic both for Feferman-Vaught decompositions and for equivalent formulas in Gaifman normal form. This supports former observations that in many aspects FO^2 is better behaved than FO^3. For instance, in contrast to FO^3 it has a finite model property and satisfiability is decidable [18]. We also prove (non-matching) lower bounds of the form $c^{\sqrt{n}}$ (for any constant c) for both Feferman-Vaught decomposition and equivalent formulas in Gaifman normal form for FO^2.

II. Preliminaries

For $i,j\in\mathbb{Z}$ let $[i,j]$ be the interval $[i,i+1,\ldots,j]$. By $\mathbb{N}=\{0,1,\ldots\}$ we denote the non-negative integers. For a set X we denote by $\mathsf{bool}(X)$ the set of boolean formulas with variables ranging over X. Let $u=u_1\cdots u_k\in\Sigma^*$ with $u_i\in\Sigma$ for each $i\in[1,k]$. By $|u|=k$ we denote the *length* of u.

A. Kripke Frames and Structures

Let us fix a countable set of *action labels* $\mathbb{A}$ and a countable set of *propositional variables* $\mathbb{P}$. For a finite set $\mathsf{A}\subseteq\mathbb{A}$ of action labels, an A-*frame* is a tuple $\mathfrak{F}=(W,\{\xrightarrow{a}\mid a\in\mathsf{A}\})$, where W is set of *worlds* and $\xrightarrow{a}\subseteq W\times W$ is a binary (accessibility) relation over W for each $a\in\mathsf{A}$. An (A,P)-*Kripke structure* (or (A,P)-*structure* for short), for a finite set $\mathsf{A}\subseteq\mathbb{A}$ of action labels and a finite set $\mathsf{P}\subseteq\mathbb{P}$ of propositional variables, is a tuple $\mathfrak{S}=(W,\{\xrightarrow{a}\mid a\in\mathsf{A}\},\{W_p\mid p\in\mathsf{P}\})$, where $(W,\{\xrightarrow{a}\mid a\in\mathsf{A}\})$ is an A-frame and $W_p\subseteq W$ is an interpretation for each propositional variable $p\in\mathsf{P}$. By $\mathfrak{F}(\mathfrak{S})\stackrel{\text{def}}{=}(W,\{\xrightarrow{a}\mid a\in\mathsf{A}\})$ we denote the underlying A-frame of $\mathfrak{S}$. By $|\mathfrak{S}|=|W|$ we denote the *size* of $\mathfrak{S}$. We say that $\mathfrak{S}$ is *finite* if W is finite. For $s\in W$ let $N_{\mathfrak{S}}(s)\stackrel{\text{def}}{=}\{u\in W\mid\exists a\in\mathsf{A}:s\xrightarrow{a}u\}$ be the set of direct successors of s in $\mathfrak{S}$. A *pointed* (A,P)-*structure* is a pair $(\mathfrak{S},s)$ where $\mathfrak{S}$ is an (A,P)-structure and s is a world of $\mathfrak{S}$. An $(\{a\},\mathsf{P})$-structure is also called *unimodal*. We write $(W,\xrightarrow{a},\{W_p\mid p\in\mathsf{P}\})$ instead of $(W,\{\xrightarrow{a}\},\{W_p\mid p\in\mathsf{P}\})$.

B. Multimodal Logic

Formulas of *multimodal logic* are defined by the following grammar, where a (resp., p) ranges over $\mathbb{A}$ (resp., $\mathbb{P}$):

$$\varphi \quad ::= \quad p \quad | \quad \neg\varphi \quad | \quad \varphi\wedge\varphi \quad | \quad \Diamond_a\varphi$$

We introduce the usual abbreviations $\top=p\vee\neg p$ for some $p\in\mathbb{P}$, $\bot=\neg\top$, $\varphi_1\vee\varphi_2=\neg(\neg\varphi_1\wedge\neg\varphi_2)$, and $\Box_a\varphi=\neg\Diamond_a\neg\varphi$. We say that φ is *over* (A,P) if the set of action labels (resp. the set of propositional variables) that appears in φ is a subset of A

(resp. P). For an (A,P)-structure $\mathfrak{S} = (W, \{\xrightarrow{a} \mid a \in \mathsf{A}\}, \{W_p \mid p \in \mathsf{P}\})$, $w \in W$, and a formula φ over (A,P), we define the satisfaction relation $(\mathfrak{S}, w) \models \varphi$ by structural induction on φ, where $a \in \mathsf{A}$ and $p \in \mathsf{P}$:

$$\begin{aligned} (\mathfrak{S}, w) \models p &\overset{\text{def}}{\Leftrightarrow} w \in W_p \\ (\mathfrak{S}, w) \models \neg\varphi &\overset{\text{def}}{\Leftrightarrow} (\mathfrak{S}, w) \not\models \varphi \\ (\mathfrak{S}, w) \models \varphi_1 \wedge \varphi_2 &\overset{\text{def}}{\Leftrightarrow} (\mathfrak{S}, w) \models \varphi_1 \text{ and } (\mathfrak{S}, w) \models \varphi_2 \\ (\mathfrak{S}, w) \models \Diamond_a\varphi &\overset{\text{def}}{\Leftrightarrow} \exists w' : w \xrightarrow{a} w' \text{ and } (\mathfrak{S}, w') \models \varphi \end{aligned}$$

Let φ be a multimodal logic formula over (A,P). An (A,P)-structure $\mathfrak{S}$ is a *model* of φ if $(\mathfrak{S}, w) \models \varphi$ for some world w of $\mathfrak{S}$. We say that φ is *satisfiable* if φ has a model.

C. Asynchronous Products and Many-Dimensional Modal Logic

Fix non-empty, finite, and pairwise disjoint sets $\mathsf{A}_1, \ldots, \mathsf{A}_d \subseteq \mathbb{A}$ of action labels and non-empty, finite, and pairwise disjoint sets $\mathsf{P}_1, \ldots, \mathsf{P}_d \subseteq \mathbb{P}$ of propositional variables. Let $\mathsf{A} = \bigcup_{i\in[1,d]} \mathsf{A}_i$ and $\mathsf{P} = \bigcup_{i\in[1,d]} \mathsf{P}_i$. For A_i-frames $\mathfrak{F}_i = (W_i, \{\xrightarrow{a}_i \mid a \in \mathsf{A}_i\})$ ($i \in [1,d]$) we define the *asynchronous product* $\prod_{i\in[1,d]} \mathfrak{F}_i \overset{\text{def}}{=} (W, \{\xrightarrow{a} \mid a \in \mathsf{A}\})$ to be the A-frame, where $W = W_1 \times \cdots \times W_d$ and where for each $\overline{v} = \langle v_1, \ldots, v_d\rangle \in W$ and $\overline{w} = \langle w_1, \ldots, w_d\rangle \in W$ we have $\overline{v} \xrightarrow{a} \overline{w}$ if and only if there is some $i \in [1,d]$ such that $a \in \mathsf{A}_i$, $v_i \xrightarrow{a}_i w_i$ and $v_j = w_j$ for each $j \in [1,d] \setminus \{i\}$. An (A,P)-structure $\mathfrak{S} = (W, \{\xrightarrow{a} \mid a \in \mathsf{A}\}, \{W_p \mid p \in \mathsf{P}\})$ is an *uninterpreted product structure* if $\mathfrak{F}(\mathfrak{S}) = \prod_{i=1}^d \mathfrak{F}_i$, where each $\mathfrak{F}_i$ is some A_i-frame. Thus, we do not make any restrictions on how atomic propositions are interpreted.

Next, let us define interpretations of atomic propositions in products, as introduced in [13]. A *(product) interpretation* is a mapping $\sigma : \mathsf{P} \to \mathsf{bool}(\mathsf{P})$. In our lower bound proofs in Section III, σ will be the *identity interpretation* id with $\mathsf{id}(p) = p$ for all $p \in \mathsf{P}$. Let $\mathfrak{S}_i = (W_i, \{\xrightarrow{a}_i \mid a \in \mathsf{A}_i\}, \{W_{p,i} \mid p \in \mathsf{P}_i\})$ be an $(\mathsf{A}_i, \mathsf{P}_i)$-structure for $i \in [1,d]$. For an interpretation σ, their *σ-product* $\prod^\sigma_{i\in[1,d]} \mathfrak{S}_i$ is defined as the (A,P)-structure $\mathfrak{S} = (W, \{\xrightarrow{a} \mid a \in \mathsf{A}\}, \{W_p \mid p \in \mathsf{P}\})$ such that $\mathfrak{F}(\mathfrak{S}) = \prod_{i\in[1,d]} \mathfrak{F}(\mathfrak{S}_i)$ and $\langle w_1, \ldots, w_d\rangle \in W_p$ if and only if $\alpha \models \sigma(p)$, where $\alpha(q) = 1$ if and only if $w_i \in W_{q,i}$ for each $i \in [1,d]$ and $q \in \mathsf{P}_i$. If no interpretation is given, we define $\prod_{i\in[1,d]} \mathfrak{S}_i \overset{\text{def}}{=} \prod^{\mathsf{id}}_{i\in[1,d]} \mathfrak{S}_i$.

