Finite Model Reasoning in Horn Description Logics

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Abstract

We study finite model reasoning in expressive Horn description logics (DLs), starting with a reduction of finite ABox consistency to unrestricted ABox consistency. The reduction relies on reversing certain cycles in the TBox, an approach that originated in database theory, was later adapted to the inexpressive DL *DL-Lite*$_\mathcal{F}$, and is shown here to extend to the expressive Horn DL Horn-$\mathcal{ALCFI}$. The model construction used to establish correctness makes the structure of finite models more explicit than existing approaches to finite model reasoning in expressive DLs that rely on solving systems of inequations over the integers. Since the reduction incurs an exponential blow-up, we then develop a dedicated consequence-based algorithm for finite ABox consistency in Horn-$\mathcal{ALCFI}$ that implements the reduction on-the-fly rather than executing it up-front. The algorithm has optimal worst-case complexity and provides a promising foundation for implementations. We next show that our approach can be adapted to finite (positive existential) query answering relative to Horn-$\mathcal{ALCFI}$ TBoxes, proving that this problem is ExpTime-complete in combined complexity and PTime-complete in data complexity. For finite satisfiability and subsumption, we also show that our techniques extend to Horn-$\mathcal{SHIQ}$.

1 Introduction

Many popular expressive description logics (DLs) include both inverse roles and some form of counting such as functionality restrictions. This combination is well-known to result in a loss of the finite model property (FMP) and, consequently, reasoning w.r.t. the class of finite models (*finite model reasoning*) does not coincide with reasoning w.r.t. the class of all models (*unrestricted reasoning*). On the one hand, this distinction is gaining importance because DLs are increasingly used in database applications, where finiteness of models and databases is a central assumption. On the other hand, finite model reasoning is rarely used when DLs are applied in practice, mainly because for many DLs that lack the FMP, no algorithmic approaches to finite model reasoning are known that lend themselves towards efficient implementation.

Among the most widely-known DLs that include both inverse roles and counting are $\mathcal{ALCFI}$, $\mathcal{ALCQI}$, $\mathcal{SHIF}$, and $\mathcal{SHIQ}$, which are prominent fragments of the OWL2 DL ontology language. While finite model reasoning in these DLs is known to have the same complexity as unrestricted reasoning, namely ExpTime-complete (Lutz, Sattler, and Tendera 2005), the algorithmic approaches are rather different when only finite models are admitted. For unrestricted reasoning, there is a wide range of applicable algorithms such as tableau and resolution calculi, which often perform rather well in practical implementations. For finite model reasoning, all known approaches rely on the construction of some system of inequations (Calvanese 1996; Lutz, Sattler, and Tendera 2005) and then solve this system over the integers; the crux is that the system of inequations is of exponential size in the best case, and consequently it is far from obvious how to come up with efficient implementations. This is also true for the two-variable fragment of first-order logic with counting quantifiers (C2), into which the mentioned DLs can be embedded (Pacholski, Szwast, and Tendera 2000; Pratt-Hartmann 2005), that is, all known approaches to finite model reasoning in C2 rely on solving (at least) exponentially large systems of inequations.

Interestingly, the situation is quite different on the other end of the expressive power spectrum. While $\mathcal{SHIQ}$ et al. belong to the family of expressive DLs, *DL-Lite*$_\mathcal{F}$ is a comparably inexpressive DL that emerged from database applications, but also includes both inverse roles and functionality restrictions and thus lacks the FMP. Building on a technique that was developed in database theory by Cosmadakis, Kanellakis, and Vardi to decide the implication of inclusion dependencies and functional dependencies over finite databases (1990), Rosati has shown that finite model reasoning in *DL-Lite*$_\mathcal{F}$ can be reduced in polynomial time to unrestricted reasoning in *DL-Lite*$_\mathcal{F}$ (2008). The reduction is conceptually simple and relies on completing the TBox by finding certain cyclic inclusions and then 'reversing' them. For example, the cycle

$$\exists r^- \sqsubseteq \exists s \qquad \exists s^- \sqsubseteq \exists r \qquad (\mathsf{funct}\ r^-) \qquad (\mathsf{funct}\ s^-)$$

that consists of existential restrictions in the 'forward direction' and functionality statements in the 'backwards direction' would lead to the addition of the reversed cycle

$$\exists s \sqsubseteq \exists r^- \qquad \exists r \sqsubseteq \exists s^- \qquad (\mathsf{funct}\ r) \qquad (\mathsf{funct}\ s).$$

As a consequence, finite model reasoning in DL-Lite$_\mathcal{F}$ does not require new algorithmic techniques and can be implemented as efficiently as unrestricted reasoning. The reduction also makes explicit the *logical consequences of finite models*; in a sense, it can be viewed as an explicit axiomatization of finiteness.

Given that DL-Lite$_\mathcal{F}$ is only a very small fragment of $\mathcal{ALCFI}$ and $\mathcal{SHIQ}$, this situation raises the question whether the cycle reversion technique extends also to larger fragments of these DLs. In particular, *DL-Lite*$_\mathcal{F}$ is a 'Horn DL', and such logics are well-known to be algorithmically much more well-behaved than non-Horn DLs such as $\mathcal{ALCFI}$ (Baader, Brandt, and Lutz 2005; Calvanese et al. 2007). Maybe, then, this is the reason why cycle reversion works for *DL-Lite*$_\mathcal{F}$?

In this paper, we show that the cycle reversion technique of Cosmadakis et al. extends all the way to the expressive DLs Horn-$\mathcal{ALCFI}$ and Horn-$\mathcal{ALCQI}$. These logics, as well as their extensions Horn-$\mathcal{SHIF}$ and Horn-$\mathcal{SHIQ}$, are popular in ontology-based data access (Hustadt, Motik, and Sattler 2007; Ortiz, Rudolph, and Šimkus 2011; Eiter et al. 2012; Bienvenu, Lutz, and Wolter 2013) and properly extend *DL-Lite*$_\mathcal{F}$ and other relevant Horn DLs such as $\mathcal{ELIF}$ (Krisnadhi and Lutz 2007). We start with showing that finite ABox consistency in Horn-$\mathcal{ALCFI}$ can be reduced to unrestricted ABox consistency in Horn-$\mathcal{ALCFI}$ by cycle reversion; it follows that the same is true for finite satisfiability, finite subsumption, and finite instance checking. While the reduction technique is conceptually similar to that for *DL-Lite*$_\mathcal{F}$, the construction of a finite model in the correctness proof is more demanding. In comparison to approaches to finite model reasoning that rely on solving systems of inequations, though, they make the structure of finite models considerably more explicit.

Another crucial difference to the *DL-Lite*$_\mathcal{F}$ case is that, when completing Horn-$\mathcal{ALCFI}$ TBoxes, the cycles that have to be considered can be of exponential length, and thus the reduction is not polynomial. Consequently, when used in a naive way it can neither be expected to perform well in practice nor be used to (re)prove tight complexity bounds. To address these shortcomings, we develop a dedicated calculus for finite ABox consistency in Horn-$\mathcal{ALCFI}$ that implements the reduction on-the-fly rather than executing it up-front. The calculus is an extension of a consequence-based procedure for unrestricted satisfiability in Horn-$\mathcal{SHIQ}$ that was introduced by Kazakov in (2009) and implemented in the highly performant reasoner CB, first to classify the notoriously difficult Galen ontology. Many other state-of-the art reasoners for Horn-DLs are also based on consequence-based procedures, including ELK (Kazakov, Krötzsch, and Simančík 2011a) and CEL (Baader, Lutz, and Suntisrivaraporn 2006). Our algorithm shares the main feature of other consequence-based procedures to carefully avoid considering 'types' (conjunctions of concept names) that are irrelevant for deciding the problem at hand. We therefore believe that it provides a very promising basis for efficient implementations of finite model reasoning in Horn-$\mathcal{ALCFI}$. It also (re)proves the optimal upper ExpTime complexity bound for finite ABox consistency in this DL. Via a reduction, the cycle reversing reduction and the consequence-based algorithm can be applied also to finite satisfiability and subsumption in Horn-$\mathcal{ALCQI}$.

We then consider the paradigm of ontology-based data access (OBDA), extending our results from finite ABox consistency to answering positive existential queries (PEQs), relative to Horn-$\mathcal{ALCFI}$ TBoxes over finite models. In particular, we show that the reduction based on cycle reversion developed for ABox consistency also works in the case of PEQ answering. The construction of (counter)models in the correctness proofs, however, becomes yet more difficult and technical, and proceeds in two stages. First, we carefully modify the models constructed for finite ABox consistency so that there are no unintended matches of acyclic conjunctive queries (CQs). And second, we take a product with a finite group of high girth to eliminate unintended matches of cyclic CQs. Based on this result, we then prove that finite PEQ entailment (the Boolean version of PEQ answering) in Horn-$\mathcal{ALCFI}$ is ExpTime-complete regarding combined complexity and PTime-complete regarding data complexity. Previously, it was only known that finite CQ answering in (non-Horn) $\mathcal{ALCQI}$ is decidable and in coNP regarding data complexity (Pratt-Hartmann 2009).

Some proof details are deferred to the appendix in the long version: http://tinyurl.com/kr14fmr

2 Preliminaries

We introduce the DLs Horn-$\mathcal{ALCFI}$ and Horn-$\mathcal{ALCQI}$, as well as the reasoning tasks studied in this paper. The original definition of these DLs is based on a notion of polarity and somewhat unwieldy (Hustadt, Motik, and Sattler 2007); alternative and more direct definitions have been proposed later, see for example (Lutz and Wolter 2012). For brevity, we directly introduce Horn-$\mathcal{ALCQI}$ TBoxes in a normal form that is convenient for our purposes and disallows syntactic nesting of operators. It is a minor variation of the normal form proposed in (Kazakov 2009).

Let $\mathsf{N_C}$, $\mathsf{N_R}$, and $\mathsf{N_I}$ be countably infinite and disjoint sets of concept names, role names, and individual names. A *role* is either a role name r or an *inverse role* r^-. A *Horn-$\mathcal{ALCQI}$ TBox* $\mathcal{T}$ is a set of *concept inclusions (CIs)* that can take the following forms:

$$K \sqsubseteq A \qquad K \sqsubseteq \bot \qquad K \sqsubseteq \exists r.K'$$
$$K \sqsubseteq \forall r.K' \qquad K \sqsubseteq (\leqslant 1\ r\ K') \qquad K \sqsubseteq (\geqslant n\ r\ K')$$

where K and K' denote a (possibly empty) conjunction of concept names, A a concept name, r a (potentially inverse) role, and $n \geq 2$. Throughout the paper, we will deliberately confuse conjunctions of concept names and sets of concept names. The empty conjunction is abbreviated by $\top$. As usual, we allow to easily switch between role names and their inverse by identifying $(r^-)^-$ and r. A *Horn-$\mathcal{ALCFI}$ TBox* is a Horn-$\mathcal{ALCQI}$ TBox that does not include CIs of the form $K \sqsubseteq (\geqslant n\ r\ K')$.

The semantics of Horn-$\mathcal{ALCQI}$ is based on interpretations as usual, see (Baader et al. 2003) for details. We write $\mathcal{T} \models C \sqsubseteq D$ if the concept inclusion $C \sqsubseteq D$ is satisfied in all models of the TBox $\mathcal{T}$, and $\mathcal{T} \models_{\mathsf{fin}} C \sqsubseteq D$ if the same holds for all finite models. A concept name A is *(finitely) satisfiable* w.r.t. a TBox $\mathcal{T}$ if $\mathcal{T}$ has a (finite) model $\mathcal{I}$ with $A^\mathcal{I} \neq \emptyset$. If $\mathcal{T} \models A \sqsubseteq B$ (resp. $\mathcal{T} \models_{\mathsf{fin}} A \sqsubseteq B$) with A and B concept names, then we say that B is *(finitely) subsumed* by A.

An *ABox* is a finite set of *concept assertions* $A(a)$ and *role assertions* $r(a, b)$ where A is a concept name, r a role name, and a, b are individual names. For simplicity, we make the *standard names assumption*, that is, every interpretation $\mathcal{I}$ interpretes all individuals as themselves; for example $\mathcal{I}$ satisfies $A(a)$ if $a \in A^{\mathcal{I}}$. The standard names assumption implies the unique name assumption (UNA). The results in this paper, however, do not depend on any of these assumptions. Throughout the paper, we sometimes write $r^-(a, b) \in \mathcal{A}$ for $r(b, a) \in \mathcal{A}$ and use $\mathsf{Ind}(\mathcal{A})$ to denote the set of all individual names that occur in $\mathcal{A}$.

We write $\mathcal{A}, \mathcal{T} \models A(a)$ if the ABox assertion $A(a)$ is satisfied in all common models of the ABox $\mathcal{A}$ and the TBox $\mathcal{T}$, and $\mathcal{A}, \mathcal{T} \models_{\mathsf{fin}} A(a)$ if the same holds for all finite models. We then say that a is a *(finite) instance* of A in $\mathcal{A}$ w.r.t. $\mathcal{T}$. An ABox $\mathcal{A}$ is *(finitely) consistent* w.r.t. $\mathcal{T}$ if there is a (finite) model $\mathcal{I}$ of $\mathcal{T}$ that satisfies all assertions in $\mathcal{A}$.

The above notions give rise to four decision problems studied in this paper, which are *finite satisfiability* (of a concept name w.r.t. a TBox), *finite subsumption* (between two concept names w.r.t. a TBox), *finite ABox consistency* (w.r.t. a TBox) and *finite instance checking* (of an ABox individual and a concept name, w.r.t. an ABox and a TBox). There are easy polynomial time reductions from satisfiability to subsumption to instance checking to ABox consistency, which work both in the finite and in the unrestricted case.

The following examples show that, in Horn-$\mathcal{ALCFI}$, finite and unrestricted reasoning do not coincide.

Example 1

$$\begin{array}{lll} \mathcal{T} = \{ & A \sqsubseteq \exists r.B, & B \sqsubseteq \exists r.B, \\ & B \sqsubseteq (\leqslant 1\, r^- \top), & A \sqcap B \sqsubseteq \bot \quad \} \end{array}$$

A is satisfiable w.r.t. $\mathcal{T}$, but not finitely satisfiable. In fact, when $d \in A^{\mathcal{I}}$ in some model $\mathcal{I}$ of $\mathcal{T}$, then there must be an infinite chain $r(d, d_1), r(d_1, d_2), \ldots$ with $d \in A^{\mathcal{I}}$, and $d_2, d_3, \cdots \in B^{\mathcal{I}}$. Since d cannot be in $B^{\mathcal{I}}$ and r is inverse functional, no two elements on the chain can be identified.

$$\begin{array}{ll} \mathcal{T}' = \{ A_1 \sqsubseteq \exists r.A_2, & A_2 \sqsubseteq \exists r.(A_1 \sqcap B), \\ & \top \sqsubseteq (\leqslant 1\, r^- \top) \quad \} \end{array}$$

The reader might want to verify that $\mathcal{T}' \not\models A_1 \sqsubseteq B$, but $\mathcal{T}' \models_{\mathsf{fin}} A_1 \sqsubseteq B$.

It follows form the observations in (Kazakov 2009) that, for the purposes of deciding satisfiability of concepts in unrestricted models, the normal form for TBoxes introduced above can be assumed without loss of generality because every Horn-$\mathcal{ALCQI}$ TBox $\mathcal{T}$ can be converted in polynomial time into a TBox $\mathcal{T}'$ in the above form such that every model of $\mathcal{T}'$ is a model of $\mathcal{T}$ and, conversely, every model of $\mathcal{T}$ can be converted into a model of $\mathcal{T}'$ by interpreting the concept names that were introduced during normalization. It follows that normal form can be assumed w.l.o.g. both for unrestricted reasoning and for finite model reasoning, and for all reasoning problems considered in this paper.

3 From Finite Models to Unrestricted Models

We show that finite ABox consistency in Horn-$\mathcal{ALCFI}$ can be reduced to unrestricted ABox consistency by reversing certain cycles in the TBox. The reduction exhibited in this section provides a novel decision procedure for finite ABox consistency in Horn-$\mathcal{ALCFI}$ and Horn-$\mathcal{ALCQI}$ (as well as for finite satisfiability, finite subsumption, and finite instance checking) and is the basis for developing a consequence-based procedure in Section 4. It also highlights the logical consequences of finite models in Horn-$\mathcal{ALCFI}$. The material in this section is an extended and improved version of the workshop paper (Ibáñez-García, Lutz, and Schneider 2013).