Let us generalize multimodal logic to higher dimensions. A *multimodal formula of dimension* $d \geq 1$ (briefly, a *multimodal* $\mathbf{K}^d$*-formula*) is a formula φ over $(\mathsf{A} = \bigcup_{i=1}^d \mathsf{A}_i, \mathsf{P} = \bigcup_{i=1}^d \mathsf{P}_i)$. If $|\mathsf{A}_i| = 1$ for all $i \in [1,d]$ then φ is a *unimodal formula of dimension* $d \geq 1$ (briefly, a *unimodal* $\mathbf{K}^d$*-formula*). For a multimodal $\mathbf{K}^d$-formula φ and $i \in [1,d]$ we define $\mathsf{rank}_i(\varphi)$ inductively: $\mathsf{rank}_i(p) = 0$ for $p \in \mathsf{P}$, $\mathsf{rank}_i(\neg\varphi) = \mathsf{rank}_i(\varphi)$, $\mathsf{rank}_i(\varphi_1 \wedge \varphi_2) = \max\{\mathsf{rank}_i(\varphi_1), \mathsf{rank}_i(\varphi_2)\}$, $\mathsf{rank}_i(\Diamond_a\varphi) = \mathsf{rank}_i(\varphi)$ for $a \in \mathsf{A} \setminus \mathsf{A}_i$, and $\mathsf{rank}_i(\Diamond_a\varphi) = \mathsf{rank}_i(\varphi) + 1$ for $a \in \mathsf{A}_i$. Finally, we define the *switching depth* of φ as $\min\{\mathsf{rank}_i(\varphi) \mid 1 \leq i \leq d\}$ [16]. An *uninterpreted product model* of φ is an uninterpreted product structure $\mathfrak{S}$ (in the above sense) such that for some world $\overline{w}$ of $\mathfrak{S}$ we have $(\mathfrak{S}, \overline{w}) \models \varphi$. For an interpretation σ, a *σ-model* is a σ-product structure $\mathfrak{S}$ such that $(\mathfrak{S}, \overline{w}) \models \varphi$ for some world $\overline{w}$ of $\mathfrak{S}$. We say φ is *uninterpreted satisfiable* (resp., *σ-satisfiable*) if φ has an uninterpreted (resp., σ-) product model. Let us introduce the following decision problems for multimodal (unimodal) $\mathbf{K}^d$:

<u>MULTIMODAL (UNIMODAL) $\mathbf{K}^d$-SAT</u>

INPUT: A multimodal (unimodal) $\mathbf{K}^d$-formula φ.
QUESTION: Is φ uninterpreted satisfiable?

We introduce the corresponding variant in the presence of an interpretation σ of the atomic propositions.

<u>MULTIMODAL (UNIMODAL) $\mathbf{K}^d_\sigma$-SAT</u>

INPUT: A multimodal (unimodal) $\mathbf{K}^d$-formula φ and some interpretation σ.
QUESTION: Is φ σ-satisfiable?

Since we mainly deal with the unimodal case in this paper, we use $\mathbf{K}^d_\sigma$-SAT as an abbreviation for UNIMODAL $\mathbf{K}^d_\sigma$-SAT. The following proposition is not hard to prove, but will be technically useful in Sections III and IV.

Proposition 1. *There is a polynomial time many-one reduction from* (MULTIMODAL) $\mathbf{K}^d_\sigma$-SAT *to* (MULTIMODAL) $\mathbf{K}^d$-SAT, *which preserves the switching depth.*

D. First-Order Logic

We assume standard definitions concerning first-order logic. Only relational signatures will be considered. For $k \geq 1$, a formula φ is an FO^k-formula if at most k different variables are used in φ. Note that a formula, in which every subformula has at most k free variables is equivalent to an FO^k-formula. The *quantifier rank* of a formula φ is the maximal nesting depth of quantifiers in φ; it is denoted by $\mathsf{qr}(\varphi)$.

E. Feferman-Vaught Decompositions

The Feferman-Vaught decomposition theorem for multimodal $\mathbf{K}^d$ can be formulated as follows, and was proven in [13].

Theorem 2 ([13])**.** *From an interpretation σ and a multimodal $\mathbf{K}^d$-formula φ over $(\bigcup_{i\in[1,d]} \mathsf{A}_i, \bigcup_{i\in[1,d]} P_i)$, one can compute a tuple $(\Psi_1, \ldots, \Psi_d, \beta)$ with $\Psi_i = \{\psi_i^j \mid j \in J_i\}$ a finite set of multimodal formulas over $(\mathsf{A}_i, \mathsf{P}_i)$ and β a positive boolean formula with variables from $X = \{x_i^j \mid i \in [1,d], j \in J_i\}$ such that for every $(\mathsf{A}_i, \mathsf{P}_i)$-structure $\mathfrak{S}_i$ and every world w_i of $\mathfrak{S}_i$ ($i \in [1,d]$):*

$$\Big(\prod^{\sigma}_{i\in[1,d]} \mathfrak{S}_i, \langle w_1, \ldots, w_n\rangle\Big) \models \varphi \qquad \Leftrightarrow \qquad \mu \models \beta$$

Here, $\mu : X \to \{0,1\}$ is defined by $\mu(x_i^j) = 1$ iff $(\mathfrak{S}_i, w_i) \models \psi_i^j$.

We call $\mathcal{D} = (\Psi_1, \ldots, \Psi_d, \beta)$ the *decomposition* of φ and define $|\mathcal{D}| = |\beta| + \sum_{i,j} |\psi_i^j|$ to be its *size*. We note that in the same way one can define decompositions for the unimodal variant and for extensions of multimodal logic.

We note that Theorem 2 only holds in the presence of an interpretation σ for the atomic propositions. We also mention that Theorem 2 has been proven in [13] for much more elaborated notions of interpretations. However, note that not every logic admits decomposition: For instance the property $\mathsf{EG}p$ meaning "there is a maximal path (a path is maximal if it is either infinite or ends in a dead-end) on which every world satisfies p" is not decomposable, as shown in [13].

An analogous theorem can be stated for first-order sentences, see [12] for a survey. In the following theorem, we view every $p \in \mathsf{P} = \bigcup_{i \in [1,d]} \mathsf{P}_i$ as a unary predicate and every $a \in \mathsf{A} = \bigcup_{i \in [1,d]} \mathsf{A}_i$ as a binary predicate.

Theorem 3 ([11]). *From an interpretation σ and an FO^k-sentence φ over the signature (A, P) one can compute a tuple $(\Psi_1, \dots, \Psi_d, \beta)$ with $\Psi_i = \{\psi_i^j \mid j \in J_i\}$ a finite set of FO^k-sentences over the signature $(\mathsf{A}_i, \mathsf{P}_i)$ and β a positive boolean formula with variables from $X = \{x_i^j \mid i \in [1,d], j \in J_i\}$ such that for every $(\mathsf{A}_i, \mathsf{P}_i)$-structure $\mathfrak{S}_i$ ($i \in [1,d]$):*

$$\prod_{i \in [1,d]}^{\sigma} \mathfrak{S}_i \models \varphi \qquad \Leftrightarrow \qquad \mu \models \beta.$$

Here, $\mu : X \to \{0,1\}$ is defined by $\mu(x_i^j) = 1$ iff $\mathfrak{S}_i \models \psi_i^j$.

III. $\mathbf{K}^2$-SAT IS HARD

The goal of this section is to show a nonelementary lower bound for $\mathbf{K}^2$-SAT and thus to close the complexity gap for this problem. As a necessary preliminary step we show how to enforce (nonelementary) big models in $\mathbf{K}^2_{\mathsf{id}}$. Using this, we prove via a standard reduction from appropriate tiling problems that $\mathbf{K}^2_{\mathsf{id}}$-SAT is nonelementary. Applying Proposition 1 yields the nonelementary lower bound for $\mathbf{K} \times \mathbf{K}$.

Recall the function $\mathsf{Tower} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined as $\mathsf{Tower}(0,n) = n$ and $\mathsf{Tower}(\ell+1,n) = 2^{\mathsf{Tower}(\ell,n)}$ for each $\ell, n \in \mathbb{N}$. In this section, we construct a family $\{\varphi_{\ell,n} \mid \ell, n \geq 1\}$ of unimodal $\mathbf{K}^2$-formulas such that for each $\ell, n \in \mathbb{N}$, (i) $|\varphi_{\ell,n}| \leq \exp(\ell) \cdot \mathsf{poly}(n)$ and (ii) if $(\mathfrak{S} \times \mathfrak{S}', \langle s, s' \rangle) \models \varphi_{\ell,n}$, then $|\mathfrak{S}|, |\mathfrak{S}'| \geq \mathsf{Tower}(\ell, n)$. Informally speaking, our intention is that if $(\mathfrak{S} \times \mathfrak{S}', \langle s, s' \rangle) \models \varphi_{\ell,n}$ then *both* $(\mathfrak{S}, s)$ *and* $(\mathfrak{S}', s')$ are of a particular structure that we will call (ℓ,n)-*treelike*. Before giving its formal definition, we provide some intuition about when a pointed structure $(\mathfrak{S}, s)$ is treelike (the definition of when $(\mathfrak{S}', s')$ is treelike will be analogous).

Intuitively, think of a pointed structure $(\mathfrak{S}, s)$ to be (ℓ,n)-treelike if it contains a tree of depth ℓ rooted in s (possibly with additional worlds and transitions) such that

- $(\mathfrak{S}, t)$ is $(\ell-1, n)$-treelike for every successor t of s,
- s has $\mathsf{Tower}(\ell, n)$ successors.

For this purpose, we additionally assign a *value* $\mathsf{val}(\mathfrak{S}, s)$ to every (ℓ,n)-treelike structure $(\mathfrak{S}, s)$ and require that for every $i \in [0,m]$ with $m = \mathsf{Tower}(\ell,n) - 1$ there is a successor s_i of s with value i (however, we cannot exclude copies of s_i). For $\ell = 0$, the value is defined by propositional variables $p_0, \dots, p_{n-1}$ which define an n-bit number, where p_0 refers to the least significant bit. For $\ell > 0$, the value is defined using an additional proposition p_b. Intuitively, the worlds $s_0, \dots, s_m$

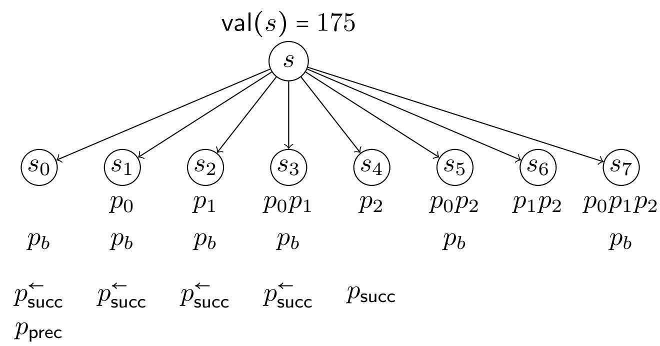

Fig. 1. Example of an $(1,3)$-treelike structure with value 175.

define a binary number (by convention, the leftmost bit is the least significant bit) $b_0 \cdots b_m$, where $b_i = 1$ precisely when the proposition p_b is satisfied in s_i. Obviously, this number is between 0 and $\mathsf{Tower}(\ell+1, n) - 1$. Figure 1 gives an example of an $(1,3)$-treelike structure with value 175. First observe that s has $8 = 2^3 = \mathsf{Tower}(1,3)$ successors $s_0, \dots, s_7$. Next note that in each s_i the evaluation of the propositions p_j ($j \in [0,2]$) gives a binary number equal to i. For instance, in s_4 only p_2 is true, hence the corresponding binary number is 001 which is 4. As indicated, the evaluation of p_b gives rise to the binary number $\vec{b} = 11110101$ which equals 175. For enforcing the described treelike structures $(\mathfrak{S}, s)$ we need additional auxiliary propositional variables p_{succ}, $p_{\overleftarrow{\mathsf{succ}}}$, p_{prec}, and $p_{\overleftarrow{\mathsf{prec}}}$. These propositions provide more information about the binary number $\vec{b} = b_0 \cdots b_{\mathsf{Tower}(\ell,n)-1}$ encoded by the successors of s:

- p_{succ} marks the first (from left to right) 0 in $\vec{b}$,
- $p_{\overleftarrow{\mathsf{succ}}}$ marks all worlds left of p_{succ},
- p_{prec} marks the first 1 in $\vec{b}$, and
- $p_{\overleftarrow{\mathsf{prec}}}$ marks all worlds left of p_{prec}.

Inuitively, p_{succ} (resp. p_{prec}) marks the maximal position that changes when $\vec{b}$ is *increased* (resp. *decreased*) by 1. In other words, increasing $\vec{b}$ by 1 can be done by flipping all bits marked with p_{succ} or $p_{\overleftarrow{\mathsf{succ}}}$ and carrying over the remaining ones. We refer again to Figure 1 for a valid evaluation of the auxiliary propositions.

It is worth mentioning that (ℓ,n)-treelike structures are similar to the trees $\widetilde{\mathcal{T}}_h(n)$ from [9, Definition 2]. As mentioned above, we add a few more unary predicates (propositional variables) since our structures will be enforced in 2-dimensional modal logic instead of first-order logic.

In the following, we formally define (ℓ,n)-treelike structures and their associated values. For this purpose, let us fix the set of action labels $\mathsf{A} = \{a, a'\}$ and for each $n \geq 1$ define the set of propositional variables $\mathsf{P}_n \stackrel{\text{def}}{=} \{p_0, \dots, p_{n-1}\} \cup \mathsf{P}_{\mathsf{aux}}$ and $\mathsf{Q}_n \stackrel{\text{def}}{=} \{q_0, \dots, q_{n-1}\} \cup \mathsf{Q}_{\mathsf{aux}}$ with $\mathsf{P}_{\mathsf{aux}} \stackrel{\text{def}}{=} \{p_b, p_{\mathsf{succ}}, p_{\overleftarrow{\mathsf{succ}}}, p_{\mathsf{prec}}, p_{\overleftarrow{\mathsf{prec}}}\}$ and $\mathsf{Q}_{\mathsf{aux}} \stackrel{\text{def}}{=} \{q_b, q_{\mathsf{succ}}, q_{\overleftarrow{\mathsf{succ}}}, q_{\mathsf{prec}}, q_{\overleftarrow{\mathsf{prec}}}\}$. For the sake of simplicity, we call $(\{a\}, \mathsf{P}_n)$-structures (resp. $(\{a'\}, \mathsf{Q}_n)$-structures) *left structures* (resp. *right structures*). We give only the definition for left pointed structures because the definition for right structures is simply obtained by replacing every proposition

p_g by q_g and $\xrightarrow{a}$ by $\xrightarrow{a'}$.

The definition of (ℓ, n)-treelike structures $(\mathfrak{S}, s)$ and their associated *values* $\mathsf{val}(\mathfrak{S}, s) \in [0, \mathsf{Tower}(\ell+1, n) - 1]$ is by induction on ℓ. Consider the left pointed structure $(\mathfrak{S}, s)$ where $\mathfrak{S} = (W, \xrightarrow{a}, \{W_p \mid p \in \mathsf{P}_n\})$. Then $(\mathfrak{S}, s)$ is $(0, n)$–treelike if $N_{\mathfrak{S}}(s) = \varnothing$. The *value* of $(\mathfrak{S}, s)$ is

$$\mathsf{val}(\mathfrak{S}, s) \stackrel{\text{def}}{=} \sum_{i=0}^{n-1} b_i 2^i \in [0, 2^n - 1],$$

where $b_i = 1$ if $s \in W_{p_i}$ and $b_i = 0$ otherwise.

For $\ell > 0$, $(\mathfrak{S}, s)$ is (ℓ, n)-*treelike* if the following hold, where $m = \mathsf{Tower}(\ell, n) - 1$:

(a) For all $u \in N_{\mathfrak{S}}(s)$, $(\mathfrak{S}, u)$ is $(\ell-1, n)$-treelike. Let $N^i_{\mathfrak{S}}(s) \stackrel{\text{def}}{=} \{u \in N_{\mathfrak{S}}(s) \mid \mathsf{val}(\mathfrak{S}, u) = i\}$ for $i \in [0, m]$.
(b) $N^i_{\mathfrak{S}}(s) \neq \varnothing$ for every $i \in [0, m]$.
(c) If $u, v \in N^i_{\mathfrak{S}}(s)$ for some $i \in [0, m]$, then $u \in W_p$ if and only if $v \in W_p$ for each $p \in \mathsf{P_{aux}}$.
(d) If $N_{\mathfrak{S}}(s) \subseteq W_{p_b}$ then $W_{p_{\mathsf{succ}}} \cap N_{\mathfrak{S}}(s) = \varnothing$ and $N_{\mathfrak{S}}(s) \subseteq W_{\overleftarrow{p}_{\mathsf{succ}}}$.
(e) If $N_{\mathfrak{S}}(s) \setminus W_{p_b} \neq \varnothing$ and $k \in [0, m]$ is minimal such that $N^k_{\mathfrak{S}}(s) \setminus W_{p_b} \neq \varnothing$, then for all $v \in N_{\mathfrak{S}}(s)$: $v \in W_{p_{\mathsf{succ}}}$ iff $\mathsf{val}(\mathfrak{S}, v) = k$ and $v \in W_{\overleftarrow{p}_{\mathsf{succ}}}$ iff $\mathsf{val}(\mathfrak{S}, v) < k$.
(f) If $N_{\mathfrak{S}}(s) \cap W_{p_b} = \varnothing$ then $W_{p_{\mathsf{prec}}} \cap N_{\mathfrak{S}}(s) = \varnothing$ and $N_{\mathfrak{S}}(s) \subseteq W_{\overleftarrow{p}_{\mathsf{prec}}}$.
(g) If $N_{\mathfrak{S}}(s) \cap W_{p_b} \neq \varnothing$ and $k \in [0, m]$ is minimal such that $N^k_{\mathfrak{S}}(s) \cap W_{p_b} \neq \varnothing$, then for all $v \in N_{\mathfrak{S}}(s)$: $v \in W_{p_{\mathsf{prec}}}$ iff $\mathsf{val}(\mathfrak{S}, v) = k$ and $v \in W_{\overleftarrow{p}_{\mathsf{prec}}}$ iff $\mathsf{val}(\mathfrak{S}, v) < k$.

Note that we make no restriction on the valuation of propositions in the world s. Moreover, also the set $W_{p_b} \cap N_{\mathfrak{S}}(s)$ is arbitrary, but this set uniquely determines the sets $W_{p_{\mathsf{succ}}} \cap N_{\mathfrak{S}}(s)$, $W_{\overleftarrow{p}_{\mathsf{succ}}} \cap N_{\mathfrak{S}}(s)$, $W_{p_{\mathsf{prec}}} \cap N_{\mathfrak{S}}(s)$, and $W_{\overleftarrow{p}_{\mathsf{prec}}} \cap N_{\mathfrak{S}}(s)$.

Finally, we define the *value* of $(\mathfrak{S}, s)$ as follows: For $i \in [0, m]$, let $b_i = 0$ if $W_{p_b} \cap N^i_{\mathfrak{S}}(s) = \varnothing$ and $b_i = 1$ otherwise. Then,

$$\mathsf{val}(\mathfrak{S}, s) \stackrel{\text{def}}{=} \sum_{i=0}^{m} b_i 2^i \in [0, 2^{m+1} - 1] = [0, \mathsf{Tower}(\ell+1, n) - 1].$$

Observe that this definition does not require a *unique* successor world s_i for each value i. In fact, one cannot enforce this in modal logic.

We will construct a family of formulas $(\varphi_{\ell,n})_{\ell,n \geq 0}$ that admit only (ℓ, n)-treelike structures as models. In order to emphasize the two dimensions that we have in formulas over $(\{a\} \uplus \{a'\}, \mathsf{P}_n \uplus \mathsf{Q}_n)$, we write $\diamondminus$ (resp. $\diamondbar$) instead of $\Diamond_a$ (resp. $\Diamond_{a'}$) to refer to the modality of the first (resp. second) dimension of the product, and similarly for box formulas.