Reversing Cycles

Let $\mathcal{T}$ be a Horn-$\mathcal{ALCFI}$ TBox. A *finmod cycle in $\mathcal{T}$* is a sequence $K_1, r_1, K_2, r_2, \ldots, r_{n-1}, K_n$, with $K_1, \ldots, K_n$ conjunctions of concept names and $r_1, \ldots, r_{n-1}$ (potentially inverse) roles such that $K_n = K_1$ and, for $1 \leq i < n$:

$$\mathcal{T} \models K_i \sqsubseteq \exists r_i.K_{i+1} \text{ and } \mathcal{T} \models K_{i+1} \sqsubseteq (\leqslant 1\, r_i^-\ K_i).$$

By *reversing* a finmod cycle $K_1, r_1, K_2, r_2, \ldots, r_{n-1}, K_n$ in a TBox $\mathcal{T}$, we mean to extend $\mathcal{T}$ with the following concept inclusions, for $1 \leq i < n$:

$$K_{i+1} \sqsubseteq \exists r_i^-.K_i \text{ and } K_i \sqsubseteq (\leqslant 1\, r_i\ K_{i+1}).$$

The *completion* $\mathcal{T}_{\mathsf{f}}$ of a TBox $\mathcal{T}$ is obtained from $\mathcal{T}$ by exhaustively reversing finmod cycles. Note that, although there may be infinitely many finmod cycles, only finitely many CIs can be added by cycle reversion (exponentially many in the size of the original TBox, in the worst case). For finding these finitely many CIs, it clearly suffices to consider finmod cycles in which all triples (r_i, K_{i+1}, r_{i+1}) are distinct. Also note that finding finmod cycles requires deciding unrestricted subsumption, which is decidable and ExpTime-complete.

Example 2 *The TBox $\mathcal{T}'$ from Example 1 entails (in unrestricted models)*

$$\begin{array}{ll} A_1 \sqcap B \sqsubseteq \exists r.A_2, & A_2 \sqsubseteq \exists r.(A_1 \sqcap B), \\ A_2 \sqsubseteq (\leqslant 1\, r^-\ A_1 \sqcap B), & A_2 \sqcap B \sqsubseteq (\leqslant 1\, r^-\ A_1). \end{array}$$

Thus, A_1, r, A_2, r, A_1, is a finmod cycle in $\mathcal{T}'$, which is reversed to

$$\begin{array}{ll} A_2 \sqsubseteq \exists r^-.A_1, & A_1 \sqsubseteq \exists r^-.A_2, \\ A_2 \sqsubseteq (\leqslant 1\, r\ A_1), & A_1 \sqsubseteq (\leqslant 1\, r\ A_2). \end{array}$$

From $A_1 \sqsubseteq \exists r^-.A_2$, $A_2 \sqsubseteq \exists r.(A_1 \sqcap B)$, and $A_2 \sqsubseteq (\leqslant 1\, r\ A_1)$, we obtain $\mathcal{T}'_{\mathsf{f}} \models A_1 \sqsubseteq B$, in correspondence with $\mathcal{T}' \models_{\mathsf{fin}} A_1 \sqsubseteq B$. Note that $\mathcal{T}'$ also contains another finmod cycle, which is $(A_1 \sqcap B), r, A_2, r, (A_1 \sqcap B)$.

The following result shows that TBox completion provides a reduction from finite ABox consistency to unrestricted ABox consistency.

Theorem 3 *Let $\mathcal{T}$ be a Horn-$\mathcal{ALCFI}$ TBox and $\mathcal{A}$ an ABox. Then $\mathcal{A}$ is finitely consistent w.r.t. $\mathcal{T}$ iff $\mathcal{A}$ is consistent w.r.t. the completion $\mathcal{T}_{\mathsf{f}}$ of $\mathcal{T}$.*

The "only if" direction of Theorem 3 is an immediate consequence of the observation that all CIs added by cycle reversion are entailed by the original TBox in finite models.

Lemma 4 *Let $K_1, r_1, \ldots, r_{n-1}, K_n$ be a finmod cycle in $\mathcal{T}$. Then $\mathcal{T} \models_{\mathsf{fin}} K_{i+1} \sqsubseteq \exists r_i^-.K_i$ and $\mathcal{T} \models_{\mathsf{fin}} K_i \sqsubseteq (\leqslant 1\, r_i\ K_{i+1})$ for $1 \leq i < n$,.*

Proof. We have to show that if $K_1, r_1, \ldots, r_{n-1}, K_n$ is a finmod cycle in $\mathcal{T}$ and $\mathcal{I}$ is a finite model of $\mathcal{T}$, then $K_i^{\mathcal{I}} \subseteq (\leqslant 1\ r_i\ K_{i+1})^{\mathcal{I}}$ and $K_{i+1}^{\mathcal{I}} \subseteq (\exists r_i^-.K_i)^{\mathcal{I}}$ for $1 \leq i < n$. We first note that, by the semantics of Horn-$\mathcal{ALCFI}$, we must have $|K_1^{\mathcal{I}}| \leq \cdots \leq |K_n^{\mathcal{I}}|$, thus $K_n = K_1$ yields $|K_1^{\mathcal{I}}| = \cdots = |K_n^{\mathcal{I}}|$. Fix some i with $1 \leq i < n$. Using $|K_i^{\mathcal{I}}| = |K_{i+1}^{\mathcal{I}}|$, $K_i^{\mathcal{I}} \subseteq (\exists r_i.K_{i+1})^{\mathcal{I}}$, and $K_{i+1}^{\mathcal{I}} \subseteq (\leqslant 1\ r_i^-\ K_i)^{\mathcal{I}}$, it is easy to verify that $K_i^{\mathcal{I}} \subseteq (\leqslant 1\ r_i\ K_{i+1})^{\mathcal{I}}$ and $K_{i+1}^{\mathcal{I}} \subseteq (\exists r_i^-.K_i)^{\mathcal{I}}$, as required. ❑

We now prove the "if" direction of Theorem 3, which is much more demanding as it requires to explicitly construct finite models.

Constructing Finite Models

Assume that $\mathcal{A}$ is consistent w.r.t. $\mathcal{T}_{\mathsf{f}}$. Our aim is to construct a finite model $\mathcal{I}$ of $\mathcal{A}$ and $\mathcal{T}_{\mathsf{f}}$ (and thus also of $\mathcal{T}$). Before we give details of the construction, we introduce some relevant preliminaries.

Let $\mathsf{CN}(\mathcal{T})$ denote the set of concept names used in $\mathcal{T}$ (or, equivalently, in $\mathcal{T}_{\mathsf{f}}$). A *type for* $\mathcal{T}_{\mathsf{f}}$ is a subset $t \subseteq \mathsf{CN}(\mathcal{T})$ such that there is a (potentially infinite) model $\mathcal{I}$ of $\mathcal{T}_{\mathsf{f}}$ and a $d \in \Delta^{\mathcal{I}}$ such that $\mathsf{tp}_{\mathcal{I}}(d) = t$, where

$$\mathsf{tp}_{\mathcal{I}}(d) := \{A \in \mathsf{CN}(\mathcal{T}) \mid d \in A^{\mathcal{I}}\}$$

is the type *realized* at d in $\mathcal{I}$. We use $\mathsf{TP}(\mathcal{T}_{\mathsf{f}})$ to denote the set of all types for $\mathcal{T}_{\mathsf{f}}$. For $t, t' \in \mathsf{TP}(\mathcal{T}_{\mathsf{f}})$ and r a role, we write

- $t \rightarrow_r t'$ if $\mathcal{T}_{\mathsf{f}} \models t \sqsubseteq \exists r.t'$ and t' is maximal with this property;
- $t \rightarrow_r^1 t'$ if $t \rightarrow_r t'$ and $\mathcal{T}_{\mathsf{f}} \models t' \sqsubseteq (\leqslant 1\ r^-\ t)$;
- $t\ {}^1{\leftrightarrow}_r^1\ t'$ if $t \rightarrow_r^1 t'$ and $t' \rightarrow_{r^-}^1 t$.

Note that when

$$t_1 \rightarrow_{r_1}^1 t_2 \rightarrow_{r_2}^1 \cdots \rightarrow_{r_{n-1}}^1 t_n = t_1 \qquad (*)$$

then $t_1, r_1, \ldots, r_{n-1}, t_n$ is a finmod cycle in $\mathcal{T}_{\mathsf{f}}$ and the fact that it has been reversed means that all '$\rightarrow^1$' in $(*)$ can be replaced with ${}^1{\leftrightarrow}^1$. Types related by ${}^1{\leftrightarrow}_r^1$ are connected very tightly by the TBox $\mathcal{T}_{\mathsf{f}}$ and are best considered together when building finite models. This is formalized by the notion of a *type class*, which is a non-empty set $P \subseteq \mathsf{TP}(\mathcal{T}_{\mathsf{f}})$ such that $t \in P$ and $t\ {}^1{\leftrightarrow}_r^1\ t'$ implies $t' \in P$, and P is minimal with this condition. Note that the set of all type classes is a partition of $\mathsf{TP}(\mathcal{T}_{\mathsf{f}})$. We set $P \prec P'$ if there are $t \in P$ and $t' \in P'$ with $t' \subsetneq t$. Let $\prec^+$ be the transitive closure of $\prec$. A proof of the following observation can be found in the appendix.

Lemma 5 *$\prec^+$ is a strict partial order.*

We construct the desired finite model $\mathcal{I}$ of $\mathcal{A}$ and $\mathcal{T}_{\mathsf{f}}$ by starting with an initial interpretation that essentially consists of the ABox $\mathcal{A}$ and then exhaustively applying three *completion rules* denoted with $(\mathbf{c1})$ to $(\mathbf{c3})$, where $(\mathbf{c1})$ is given preference over $(\mathbf{c2})$. Completion repeatedly introduces elements whose existence is required by CIs $K \sqsubseteq \exists r.C$, carefully distinguishing several cases to ensure that no functionality restrictions are violated. We will prove that rule application terminates after finitely many steps, producing a finite model.

During the construction of $\mathcal{I}$, we will make sure that the following invariants are satisfied:

(i1) $\mathsf{tp}_{\mathcal{I}}(d) \in \mathsf{TP}(\mathcal{T}_{\mathsf{f}})$ for all $d \in \Delta^{\mathcal{I}}$;

(i2) if $(d, d') \in r^{\mathcal{I}} \setminus (\mathsf{Ind}(\mathcal{A}) \times \mathsf{Ind}(\mathcal{A}))$, then we have $\mathsf{tp}_{\mathcal{I}}(d) \rightarrow_r \mathsf{tp}_{\mathcal{I}}(d')$ or $\mathsf{tp}_{\mathcal{I}}(d') \rightarrow_{r^-} \mathsf{tp}_{\mathcal{I}}(d)$;

(i3) if $\mathcal{T}_{\mathsf{f}} \models K \sqsubseteq (\leqslant 1\ r\ K')$, then $\mathcal{I} \models K \sqsubseteq (\leqslant 1\ r\ K')$.

The initial version of $\mathcal{I}$ is defined by introducing an element for every ABox individual, and an element d_t for each $t \in \mathsf{TP}(\mathcal{T}_{\mathsf{f}})$. In detail, we set

$$\begin{aligned}
\Delta^{\mathcal{I}} &= \mathsf{Ind}(\mathcal{A}) \cup \{\ d_t \mid t \in \mathsf{TP}(\mathcal{T})_{\mathsf{f}}\ \} \\
A^{\mathcal{I}} &= \{\ a \in \mathsf{Ind}(\mathcal{A}) \mid A \in \mathsf{tp}_{\mathcal{A}}(a)\ \} \cup \{d_t \mid A \in t\ \} \\
r^{\mathcal{I}} &= \{\ (a, b) \mid r(a, b) \in \mathcal{A}\ \}
\end{aligned}$$

where

$$\mathsf{tp}_{\mathcal{A}}(a) := \{A \in \mathsf{CN}(\mathcal{T}) \mid \mathcal{A}, \mathcal{T}_{\mathsf{f}} \models A(a)\}.$$

The completion rules are described in detail below.

(c1) Choose a $d \in \Delta^{\mathcal{I}}$ such that $\mathsf{tp}_{\mathcal{I}}(d) \rightarrow_r^1 t$, $t \not\rightarrow_{r^-}^1 \mathsf{tp}_{\mathcal{I}}(d)$, and $d \notin (\exists r.t)^{\mathcal{I}}$. Add a fresh domain element e, and modify the extension of concept and role names such that $\mathsf{tp}_{\mathcal{I}}(e) = t$ and $(d, e) \in r^{\mathcal{I}}$.

(c2) Choose a type class P that is minimal w.r.t. the order $\prec^+$, a $\lambda = s\ {}^1{\leftrightarrow}_r^1\ s'$ with $s \in P$, and an element $d \in s^{\mathcal{I}} \setminus (\exists r.s')^{\mathcal{I}}$.

For each $\lambda = s\ {}^1{\leftrightarrow}_r^1\ s'$ with $s \in P$, set

$$X_{\lambda,1}^{\mathcal{I}} = s^{\mathcal{I}} \setminus (\exists r.s')^{\mathcal{I}} \qquad X_{\lambda,2}^{\mathcal{I}} = s'^{\mathcal{I}} \setminus (\exists r^-.s)^{\mathcal{I}}.$$

Take (i) a fresh set Δ_s for each $s \in P$ such that $|\biguplus_{s \in P} \Delta_s| \leq 2^{|\mathcal{T}|} \cdot |\Delta^{\mathcal{I}}|$ and (ii) a bijection π_λ between $X_{\lambda,1}^{\mathcal{I}} \cup \Delta_s$ and $X_{\lambda,2}^{\mathcal{I}} \cup \Delta_{s'}$ for each $\lambda = s\ {}^1{\leftrightarrow}_r^1\ s'$ with $s, s' \in P$ and r a role name (the concrete construction is detailed below). Now extend $\mathcal{I}$ as follows:

- add all domain elements in $\biguplus_{s \in P} \Delta_s$;
- extend $r^{\mathcal{I}}$ with π_λ, for each $\lambda = s\ {}^1{\leftrightarrow}_r^1\ s'$ with $s, s' \in P$ and r a role name;
- interpret concept names so that $\mathsf{tp}_{\mathcal{I}}(d) = s$ for all $d \in \Delta_s$, $s \in P$.

(c3) Choose a $d \in \Delta^{\mathcal{I}}$ such that $\mathsf{tp}_{\mathcal{I}}(d) \rightarrow_r t$, $\mathsf{tp}_{\mathcal{I}}(d) \not\rightarrow_r^1 t$, and $d \notin (\exists r.t)^{\mathcal{I}}$. Add the edge (d, d_t) to $r^{\mathcal{I}}$, where d_t is the element introduced for type t in the initial version of $\mathcal{I}$.

To complete the description of the rules, we have to show that, in $(\mathbf{c2})$, the sets Δ_s and bijections π_λ indeed exist. Let $n_{\max} = \max\{|s^{\mathcal{I}}| \mid s \in P\}$. For each $s \in P$, set $\Delta_s := \{d_{s,i} \mid |s^{\mathcal{I}}| < i \leq n_{\max}\}$ and define the set of *s-instances* $I_s := s^{\mathcal{I}} \cup \Delta_s$. For each $\lambda = s\ {}^1{\leftrightarrow}_r^1\ s'$ with $s, s' \in P$, define

$$R_\lambda := \{(d, e) \in r^{\mathcal{I}} \mid d \in s^{\mathcal{I}} \text{ and } e \in s'^{\mathcal{I}}\}.$$

We first note that it is a consequence of invariant $(\mathbf{i3})$ that

$(*)$ the relation R_λ is functional and inverse functional.

In fact, $(d, e_1), (d, e_2) \in R_\lambda$ implies $(d, e_1), (d, e_2) \in r^{\mathcal{I}}$, $d \in s^{\mathcal{I}}$, and $e_1, e_2 \in s'^{\mathcal{I}}$. By λ, $\mathcal{T}_{\mathsf{f}} \models s \sqsubseteq (\leqslant 1\ r\ s')$. Thus, $(\mathbf{i3})$ yields $e_1 = e_2$. Inverse functionality can be shown analogously.