Before we define the formulas $\varphi_{\ell,n}$, we introduce auxiliary formulas $\mathsf{eq}_{\ell,n}$, $\mathsf{first}_{\ell,n}$, $\mathsf{last}_{\ell,n}$, and $\mathsf{succ}_{\ell,n}$ whose names indicate their intended purposes. For $\ell = 0$ they are as follows:

$$\begin{aligned}
\mathsf{eq}_{0,n} &\stackrel{\text{def}}{=} \textstyle\bigwedge_{i \in [0,n-1]} p_i \leftrightarrow q_i \\
\mathsf{first}_{0,n} &\stackrel{\text{def}}{=} \textstyle\bigwedge_{i \in [0,n-1]} \neg p_i \wedge \neg q_i \\
\mathsf{last}_{0,n} &\stackrel{\text{def}}{=} \textstyle\bigwedge_{i \in [0,n-1]} p_i \wedge q_i \\
\mathsf{succ}_{0,n} &\stackrel{\text{def}}{=} \textstyle\bigvee_{i \in [0,n-1]} (\neg p_i \wedge q_i \wedge \\
&\qquad \textstyle\bigwedge_{j \in [0,i-1]} (p_j \wedge \neg q_j) \wedge \bigwedge_{j \in [i+1,n-1]} p_j \leftrightarrow q_j)
\end{aligned}$$

For $\ell > 0$ we define them as follows:

$$\begin{aligned}
\mathsf{eq}_{\ell,n} &\stackrel{\text{def}}{=} \boxminus \boxbar \left(\mathsf{eq}_{\ell-1,n} \rightarrow (p_b \leftrightarrow q_b)\right) \\
\mathsf{first}_{\ell,n} &\stackrel{\text{def}}{=} \boxminus \neg p_b \wedge \boxbar \neg q_b \\
\mathsf{last}_{\ell,n} &\stackrel{\text{def}}{=} \boxminus p_b \wedge \boxbar q_b \\
\mathsf{succ}_{\ell,n} &\stackrel{\text{def}}{=} \diamondminus \neg p_b \wedge \boxminus \boxbar \left(\mathsf{eq}_{\ell-1,n} \rightarrow\right. \\
&\quad \left.(p_{\mathsf{succ}} \leftrightarrow q_{\mathsf{prec}}) \wedge ((\neg \overleftarrow{p}_{\mathsf{succ}} \wedge \neg p_{\mathsf{succ}}) \rightarrow (p_b \leftrightarrow q_b))\right)
\end{aligned}$$

In order to show the intuition of the introduced formulas we prove the following lemma.

Lemma 4. *Let $\ell \geq 0$ and let $(\mathfrak{S}, s)$ and $(\mathfrak{S}', s')$ be left and right (ℓ, n)-treelike structures. Then the following holds:*

(a) $(\mathfrak{S} \times \mathfrak{S}', \langle s, s' \rangle) \models \mathsf{eq}_{\ell,n}$ *iff* $\mathsf{val}(\mathfrak{S}, s) = \mathsf{val}(\mathfrak{S}', s')$.
(b) $(\mathfrak{S} \times \mathfrak{S}', \langle s, s' \rangle) \models \mathsf{first}_{\ell,n}$ *iff* $\mathsf{val}(\mathfrak{S}, s) = \mathsf{val}(\mathfrak{S}', s') = 0$.
(c) $(\mathfrak{S} \times \mathfrak{S}', \langle s, s' \rangle) \models \mathsf{last}_{\ell,n}$ *iff* $\mathsf{val}(\mathfrak{S}, s) = \mathsf{val}(\mathfrak{S}', s') = \mathsf{Tower}(\ell+1, n) - 1$.
(d) $(\mathfrak{S} \times \mathfrak{S}', \langle s, s' \rangle) \models \mathsf{succ}_{\ell,n}$ *iff* $\mathsf{val}(\mathfrak{S}', s') = \mathsf{val}(\mathfrak{S}, s) + 1$

Now we give a family of formulas $\varphi_{\ell,n}$ with the idea that every model of $\varphi_{\ell,n}$ is the product of a left (ℓ, n)-treelike structure and a right (ℓ, n)-treelike structure with the same value.

Definition 5. *Set $\varphi_{0,n} = \mathsf{eq}_{0,n} \wedge \boxminus \bot \wedge \boxbar \bot$ and define $\varphi_{\ell,n}$, by induction on ℓ, as the conjunction of the following formulas:*

(1) $\diamondminus \diamondbar (\varphi_{\ell-1,n} \wedge \mathsf{first}_{\ell-1,n})$
(2) $\boxminus \diamondbar \varphi_{\ell-1,n}$
(3) $\boxbar \diamondminus \varphi_{\ell-1,n}$
(4) $\boxminus (\boxbar \neg \mathsf{last}_{\ell-1,n} \rightarrow \diamondbar \mathsf{succ}_{\ell-1,n})$
(5) $\boxminus \boxbar (\mathsf{eq}_{\ell-1,n} \rightarrow \bigwedge_{p_g \in \mathsf{P_{aux}}} (p_g \leftrightarrow q_g))$
(6) $\boxminus \boxbar (((p_{\mathsf{succ}} \vee \overleftarrow{p}_{\mathsf{prec}}) \rightarrow \neg p_b) \wedge ((\overleftarrow{p}_{\mathsf{succ}} \vee p_{\mathsf{prec}}) \rightarrow p_b)))$
(7) $\boxminus \boxbar (\mathsf{succ}_{\ell-1,n} \rightarrow \bigwedge_{x \in \{\mathsf{succ}, \mathsf{prec}\}} ((q_x \vee \overleftarrow{q}_x) \rightarrow \overleftarrow{p}_x) \wedge (\overleftarrow{p}_x \rightarrow (\overleftarrow{q}_x \vee q_x)))$
(8) $\diamondminus \diamondbar (p_{\mathsf{succ}} \vee \overleftarrow{p}_{\mathsf{succ}}) \;\wedge\; \diamondminus \diamondbar (p_{\mathsf{prec}} \vee \overleftarrow{p}_{\mathsf{prec}})$

Some remarks regarding the intuition of the formulas are appropriate. In the following explanation we will, in analogy to left and right structures, distinguish left and right worlds.

Formulas (2) and (3) together imply inductively condition (a) from the definiton of (ℓ, n)-treelike structures (every successor is $(\ell-1, n)$-treelike). Condition (b), the existence of successor worlds for each value $k \in [0, \mathsf{Tower}(\ell+1, n) - 1]$, is enforced by induction on k: Formula (1) enforces a left $(\ell-1, n)$-treelike structure with value 0, thus establishing the induction base. Formula (4) enforces for every left world with value k a right world with value $k+1$. Formula (3) enforces a left world having the same value $k+1$; this yields the induction step. Formula (5) enforces condition (c). The remaining conditions (d)-(g) from the definiton of (ℓ, n)-treelike structures can be reformulated as follows:

(i) If a world satisfies p_{succ} or $\overleftarrow{p}_{\mathsf{prec}}$ (resp., $\overleftarrow{p}_{\mathsf{succ}}$ or p_{prec}), then it does not satisfy p_b (resp., it satisfies p_b).
(ii) If p_{succ} or $\overleftarrow{p}_{\mathsf{succ}}$ (resp., p_{prec} or $\overleftarrow{p}_{\mathsf{prec}}$) is satisfied in a left world of value $k > 0$, then $\overleftarrow{p}_{\mathsf{succ}}$ (resp., $\overleftarrow{p}_{\mathsf{prec}}$) is satisfied in all left worlds with value $k-1$.

(iii) If $p^{\leftarrow}_{\mathsf{succ}}$ (resp., $p^{\leftarrow}_{\mathsf{prec}}$) is satisfied in a left world of value $k < \mathsf{Tower}(\ell, n) - 1$, then $p^{\leftarrow}_{\mathsf{succ}}$ or p_{succ} (resp., $p^{\leftarrow}_{\mathsf{prec}}$ or p_{prec}) is satisfied in every left world of value $k+1$.

(iv) There is a successor world satisfying either p_{succ} or $p^{\leftarrow}_{\mathsf{succ}}$ (resp. p_{prec} or $p^{\rightarrow}_{\mathsf{prec}}$).

Clearly, (i) (resp. (iv)) is expressed by formula (6) (resp. (8)). Finally, formula (5) and (7) yield (ii) and (iii). For instance, if a left world with value $k > 0$ satisfies $p^{\leftarrow}_{\mathsf{succ}}$ or p_{succ}, then by formula (5) the corresponding right world satisfies $q^{\leftarrow}_{\mathsf{succ}}$ or q_{succ}. Formula (7) implies that $p^{\leftarrow}_{\mathsf{succ}}$ is satisfied in every left world with value $k-1$. We are now ready to present our main theorem.

Theorem 6. *For every $\ell \geq 0$ the following holds:*

(a) $(\mathfrak{S} \times \mathfrak{S}', \langle s, s' \rangle) \models \varphi_{\ell,n}$ *iff* $(\mathfrak{S}, s)$ *and* $(\mathfrak{S}', s')$ *are* (ℓ, n)-*treelike structures with* $\mathsf{val}(\mathfrak{S}, s) = \mathsf{val}(\mathfrak{S}', s')$.

(b) $|\varphi_{\ell,n}| \leq 3^{\ell} \cdot \mathsf{poly}(\ell, n)$ *and the formula* $\varphi_{\ell,n}$ *is computable in time* $3^{\ell} \cdot \mathsf{poly}(\ell, n)$.

(c) The switching depth of $\varphi_{\ell,n}$ *is* ℓ.

By making use of the models that are be enforced by Theorem 6, we can finally prove a nonelementary lower bound for $\mathbf{K}^2_{\mathsf{id}}$-SAT via a standard reduction from an appropriately chosen tiling problem. Let ℓ-NEXP = $\mathsf{NTIME}(\mathsf{Tower}(\ell, \mathsf{poly}(n)))$.

Proposition 7. *For each* $\ell \geq 1$, $\mathbf{K}^2_{\mathsf{id}}$-SAT *restricted to formulas of switching depth* ℓ *is* ℓ-NEXP-*hard under polynomial time reductions. In particular,* $\mathbf{K}^2_{\mathsf{id}}$-SAT *is nonelementary.*

The following theorem is an immediate consequence of Proposition 1 and Proposition 7.

Corollary 8. *For each* $\ell \geq 1$, $\mathbf{K}^2$-SAT *restricted to formulas of switching depth* ℓ *is* ℓ-NEXP-*hard under polynomial time reductions. In particular,* $\mathbf{K}^2$-SAT *is nonelementary.*

IV. Hardness Results for $\mathbf{K4} \times \mathbf{K}$ and $\mathbf{S5}_2 \times \mathbf{K}$

In this section, we prove further nonelemementary lower bound results for the satisfiability problem of two-dimensional modal logics on restricted classes of frames. We hereby close nonelementary complexity gaps that were stated as open problems in [19]. Although in [19] uninterpreted product models for these logics are of interest, we prove our lower bounds for the id-interpretation only: For each of the logics studied here, the id-interpretation case can be reduced in polynomial time to the uninterpreted case in analogy to Proposition 1.

We define the following logics:

- $\mathbf{K4} \times \mathbf{K}$: Two-dimensional unimodal logic restricted to product models $\mathfrak{S} \times \mathfrak{S}'$, where $\mathfrak{F}(\mathfrak{S})$ is *transitive*.
- $\mathbf{S5} \times \mathbf{K}$: Two-dimensional unimodal logic restricted to product models $\mathfrak{S} \times \mathfrak{S}'$ such that if $\mathfrak{F}(\mathfrak{S}) = (W, \equiv)$, then $\equiv$ is an *equivalence relation*.
- $\mathbf{S5}_2 \times \mathbf{K}$: Two-dimensional modal logic that is bimodal in the first dimension and unimodal in the second dimension restricted to models $\mathfrak{S} \times \mathfrak{S}'$ such that if $\mathfrak{F}(\mathfrak{S}) = (W, \equiv_1, \equiv_2)$, then both $\equiv_1$ and $\equiv_2$ are equivalence relations.

Let us start with $\mathbf{K4} \times \mathbf{K}$. We adapt the straightforward reduction from $\mathbf{K}$ to $\mathbf{K4}$ to the two-dimensional case. When following a transition in a $\mathbf{K4}$-frame one has no control over how far one is actually going due to transitivity of the frame. The idea for the reduction is to introduce additional propositions $h_0, \ldots, h_n$ and enforce *levels* in the models. Intuitively, h_i is true in w' precisely when w' is in level i seen from the world w where the formula is evaluated. Following a transition is then restricted to increase the level only by 1.

Let φ be a unimodal $\mathbf{K}^2$-formula with $\mathsf{rank}_1(\varphi) = r$ and define for every $0 \leq k \leq r$ the translation function t_k by taking

$$\begin{array}{rcl} t_k(p) & = & H_k \wedge p \\ t_k(\neg\psi) & = & H_k \wedge \neg t_k(\psi) \\ t_k(\psi_1 \wedge \psi_2) & = & t_k(\psi_1) \wedge t_k(\psi_2) \\ t_k(\varPhi\psi) & = & \varPhi t_k(\psi) \\ t_k(\diamondminus\psi) & = & H_k \wedge \diamondminus(H_{k+1} \wedge t_{k+1}(\psi)) \end{array}$$

where $H_k \stackrel{\text{def}}{=} h_k \wedge \bigwedge_{i \neq k} \neg h_i$ and $k < r$ in the definition of $t_k(\diamondminus\psi)$. We show that the translation is satisfiability preserving. More precisely, we prove the following lemma.

Lemma 9. *For every unimodal* $\mathbf{K}^2$-*formula* φ *we have:* φ *is* id-*satisfiable in* $\mathbf{K}^2$ *iff* $t_0(\varphi)$ *is* id-*satisfiable in* $\mathbf{K4} \times \mathbf{K}$.

It is easy to see that Lemma 9 provides a reduction of $\mathbf{K}^2_{\mathsf{id}}$-SAT to id-satisfiability in $\mathbf{K4} \times \mathbf{K}$. Finally, an adaption of Proposition 1 to the logic $\mathbf{K4} \times \mathbf{K}$ together with Proposition 7 yields the following result.

Theorem 10. *Satisfiability in* $\mathbf{K4} \times \mathbf{K}$ *is nonelementary.*

Next, we study combinations of $\mathbf{K}$ with $\mathbf{S5}$ and $\mathbf{S5}_2$. It is well-known that the complexity for checking satisfiability jumps from NP for $\mathbf{S5}$ to PSPACE for $\mathbf{S5}_2$. We will show that also the complexity for deciding satisfiability in the product logics $\mathbf{S5} \times \mathbf{K}$ and $\mathbf{S5}_2 \times \mathbf{K}$, respectively, differs. In particular, we will again reduce from $\mathbf{K}^2_{\mathsf{id}}$-SAT in order to show a nonelementary lower bound for the latter logic, which is in sharp contrast to the following result by Marx [20].

Theorem 11 ([20]). *Satisfiability in* $\mathbf{S5} \times \mathbf{K}$ *is* NEXP-*complete.*

PSPACE-hardness for satisfiability in $\mathbf{S5}_2$ is shown by a straightforward reduction from $\mathbf{K}$ [7]. We adapt this reduction to the two-dimensional case by defining a translation † by

$$\begin{array}{rcl} q^{\dagger} & = & p^* \wedge q \\ (\varphi_1 \wedge \varphi_2)^{\dagger} & = & \varphi_1^{\dagger} \wedge \varphi_2^{\dagger} \\ (\neg\varphi)^{\dagger} & = & p^* \wedge \neg(\varphi^{\dagger}) \\ (\varPhi\varphi)^{\dagger} & = & \varPhi(\varphi^{\dagger}) \\ (\diamondminus\varphi)^{\dagger} & = & p^* \wedge \diamondminus_1(\neg p^* \wedge \diamondminus_2(p^* \wedge \varphi^{\dagger})) \end{array}$$

where $\diamondminus_1$ and $\diamondminus_2$ refer to the two modalities in $\mathbf{S5}_2$ and p^* is a fresh propositional variable in the left signature. Intuitively, *one* transition in $\mathbf{K}$ is simulated by *two* transitions in $\mathbf{S5}_2$. This is possible since the composition of two equivalence relations is neither symmetric nor transitive in general and using the fresh variable p^* we can enforce a non-trivial transition, i.e., no loops. It can be proven along the lines of the proof in [7] that † preserves id-satisfiability.

Lemma 12. *For every unimodal* $\mathbf{K}^2$*-formula* φ *we have:* φ *is* id*-satisfiable in* $\mathbf{K}^2$ *iff* $\varphi^\dagger$ *is* id*-satisfiable in* $\mathbf{S5}_2 \times \mathbf{K}$.

The following theorem is an immediate consequence of Lemma 12, Proposition 7, and an adaption of Proposition 1 to $\mathbf{S5}_2 \times \mathbf{K}$.

Theorem 13. *Satisfiability in* $\mathbf{S5}_2 \times \mathbf{K}$ *is nonelementary.*

V. Feferman-Vaught Decompositions for Products

Having enforced nonelementarily branching trees with small 2-dimensional unimodal formulas (Theorem 6) allows us to prove a nonelementary lower bound for the sizes of Feferman-Vaught decompositions for 2-dimensional unimodal logic. Without making this explicit in the statement, our lower bound is more general than the nonelementary lower bound for 2-dimensional unimodal logic from [17] in the following sense. We provide a family of small formulas which are "inherently hard to decompose": When assuming, by contradiction, the existence of small decompositions for our formulas, *any* model for them can be used to deduce the desired contradiction, whereas in [17] *appropriately chosen models* had to be defined for this. Our proof strategy is similar to the proof of Theorem 5.1. in [9].

Theorem 14. *Feferman-Vaught decompositions for unimodal logic w.r.t. asynchronous product are inherently nonelementary. More precisely, for every elementary function* $f(n)$ *there exists* $\ell \geq 1$ *such that the unimodal* $\mathbf{K}^2$*-formula* $\varphi_{\ell,2}$ *from Definition 5 has no decomposition* $\mathcal{D}_\ell$ *in the sense of Theorem 2 with* $|\mathcal{D}_\ell| \leq f(|\varphi_{\ell,2}|)$*. The same lower bound holds when relativized to product structures* $\mathfrak{T}\times\mathfrak{T}'$*, where* $\mathfrak{F}(\mathfrak{T})$ *and* $\mathfrak{F}(\mathfrak{T}')$ *are finite trees.*

Proof: Assume by contradiction that there is an elementary function $f : \mathbb{N} \to \mathbb{N}$ such that for each $\ell \geq 1$ there is a decomposition $\mathcal{D}_\ell = (\Psi_1^{(\ell)}, \Psi_2^{(\ell)}, \beta_\ell)$ of $\varphi_{\ell,2}$ in the sense of Theorem 2 with $|\mathcal{D}_\ell| \leq f(|\varphi_{\ell,2}|)$. In particular, $|\beta_\ell| \leq f(|\varphi_{\ell,2}|)$. Since $|\varphi_{\ell,2}| \leq \exp(\ell)$ by Theorem 6(b), there exists an elementary function g such that $|\beta_\ell| \leq g(\ell)$ for all $\ell \geq 0$. Thus, there exists an $h \geq 0$ with $2^{g(h-1)} < \mathsf{Tower}(h,2)$; let us fix such an h.