Let R^1_λ be the domain of R_λ, and let R^2_λ be its range. By $(*)$, we have $|R^1_\lambda| = |R^2_\lambda|$. By definition of the sets Δ_s, we have $|I_s| = |I_{s'}|$. Moreover, $R^1_\lambda \subseteq I_s$ and $R^2_\lambda \subseteq I_{s'}$. We can thus choose a bijection π_λ between $I_s \setminus R^1_\lambda$ and $I_{s'} \setminus R^2_\lambda$, which is as required since $I_s \setminus R^1_\lambda = X^{\mathcal{I}}_{\lambda,1} \cup \Delta_s$ and $I_{s'} \setminus R^2_\lambda = X^{\mathcal{I}}_{\lambda,2} \cup \Delta_{s'}$. The construction of the sets Δ_s clearly ensures that their union has the required cardinality.

The following theorem summarizes the statements that remain to be proved in order to show that the construction of $\mathcal{I}$ is well-defined and yields a finite model of $\mathcal{A}$ and $\mathcal{T}_{\mathsf{f}}$.

Theorem 6

1. *Applying* $(\mathbf{c1})$ *to* $(\mathbf{c3})$ *preserves invariants* $(\mathbf{i1})$ *to* $(\mathbf{i3})$*;*
2. *Application of* $(\mathbf{c1})$ *to* $(\mathbf{c3})$ *terminates;*
3. $\mathcal{I}$ *is a model of* $\mathcal{A}$ *and* $\mathcal{T}_{\mathsf{f}}$.

Proof. We refer to the appendix for full proofs and only sketch the central idea in the proof of Point 2 here, going back to (Cosmadakis, Kanellakis, and Vardi 1990). The main issue in the termination proof is to show that no infinite role chain $r_0(d_0, d_1), r_1(d_1, d_2), \ldots$ is generated in which all the elements d_i are pairwise distinct. Since every application of a completion rule generates only finitely many elements, any such chain must be generated by infinitely many rule applications. As there are only finitely many types, we must find elements d_i and d_j with $\mathsf{tp}_{\mathcal{I}}(d_i) = \mathsf{tp}_{\mathcal{I}}(d_j)$ and such that d_i and d_j were generated by different rule applications. It can be shown that, w.l.o.g., we can assume that the elements on the chain are ordered so that if $j > i$, then d_j was not generated by an earlier rule application than d_i. Analysing the completion rules, it is easy to see that this implies $\mathsf{tp}_{\mathcal{I}}(d_i) \rightarrow^1_{r_i} \mathsf{tp}_{\mathcal{I}}(d_{i+1}) \rightarrow^1_{r_{i+1}} \cdots \rightarrow^1_{r_{j-1}} \mathsf{tp}_{\mathcal{I}}(d_j)$. Since $\mathsf{tp}_{\mathcal{I}}(d_i) = \mathsf{tp}_{\mathcal{I}}(d_j)$, this is a finmod cycle, which has been reversed when constructing $\mathcal{T}_{\mathsf{f}}$, and thus all arrows $\rightarrow^1_{r_{i+\ell}}$ can be replaced with ${}^1\!\leftrightarrow^1_{r_{i+\ell}}$. By definition of the completion rules, this means that all of $d_i, \ldots, d_j$ were introduced in the same application of $(\mathbf{c2})$, which is a contradiction to d_i and d_j being generated by different rule applications. ❏

4 Consequence-Driven Procedure

While completing TBoxes with reversed cycles yields a reduction of finite model reasoning to infinite model reasoning, it blows up the TBox exponentially and is thus not suited for direct implementation. In this section, we build on the results from the previous section to devise a calculus for ABox consistency in Horn-$\mathcal{ALCFI}$ that does not require TBox completion to be carried out up-front, but instead reverses cycles 'on the fly'; moreover, the calculus implicitly groups together cycles that are closely related, potentially reversing a very large number of cycles in only a few steps (see Example 7 below). Our calculus belongs to a family of algorithms that are known as consequence-driven procedures and underly modern and highly efficient reasoners for Horn DLs such as CEL, CB, and ELK (Baader, Lutz, and Suntisrivaraporn 2006; Kazakov 2009; Kazakov, Krötzsch, and Simančík 2011b). It thus establishes a promising foundation for actual implementations of finite-model reasoning in Horn-$\mathcal{ALCFI}$ and, via the reduction in Section 6, in Horn-$\mathcal{ALCQI}$. For simplicity, we start with a calculus for finite satisfiability and finite subsumption. An expansion to finite ABox consistency (and thus to finite instance checking) is sketched afterwards.

R1 $$\frac{}{K \sqcap A \sqsubseteq A}$$

R2 $$\frac{}{K \sqsubseteq \top}$$

R3 $$\frac{K \sqsubseteq A_i \quad \bigsqcap A_i \sqsubseteq C}{K \sqsubseteq C}$$

R4 $$\frac{K \sqsubseteq \exists r.K' \quad K' \sqsubseteq \forall r^-.A}{K \sqsubseteq A}$$

R5 $$\frac{K \sqsubseteq \exists r.K' \quad K \sqsubseteq \forall r.A}{K \sqsubseteq \exists r.(K' \sqcap A)}$$

R6 $$\frac{K \sqsubseteq \exists r.K' \quad K' \sqsubseteq \bot}{K \sqsubseteq \bot}$$

R7 $$\frac{\begin{array}{lll} K \sqsubseteq \exists r.K_1 & K \sqsubseteq \exists r.K_2 & K_1 \sqsubseteq A \\ K \sqsubseteq (\leqslant 1\ r\ A) & & K_2 \sqsubseteq A \end{array}}{K \sqsubseteq \exists r.(K_1 \sqcap K_2)}$$

R8 $$\frac{\begin{array}{lll} K \sqsubseteq \exists r.K' & K' \sqsubseteq \exists r^-.K_1 & K \sqsubseteq A \\ K' \sqsubseteq (\leqslant 1\ r^-\ A) & & K_1 \sqsubseteq A \end{array}}{K \sqsubseteq A_1 \quad \text{for any } A_1 \in K_1}$$

R9 $$\frac{\begin{array}{lll} K_i \sqsubseteq \exists r_i.K_{i \oplus_n 1} & & \\ K_{i \oplus_n 1} \sqsubseteq (\leqslant 1\ r_i^-\ A_i) & K_i \sqsubseteq A_i & i < n \end{array}}{K_1 \sqsubseteq \exists r_0^-.K_0 \quad K_0 \sqsubseteq (\leqslant 1\ r_0\ A_1)}$$

Figure 1: Inference Rules

The calculus starts with a given TBox $\mathcal{T}$ and then exhaustively applies a set of inference rules. To ease their presentation, we assume that $\mathcal{T}$ is in a normal form that is slightly stricter than the one introduced in Section 2: in CIs $K \sqsubseteq \forall r.K'$ and $K \sqsubseteq (\leqslant 1\ r\ K)'$, K' must be a concept name A. The inference rules are displayed in Figure 1. They preserve the normal form and are applied in the sense that, if the concept inclusions in the precondition (above the line) are already present, then those in the postcondition (below the line) are added. Recall that K stands for a conjunction of concept names, which we read here modulo commutativity. Rule **R1** is applied only if $K \sqcap A$ *occurs* in the current (partially completed) TBox, that is, there is a CI of the form $K \sqcap A \sqsubseteq C$ or $K' \sqsubseteq \exists r.(K \sqcap A)$. The same is true for rule **R2** with K in place of $K \sqcap A$. In rule **R9**, $\oplus_n$ means addition modulo n.

We point out that rules **R1** to **R8** are minor variations of the corresponding rules in the calculus presented by Kazakov (2009), the main difference being that our language does not include role hierarchies. Rule **R9** is novel and deals with reversing cycles on the fly. Note that only the 'first edge' of each cycle is reversed, and that this is sufficient because the cycle can be rotated to make any edge the 'first' one.

Example 7 *Consider the TBox*

$$\mathcal{T} = \{A \sqsubseteq \exists r.(A \sqcap A_1 \sqcap \cdots \sqcap A_n), \quad (1)$$
$$A \sqsubseteq (\leqslant 1\, r^- A)\,\}. \quad (2)$$

Cycle reversion from Section 3 reverses all of the exponentially many cycles K, r, K with $K \subseteq S := \{A, A_1, \ldots, A_n\}$ and $A \in K$, adding $K \sqsubseteq \exists r^-.K$ and $K \sqsubseteq (\leqslant 1\ r\ K)$ for all such K. In contrast, the calculus avoids introducing 'irrelevant' conjunctions K and instead jointly reverses all these cycles by generating $A \sqsubseteq \exists r^-.S$ and $A \sqsubseteq (\leqslant 1\ r\ A)$:

S	$\sqsubseteq$	A	*from* **R1**	(3)
A	$\sqsubseteq$	A	*from* **R1**	(4)
S	$\sqsubseteq$	$\exists r.S$	*from* (1), (3), **R3**	(5)
S	$\sqsubseteq$	$(\leqslant 1\, r^-\ A)$	*from* (2), (3), **R3**	(6)
S	$\sqsubseteq$	$\exists r^-.S$	*and*	(7)
S	$\sqsubseteq$	$(\leqslant 1\ r\ A)$	*from* (3), (5), (6), **R9**	(8)
A	$\sqsubseteq$	A_i	*from* (1), (3), (4), (6), (7), **R8**	(9)
A	$\sqsubseteq$	$\exists r^-.S$	*from* (7), (9), **R3**	(10)
A	$\sqsubseteq$	$(\leqslant 1\ r\ A)$	*from* (8), (9), **R3**	(11)

Note that avoiding to introduce 'irrelevant' conjunctions K as illustrated by Example 7 is a main feature of consequence-based procedures which enables the excellent practical performance typically observed for this class of calculi.

The algorithm terminates after at most exponentially many rule applications since there are only exponentially many different concept inclusions that use the concept and role names of the original TBox. Each rule application can be performed in polynomial time, which is easy to see for the rules **R1**–**R8**. For **R9**, the crucial observation is that it suffices to consider all conjunctions K_0, K_1 and to check whether they are involved in *any* cycle. The latter can easily be done by a variation of directed graph reachability, where the nodes of the graph are the conjunctions that occur in the current TBox and the edges come from inclusions $K \sqsubseteq \exists r.K'$.

The following theorem, which is the main result of this section, states that the calculus is sound and complete.

Theorem 8 *Let $\mathcal{T}$ be a Horn-$\mathcal{ALCFI}$ TBox, $\widehat{\mathcal{T}}$ be obtained by exhaustively applying Rules* **R1**–**R9**, *and let A_0 be a concept name. Then A_0 is finitely satisfiable w.r.t. $\mathcal{T}$ iff $A_0 \sqsubseteq \bot \notin \widehat{\mathcal{T}}$.*

While Theorem 8 is formulated only for finite satisfiability, the algorithm can of course also be used to decide finite subsumption via the usual reduction to finite satisfiability. The following continues Example 7.

Example 9 *Let $\mathcal{T}$ be the TBox from Example 7 and*

$$\mathcal{T}' = \mathcal{T} \cup \{ \quad A \sqsubseteq \exists r.(A \sqcap X_1), \quad (12)$$
$$A \sqsubseteq \exists r.(A \sqcap X_2), \quad (13)$$
$$X_1 \sqcap X_2 \sqsubseteq \bot \quad \} \quad (14)$$

The calculus derives $A \sqsubseteq \bot$, thus A is finitely unsatisfiable w.r.t. $\mathcal{T}'$:[1]

$A \sqcap X_i \sqsubseteq A$	*from* **R1**	(15)
$A \sqsubseteq \exists r.(A \sqcap X_1 \sqcap X_2)$	*from* (11)–(13), (15), **R7**	(16)
$A \sqsubseteq \bot$	*from* (14), (16), **R6**	(17)

[1] A is obviously satisfiable w.r.t. $\mathcal{T}'$ in unrestricted models.

We now prove Theorem 8. The "only if" direction (soundness) is straightforward by verifying that each rule is sound in finite models. In contrast, the "if" direction (completeness) turns out to be surprisingly subtle to establish. The proof strategy is as follows. Assume that $A_0 \sqsubseteq \bot \notin \widehat{\mathcal{T}}$. We construct a (possibly infinite) model $\widehat{\mathcal{I}}$ of $\widehat{\mathcal{T}}$ with $A_0^{\widehat{\mathcal{I}}} \neq \emptyset$ and show that $\widehat{\mathcal{I}}$ is actually a model of $\mathcal{T}_\mathsf{f}$. By Theorem 3, it follows that A_0 is finitely satisfiable w.r.t. $\mathcal{T}$. From now on, assume w.l.o.g. that A_0 actually occurs in $\mathcal{T}$.

To construct $\widehat{\mathcal{I}}$, let $\mathsf{KON}(\widehat{\mathcal{T}})$ denote the set of all conjunctions K such that K occurs in $\widehat{\mathcal{T}}$ (in the sense explained above) and $K \sqsubseteq \bot \notin \widehat{\mathcal{T}}$. The domain $\Delta^{\widehat{\mathcal{I}}}$ consists of finite words $d = K_1 K_2 \cdots K_n \in \mathsf{KON}(\widehat{\mathcal{T}})^*$, and we use $\mathsf{tail}(d)$ to denote K_n. Define $\widehat{\mathcal{I}}$ by starting with

$$\Delta^{\widehat{\mathcal{I}}} = \mathsf{KON}(\widehat{\mathcal{T}})$$
$$A^{\widehat{\mathcal{I}}} = \{K \in \mathsf{KON}(\widehat{\mathcal{T}}) \mid K \sqsubseteq A \in \widehat{\mathcal{T}}\}$$
$$r^{\widehat{\mathcal{I}}} = \emptyset$$

Observe that since A_0 occurs in $\widehat{\mathcal{T}}$ and $A_0 \sqsubseteq \bot \notin \widehat{\mathcal{T}}$, $\Delta^{\widehat{\mathcal{I}}}$ contains the conjunction $K = A_0$ and thus $A_0^{\widehat{\mathcal{I}}} \neq \emptyset$. We finish the construction of $\widehat{\mathcal{I}}$ by exhaustively applying the following rule: if there is some $d \in \Delta^{\widehat{\mathcal{I}}}$ with $\mathsf{tail}(d) \sqsubseteq \exists r.K' \in \widehat{\mathcal{T}}$, K' maximal with this property, and $d \notin (\exists r.K')^{\widehat{\mathcal{I}}}$, then add a fresh element $e = dK'$ to $\Delta^{\widehat{\mathcal{I}}}$, add (d, K') to $r^{\widehat{\mathcal{I}}}$, and add dK' to $A^{\widehat{\mathcal{I}}}$ whenever $K' \sqsubseteq A \in \widehat{\mathcal{T}}$.

We first show that $\widehat{\mathcal{I}}$ is a model of $\widehat{\mathcal{T}}$, which amounts to a case distinction over the forms of CIs that can be present in $\widehat{\mathcal{T}}$, in each case relying on the fact that $\widehat{\mathcal{T}}$ is closed under the rules of the calculus. Details are provided in the appendix.

Lemma 10 $\widehat{\mathcal{I}} \models \widehat{\mathcal{T}}$.

It remains to show that $\widehat{\mathcal{I}}$ is a model of $\mathcal{T}_\mathsf{f}$, which is significantly more difficult to prove than Lemma 10 due to the fact that $\mathcal{T}_\mathsf{f}$ is obtained by reversing all cycles in $\mathcal{T}$ whereas the calculus is more careful to reverse only the 'relevant' ones, as explained above. We start with the observation that, when constructing $\mathcal{T}_\mathsf{f}$, it suffices to close only maximal cycles. More precisely, a cycle $K_1, r_1, K_2, \ldots, K_n$ in a TBox $\mathcal{T}$ is *maximal* if K_{j+1} is maximal with $\mathcal{T} \models K_j \sqsubseteq \exists r_j.K_{j+1}$, for $1 \leq j < n$. Let $\mathcal{T}_\mathsf{f}^{\max}$ be the variation of $\mathcal{T}_\mathsf{f}$ that is obtained by reversing only maximal cycles.