By Theorem 6(a), $\varphi_{h,2}$ is id-satisfiable. Assume that $(\mathfrak{S} \times \mathfrak{S}', \langle w, w'\rangle) \models \varphi_{h,2}$ for some left pointed structure $(\mathfrak{S}, w)$ and some right pointed structure $(\mathfrak{S}', w')$. By Theorem 6(a), $(\mathfrak{S}, w)$ and $(\mathfrak{S}', w')$ are $(h,2)$-treelike and $\mathsf{val}(\mathfrak{S}, w) = \mathsf{val}(\mathfrak{S}', w') = k$ for some $k \in [0, \mathsf{Tower}(h+1,2)-1]$. By the definition of $(h,2)$-treelike structures, there exist for each $i \in [0, \mathsf{Tower}(h,2)-1]$ worlds $v_i \in N_{\mathfrak{S}}(w)$ and $v_i' \in N_{\mathfrak{S}'}(w')$ such that $(\mathfrak{S}, v_i)$ and $(\mathfrak{S}', v_i')$ are $(h-1,2)$-treelike and $\mathsf{val}(\mathfrak{S}, v_i) = \mathsf{val}(\mathfrak{S}', v_i') = i$. Also note that

$$(\mathfrak{S} \times \mathfrak{S}', \langle v_i, v_j'\rangle) \models \varphi_{h-1,2} \;\Leftrightarrow\; i = j \tag{1}$$

for all $i, j \in [0, \mathsf{Tower}(h,2)-1]$. Consider our decomposition $\mathcal{D}_{h-1} = (\Psi_1^{(h-1)}, \Psi_2^{(h-1)}, \beta_{h-1})$ of $\varphi_{h-1,2}$. Assume that $\Psi_1^{(h-1)} = \{\psi_j \mid j \in J\}$ and $\Psi_2^{(h-1)} = \{\psi_j' \mid j \in J'\}$. Recall that β_{h-1} is a positive boolean formula with variables from $X = \{x_j \mid j \in J\} \cup \{x_j' \mid j \in J'\}$ and that $|\beta_{h-1}| \leq g(h-1)$. Hence, $|X| \leq g(h-1)$.

For each $r \in [0, \mathsf{Tower}(h,2)-1]$ we define a truth assignment $\mu_r : X \to \{0,1\}$ as follows:

$$\mu_r(x_j) = 1 \quad\Leftrightarrow\quad (\mathfrak{S}, v_r) \models \psi_j$$
$$\mu_r(x_j') = 1 \quad\Leftrightarrow\quad (\mathfrak{S}', v_r') \models \psi_j'$$

Since for β_{h-1} there are $2^{|X|} \leq 2^{g(h-1)} < \mathsf{Tower}(h,2)$ many truth assignments, there exist $0 \leq r < s < \mathsf{Tower}(h,2)$ with $\mu_r = \mu_s$. Since $(\mathfrak{S} \times \mathfrak{S}', \langle v_r, v_r'\rangle) \models \varphi_{h-1,2}$, this implies $(\mathfrak{S} \times \mathfrak{S}', \langle v_r, v_s'\rangle) \models \varphi_{h-1,2}$. But this contradicts (1).

Our lower bound also holds when only products of finite trees are allowed as models, since for every ℓ, n, there exists an (ℓ, n)-treelike structure $\mathfrak{S}$ such that $\mathfrak{F}(\mathfrak{S})$ is a finite tree (of height ℓ). ∎

Note that the lower bound from Theorem 14 would even hold if we defined the size of a decomposition $(\Psi_1, \ldots, \Psi_d, \beta)$ as the size of the boolean formula β only (and not accounting for the sizes of the Ψ_i); the same proof works for this variant. In contrast to [17] the proof of Theorem 14 allows to derive nonelementary lower bounds on decompositions for any decomposable logic (in the sense of Theorem 2) that is at least as expressive as unimodal logic and only elementarily less succinct than unimodal logic.

Corollary 15. *Every logic that is at least as expressive as and at most elementary less succinct as unimodal logic does not have elementary sized Feferman-Vaught decompositions with respect to asynchronous product.*

VI. Feferman-Vaught Decompositions for Sums

So far, we only considered Feferman-Vaught decompositions for asynchronous products. Another important and natural operation on structures is the disjoint sum. Let us fix a relational signature τ and for $i \in [1,d]$ let $\mathfrak{S}_i = (D_i, \{P_{i,a} \mid a \in \tau\})$ be a τ-structure such that $D_i \cap D_j = \emptyset$ for $i \neq j$. Let $A_i \notin \tau$ ($i \in [1,d]$) be a fresh unary predicate symbol. The the *disjoint sum* $\sum_{i=1}^d \mathfrak{S}_i$ is the following structure over the signature $\tau \cup \{A_1, \ldots, A_d\}$:

$$\sum_{i=1}^d \mathfrak{S}_i \overset{\text{def}}{=} \Big(\bigcup_{i\in[1,d]} D_i, \{\bigcup_{i\in[1,d]} P_{i,a} \mid a \in \tau\} \cup \{D_i \mid i \in [1,d]\}\Big).$$

Here, $\bigcup_{i\in[1,d]} P_{i,a}$ is the interpretation for $a \in \tau$ and D_i is the interpretation for the fresh symbol A_i. The following result is again classical [11], [12].

Theorem 16. *For every* FO^k*-sentence* φ *over the signature* $\tau \uplus \{A_1, \ldots, A_d\}$ *one can compute a tuple* $(\Psi_1, \ldots, \Psi_d, \beta)$*, where each* $\Psi_i = \{\psi_i^j \mid j \in J_i\}$ *is a finite set of* FO^k*-sentences over the signature* τ *and where* β *is a positive boolean formula with variables from* $X = \{x_i^j \mid i \in [1,d], j \in J_i\}$ *such that for all* τ*-structures* $\mathfrak{S}_1, \ldots, \mathfrak{S}_d$*:*

$$\sum_{i=1}^d \mathfrak{S}_i \models \varphi \qquad \text{if and only if} \qquad \mu \models \beta.$$

Here, $\mu : X \to \{0,1\}$ *is defined by:* $\mu(x_i^j) = 1$ *iff* $\mathfrak{S}_i \models \psi_i^j$.

The following result is a simple corollary of Corollary 15.

Corollary 17. *For every $k \geq 3$, there is no elementary function $f(n)$ such that every FO^k-formula φ has a Feferman-Vaught decomposition w.r.t. disjoint sum of size $f(|\varphi|)$.*

Corollary 17 raises the question whether even Feferman-Vaught decompositions for FO^2 w.r.t. disjoint sum become nonelementary. We give a negative answer to this question.

Theorem 18. *The following is computable in doubly exponential time:*
INPUT: An FO^2-sentence φ over $\tau \uplus \{A_1, \dots, A_d\}$
OUTPUT: A decomposition $(\Psi_1, \dots, \Psi_d, \beta)$, where $\Psi_i = \{\psi_i^j \mid j \in J_i\}$ is a finite set of FO^2-sentences over τ and β is a positive boolean formula with variables from $X = \{x_i^j \mid i \in [1,d], j \in J_i\}$ such that for all τ-structures $\mathfrak{S}_1, \dots, \mathfrak{S}_d$:

$$\sum_{i=1}^{d} \mathfrak{S}_i \models \varphi \qquad \text{if and only if} \qquad \mu \models \beta.$$

Here, $\mu : X \to \{0,1\}$ is defined by: $\mu(x_i^j) = 1$ iff $\mathfrak{S}_i \models \psi_i^j$.

We will prove Theorem 18 only for the case $d = 2$. Hence, let us fix a signature τ of relational symbols and let $A_1, A_2 \notin \tau$ be two additional unary symbols. Let $\mathfrak{S}_1$ and $\mathfrak{S}_2$ be relational structures over the signature τ.

We define a partial order $\preceq$ on the set of all first-order formulas by setting $\psi_1 \preceq \psi_2$ if and only if ψ_1 is a subformula of ψ_2. For a formula φ we denote with $\mathcal{Q}_\varphi$ the set of all subformulas of φ that start with a quantifier. With $\mathcal{Q}_\varphi^{\mathsf{cl}}$ we denote the set of those formulas in $\mathcal{Q}_\varphi$ that are closed, i.e., do not have free variables. In a formula $\exists x : A_i(x) \wedge \psi$ (resp. $\forall x : A_i(x) \to \psi$), where $i \in \{1,2\}$, we say that x *is relativized to* A_i, and for better readability we write $\exists x \in A_i : \psi$ (resp. $\forall x \in A_i : \psi$) for that formula.

A formula φ over the signature $\tau \cup \{A_1, A_2\}$ is called *pure* if φ is a boolean combination of formulas $\varphi_1, \dots, \varphi_n$ such that for every $1 \leq i \leq n$ there exists $j \in \{1,2\}$ such that for every $(Qx : \psi) \in \mathcal{Q}_{\varphi_i}$ (where $Q \in \{\exists, \forall\}$), x is relativized to A_j in $Qx : \psi$. Equivalently, φ is pure, if for all $(Q_1 x : \psi_1), (Q_2 y : \psi_2) \in \mathcal{Q}_\varphi$ with $(Q_1 x : \psi_1) \preceq (Q_2 y : \psi_2)$, x is relativized in $(Q_1 x : \psi_1)$ to the same A_i as y in $(Q_2 y : \psi_2)$. To prove Theorem 18 (for $d = 2$), it suffices to transform an FO^2 sentence over the signature $\tau \cup \{A_1, A_2\}$ in doubly exponential time into an equivalent pure FO^2-sentence over the signature $\tau \cup \{A_1, A_2\}$.

A formula φ over the signature $\tau \cup \{A_1, A_2\}$ is called *almost pure* if it satisfies the following conditions:

- For all $(Qx : \psi) \in \mathcal{Q}_\varphi$, x is relativized in $(Qx : \psi)$ to either A_1 or A_2.
- If $(Q_1 x : \psi_1), (Q_2 y : \psi_2) \in \mathcal{Q}_\varphi$ with $(Q_1 x : \psi_1) \preceq (Q_2 y : \psi_2)$, then either x is relativized in $(Q_1 x : \psi_1)$ to the same A_i as y in $(Q_2 y : \psi_2)$, or there exists $\theta \in \mathcal{Q}_\varphi^{\mathsf{cl}}$ with $(Q_1 x : \psi_1) \preceq \theta \prec (Q_2 y : \psi_2)$.