Lemma 11 *$\mathcal{T}_\mathsf{f}$ is equivalent to $\mathcal{T}_\mathsf{f}^{max}$.*

To finish the proof of Theorem 8, let $\mathcal{T}_\mathsf{f}^0, \mathcal{T}_\mathsf{f}^1, \ldots$ be the sequence of TBoxes obtained by starting with $\mathcal{T}_\mathsf{f}^0 = \mathcal{T}$ and then exhaustively closing maximal cycles, that is, $\mathcal{T}_\mathsf{f}^{\max}$ is the limit of this sequence. In the appendix, we prove by induction on i that $\widehat{\mathcal{I}}$ is a model of each $\mathcal{T}_\mathsf{f}^i$, thus of $\mathcal{T}_\mathsf{f}$.

We now briefly consider an extension of our algorithm to ABox consistency, with Figure 2 showing the additional rules. Instead of starting with only a TBox $\mathcal{T}$, the algorithm now begins with a set $\mathcal{T} \cup \mathcal{A}$, where $\mathcal{T}$ is a TBox and $\mathcal{A}$ an ABox, and then exhaustively applies rules **R1** to **R12**. In rules **R10**

$$\textbf{R10}\ \frac{K(a) \quad K \sqsubseteq A}{A(a)} \qquad \textbf{R11}\ \frac{K(a) \quad r(a,b) \quad K \sqsubseteq \forall r.K'}{K'(b)}$$

$$\textbf{R12}\ \frac{\begin{array}{c} K_1(a) \quad K_2(a) \quad r(a,b) \quad K(b) \quad K_1 \sqsubseteq (\leqslant 1\ r\ A) \\ K_2 \sqsubseteq \exists r.K' \quad K \sqsubseteq A \quad K' \sqsubseteq A \end{array}}{K'(b)}$$

Figure 2: Additional Inference Rules

to **R12**, $K(a)$ is an abbreviation for $A_1(a) \cdots A_k(a)$ when $K = \{A_1, \dots, A_k\}$. Recall that rules **R1** and **R2** only apply when the conjunction in their precondition occurs in the partially completed TBox. For the extension with ABoxes, an additional way for K to *occur* is that, for some ABox individual a, $K = \{A \mid A(a) \text{ is in the partial completion}\}$. It is easy to see that rule application still terminates after exponentially many steps. Let Γ be the set of concept inclusions and ABox assertions finally generated. The algorithm is sound and complete in the sense that $\mathcal{A}$ is finitely inconsistent w.r.t. $\mathcal{T}$ iff there is an ABox individual a and a conjunction K such that Γ contains both $K(a)$ and $K \sqsubseteq \bot$. To prove this, one updates the construction of $\widehat{\mathcal{I}}$ by starting with an initial interpretation defined by setting $\Delta^{\widehat{\mathcal{I}}} = \mathsf{Ind}(\mathcal{A})$, $r^{\widehat{\mathcal{I}}} = \{(a,b) \mid r(a,b) \in \mathcal{A}\}$, and $A^{\widehat{\mathcal{I}}} = \{a \in \mathsf{Ind}(\mathcal{A}) \mid A(a) \in \Gamma\}$. The rest of the construction of $\widehat{\mathcal{I}}$ is as before. It is not hard to adapt the proof of Lemma 10 to show that $\widehat{\mathcal{I}}$ satisfies all inclusions and assertions in Γ. As in the case of finite satisfiability, it thus remains to prove that $\widehat{\mathcal{I}}$ is a model of $\mathcal{T}_\mathsf{f}$. Fortunately, the proof of goes through without modification.

Apart from providing a basis for practical implementations, our algorithm also yields an ExpTime upper bound for finite ABox consistency in Horn-$\mathcal{ALCFI}$. This result is known from (Lutz, Sattler, and Tendera 2005), where it is shown that ABox consistency in the non-Horn version of $\mathcal{ALCQI}$ is in ExpTime. A matching lower bound can be derived from (Baader, Brandt, and Lutz 2008) where an ExpTime lower bound is established for unrestricted subsumption in (the $\mathcal{ELI}$ fragment of) Horn-$\mathcal{ALCFI}$; the proof can easily be adapted to finite satisfiability.

Theorem 12 *Finite satisfiability and finite ABox consistency in Horn-$\mathcal{ALCQI}$ are* ExpTime*-complete.*

5 Query Answering in the Finite

In the ontology-based data access (OBDA) paradigm, the central reasoning problem is answering database-style queries over ABoxes in the presence of a DL TBox. In this section, we study the finite model version of this problem, assuming that queries are positive existential queries (PEQs) and that TBoxes are formulated in Horn-$\mathcal{ALCFI}$. We show that, as in the case of ABox consistency, finite PEQ answering can be reduced to unrestricted PEQ answering by reversing finmod cycles in the TBox. This result enables the use of algorithms for unrestricted PEQ answering also in the finite case. It also allows us to show that finite PEQ answering w.r.t. Horn-$\mathcal{ALCFI}$ TBoxes is ExpTime-complete regarding combined complexity, and PTime-complete regarding data complexity.

We start with a brief introduction of positive existential queries and of query answering. For simplicity, we concentrate on Boolean queries, that is, queries without answer variables. It is, however, easy to adapt all techniques established in this section to the case of queries with answer variables. A *(Boolean) positive existential query (PEQ)* q takes the form $\exists \mathbf{x}\, \varphi(\mathbf{x})$ where φ is built from atoms of the form $A(x)$ and $r(x,y)$ using conjunction and disjunction, with x, y variables from $\mathbf{x}$, A a concept name, and r a role name. Let $\mathcal{I}$ be an interpretation and $q = \exists \mathbf{x}\, \varphi$ a PEQ. A *match* of q in $\mathcal{I}$ is a mapping $\pi : \mathbf{x} \rightarrow \Delta^\mathcal{I}$ such that φ evaluates to true unter the valuation that assigns true to an atom $A(x)$ in φ iff $\pi(x) \in A^\mathcal{I}$ and true to an atom $r(x,y)$ in φ iff $(\pi(x), \pi(y)) \in r^\mathcal{I}$. We write $\mathcal{I} \models q$ if there is a match of q in $\mathcal{I}$. For an ABox $\mathcal{A}$ and a TBox $\mathcal{T}$, we write $\mathcal{A}, \mathcal{T} \models q$ (resp. $\mathcal{A}, \mathcal{T} \models_\mathsf{fin} q$) if $\mathcal{I} \models q$ for all models (resp. finite models) $\mathcal{I}$ of $\mathcal{T}$ and $\mathcal{A}$. We then say that $\mathcal{A}, \mathcal{T}$ *entails* (resp. *finitely entails*) q. The problem that we are interested in is *finite query entailment*, that is, given an ABox $\mathcal{A}$, a TBox $\mathcal{T}$, and a query q, to decide whether $\mathcal{A}, \mathcal{T} \models_\mathsf{fin} q$. We will study both the combined complexity and the data complexity of this problem. When studying combined complexity, all of $\mathcal{A}$, $\mathcal{T}$, and q are considered an input. In the case of data complexity, $\mathcal{T}$ and q are assumed to be fixed and q is the only input.

The main result of this section is the following theorem, where $\mathcal{T}_\mathsf{f}$ is the TBox obtained from $\mathcal{T}$ by exhaustively reversing finmod cycles, exactly as in Section 3.

Theorem 13 *Let $\mathcal{T}$ be a Horn-$\mathcal{ALCFI}$ TBox and $\mathcal{A}$ an ABox that is finitely consistent w.r.t. $\mathcal{T}$. For any PEQ q,*

$$\mathcal{A}, \mathcal{T} \models_\mathsf{fin} q \text{ iff } \mathcal{A}, \mathcal{T}_\mathsf{f} \models q$$

The proof of the "$\Leftarrow$" direction is trivial. Indeed, if $\mathcal{A}, \mathcal{T} \not\models_\mathsf{fin} q$, then there is a finite model $\mathcal{I}$ of $\mathcal{A}$ and $\mathcal{T}$ such that $\mathcal{I} \not\models q$. Since every finite model of $\mathcal{T}$ is also a model of $\mathcal{T}_\mathsf{f}$ by Lemma 4, it follows that $\mathcal{A}, \mathcal{T}_\mathsf{f} \not\models q$.

For the proof of the "$\Rightarrow$" direction, we use a well-known (infinite) canonical model $\mathcal{U}$ of $\mathcal{A}$ and $\mathcal{T}_\mathsf{f}$, constructed by starting with the following initial interpretation

$$\begin{aligned} \Delta^\mathcal{U} &= \mathsf{Ind}(\mathcal{A}) \\ A^\mathcal{U} &= \{a \in \mathsf{Ind}(\mathcal{A}) \mid \mathcal{A}, \mathcal{T}_\mathsf{f} \models A(a)\} \\ r^\mathcal{U} &= \{(a,b) \mid r(a,b) \in \mathcal{A}\} \end{aligned}$$

and then exhaustively applying the following completion rule: for all $d \in \Delta^\mathcal{U}$ such that $\mathcal{T}_\mathsf{f} \models \mathsf{tp}_\mathcal{U}(d) \sqsubseteq \exists r.t'$, where t' is maximal with this property and $d \notin (\exists r.t')^\mathcal{U}$, add a fresh element d' to $\Delta^\mathcal{U}$, the edge (d, d') to $r^\mathcal{U}$, and d' to the interpretation $A^\mathcal{U}$ of all concept names $A \in t'$.

The following properties of $\mathcal{U}$ are well-known and the reason for why $\mathcal{U}$ is called canonical (Krisnadhi and Lutz 2007; Eiter et al. 2008; Ortiz, Rudolph, and Šimkus 2011).

Lemma 14

1. *$\mathcal{U}$ is a model of $\mathcal{A}$ and of $\mathcal{T}_\mathsf{f}$;*
2. *For any PEQ q, we have that $\mathcal{A}, \mathcal{T}_\mathsf{f} \models q$ iff $\mathcal{U} \models q$.*

By Point 2 of Lemma 14, we can establish the "⇒" direction of Theorem 13 by showing that $\mathcal{A}, \mathcal{T} \models_{\mathsf{fin}} q$ implies $\mathcal{U} \models q$. The proof makes intense use of homomorphisms. For interpretations $\mathcal{I}_1, \mathcal{I}_2$, a *homomorphism from $\mathcal{I}_1$ to $\mathcal{I}_2$* is a function $h : \Delta^{\mathcal{I}_1} \to \Delta^{\mathcal{I}_2}$ such that

1. $h(a) = a$ for all $a \in \mathsf{N_I}$;
2. $d \in A^{\mathcal{I}_1}$ implies $h(d) \in A^{\mathcal{I}_2}$ for all concept names A;
3. $(d, e) \in r^{\mathcal{I}_1}$ implies $(h(d), h(e)) \in r^{\mathcal{I}_2}$ for all (possibly inverse) roles r.

For $n > 0$, an *n-substructure* of an interpretation $\mathcal{I}$ is an interpretation $\mathcal{I}'$ obtained from $\mathcal{I}$ by selecting a domain $\Delta^{\mathcal{I}'} \subseteq \Delta^{\mathcal{I}}$ with at most n elements and restricting $\mathcal{I}$ to $\Delta^{\mathcal{I}'}$. To show that $\mathcal{A}, \mathcal{T} \models_{\mathsf{fin}} q$ implies $\mathcal{U} \models q$, it suffices to establish the following.

Proposition 15 *For every $n_0 > 0$, there is a finite model $\mathcal{J}_{n_0}$ of $\mathcal{A}$ and $\mathcal{T}$ such that there is a homomorphism from any n_0-substructure of $\mathcal{J}_{n_0}$ to $\mathcal{U}$.*

In fact, $\mathcal{A}, \mathcal{T} \models_{\mathsf{fin}} q$ implies $\mathcal{J}_{n_0} \models q$ and thus there is an n_0-substructure $\mathcal{J}$ of $\mathcal{J}_{n_0}$ with $\mathcal{J} \models q$, where n_0 is the number of variables in q. The latter is witnessed by a match π. By Proposition 15, there is a homomorphism h from $\mathcal{J}$ to $\mathcal{U}$ and thus a match of q in $\mathcal{U}$ can be found by composing π with h.

We construct the model $\mathcal{J}$ from Proposition 15 by modifying the finite model $\mathcal{I}$ constructed in Section 3. For two reasons, the finite model $\mathcal{I}$ constructed in Section 3 need not satisfy the condition formulated for $\mathcal{J}_{n_0}$ in Proposition 15.

1. $\mathcal{I}$ can contain paths of length $\leq n_0$ that do not exist in $\mathcal{U}$.
2. $\mathcal{I}$ can contain cycles that do not exclusively consist of ABox elements, while no such cycles are present in $\mathcal{U}$.

Let us start with Problem 1 above. There are, in turn, two sources for paths in $\mathcal{I}$ that we cannot reproduce in $\mathcal{U}$.

(i) Application of $(\mathbf{c3})$ can generate a path $(d_1, d) \in r^{\mathcal{I}}$, $(d, d_2) \in s^{\mathcal{I}}$ such that $\mathsf{tp}_{\mathcal{I}}(d_1) \to_r \mathsf{tp}_{\mathcal{I}}(d) \;{}_s\!\!\leftarrow \mathsf{tp}_{\mathcal{I}}(d_2)$ and d is not identified by an ABox element. Such situations are not necessarily reproducible in $\mathcal{U}$. As a concrete example, consider

$$\mathcal{A} = \{\, B_1(a), B_2(b) \,\}$$
$$\mathcal{T} = \{\, B_1 \sqsubseteq \exists r.A,\ B_2 \sqsubseteq \exists r.A \,\}.$$

The problematic path is $(a, d_t) \in r^{\mathcal{I}}$, $(d_t, b) \in (r^-)^{\mathcal{I}}$ with $t = \{A\}$.

(ii) Application of $(\mathbf{c2})$ can result in similar a situation as above, but where the middle element d is replaced with a sequence of elements $e_0, \dots, e_k$ such that $(e_i, e_{i+1}) \in r_i^{\mathcal{I}}$ for all $i < k$ (for some roles $r_0, \dots, r_{k-1}$) and

$$\mathsf{tp}_{\mathcal{I}}(e_0)\ {}^1\!\!\leftrightarrow^1_{r_1} \cdots\ {}^1\!\!\leftrightarrow^1_{r_{k-1}} \mathsf{tp}_{\mathcal{I}}(e_k). \tag{18}$$

For a very simple example, take

$$\mathcal{A} = \{\, B_1(a),\ B_2(b) \,\}$$

and assume that $\mathcal{T}$ is such that $B_1\ {}^1\!\!\leftrightarrow^1_r\ B_2$. Then an application of $(\mathbf{c2})$ will simply add $r(a, b)$, an edge that does not exist in $\mathcal{U}$.

To obtain the desired model $\mathcal{J}_{n_0}$ from Proposition 15, we first solve Problems (i) and (ii) above, and then Problem 2. To make precise what we mean by this, we introduce bounded simulations, a weakening of homomorphisms. A *bounded simulation of $\mathcal{I}_1$ in $\mathcal{I}_2$* is a relation $\rho \subseteq \Delta^{\mathcal{I}_1} \times \mathbb{N} \times \Delta^{\mathcal{I}_2}$ such that for all $(d, i, e) \in \rho$, the following conditions are satisfied:

1. if $d \in A^{\mathcal{I}_1}$, then $e \in A^{\mathcal{I}_2}$;
2. if $i > 0$ and $(d, d') \in r^{\mathcal{I}_1}$ for some (possibly inverse) role r, then there is an $e' \in \Delta^{\mathcal{I}_2}$ with $(e, e') \in r^{\mathcal{I}_2}$ and $(d', i-1, e') \in \rho$.

We write $(\mathcal{I}_1, d) \preceq_k (\mathcal{I}_2, e)$, for $d \in \Delta^{\mathcal{I}_1}$ and $e \in \Delta^{\mathcal{I}_2}$, if there is a bounded simulation of $\mathcal{I}_1$ in $\mathcal{I}_2$ such that $(d, k, e) \in \rho$ and for all $a \in \mathsf{N_I} \cap \Delta^{\mathcal{I}_1}$, we have $(a, k, a) \in \rho$. Then $\mathcal{I}_1 \preceq_k \mathcal{I}_2$ denotes that for every $d \in \Delta^{\mathcal{I}_1}$, there is an $e \in \Delta^{\mathcal{I}_2}$ with $(\mathcal{I}_1, d) \preceq_k (\mathcal{I}_2, e)$. We write $(\mathcal{I}_1, d) \sim_k (\mathcal{I}_2, e)$ if $(\mathcal{I}_1, d) \preceq_k (\mathcal{I}_2, e)$ and vice versa.

With solving Problems (i) and (ii), we mean to establish the following intermediate result.

Proposition 16 *For every $n_0 > 0$, there is a finite model $\mathcal{I}_{n_0}$ of $\mathcal{A}$ and $\mathcal{T}$ such that $\mathcal{I}_{n_0} \preceq_{n_0} \mathcal{U}$.*

To remove the undesired paths illustrated in (i) above, we modify the construction of $\mathcal{I}$ by replacing the elements d_t, $t \in \mathsf{TP}(\mathcal{T}_{\mathsf{f}})$, that are introduced at the beginning of the construction of $\mathcal{I}$ and used as 'targets' for role edges introduced by applications of $(\mathbf{c3})$. In the modified construction, we instead introduce one $(\mathbf{c3})$-target for each *n_0-bounded simulation type*, which is an equivalence class of $\sim_{n_0}$ on the set of all pointed interpretations $(\mathcal{I}_1, d)$. In the example given in (i) above, the result is that the two existential restrictions would no longer be witnessed by the same d_t because the 1-simulation type of the witnesses are different (one has an r-predecessor in B_1, the other in B_2). Since simulations need only to consider symbols that occur in the (fixed) ABox $\mathcal{A}$ and (fixed) TBox $\mathcal{T}$, there are only finitely many n_0-simulation types and thus finiteness of $\mathcal{I}$ is not compromised.

Undesired paths of type (ii) are avoided by modifying the $(\mathbf{c2})$ rule so that the sequences (18) are of length exceeding n_0 and thus the highlighted problem which involves both ends of the sequence is not 'visible' in n_0-substructures. We also include an initial piece of the canonical model $\mathcal{U}$ for $\mathcal{A}$ and $\mathcal{T}_{\mathsf{f}}$ of depth n_0 in the initial version of $\mathcal{I}$ to avoid the undesired 'shortcuts' between ABox elements illustrated by the example given in (ii) above.

The construction is spelled out in full detail in the appendix. We have actually omitted some aspects in the overview above for the sake of a clearer exposition, such as the fact that we first exhaustively apply rules $(\mathbf{c1})$ and $(\mathbf{c2})$, followed by exhaustive application of $(\mathbf{c3})$ (the latter two in their modified versions), and that we actually cannot include in the initial $\mathcal{I}$ *all* n_0-bounded simulation types, but must select only the 'relevant' ones. This finishes the proof of Proposition 16.

To solve Problem 2 above and thus obtain the model $\mathcal{J}_{n_0}$ stipulated by Proposition 15, we have to eliminate all non-ABox-cycles of size at most n_0 in the model $\mathcal{I}_{n_0}$ delivered by Proposition 16. This is achieved by taking the product with a suitable finite group of high girth, a technique championed by Otto (2012). Details are provided in the appendix. This finishes the proof of Theorem 13.

Apart from enabling the use of algorithms for unrestricted PEQ answering also in the finite case, Theorem 13 yields tight complexity bounds for finite PEQ entailment.

Theorem 17 *Finite PEQ entailment in Horn-$\mathcal{ALCFI}$ is decidable,* ExpTime*-complete in combined complexity, and* PTime*-complete in data complexity.*

Proof.(sketch) For the unrestricted case, an ExpTime lower bound is in (Baader, Brandt, and Lutz 2008) and a PTime one in (Calvanese et al. 2006). Both results can easily be adapted to the finite case. The upper bounds can be proved using the following straightforward algorithm for PEQ entailment, which resembles existing algorithms such as those presented in (Krisnadhi and Lutz 2007; Eiter et al. 2008; Calì, Gottlob, and Lukasiewicz 2009; Ortiz, Rudolph, and Šimkus 2011). Assume that an input ABox $\mathcal{A}$, TBox $\mathcal{T}$, and PEQ q are given, and let n_0 be the number of variables in q. As a consequence of Theorem 3, finite satisfiability w.r.t. $\mathcal{T}$ coincides with unrestricted satisfiability w.r.t. $\mathcal{T}_\mathsf{f}$. Using our algorithm for computing finite satisfiability in Horn-$\mathcal{ALCFI}$ in ExpTime, we can thus compute the set $\mathsf{TP}(\mathcal{T}_\mathsf{f})$ of types for $\mathcal{T}_\mathsf{f}$ without computing $\mathcal{T}_\mathsf{f}$ or explicitly reasoning w.r.t. this exponentially large TBox. Let $\mathcal{A}'$ be the extension of $\mathcal{A}$ with assertions $\{A(a_t) \mid A \in t\}$ for each $t \in \mathsf{TP}(\mathcal{A})$. Now compute an initial piece $\mathcal{U}'$ of the canonical model $\mathcal{U}$ of $\mathcal{A}'$ and $\mathcal{T}_\mathsf{f}$, namely its restriction to depth n_0. Similar to the computation of $\mathsf{TP}(\mathcal{T}_\mathsf{f})$ above, we can do this by using finite subsumption w.r.t. $\mathcal{T}$ instead of unrestricted subsumption w.r.t. $\mathcal{T}_\mathsf{f}$. It is not difficult to prove that $\mathcal{U}' \models q$ iff $\mathcal{U} \models q$. To check whether $\mathcal{U}' \models q$ within the desired time bounds, we can simply enumerate all possible maps of variables in q to elements of $\mathcal{U}'$ and check whether any such map is a match. ❏

Note that decidability of PEQ entailment in Horn-$\mathcal{ALCFI}$ was expected given a result by Pratt-Hartmann which states that finite CQ answering for the two-variable guarded fragment of first-order logic extended with counting quantifiers is decidable (Pratt-Hartmann 2009). We assume that his proof can be extended to unions of conjunctive queries (UCQs), thus to PEQs. Pratt-Hartmann also analyses the data complexity of finite CQ answering in his logic, but finds it to be coNP-complete. He does not analyse combined complexity. Theorem 17 suggests that PEQ entailment in Horn-$\mathcal{ALCFI}$ has the same complexity in finite and in unrestricted models. For the unrestricted case, PTime-completeness in data complexity follows from the results in (Hustadt, Motik, and Sattler 2007), and ExpTime-completeness in combined complexity is proved in (Eiter et al. 2008) for UCQs. We assume that the techniques in that paper extend to PEQs.

6 From Horn-$\mathcal{ALCFI}$ to Horn-$\mathcal{ALCQI}$

Our results for finite satisfiability and finite subsumption (the reasoning tasks that do not involve ABoxes) extend in a straightforward way from Horn-$\mathcal{ALCFI}$ to Horn-$\mathcal{ALCQI}$. In particular, we can convert a Horn-$\mathcal{ALCQI}$ TBox $\mathcal{T}$ into a Horn-$\mathcal{ALCFI}$ TBox $\mathcal{T}'$ such that finite (un)satisfiability is preserved by replacing each CI $K \sqsubseteq (\geqslant n\ r\ K')$ in $\mathcal{T}$ with the following inclusions, for $1 \leq i < j \leq n$:

$$K \sqsubseteq \exists r.B_i, \quad B_i \sqsubseteq K', \quad B_i \sqcap B_j \sqsubseteq \bot \qquad (*)$$

While an easy unraveling argument can be used to prove that this reduction is correct in the presence of infinite models, more care is required in the finite case (see appendix).

Proposition 18 *$\mathcal{T}$ is finitely satisfiable iff $\mathcal{T}'$ is finitely satisfiable.*

It follows from Proposition 18 and Theorem 3 that a Horn-$\mathcal{ALCQI}$ TBox $\mathcal{T}$ is finitely satisfiable iff $(\mathcal{T}')_\mathsf{f}$ is satisfiable. Actually, it is not hard to see that this is the case iff $\mathcal{T}_\mathsf{f}$ (the result of applying cycle reversion directly to the Horn-$\mathcal{ALCQI}$ TBox, ignoring all inclusions $A \sqsubseteq (\geqslant n\ r\ C)$) is satisfiable because if any of the existential restrictions in $\mathcal{T}' \setminus \mathcal{T}$ is involved in a finmod cycle, then a simple semantic argument shows that both $\mathcal{T}_\mathsf{f}$ and $(\mathcal{T}')_\mathsf{f}$ are unsatisfiable. Proposition 18 also enables the use of our consequence-based procedure for deciding finite satisfiability in Horn-$\mathcal{ALCQI}$.

It is not immediately obious how to extend $(*)$ and Proposition 18 to ABox consistency and instance checking. We believe, though, that it is not too hard to modify the proof of Theorem 3 for Horn-$\mathcal{ALCQI}$, to adapt the consequence-based procedure to allow a direct treatment of Horn-$\mathcal{ALCQI}$ TBoxes without prior reduction to Horn-$\mathcal{ALCFI}$, and to extend all model constructions underlying our results about PEQ entailment to Horn-$\mathcal{ALCQI}$. In particular, such a direct approach should yield ExpTime/ PTime upper bounds for PEQ entailment in Horn-$\mathcal{ALCQI}$ even when the numbers in at least restrictions are coded in binary (note that, in this case, the translation $(*)$ incurs an exponential blowup).

7 Future Work

As future research, it would be interesting to extend the results in this paper to Horn-$\mathcal{SHIQ}$, that is, to add role hierarchies and transitive roles. Reducing out role hierarchies does not seem easily possible in the finite,[2] so they would have to be built directly into all constructions. For query entailment, we expect transitive roles to cause significant additional challenges, see for example (Eiter et al. 2009; Mosurovic et al. 2013). In particular, transitive roles result in an additional way in which the finite model property is lost, illustrated by the TBox $\mathcal{T} = \{A \sqsubseteq \exists r.A, \mathsf{trans}(r)\}$ and the conjunctive query $q = \exists x\, r(x,x)$. We have $\{A(a)\}, \mathcal{T} \not\models q$, but $\{A(a)\}, \mathcal{T} \models_\mathsf{fin} q$ although neither counting nor inverse roles are present (the TBox $\mathcal{T}$ is formulated in the DL $\mathcal{EL}_\mathsf{trans}$). Finite model reasoning in versions of Datalog$^\pm$ that extend $\mathcal{EL}_\mathsf{trans}$ has recently been studied in (Gogacz and Marcinkowski 2013b; 2013a).

In this paper, we have not analyzed the size of finite models. It is, however, easy to prove a double exponential lower bound on the size of finite models for satisfiability in Horn-$\mathcal{ALCFI}$ by enforcing a tree of exponential depth in which no two elements can be identical. A matching upper bound follows from Pratt-Hartmann's result that every finitely satisfiable formula in first-order logic with two variables and

[2] In contrast to what we have claimed in the workshop predecessor of this paper (Ibáñez-García, Lutz, and Schneider 2013).

counting quantifiers has a model of at most double exponential size (Pratt-Hartmann 2005). Analyzing the size of finite (counter)models for query entailment is left as future work.

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A Proofs for Section 3

Lemma 5. $\prec^+$ *is a strict partial order.*

Proof. Since $\prec^+$ is transitive by definition, it remains to establish irreflexivity and asymmetry. To this end, it suffices to show that $\prec$ is acyclic in the sense that there are no type partitions $P_0, \dots, P_n$, $n \geq 0$, such that $P_0 \prec \cdots \prec P_n = P_0$. Assume to the contrary that there are such $P_0, \dots, P_n$. By reversing the order, we can assume that $P_0 \succ \cdots \succ P_n = P_0$. Then there are, for each $i < n$, types $t_i \in P_i$ and $t'_{i+1} \in P_{i+1}$ such that $t_i \subsetneq t'_{i+1}$. For uniformity, set $t_n = t_0$ and $t'_0 = t'_n$.

Let $i < n$. By definition of type partitions and since $t \;{}^1{\leftrightarrow}^1_r\; t'$ implies $t' \;{}^1{\leftrightarrow}^1_{r^-}\; t$ for all types t, t' and roles r, we can derive from $t_i, t'_i \in P_i$ the existence of types $s_{0,i}, \dots, s_{k_i,i} \in P_i$, $k_i \geq 0$, and roles $r_{0,i}, \dots, r_{k_i-1,i}$ such that

$$t_i = s_{0,i} \;{}^1{\leftrightarrow}^1_{r_{0,i}}\; s_{1,i} \;{}^1{\leftrightarrow}^1_{r_{1,i}} \cdots {}^1{\leftrightarrow}^1_{r_{k_i-1,i}}\; s_{k_i,i} = t'_i.$$

For each i, we thus find a sequence

$$t_i, r_{0,i}, s_{1,i}, \dots, s_{k_i-1,i}, r_{k_i-1,i}, t'_i \qquad (*)$$

that satisfies the prerequisites for finmod cycles, namely

$$\mathcal{T} \models s_{j,i} \sqsubseteq \exists r_{j,i}.s_{j+1,i} \qquad (19)$$

$$\mathcal{T} \models s_{j+1,i} \sqsubseteq (\leqslant 1\; r_{j,i}\; s_{j,i}) \qquad (20)$$

for all $j = 0, \dots, k_i$ (but this sequence need not be a finmod cycle since $t_i = t'_i$ is not guaranteed). Note that we cannot have $k_i = 0$ for all i, since then

$$t_0 \subsetneq t'_1 = t_1 \subsetneq t'_2 = t_2 \subsetneq \cdots \subsetneq t'_n = t_n,$$

in contradiction to $t_n = t_0$. In the following, we can thus assume that $k_i > 0$ for at least one i.

Because of (20), we have $\mathcal{T}_{\mathsf{f}} \models t_i \sqsubseteq \exists r_{0,i}.s_{1,i}$ and $\mathcal{T}_{\mathsf{f}} \models s_{1,i} \sqsubseteq (\leqslant 1\; r^-_{0,i}\; t_i)$. Because of $t_i \subsetneq t'_{i+1}$, we thus obtain $\mathcal{T}_{\mathsf{f}} \models t'_{i+1} \sqsubseteq \exists r_{0,i}.s_{1,i}$ and $\mathcal{T}_{\mathsf{f}} \models s_{1,i} \sqsubseteq (\leqslant 1\; r^-_{0,i}\; t'_{i+1})$. Consequently, the following sequences also satisfy conditions (19) and (20):

$$t'_n, r_{0,n-1}, s_{1,n-1}, \dots, s_{k_{n-1}-1,n-1}, r_{k_{n-1}-1,n-1}, t'_{n-1}$$
$$t'_{n-1}, r_{0,n-2}, s_{1,n-2}, \dots, s_{k_{n-1}-1,n-2}, r_{k_{n-1}-1,n-2}, t'_{n-2}$$
$$\vdots$$
$$t'_1, r_{0,0}, s_{1,0}, \dots, s_{k_0-1,0}, r_{k_0-1,0}, t'_0.$$

Since $t'_0 = t'_n$, we can concatenate all these sequences to a finmod cycle. As $k_i > 0$ for at least one i, this cycle is non-empty, and the construction of $\mathcal{T}_{\mathsf{f}}$ ensures that the reversed cycle is also present in $\mathcal{T}_{\mathsf{f}}$. Let us assume w.l.o.g. that $k_{n-1} > 0$. The presence of the reversed cycle yields $\mathcal{T}_{\mathsf{f}} \models s_{1,n-1} \sqsubseteq \exists r^-_{0,n-1}.t'_n$. Since $t_n \;{}^1{\leftrightarrow}^1_{r_{0,n-1}}\; s_{1,n-1}$, we have $\mathcal{T}_{\mathsf{f}} \models s_{1,n-1} \sqsubseteq \exists r^-_{0,n-1}.t_{n-1}$ and t_{n-1} is maximal with this property. This is a contradiction to $t_{n-1} \supsetneq t'_n$. ❑

Proof of Theorem 6

To eliminate case distinctions later on, for each $\lambda = s \;{}^1{\leftrightarrow}^1_r\; s'$ let λ^- denote $s' \;{}^1{\leftrightarrow}^1_{r^-}\; s$. We start with a technical lemma that will be used below to show that applications of $(\mathbf{c2})$ preserve all invariants. The statement by the lemma is meant to refer to a concrete application of $(\mathbf{c2})$.