In other words, whenever a chain of subformulas $(Q_1 x : \psi_1) \preceq (Q_2 y : \psi_2) \preceq \varphi$ does not satisfy the pureness condition, then $(Q_1 x : \psi_1)$ occurs within a proper subsentence of $(Q_2 y : \psi_2)$ that moreover starts with a quantifier. Clearly, every pure formula is almost pure. Vice versa, we have:

Lemma 19. *From a given almost pure formula φ over the signature $\tau \cup \{A_1, A_2\}$ one can compute a logically equivalent pure formula φ' of size $2^{|\mathcal{Q}_\varphi^{\mathsf{cl}}|} \cdot O(|\varphi|)$. If φ is an FO^2-formula then φ' is an FO^2-formula as well.*

Proof: The idea is to replace the topmost occurrences of sentences from the set $\mathcal{Q}_\varphi^{\mathsf{cl}}$ by truth values in all possible ways in a big disjunction over all possible truth assignments. Since sentences from $\mathcal{Q}_\varphi^{\mathsf{cl}}$ may also violate the pureness condition, we have to iterate this replacement step.

Let φ be almost pure and let $\mathcal{F}$ be the set of all mappings from $\mathcal{Q}_\varphi^{\mathsf{cl}}$ to $\{\mathsf{true}, \mathsf{false}\}$. For $f \in \mathcal{F}$ and a formula θ let $\theta[f]$ be the formula that results from θ by replacing every $\preceq$-maximal formula ψ from the set $(\mathcal{Q}_\theta \setminus \{\theta\}) \cap \mathcal{Q}_\varphi^{\mathsf{cl}}$ by the truth value $f(\psi)$. Then, we define φ' as the disjunction

$$\bigvee_{f \in \mathcal{F}} \Big(\varphi[f] \wedge \bigwedge_{\psi \in \mathcal{Q}_\varphi^{\mathsf{cl}}} \big(f(\psi) \leftrightarrow \psi[f] \big) \Big).$$

Clearly, φ' is equivalent to φ and φ' is pure. ∎

Lemma 20. *From a given FO^2-formula $\varphi(x)$ over the signature $\tau \cup \{A_1, A_2\}$ with at most one free variable x, one can compute FO^2-formulas $\varphi'(x)$ and $\varphi''(x)$ of size $2^{O(|\varphi|^2)}$ such that the following holds for all structures $\mathfrak{S}_1$ and $\mathfrak{S}_2$ over the signature τ.*

- *$Qx \in A_1 : \varphi'(x)$ and $Qx \in A_2 : \varphi''(x)$ are almost pure (where $Q \in \{\forall, \exists\}$).*
- *For all $a \in \mathfrak{S}_1$, $\mathfrak{S}_1 + \mathfrak{S}_2 \models \varphi(a)$ iff $\mathfrak{S}_1 + \mathfrak{S}_2 \models \varphi'(a)$.*
- *For all $a \in \mathfrak{S}_2$, $\mathfrak{S}_1 + \mathfrak{S}_2 \models \varphi(a)$ iff $\mathfrak{S}_1 + \mathfrak{S}_2 \models \varphi''(a)$.*

Moreover, $|\mathcal{Q}_{\varphi'(x)}^{\mathsf{cl}}| \in 2^{O(|\varphi|)}$ and $|\mathcal{Q}_{\varphi''(x)}^{\mathsf{cl}}| \in 2^{O(|\varphi|)}$.

Proof: Let us construct the formula $\varphi'(x)$ ($\varphi''(x)$ is constructed analogously) by induction over the structure of the formula $\varphi(x)$. The case that the top-most operator in $\varphi(x)$ is a boolean operator is clear, e.g., set $(\varphi_1 \wedge \varphi_2)' = \varphi_1' \wedge \varphi_2'$.

Now, assume that $\varphi(x) = \exists y : \psi(x,y)$. Since $\varphi(x)$ is an FO^2-formula, the formula $\psi(x,y)$ can be obtained from a positive boolean formula $B(p_1, \dots, p_k)$ by replacing every propositonal variable p_i by

(a) some $\alpha(x) \in \mathcal{Q}_\varphi$, where only x may occur freely, or by
(b) some $\beta(y) \in \mathcal{Q}_\varphi$, where only y may occur freely, or by
(c) a possibly negated atomic formula (i.e., a literal) that involves a subset of the variables $\{x, y\}$.

Let $\psi'(x,y)$ be the formula that results from $\psi(x,y)$ by replacing every subformula $\alpha(x)$ (resp. $\beta(y)$) of type (a) (resp. (b)) by $\alpha'(x)$ (resp. $\beta'(y)$). We can write B as a DNF formula $B = \bigvee_{i=1}^{r} B_i$ of size $2^{O(|B|)}$, where every B_i is a conjunction of formulas of the types (a)–(c). Hence, we can write B_i as

$$B_i = \alpha_i(x) \wedge \beta_i(y) \wedge \gamma_i(x,y),$$

where α_i is a conjunction of type-(a) formulas, β_i is a conjunction of type-(b) formulas, and $\gamma_i(x,y)$ is a conjunction of type-(c) formulas.

Clearly, over a structure $\mathfrak{S}_1 + \mathfrak{S}_2$, the formula $\exists y : \psi(x,y)$ is equivalent to $\exists y \in A_1 : \psi(x,y) \vee \exists y \in A_2 : \psi(x,y)$, i.e., to

$$\exists y \in A_1 : \psi(x,y) \vee \bigvee_{i=1}^{r} \exists y \in A_2 : (\alpha_i(x) \vee \beta_i(y) \vee \gamma_i(x,y)).$$

By induction, for all $x \in \mathfrak{S}_1$, this formula is equivalent to

$$\exists y \in A_1 : \psi'(x,y) \vee \tag{2}$$

$$\bigvee_{i=1}^{r} \exists y \in A_2 : (\alpha'_i(x) \wedge \beta''_i(y) \wedge \gamma_i(x,y)). \tag{3}$$

In line (3), every occurrence of a literal in $\gamma_i(x,y)$, in which both x and y occur, can be replaced either by true (if the literal is negative) or false (if the literal is positive). The reason for this is that no atomic relations of $\mathfrak{S}_1 + \mathfrak{S}_2$ involve both elements of $\mathfrak{S}_1$ and $\mathfrak{S}_2$. We therefore obtain an equivalent formula of the form

$$\exists y \in A_1 : \psi'(x,y) \vee \bigvee_{i=1}^{r} (\alpha'_i(x) \wedge \delta_{i,1}(x) \wedge \exists y \in A_2 : (\beta''_i(y) \wedge \delta_{i,2}(y))).$$

Here $\delta_{i,1}(x)$ (resp. $\delta_{i,2}(y)$) is the conjunction of all literals in $\gamma_i(x,y)$ that only involve the variable x (resp. y). Let $\varphi'(x)$ be the above formula. We have to show that the formula

$$\exists x \in A_1 \left(\begin{array}{l} \exists y \in A_1 : \psi'(x,y) \vee \\ \bigvee_{i=1}^{r} (\alpha'_i(x) \wedge \delta_{i,1}(x) \wedge \exists y \in A_2 : (\beta''_i(y) \wedge \delta_{i,2}(y))) \end{array} \right)$$

is almost pure. This follows inductively from the fact that $\exists x \in A_1 \exists y \in A_1 : \psi'(x,y)$, $\exists x \in A_1 : \alpha'_i(x)$, and $\exists y \in A_2 : \beta''_i(y)$ are almost pure, and the fact that $\exists y \in A_2 : (\beta''_i(y) \wedge \delta_{i,2}(y))$ is closed. This concludes the case $\varphi(x) = \exists y : \psi(x,y)$. The case $\varphi(x) = \forall y : \psi(x,y)$ can be treated analogously.

If we allow $\wedge$'s and $\vee$'s of arbitrary width, then the depth (i.e., the height of the syntax tree) of $\varphi'(x)$ is bounded by $O(|\varphi|)$. Due to forming CNFs and DNFs, the width of $\wedge$'s and $\vee$'s can be bounded by $2^{|\varphi(x)|}$. Hence, the syntax tree of $\varphi'(x)$ has height $O(|\varphi|)$ and branching degree $2^{|\varphi(x)|}$, and therefore has $2^{O(|\varphi|^2)}$ many nodes. Replacing $\wedge$'s and $\vee$'s of arbitrary width $\leq 2^{|\varphi(x)|}$ by 2-ary $\wedge$'s and $\vee$'s only multiplies the number of nodes by $2^{|\varphi(x)|}$. Hence, $\varphi'(x)$ is of size $2^{O(|\varphi|^2)}$.

For the bound $|\mathcal{Q}^{\mathsf{cl}}_{\varphi'(x)}| \in 2^{O(|\varphi|)}$ note that in the above construction, the number of closed subformulas that start with a quantifier is increased by at most $r+1$ (due to the formulas $\exists y \in A_2 : (\beta''_i(y) \wedge \delta_{i,2}(y))$ for $i \in [1,r]$). Since r is exponential in the size of the boolean formula B, the bound $|\mathcal{Q}^{\mathsf{cl}}_{\varphi'(x)}| \in 2^{O(|\varphi|)}$ follows. ∎

Theorem 21. *From a given closed* FO^2*-formula* φ *over the signature* $\tau \cup \{A_1, A_2\}$ *one can compute a pure closed* FO^2*-formula* ψ *of size* $2^{2^{O(|\varphi|)}}$ *such that for all structures* $\mathfrak{S}_1$ *and* $\mathfrak{S}_2$ *over the signature* τ, $\mathfrak{S}_1 + \mathfrak{S}_2 \models \varphi$ *iff* $\mathfrak{S}_1 + \mathfrak{S}_2 \models \psi$.