Lemma 19 *If $\lambda = s \;{}^1{\leftrightarrow}^1_r\; s'$ with $s, s' \in P$ and $(d, d') \in \pi_\lambda$, then $\mathsf{tp}_{\mathcal{I}}(d) = s$ and $\mathsf{tp}_{\mathcal{I}}(d') = s'$.*

Proof. Let $\lambda = s \;{}^1{\leftrightarrow}^1_r\; s'$ and $(d, d') \in \pi_\lambda$. We first show that $\mathsf{tp}_{\mathcal{I}}(d) = s$. Since d is in the domain of π_λ, we have $d \in \Delta_s$ or $d \in X^{\mathcal{I}}_{\lambda,1}$. In the former case, $\mathsf{tp}_{\mathcal{I}}(d) = s$ is immediate by construction of $\mathcal{I}$. Thus assume that $d \in X^{\mathcal{I}}_{\lambda,1}$. Then $s \subseteq \mathsf{tp}_{\mathcal{I}}(d)$. From λ, we obtain $\mathcal{T}_{\mathsf{f}} \models \mathsf{tp}_{\mathcal{I}}(d) \sqsubseteq \exists r.s'$. Let $\hat{s}' \supseteq s'$ be maximal such that $\mathcal{T}_{\mathsf{f}} \models \mathsf{tp}_{\mathcal{I}}(d) \sqsubseteq \exists r.\hat{s}'$. Note that, by λ and since $s \subseteq \mathsf{tp}_{\mathcal{I}}(d)$ and $s' \subseteq \hat{s}'$, we have $\mathcal{T}_{\mathsf{f}} \models \hat{s}' \sqsubseteq (\leqslant 1\; r\; \mathsf{tp}_{\mathcal{I}}(d))$. Thus $\mathsf{tp}_{\mathcal{I}}(d) \to^1_r \hat{s}'$.

Next, observe that $\hat{s}' \to^1_{r^-} \mathsf{tp}_{\mathcal{I}}(d)$. If this was not the case, then $(\mathbf{c1})$ would be applicable to d and, since its application is preferred over applications of $(\mathbf{c2})$, generate an $e \in \Delta^{\mathcal{I}}$ such that $(d, e) \in r^{\mathcal{I}}$ and $e \in (\hat{s}')^{\mathcal{I}}$ before the $(\mathbf{c2})$ application considered here. This contradicts the fact that $d \in X^{\mathcal{I}}_{\lambda_1}$, which implies that $d \notin (\exists r.s)^{\mathcal{I}}$ by the time when $(\mathbf{c2})$ was applied.

In summary, we have established that $\lambda' = \mathsf{tp}_{\mathcal{I}}(d) \;{}^1{\leftrightarrow}^1_r\; \hat{s}'$ holds. Now assume to the contrary of what we have to show that $s \subsetneq \mathsf{tp}_{\mathcal{I}}(d)$. Recall that P is the type class that the current $(\mathbf{c2})$ application treats, and that $s, s' \in P$. By λ', there is a type class P' with $\mathsf{tp}_{\mathcal{I}}(d), \hat{s}' \in P'$. Since $s \subsetneq \mathsf{tp}_{\mathcal{I}}(d)$, we have $P' \prec P$. Since $d \in X^{\mathcal{I}}_{\lambda,1}$, we had $d \notin (\exists r.s')^{\mathcal{I}}$ before the current rule application, thus also $d \notin (\exists r.\hat{s}')^{\mathcal{I}}$. Summing up, before the current rule application we had $\mathsf{tp}_{\mathcal{I}}(d), \hat{s}' \in P'$, $\lambda' = \mathsf{tp}_{\mathcal{I}}(d) \;{}^1{\leftrightarrow}^1_r\; \hat{s}'$, $d \in \mathsf{tp}_{\mathcal{I}}(d)$, and $d \notin (\exists r.\hat{s}')^{\mathcal{I}}$. Consequently, rule $(\mathbf{c2})$ was applicable also to type class P'. Since $P' \prec P$ and with the preference order that $(\mathbf{c2})$ imposes on type classes, this contradicts that the current application is treating P.

It remains to show that $\mathsf{tp}_{\mathcal{I}}(d') = s'$. The argument is exactly the same as above, with r^- playing the role of r, s' playing the role of s and vice versa, λ^- playing the role of λ, and $\mathsf{tp}_{\mathcal{I}}(d')$ playing the role of $\mathsf{tp}_{\mathcal{I}}(d)$ and vice versa. ❑

Satisfaction of Invariants It is easy to verify that the initial interpretation $\mathcal{I}$ satisfies all invariants. Indeed, $(\mathbf{i1})$ is trivially satisfied. Since $r^{\mathcal{I}} \setminus (\mathsf{Ind}(\mathcal{A}) \times \mathsf{Ind}(\mathcal{A})) = \emptyset$, $(\mathbf{i2})$ is satisfied, too. Moreover, since $\mathcal{A}$ is consistent w.r.t. $\mathcal{T}_{\mathsf{f}}$, the standard names assumption ensures that $(\mathbf{i3})$ is satisfied. It remains to show that each of the rules $(\mathbf{c1})$ to $(\mathbf{c3})$ preserves the invariants.

Application of (c1) preserves all invariants. It is obvious that the invariants $(\mathbf{i1})$ and $(\mathbf{i2})$ are preserved with each single application of $(\mathbf{c1})$. We have to show that the same is true for $(\mathbf{i3})$. Assume that completion processed $d \in \Delta^{\mathcal{I}}$ with $\mathsf{tp}_{\mathcal{I}}(d) \to^1_r t$ and $t \not\to^1_{r^-} \mathsf{tp}_{\mathcal{I}}(d)$, and that after the application $(d, d_1) \in r^{\mathcal{I}}$, with d_1 the fresh domain element added. Assume to the contrary of what is to be proved that $\mathcal{T}_{\mathsf{f}} \models K \sqsubseteq (\leqslant 1\; r\; K')$ and there is a $d_2 \in \Delta^{\mathcal{I}}$ distinct from

d_1 such that $d \in K^{\mathcal{I}}$, $(d, d_2) \in r^{\mathcal{I}}$, and $d_1, d_2 \in {K'}^{\mathcal{I}}$. We aim to show that if such d_2 exists then $t \subseteq \mathsf{tp}_{\mathcal{I}}(d_2)$, which establishes a contradiction to the fact that $d \notin (\exists r.t)^{\mathcal{I}}$ was true before the rule application. According to (**i2**), we can distinguish the following cases:

- $\mathsf{tp}_{\mathcal{I}}(d) \rightarrow_r \mathsf{tp}_{\mathcal{I}}(d_2)$. Then $\mathcal{T}_{\mathsf{f}} \models \mathsf{tp}_{\mathcal{I}}(d) \sqsubseteq \exists r.\mathsf{tp}_{\mathcal{I}}(d_2)$ and $\mathsf{tp}_{\mathcal{I}}(d_2)$ is maximal with this property. From $\mathsf{tp}_{\mathcal{I}}(d) \rightarrow_r t$, we additionally get $\mathcal{T}_{\mathsf{f}} \models \mathsf{tp}_{\mathcal{I}}(d) \sqsubseteq \exists r.t$. Furthermore, since $K \subseteq \mathsf{tp}_{\mathcal{I}}(d)$ and $d_1, d_2 \in {K'}^{\mathcal{I}}$ implies $K' \subseteq \mathsf{tp}_{\mathcal{I}}(d_2) \cap t$, a simple semantic argument shows that $\mathcal{T}_{\mathsf{f}} \models K \sqsubseteq \exists r.(\mathsf{tp}_{\mathcal{I}}(d_2) \cup t)$. The maximality of $\mathsf{tp}_{\mathcal{I}}(d_2)$ thus implies $t \subseteq \mathsf{tp}_{\mathcal{I}}(d_2)$.

- $\mathsf{tp}_{\mathcal{I}}(d_2) \rightarrow_{r^-} \mathsf{tp}_{\mathcal{I}}(d)$. Then $\mathcal{T}_{\mathsf{f}} \models \mathsf{tp}_{\mathcal{I}}(d_2) \sqsubseteq \exists r^-.\mathsf{tp}_{\mathcal{I}}(d)$ and, additionally, we have $\mathcal{T}_{\mathsf{f}} \models \mathsf{tp}_{\mathcal{I}}(d) \sqsubseteq \exists r.t$. Since $K \subseteq \mathsf{tp}_{\mathcal{I}}(d)$ and $K' \subseteq \mathsf{tp}_{\mathcal{I}}(d_2) \cap t$, a simple semantic argument shows that $\mathcal{T}_{\mathsf{f}} \models \mathsf{tp}_{\mathcal{I}}(d_2) \sqsubseteq t$. Since $\mathsf{tp}_{\mathcal{I}}(d_2)$ is a type for $\mathcal{T}_{\mathsf{f}}$ by (**i1**), it follows that $t \subseteq \mathsf{tp}_{\mathcal{I}}(d_2)$.

- $r(d, d_2) \in \mathcal{A}$. Then $\mathsf{tp}_{\mathcal{A}}(d) = \mathsf{tp}_{\mathcal{I}}(d)$ by construction of the initial interpretation $\mathcal{I}$. Since $\mathsf{tp}_{\mathcal{I}}(d) \rightarrow_r t$, we thus have $\mathcal{T}_{\mathsf{f}} \models \mathsf{tp}_{\mathcal{A}}(d) \sqsubseteq \exists r.t$. With $r(d, d_2) \in \mathcal{A}$ and by the semantics, $t \subseteq \mathsf{tp}_{\mathcal{A}}(d_2) = \mathsf{tp}_{\mathcal{I}}(d_2)$.

Application of (c2) preserves all invariants. Invariant (**i1**) is clearly preserved by each single application of (**c2**). We have to prove that the same is true for (**i2**) and (**i3**).

Assume that (**c2**) is applied to a type class P. Note that λ holds if and only if λ^- does. Then define π_{λ^-} to be the converse of π_λ, for all $\lambda = s \; {}^1\!\!\leftrightarrow^1_r \; s'$ with r an inverse role. Note that π_λ is a bijection from $X^{\mathcal{I}}_{\lambda,1} \cup \Delta_s$ to $X^{\mathcal{I}}_{\lambda,2} \cup \Delta_{s'}$, just as in the case where r is a role name. Also note that whenever $(d, e) \in r^{\mathcal{I}}$ is added by the current application of (**c2**) with r a (potentially inverse) role, then there is a $\lambda = s \; {}^1\!\!\leftrightarrow^1_r \; s'$ such that $(d, d') \in \pi_\lambda(d)$.

To show that (**i2**) is preserved by (**c2**), consider a (potentially inverse) role r and a pair $(d, d') \in r^{\mathcal{I}}$ that has been added in a (**c2**) application. Take $\lambda = s \; {}^1\!\!\leftrightarrow^1_r \; s'$ such that $(d, d') \in \pi_\lambda(d)$. From Lemma 19, we obtain $\mathsf{tp}_{\mathcal{I}}(d) = s$ and $\mathsf{tp}_{\mathcal{I}}(d') = s'$. Consequently, $s \; {}^1\!\!\leftrightarrow^1_r \; s'$ yields $\mathsf{tp}_{\mathcal{I}}(d) \rightarrow_r \mathsf{tp}_{\mathcal{I}}(d')$ *and* $\mathsf{tp}_{\mathcal{I}}(d') \rightarrow_{r^-} \mathsf{tp}_{\mathcal{I}}(d)$.

We now show that (**i3**) is preserved by (**c2**). Let $\mathcal{T}_{\mathsf{f}} \models K \sqsubseteq (\leqslant 1 \; r \; K')$, and assume to the contrary of what is to be shown that, after some application of (**c2**), there are $(d, d_1), (d, d_2) \in r^{\mathcal{I}}$ with $d \in K^{\mathcal{I}}$, $d_1, d_2 \in {K'}^{\mathcal{I}}$, and $d_1 \neq d_2$. We distinguish the following cases:

- (d, d_1) was added by an application of (**c2**), (d, d_2) was not. By the former, there is $\lambda = s \; {}^1\!\!\leftrightarrow^1_r \; s'$ such that $(d, d_1) \in \pi_\lambda$. By Lemma 19, $\mathsf{tp}_{\mathcal{I}}(d) = s$ and $\mathsf{tp}_{\mathcal{I}}(d_1) = s'$.

  We aim to show that $s' \subseteq \mathsf{tp}_{\mathcal{I}}(d_2)$ because this means that $d \in (\exists r.s')^{\mathcal{I}}$ was true before the current appliction of (**c2**), in contradiction to d being in the domain of π_λ. Since (d, d_2) was not added by (**c2**), by (**i2**) we can distinguish the following subcases:

  – $\mathsf{tp}_{\mathcal{I}}(d) \rightarrow_r \mathsf{tp}_{\mathcal{I}}(d_2)$. Thus $\mathcal{T}_{\mathsf{f}} \models \mathsf{tp}_{\mathcal{I}}(d) \sqsubseteq \exists r.\mathsf{tp}_{\mathcal{I}}(d_2)$ and $\mathsf{tp}_{\mathcal{I}}(d_2)$ is maximal with this property. Since $\mathsf{tp}_{\mathcal{I}}(d) = s$ and by λ, $\mathcal{T}_{\mathsf{f}} \models \mathsf{tp}_{\mathcal{I}}(d) \sqsubseteq \exists r.s'$. Using the facts that $\mathcal{T}_{\mathsf{f}} \models K \sqsubseteq (\leqslant 1 \; r \; K')$, $K \subseteq \mathsf{tp}_{\mathcal{I}}(d) = s$, $K' \subseteq \mathsf{tp}_{\mathcal{I}}(d_2)$, and $K' \subseteq \mathsf{tp}_{\mathcal{I}}(d_1) = s'$, an easy semantic argument shows that $\mathcal{T}_{\mathsf{f}} \models \mathsf{tp}_{\mathcal{I}}(d) \sqsubseteq \exists r.(\mathsf{tp}_{\mathcal{I}}(d_2) \cup s')$. The maximality of $\mathsf{tp}_{\mathcal{I}}(d_2)$ thus yields $s' \subseteq \mathsf{tp}_{\mathcal{I}}(d_2)$.

  – $\mathsf{tp}_{\mathcal{I}}(d_2) \rightarrow_{r^-} \mathsf{tp}_{\mathcal{I}}(d)$. Then $\mathcal{T}_{\mathsf{f}} \models \mathsf{tp}_{\mathcal{I}}(d_2) \sqsubseteq \exists r^-.s$. By λ, we have $\mathcal{T}_{\mathsf{f}} \models s \sqsubseteq \exists r.s'$. Since $K \subseteq s$, $K \subseteq \mathsf{tp}_{\mathcal{I}}(d_2)$, $K \subseteq \mathsf{tp}_{\mathcal{I}}(d_1) = s'$, and $\mathcal{T}_{\mathsf{f}} \models K \sqsubseteq (\leqslant 1 \; r \; K')$, a simple semantic argument shows that $s' \subseteq \mathsf{tp}_{\mathcal{I}}(d_2)$.

  – $r(d, d_2) \in \mathcal{A}$. Since $d \in K^{\mathcal{I}}$ and $d_2 \in {K'}^{\mathcal{I}}$, we have $K \subseteq \mathsf{tp}_{\mathcal{A}}(d)$ and $K' \subseteq \mathsf{tp}_{\mathcal{A}}(d_2)$ by definition of the initial interpretation $\mathcal{I}$. Also, $\mathsf{tp}_{\mathcal{A}}(d) = s$. By λ, we thus have $\mathcal{T}_{\mathsf{f}} \models \mathsf{tp}_{\mathcal{A}}(d) \sqsubseteq \exists r.s'$. With $\mathcal{T}_{\mathsf{f}} \models K \sqsubseteq (\leqslant 1 \; r \; K')$ and $r(d, d_2) \in \mathcal{A}$, the semantics yields $s' \subseteq \mathsf{tp}_{\mathcal{A}}(d_2)$, thus $s' \subseteq \mathsf{tp}_{\mathcal{I}}(d_2)$.