Proof: We first apply Lemma 20 to φ and obtain a closed almost pure FO^2-formula θ such that $\mathfrak{S}_1 + \mathfrak{S}_2 \models \varphi$ iff $\mathfrak{S}_1 + \mathfrak{S}_2 \models \theta$. The size of θ is bounded by $2^{O(|\varphi|^2)}$. Finally, we apply Lemma 19 to θ and obtain an equivalent pure FO^2-formula ψ of size $2^{|\mathcal{Q}^{\mathsf{cl}}_\theta|} \cdot O(|\theta|)$. Since $|\theta| \in 2^{O(|\varphi|^2)}$ and $|\mathcal{Q}^{\mathsf{cl}}_\theta| \in 2^{O(|\varphi|)}$ this yields the upper bound $2^{2^{O(|\varphi|)}}$ for the size of ψ. ∎

Let us conclude this section with a (non-matching) lower bound on Feferman-Vaught decompositions for FO^2.

Proposition 22. *There is no function* $f(n) \in o(\sqrt{n})$ *and* $c > 1$ *such that every* FO^2*-formula* φ *has a Feferman-Vaught decompositions w.r.t. disjoint sum of size* $c^{o(\sqrt{|\varphi|})}$.

VII. Gaifman Normal Form

Our technique from the proof of Theorem 18 can be used to prove a doubly exponential upper bound on the size (and construction) of Gaifman normal forms [14]. Let us start with a few definitions.

Let $\mathfrak{S} = (D, \{P_a \mid a \in \tau\})$ be a structure over a relational signature τ. Then the *Gaifman graph* of $\mathfrak{S}$ is the undirected graph $G(\mathfrak{S}) = (D, E)$, where the edge relation E contains a pair $(u,v) \in D \times D$ with $u \neq v$ if and only if there exists a relation P_a $(a \in \tau)$ of arity say n and a tuple $(u_1, \ldots, u_n) \in P_a$ such that $u, v \in \{u_1, \ldots, u_n\}$. For $u, v \in D$, the distance $d_{\mathfrak{S}}(u,v)$ is the length (number of edges) of a shortest path from u to v in $G(\mathfrak{S})$. For a tuple $\overline{u} = (u_1, \ldots, u_n)$ and v, let $d_{\mathfrak{S}}(\overline{u}, v) = \min\{d_{\mathfrak{S}}(u_i, v) \mid 1 \leq i \leq n\}$. For $n \in \mathbb{N}$, the n*-sphere* around $\overline{u}$ is $S_{\mathfrak{S},n}(\overline{u}) = \{v \in D \mid d_{\mathfrak{S}}(\overline{u}, v) \leq n\}$. We write $S_n(\overline{u})$ for $S_{\mathfrak{S},n}(\overline{u})$, if $\mathfrak{S}$ is clear from the context.

Note that for every $n \in \mathbb{N}$, there exists a first-order formula $d_n(\overline{x}, y)$ such that for all structures $\mathfrak{S}$ and all elements $\overline{u}, v$ of $\mathfrak{S}$, $\mathfrak{S} \models d_n(\overline{u}, v)$ if and only if $d_{\mathfrak{S}}(\overline{u}, v) \leq n$. For better readability, we write $d(\overline{x}, y) \leq n$ instead of $d_n(\overline{x}, y)$. The formula $d(\overline{x}, y) > n$ should be understood similarly. In a formula of the form $\exists y : d(\overline{x}, y) \leq r \wedge \psi$ or $\forall y : d(\overline{x}, y) \leq r \rightarrow \psi$, we say that the variable y is *relativized* to $S_r(\overline{x})$. A formula φ is called r*-local around* $\overline{x}$ if for every occurrence of a subformula $(Qy : \psi) \in \mathcal{Q}_\varphi$, the variable y is relativized in $(Qy : \psi)$ to a sphere $S_q(\overline{x})$ for some $q \leq r$. A sentence ψ is called an r*-local Gaifman-sentence* if it is of the form

$$\exists x_1, \ldots, x_n : \bigwedge_{1 \leq i < j \leq n} d(x_i, x_j) > 2q \wedge \bigwedge_{1 \leq i \leq n} \varphi_i(x_i),$$

where every $\varphi_i(x_i)$ is q-local around (the single variable) x_i for some $q \leq r$.

Theorem 23 (Gaifman's theorem [14])**.** *Every first-order formula* $\varphi(\overline{x})$ *is equivalent to a boolean combination* $\psi(\overline{x})$ *of* r*-local formulas around* $\overline{x}$ *and* q*-local Gaifman-sentences for suitable* r *and* q *(that are exponential in the size of* $\varphi(\overline{x})$*).*

We call the formula $\psi(\overline{x})$ from Theorem 23 a *Gaifman normal form* for $\varphi(\overline{x})$. In [9] it was shown that (for FO^4-formulas already) the size of equivalent formulas in Gaifman normal form cannot be bounded elementarily. By using our formulas $\varphi_{\ell,n}$ from Section III and analogous ideas as in [9], we can strengthen the latter result to FO^3.

Proposition 24. *There is no elementary function* $f(n)$ *such that every* FO^3*-formula* φ *has an equivalent formula in Gaifman normal form of size* $f(|\varphi|)$.

Next, we show that for the fragment FO^2 such an elementary (in fact, doubly exponential) bound is possible:

Theorem 25. *Every FO^2-formula $\varphi(x)$ is equivalent to a boolean combination $\psi(x)$ of r-local formulas around x and q-local Gaifman-sentences with $r \leq 3\mathsf{qr}(\varphi)$, $q \leq 6\mathsf{qr}(\varphi)$, and $|\psi| \leq 2^{2^{O(|\varphi|)}}$.*

In Theorem 25, x is a single variable. This is no restriction, since every FO^2-formula can be written as a boolean combination of formulas that (i) start with a quantifier, and (ii) that have at most one free variable. In the rest of this section, all r-local formulas will be r-local around a single variable x. For the proof of Theorem 25 it is useful to define *almost r-local formulas* around x and *almost r-local Gaifman-sentences*. We do this by simultaneous induction:

- Every formula that is built up from atomic formulas and almost p-local Gaifman-sentences (for arbitrary p) using boolean operators and quantifiers relativized to $S_q(x)$ for arbitrary $q \leq r$ is an almost r-local formula around x (hence, every r-local formula around x is almost r-local around x).
- If for some $q \leq r$ every formula $\varphi_i(x_i)$ is almost q-local around x_i ($1 \leq i \leq n$), then the sentence

$$\exists x_1, \ldots, x_n : \bigwedge_{1 \leq i < j \leq n} d(x_i, x_j) > 2q \wedge \bigwedge_{1 \leq i \leq n} \varphi_i(x_i)$$

  is an almost r-local Gaifman-sentence.

For a formula φ, let $G(\varphi)$ be the set of all almost p-local Gaifman-sentences ψ (for arbitrary p) with $\psi \leq \varphi$.

Lemma 26. *From an almost r-local formula $\varphi(x)$ (around x) one can compute a logically equivalent Boolean combination $\varphi'(x)$ of r-local formulas around x and q-local Gaifman sentences. Here, the size of $\varphi'(x)$ is bounded by $2^{|G(\varphi)|} \cdot O(|\varphi|)$ and q is the maximum of all p such that $G(\varphi)$ contains an almost p-local Gaifman sentence.*

Proof: Let $\varphi(x)$ be almost r-local around x and let $\mathcal{F}$ be the set of all mappings from $G(\varphi)$ to $\{\mathsf{true}, \mathsf{false}\}$. For $f \in \mathcal{F}$ and a formula θ let $\theta[f]$ be the formula that results from θ by replacing every $\leq$-maximal formula ψ from the set $(G(\theta) \setminus \{\theta\}) \cap G(\varphi)$ by the truth value $f(\psi)$. Then, we define φ' as the disjunction

$$\bigvee_{f \in \mathcal{F}} \Big(\varphi[f] \wedge \bigwedge_{\psi \in G(\varphi)} \big(f(\psi) \leftrightarrow \psi[f]\big)\Big).$$

Clearly, φ' is equivalent to φ and φ' is r-local around x. ■

Lemma 27. *From an FO^2-formula $\varphi(x)$ with at most one free variable x, one can compute an equivalent almost r-local formula $\varphi^\ell(x)$ of size $2^{O(|\varphi|^2)}$ with $r \leq 3\mathsf{qr}(\varphi)$, $|G(\varphi^\ell)| \leq 2^{O(|\varphi|)}$, and every $\psi \in G(\varphi^\ell)$ is an almost $2r$-local Gaifman sentence.*

Let us finally prove Theorem 25. We first apply Lemma 27 to $\varphi(x)$ and obtain an equivalent almost r-local formula $\theta(x)$ with $|\theta| \leq 2^{O(|\varphi|^2)}$. Moreover $r \leq 3\mathsf{qr}(\varphi)$ and every sentence in $G(\theta)$ is an almost $2r$-local Gaifman sentence. Finally, we apply Lemma 26 to θ and obtain an equivalent Boolean combination $\psi(x)$ of r-local formulas around x and $2r$-local Gaifman sentences. The size of $\psi(x)$ is bounded by $2^{|G(\theta)|} \cdot O(|\theta|)$. Since $|\theta| \leq 2^{O(|\varphi|^2)}$ and $|G(\theta)| \leq 2^{O(|\varphi|)}$, this yields the upper bound $2^{2^{O(|\varphi|)}}$ for the size of $\psi(x)$. □

Finally, we give a (non-matching) lower bound on the size of equivalent formulas in Gaifman normal form for FO^2; the proof is again based on techniques from [9].

Proposition 28. *There is no function $f(n) \in o(\sqrt{n})$ and $c > 1$ such that every FO^2-formula φ has an equivalent formula in Gaifman normal form of size $c^{f(|\varphi|)}$.*

VIII. Open Problems

The main open problem concerns the size of Feferman-Vaught decompositions (w.r.t. disjoint sum) and equivalent formulas in Gaifman normal form for FO^2. For both formalisms, we proved a doubly exponential upper bound and a lower bound of the form $c^{o(\sqrt{n})}$ (for any constant $c > 1$). We conjecture that the upper bound can be improved to a singly exponential bound.

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