- both (d, d_1) and (d, d_2) were added by an application of (**c2**). Then there are λ_1 and λ_2, such that, for $i \in \{1, 2\}$, we have $\lambda_i = s_i \; {}^1\!\!\leftrightarrow^1_r \; s'_i$ and $(d, d_i) \in \pi_{\lambda_i}$. Applying Lemma 19 to λ_i yields $s_i = \mathsf{tp}_{\mathcal{I}}(d)$, for $i \in \{1, 2\}$. Consequently, $s_1 = s_2$. We next show $s'_1 = s'_2$, thus $\lambda_1 = \lambda_2$. For uniformity, we use s to denote s_1 and s_2. From λ_i, we obtain $\mathcal{T}_{\mathsf{f}} \models s \sqsubseteq \exists r.s'_i$ and s'_i is maximal with this property, for $i \in \{1, 2\}$. Lemma 19 yields $\mathsf{tp}_{\mathcal{I}}(d_i) = s'_i$. Using the facts that $\mathcal{T}_{\mathsf{f}} \models s \sqsubseteq \exists r.s'_i$ for $i \in \{1, 2\}$, $K \subseteq \mathsf{tp}_{\mathcal{I}}(d) = s$, $K' \subseteq \mathsf{tp}_{\mathcal{I}}(d_i) = s'_i$ for $i \in \{1, 2\}$, and $\mathcal{T}_{\mathsf{f}} \models K \sqsubseteq (\leqslant 1 \; r \; K')$, an easy semantic argument shows that $\mathcal{T}_{\mathsf{f}} \models s \sqsubseteq \exists r.(s'_1 \cup s'_2)$. The maximality of s'_1 and s'_2 thus implies $s'_1 = s'_2$ as desired.

  Hence, $\lambda_1 = \lambda_2$ and $(d, d_1), (d, d_2) \in \pi_{\lambda_1}$. Since π_{λ_1} is a bijection, we obtain $d_1 = d_2$, a contradiction.

Application of (c3) preserves all invariants. It is obvious that the invariants (**i1**) and (**i2**) are preserved with each single application of (**c3**). It thus remains to treat (**i3**). Assume that completion processed $d \in \Delta^{\mathcal{I}}$ with $\mathsf{tp}_{\mathcal{I}}(d) \rightarrow_r t$ and $\mathsf{tp}_{\mathcal{I}}(d) \not\rightarrow^1_r t$, adding the edge (d, d_t) to $r^{\mathcal{I}}$. Since $\mathsf{tp}_{\mathcal{I}}(d_t) = t$ and $\mathsf{tp}_{\mathcal{I}}(d) \not\rightarrow^1_r t$, there is no $K \sqsubseteq (\leqslant 1 \; r^- \; K') \in \mathcal{T}_{\mathsf{f}}$ such that $K \subseteq t$ and $K' \subseteq \mathsf{tp}_{\mathcal{I}}(d)$. Take a $K \sqsubseteq (\leqslant 1 \; r \; K') \in \mathcal{T}_{\mathsf{f}}$ with $K \subseteq \mathsf{tp}_{\mathcal{I}}(d)$ and $K' \subseteq t$. We have to prove that there is no $e \in \Delta^{\mathcal{I}}$ distinct from d_t such that $(d, e) \in r^{\mathcal{I}}$ and $e \in {K'}^{\mathcal{I}}$. This can be done exactly as in the case of the completion rule (**c1**).

Termination of Model Construction

We show that the constructed interpretation $\mathcal{I}$ is indeed finite.

Proposition 20 *$\Delta^{\mathcal{I}}$ is finite.*

Proof. To analyze the termination of the construction of $\mathcal{I}$, we associate a certain directed tree $T = (V, E)$ with the model $\mathcal{I}$ that makes more explicit the way in which $\mathcal{I}$ was constructed. Note that only the completion rules (**c1**) and (**c2**) introduce new domain elements and that (**c1**) introduces a single new element with each application whereas (**c2**) introduces a whole (finite) set of fresh elements. Also note that each application of a completion rule is triggered by a single domain element d for which some existential restriction is not yet satisfied.[3] Now, the tree T is defined as follows:

- V consists of all subsets of $\Delta^{\mathcal{I}}$ that were introduced together by a single application of one of the completion rules (**c1**) and (**c2**); additionally, the set of all elements in the initial interpretation $\mathcal{I}$ is a node in V (in fact, the root node);
- the edge (v, v') is included in E if the elements in v' were introduced by an application of a completion rule to an element d of v. We call this element the *trigger* of v' and denote it with $d_{v'}$.

To show that $\Delta^{\mathcal{I}}$ is finite, it clearly suffices to show that V is finite. The outdegree of T is finite since every rule application introduces only finitely many elements. By König's Lemma, it thus remains to show that T is of finite depth. We first note that an easy analysis of (**c1**) and (**c2**) reveals the following property:

($*$) if $(v_1, v_2), (v_2, v_3) \in E$, then there are $d_0, \ldots, d_k$ and roles $r_0, \ldots, r_{k-1}$ s.t.
  - $d_0 = d_{v_2} \in v_1$ and $d_1, \ldots, d_k = d_{v_3} \in v_2$;
  - $\mathsf{tp}_{\mathcal{I}}(d_i) \to^1_{r_i} \mathsf{tp}_{\mathcal{I}}(d_{i+1})$ for all $i < k$.

Now assume towards a contradiction that the depth of T is larger than $2|\mathsf{TP}(\mathcal{T}_\mathsf{f})|+1$ and choose a concrete path $v_1 \cdots v_n$ with v_1 the root of T and $n > 2|\mathsf{TP}(\mathcal{T}_\mathsf{f})|+1$. This path gives rise to a corresponding sequence of triggers $d_{v_1}, \ldots, d_{v_n}$. Since the length of this sequence exceeds $2|\mathsf{TP}(\mathcal{T}_\mathsf{f})|$, there must be i, j with $2 \leq i < j \leq n$ and such that $\mathsf{tp}_{\mathcal{I}}(d_{v_i}) = \mathsf{tp}_{\mathcal{I}}(d_{v_j})$ and $j > i+1$. By applying ($*$) multiple times, we obtain a sequence of domain elements $d_0, \ldots, d_k$ and roles $r_0, \ldots, r_{k-1}$ such that

1. $d_0 = d_{v_i} \in v_{i-1}$, $d_1 \in v_i$, and $d_k = d_{v_j} \in v_{j-1}$;
2. $\mathsf{tp}_{\mathcal{I}}(d_\ell) \to^1_{r_\ell} \mathsf{tp}_{\mathcal{I}}(d_{\ell+1})$ for $\ell < k$.
3. $d_0, \ldots, d_k$ contains all elements $d_{v_i}, d_{v_{i+1}} \ldots, d_{v_j}$.

Since $\mathsf{tp}_{\mathcal{I}}(d_{v_i}) = \mathsf{tp}_{\mathcal{I}}(d_{v_j})$ and by Point 2, we have that $\mathsf{tp}_{\mathcal{I}}(d_0), r_0, \ldots, r_{k-1}, \mathsf{tp}_{\mathcal{I}}(d_k)$ is a finmod cycle in $\mathcal{T}_\mathsf{f}$. Since all finmod cycles in $\mathcal{T}_\mathsf{f}$ have been reversed, we have

$$\mathsf{tp}_{\mathcal{I}}(d_0) \;{}^1{\leftrightarrow}^1_{r_0}\; \mathsf{tp}_{\mathcal{I}}(d_1) \;{}^1{\leftrightarrow}^1_{r_1} \cdots {}^1{\leftrightarrow}^1_{r_{k-1}}\; \mathsf{tp}_{\mathcal{I}}(d_k). \quad (\dagger)$$

We prove the following claim:

Claim. If (**c1**) is triggered by $d \in \Delta^{\mathcal{I}}$ generating a new element $e \in \Delta^{\mathcal{I}}$, then there is no role r such that $\mathsf{tp}_{\mathcal{I}}(d) \;{}^1{\leftrightarrow}^1_r\; \mathsf{tp}_{\mathcal{I}}(e)$.

Since $d_{v_i} \in v_{i-1}$ and $d_1 \in v_i$, d_1 was generated by the application of a completion rule triggered by d_0. By ($\dagger$) and the claim, this completion rule must be (**c2**). By definition of (**c2**) and ($\dagger$), all elements $d_1, \ldots, d_k$ have been introduced by exactly this application of (**c2**). This leads to a contradiction: we have $d_1 \in v_i$ and $d_k \in v_{j-1}$, and since $j > i+1$, $v_i \neq v_{j-1}$. Consequently, d_1 and d_k were introduced by different applications of completion rules. ❑

[3] In the case of (**c2**), there are potentially many domain elements that trigger the same application. In such a case, we choose one element as the actual trigger; see formulation of (**c2**).

The following is the remaining ingredient to the termination proof (Claim in the proof of Proposition 20).

Lemma 21 *If (**c1**) is triggered by $d \in \Delta^{\mathcal{I}}$ and generates a new element $e \in \Delta^{\mathcal{I}}$, then there is no role r such that $\mathsf{tp}_{\mathcal{I}}(d) \;{}^1{\leftrightarrow}^1_r\; \mathsf{tp}_{\mathcal{I}}(e)$.*

Proof. Observe that if e is introduced by an application of ($\mathbf{c_1}$) to $d \in \Delta^{\mathcal{I}}$, then $\mathcal{T}_\mathsf{f}$ entails that $\mathsf{tp}_{\mathcal{I}}(d) \to^1_s \mathsf{tp}_{\mathcal{I}}(e)$ for some role s. Assume towards a contradiction that there is a role r such that $\mathsf{tp}_{\mathcal{I}}(d) \;{}^1{\leftrightarrow}^1_r\; \mathsf{tp}_{\mathcal{I}}(e)$. Then, the finmod cycle $\mathsf{tp}_{\mathcal{I}}(d), s, \mathsf{tp}_{\mathcal{I}}(e), r^-, \mathsf{tp}_{\mathcal{I}}(d)$ occurs in $\mathcal{T}_\mathsf{f}$. Since every finmod cycle in $\mathcal{T}_\mathsf{f}$ is reversed, we have $\mathsf{tp}_{\mathcal{I}}(e) \to^1_{s^-} \mathsf{tp}_{\mathcal{I}}(d)$. This is in contradiction to the assumption that e was introduced by an application of ($\mathbf{c_1}$). ❑

Correctness of Model Construction

To complete the proof of the "if" direction of Theorem 3, it remains to show the following.

Proposition 22 *$\mathcal{I}$ is a model of $\mathcal{A}$ and $\mathcal{T}_\mathsf{f}$.*

Proof. First, we show that for every assertion $\alpha \in \mathcal{A}, \mathcal{I} \models \alpha$. This is a consequence of the definition of $\mathcal{I}$. Indeed, for every individual a, if $\alpha = A(a) \in \mathcal{A}$, then $A \in \mathsf{tp}_{\mathcal{A}}(a)$ which by the definition of $\mathcal{I}$ implies that $a \in A^{\mathcal{I}}$. Further, if $\alpha = r(a,b) \in \mathcal{A}$ then $(a,b) \in r^{\mathcal{I}}$.

Next, we show that for every axiom $K \sqsubseteq C \in \mathcal{T}_\mathsf{f}$, we have that $\mathcal{I} \models K \sqsubseteq C$. We distinguish the following cases:

- $C = A$. Let $d \in K^{\mathcal{I}}$. Then $K \subseteq \mathsf{tp}_{\mathcal{I}}(d)$ and by (**i1**) $\mathsf{tp}_{\mathcal{I}}(d) \in \mathsf{TP}(\mathcal{T}_\mathsf{f})$. Since $\mathcal{T}_\mathsf{f} \models K \sqsubseteq A$, this yields $A \in \mathsf{tp}_{\mathcal{I}}(d)$ and thus $d \in A^{\mathcal{I}}$.
- $C = \bot$. Follows from (**i1**). Indeed since for every $d \in \Delta^{\mathcal{I}}$, $\mathsf{tp}_{\mathcal{I}}(d) \in \mathsf{TP}(\mathcal{T}_\mathsf{f})$, $K^{\mathcal{I}} = \emptyset$.
- $C = \exists r.K'$. Let $d \in K^{\mathcal{I}}$. Then we have that $K \subseteq \mathsf{tp}_{\mathcal{I}}(d)$. Since $\mathcal{T}_\mathsf{f} \models K \sqsubseteq \exists r.K'$, we have that $\mathsf{tp}_{\mathcal{I}}(d) \to_r t'$ for some t' with $K' \subseteq t'$. It suffices to show that there is some d' with $(d, d') \in r^{\mathcal{I}}$ and $\mathsf{tp}_{\mathcal{I}}(d') = t'$. The easiest case is that such a d' already exists in the initial $\mathcal{I}$. Assume that this is not the case. Note that one of the following cases must apply: (1) $\mathsf{tp}_{\mathcal{I}}(d) \to^1_r t'$ and $t' \not\to^1_{r^-} \mathsf{tp}_{\mathcal{I}}(d)$, (2) $\mathsf{tp}_{\mathcal{I}}(d) \to^1_r t'$ and $t' \to^1_{r^-} \mathsf{tp}_{\mathcal{I}}(d)$, and (3) $\mathsf{tp}_{\mathcal{I}}(d) \not\to^1_r t'$. These cases correspond exactly to the completion rules (**c1**) to (**c3**). Thus, one of these rules will add the required successor.
- $C = \forall r.K'$. Let $d \in K^{\mathcal{I}}$ and $(d, d') \in r^{\mathcal{I}}$, We have that $K \subseteq \mathsf{tp}_{\mathcal{I}}(d)$. Further, by (**i2**), we can distinguish the following cases:
  - $\mathsf{tp}_{\mathcal{I}}(d) \to_r \mathsf{tp}_{\mathcal{I}}(d')$. Then $\mathcal{T}_\mathsf{f} \models \mathsf{tp}_{\mathcal{I}}(d) \sqsubseteq \exists r.\mathsf{tp}_{\mathcal{I}}(d')$ and $\mathsf{tp}_{\mathcal{I}}(d')$ is maximal with this property. Since $\mathcal{T}_\mathsf{f} \models K \sqsubseteq \forall r.K'$, we have that $\mathcal{T}_\mathsf{f} \models \mathsf{tp}_{\mathcal{I}}(d) \sqsubseteq \exists r.\mathsf{tp}_{\mathcal{I}}(d') \cup K'$, and the maximality of $\mathsf{tp}_{\mathcal{I}}(d')$ yields $K' \subseteq \mathsf{tp}_{\mathcal{I}}(d')$, and thus $d' \in K'^{\mathcal{I}}$.
  - $\mathsf{tp}_{\mathcal{I}}(d') \to_{r^-} \mathsf{tp}_{\mathcal{I}}(d)$. Then we have $\mathcal{T}_\mathsf{f} \models \mathsf{tp}_{\mathcal{I}}(d') \sqsubseteq \exists r^-.\mathsf{tp}_{\mathcal{I}}(d)$. Together with $\mathcal{T}_\mathsf{f} \models K \sqsubseteq \forall r.K'$, we obtain $\mathcal{T}_\mathsf{f} \models \mathsf{tp}_{\mathcal{I}}(d') \sqsubseteq K'$. Since $\mathsf{tp}_{\mathcal{I}}(d') \in \mathsf{TP}(\mathcal{T}_\mathsf{f})$ by (**i1**), we obtain $K' \subseteq \mathsf{tp}_{\mathcal{I}}(d')$ and thus $d' \in K'^{\mathcal{I}}$.

– $r(d, d') \in \mathcal{A}$. Then $K \subseteq \mathsf{tp}_{\mathcal{A}}(d)$ by definition of the initial $\mathcal{I}$. By the semantics, we thus have $K' \subseteq \mathsf{tp}_{\mathcal{A}}(d') = \mathsf{tp}_{\mathcal{I}}(d')$, thus $d' \in K'^{\mathcal{I}}$.

- $C = (\leqslant 1\ r\ K)'$. Follows from (**i3**).

❑

B Proofs for Section 4

We continue to assume that the original TBox $\mathcal{T}$ is in the stricter normal form introduced at the begin of Section 4. Before we can prove $\widehat{\mathcal{I}} \models \mathcal{T}_{\mathsf{f}}$, we will proceed as outlined in Section 4, showing first that $\widehat{\mathcal{I}}$ is a model of $\widehat{\mathcal{T}}$ and that it suffices to close only maximal cycles when constructing $\widehat{\mathcal{T}}$. From here on, we will write $K \vdash_{\widehat{\mathcal{T}}} K'$ as a shortcut for $K \sqsubseteq A \in \widehat{\mathcal{T}}$ for all $A \in K'$.

Lemma 10 $\widehat{\mathcal{I}} \models \widehat{\mathcal{T}}$.

Proof. Let $K \sqsubseteq C \in \widehat{\mathcal{T}}$. We distinguish the following cases.

- $C = \bot$.
  If there were some $d \in K^{\widehat{\mathcal{I}}}$, then the construction of $\widehat{\mathcal{I}}$ would ensure that $\mathsf{tail}(d) \vdash_{\widehat{\mathcal{T}}} K$, and from $K \sqsubseteq \bot \in \widehat{\mathcal{T}}$ and **R3** we would get $\mathsf{tail}(d) \sqsubseteq \bot \in \widehat{\mathcal{T}}$, which is impossible, as the following inductive argument shows. If $d = K \in \Sigma$, the claim follows from the construction in the initial step. If $|d| > 1$, then d was added due to some element d' with $\mathsf{tail}(d') \sqsubseteq \exists r.\mathsf{tail}(d) \in \widehat{\mathcal{T}}$. Now $\mathsf{tail}(d) \sqsubseteq \bot \in \widehat{\mathcal{T}}$ implies $\mathsf{tail}(d') \sqsubseteq \bot \in \widehat{\mathcal{T}}$ due to **R6**, contradicting the inductive hypothesis.
- $C = A$.
  Let $d \in K^{\widehat{\mathcal{I}}}$. Then $\mathsf{tail}(d) \vdash_{\widehat{\mathcal{T}}} K$ by construction of $\widehat{\mathcal{I}}$. Together with $K \sqsubseteq A \in \widehat{\mathcal{T}}$, Rule **R3** yields $\mathsf{tail}(d) \sqsubseteq A \in \widehat{\mathcal{T}}$, hence $d \in A^{\widehat{\mathcal{I}}}$ by construction of $\widehat{\mathcal{I}}$.
- $C = \exists r.K'$.
  Let $d \in K^{\widehat{\mathcal{I}}}$ and assume $\mathsf{tail}(d) = K''$. By construction of $\widehat{\mathcal{I}}$, we have $K'' \vdash_{\widehat{\mathcal{T}}} K$. Further, from $K \sqsubseteq \exists r.K' \in \widehat{\mathcal{T}}$, by **R3**, $K'' \sqsubseteq \exists r.K' \in \widehat{\mathcal{T}}$. Then the construction ensures that $d' \in (\exists r.K')^{\widehat{\mathcal{I}}}$ as required.
- $C = \forall r.A$.
  Let $d \in K^{\widehat{\mathcal{I}}}$ and $(d, d') \in r^{\widehat{\mathcal{I}}}$. Further, let $\mathsf{tail}(d) = K_1$. Since $K \sqsubseteq \forall r.A \in \widehat{\mathcal{T}}$ and $K_1 \vdash_{\widehat{\mathcal{T}}} K$, we get by Rule **R3** that $K_1 \sqsubseteq \forall r.A \in \widehat{\mathcal{T}}$. We distinguish the following cases.
  (i) $d' = dK_2$ i.e., d' was added after d because of some $K_1 \sqsubseteq \exists r.K_2 \in \widehat{\mathcal{T}}$, with K_2 maximal with this property. Since $K_1 \sqsubseteq \forall r.A \in \widehat{\mathcal{T}}$ and $K_1 \sqsubseteq \exists r.K_2 \in \widehat{\mathcal{T}}$, we get by **R5** that $K_1 \sqsubseteq \exists r.(K_2 \sqcap A) \in \widehat{\mathcal{T}}$. Maximality of K_2 implies that $A \in K_2$. By **R1**, we have $K_2 \sqsubseteq A \in \widehat{\mathcal{T}}$ and thus $d' \in A^{\widehat{\mathcal{I}}}$ by construction of $\widehat{\mathcal{I}}$.
  (ii) $d = d'K_1$ i.e., d was added after d' because of some $K_2 \sqsubseteq \exists r^-.K_1 \in \widehat{\mathcal{T}}$ with $K_2 = \mathsf{tail}(d')$. Since $K_1 \sqsubseteq \forall r.A \in \widehat{\mathcal{T}}$ and $K_2 \sqsubseteq \exists r^-.K_1 \in \widehat{\mathcal{T}}$, we get by rule **R4** that $K_2 \sqsubseteq A \in \widehat{\mathcal{T}}$. Thus, $d' \in A^{\widehat{\mathcal{I}}}$ by construction of $\widehat{\mathcal{I}}$.
- $C = (\leqslant 1\ r\ A)$.
  Let $d \in K^{\widehat{\mathcal{I}}}$ and let $K'' = \mathsf{tail}(d)$. Assume that there are e_1, e_2 with $(d, e_i) \in r^{\widehat{\mathcal{I}}}$ and $e_i \in A^{\widehat{\mathcal{I}}}$ for $i = 1, 2$. We have $K'' \vdash_{\widehat{\mathcal{T}}} K$ by construction of $\widehat{\mathcal{I}}$ and thus, by Rule **R3**, $K'' \sqsubseteq (\leqslant 1\ r\ A) \in \widehat{\mathcal{T}}$.
  Let $K_i = \mathsf{tail}(e_i)$; hence $K_i \sqsubseteq A \in \widehat{\mathcal{T}}$, $i = 1, 2$. We distinguish two cases according to the construction of $\widehat{\mathcal{I}}$.
  (i) Each e_i was added by $K'' \sqsubseteq \exists r.K_i$ and K_i is maximal with this property. Hence $e_i = dK_i$. Since $K'' \sqsubseteq (\leqslant 1\ r\ A) \in \widehat{\mathcal{T}}$ and $K_i \sqsubseteq A \in \widehat{\mathcal{T}}$, we have by **R7** that $K'' \sqsubseteq \exists r.(K_1 \sqcap K_2) \in \widehat{\mathcal{T}}$. The maximality conditions on both K_i imply $K_1 \subseteq K_2$ and $K_2 \subseteq K_1$. Hence, $e_1 = e_2$, and $d \in (\leqslant 1\ r\ A)^{\widehat{\mathcal{I}}}$ as required.
  (ii) $d = e_1K''$ and $e_2 = dK_2$. Hence d is added after e_1 due to some $K_1 \sqsubseteq \exists r^-.K'' \in \widehat{\mathcal{T}}$ with K'' maximal, and e_2 is added after d due to some $K'' \sqsubseteq \exists r.K_2 \in \widehat{\mathcal{T}}$ with K_2 maximal. Since $K'' \sqsubseteq (\leqslant 1\ r\ A) \in \widehat{\mathcal{T}}$ and $K_i \sqsubseteq A \in \widehat{\mathcal{T}}$, we get by rule **R8** that $K_1 \sqsubseteq A' \in \widehat{\mathcal{T}}$ for every $A' \in K_2$. Then, by construction of $\widehat{\mathcal{I}}$, we have $e_1 \in K_2^{\widehat{\mathcal{I}}}$ and thus e_2 cannot be added as an r-successor of d. Hence, $d \in (\leqslant 1\ r\ A)^{\widehat{\mathcal{I}}}$.

❑

Lemma 11. $\mathcal{T}_{\mathsf{f}}$ is equivalent to $\mathcal{T}_{\mathsf{f}}^{\max}$.

Proof. It suffices to show that, for every cycle C in a TBox $\mathcal{S}$, there is a maximal cycle $\widehat{C}$ in $\mathcal{S}$ whose reversal implies the reversal of C. More precisely, let $C = K_1, r_1, K_2, \ldots, K_n$ be a cycle in $\mathcal{S}$. We show that there is a maximal cycle $\widehat{C} = \widehat{K}_1, r_1, \widehat{K}_2, \ldots, \widehat{K}_n$ whose reversal – that is, adding the axioms $\widehat{K}_{j+1} \sqsubseteq \exists r_j^-.\widehat{K}_j$ and $\widehat{K}_j \sqsubseteq (\leqslant 1\ r_j\ \widehat{K}_{j+1})$ to $\mathcal{S}$– will lead to $\mathcal{S}$ implying the reversal of C. We proceed in three steps.

First, we construct $\widehat{C} = \widehat{K}_1, r_1, \widehat{K}_2, \ldots, \widehat{K}_n$ iteratively as follows. Initially, set $\widehat{K}_j = K_j$ for every $j = 1, \ldots, n$. Then exhaustively apply the following step.

While there is some $\widehat{L}_{j+1} \supsetneq \widehat{K}_{j+1}$ maximal with $\mathcal{S} \models \widehat{K}_j \sqsubseteq \exists r_j.\widehat{L}_{j+1}$ for some $j = 1, \ldots, n-1$, set $\widehat{K}_{j+1} = \widehat{L}_{j+1}$.

The iteration terminates because the supply of conjunctions is bounded and C's length is fixed.

Second, we verify that $\widehat{C}$ is indeed a cycle. It suffices to show that one application of the construction step does not destroy the cycle property, i.e., by replacing $\widehat{K}_{j+1}$ with the larger $\widehat{L}_{j+1}$, the four subsumptions involving $\widehat{K}_{j+1}$ now hold for $\widehat{L}_{j+1}$:

- $\mathcal{S} \models \widehat{K}_j \sqsubseteq \exists r_j.\widehat{L}_{j+1}$ holds due to the step's precondition.
- $\mathcal{S} \models \widehat{L}_{j+1} \sqsubseteq \exists r_j.\widehat{K}_{j+2}$ holds because $\mathcal{S} \models \widehat{L}_{j+1} \sqsubseteq \widehat{K}_{j+1} \sqsubseteq \exists r_j.\widehat{K}_{j+2}$.
- $\mathcal{S} \models \widehat{L}_{j+1} \sqsubseteq (\leqslant 1\ r_j^-\ \widehat{K}_j)$ holds because $\mathcal{S} \models \widehat{L}_{j+1} \sqsubseteq \widehat{K}_{j+1} \sqsubseteq (\leqslant 1\ r_j^-\ \widehat{K}_j)$.

- $\mathcal{S} \models \widehat{K}_{j+2} \sqsubseteq (\leqslant 1\ r^-_{j+1}\ \widehat{L}_{j+1})$ holds because $\mathcal{S} \models \widehat{K}_{j+2} \sqsubseteq (\leqslant 1\ r^-_{j+1}\ \widehat{K}_{j+1})$ and $\mathcal{S} \models \widehat{L}_{j+1} \sqsubseteq \widehat{K}_{j+1}$.

Third, we show that the reversal of $\widehat{C}$ implies the reversal of C. Again, it suffices to show that this is the case when $\widehat{C}$ is obtained from C applying one single construction step. Let $\mathcal{S}^+$ be the TBox obtained from $\mathcal{S}$ after reversing $\widehat{C}$, that is, $\mathcal{S}^+$ equals $\mathcal{S}$ plus the following $2j$ axioms.

$$\ldots\ \widehat{L}_{j+1} \sqsubseteq \exists r^-_j.\widehat{K}_j \qquad \widehat{K}_{j+2} \sqsubseteq \exists r^-_{j+1}.\widehat{L}_{j+1} \quad (*)\ \ldots$$
$$\ldots\quad \widehat{K}_j \sqsubseteq (\leqslant 1\ r_j\ \widehat{L}_{j+1}) \quad \widehat{L}_{j+1} \sqsubseteq (\leqslant 1\ r_{j+1}\ \widehat{K}_{j+2})\ \ldots$$

To prove that all $2j$ axioms that would be added by reversing C are implied by $\mathcal{S}^+$, it suffices to show that $\mathcal{S}^+ \models \widehat{K}_{j+1} \sqsubseteq \widehat{L}_{j+1}$ (which implies $\mathcal{S}^+ \models \widehat{K}_{j+1} \equiv \widehat{L}_{j+1}$). Consider an arbitrary model $\widehat{\mathcal{I}} \models \mathcal{S}^+$ and an instance d of $\widehat{K}_{j+1}$ in $\widehat{\mathcal{I}}$. Since $\widehat{C}$ is a cycle, there is some e with $(d,e) \in r^{\mathcal{I}}_{j+1}$ and $e \in \widehat{K}^{\mathcal{I}}_{j+2}$. Then, due to the above axiom $(*)$ in $\mathcal{S}^+$, there is some d' with $(d',e) \in r^{\mathcal{I}}_{j+1}$ and $d' \in \widehat{L}^{\mathcal{I}}_{j+1}$. Now, since C is a cycle in $\mathcal{S}$ – i.e., $\widehat{K}_{j+2} \sqsubseteq (\leqslant 1\ r^-_{j+1}\ \widehat{K}_{j+1}) \in \mathcal{S}$ – and because $\widehat{L}_{j+1} \supseteq \widehat{K}_{j+1}$, we obtain that $d' = d$. Hence d is an instance of $\widehat{L}_{j+1}$. ❑

When proving $\widehat{\mathcal{I}} \models \mathcal{T}_{\mathsf{f}}^{\max}$, we have to deal with the complication that $\mathcal{T}_{\mathsf{f}}^{\max} \subseteq \widehat{\mathcal{T}}$ need not hold. As illustrated by Example 7, this is a actually a main feature of our calculus because we are avoiding to introduce conjunctions K that are 'irrelevant' for the reasoning task at hand.

We address this issue by showing that the *relevant consequences* of all concept inclusions in $\mathcal{T}_{\mathsf{f}}^{\max} \setminus \widehat{\mathcal{T}}$ are reflected in $\widehat{\mathcal{T}}$, even if the inclusions themselves are missing. To make this more precise, note that $\mathcal{T}_{\mathsf{f}}^{\max} \setminus \widehat{\mathcal{T}}$ only contains CIs of the form

(i) $K \sqsubseteq \exists r.K'$ and

(ii) $K \sqsubseteq (\leqslant 1\ r\ K')$.

Note that, while $\mathcal{T}$ is in the stricter normal form, cycle reversion may have introduced CIs of the form (ii) with arbitrary conjunctions K'.

For CIs of the form (i), we observe that K, K' may be irrelevant: they may not occur in $\widehat{\mathcal{T}}$. We show that there is some conjunction $\widehat{K}'$ that satisfies $\widehat{K}' \vdash_{\widehat{\mathcal{T}}} K'$ and which intuitively replaces K' such that for all relevant conjunctions $\widehat{K}$ with $\widehat{K} \vdash_{\widehat{\mathcal{T}}} K$, the inclusion $\widehat{K} \sqsubseteq \exists r.\widehat{K}'$ is contained in $\widehat{\mathcal{T}}$.

For CIs of the form (ii), we show analogously that there is a replacement A of K' such that for all relevant conjunctions $\widehat{K}$ with $\widehat{K} \vdash_{\widehat{\mathcal{T}}} K$, $\widehat{\mathcal{T}}$ contains $\widehat{K} \sqsubseteq (\leqslant 1\ r\ A)$. However, in this case the fact that A is a replacement of K' has to be formalized even a bit more carefully. We cannot require that $K' \vdash_{\widehat{\mathcal{T}}} A$, again because K' may be irrelevant. Instead, we need that $\widetilde{K}' \vdash_{\widehat{\mathcal{T}}} K'$ implies $\widetilde{K}' \sqsubseteq A \in \widehat{\mathcal{T}}$ for all relevant $\widetilde{K}'$.

What we have just discussed is Lemma 24 below. In order to show that the above concept inclusions $\widehat{K} \sqsubseteq \exists r.\widehat{K}'$ and $\widehat{K} \sqsubseteq (\leqslant 1\ r\ A)$ all are in $\widehat{\mathcal{T}}$, we consider the sequence of TBoxes $\mathcal{T}_{\mathsf{f}}^0, \mathcal{T}_{\mathsf{f}}^1, \ldots$ that are obtained by repeatedly reversing maximal cycles and whose limit is $\mathcal{T}_{\mathsf{f}}^{\max}$. Note that $\mathcal{T}_{\mathsf{f}}^{i+1}$ is produced from $\mathcal{T}_{\mathsf{f}}^i$ by reversing one cycle, and that cycles are defined in terms of *semantic entailment* of CIs of the form (i) and (ii) by $\mathcal{T}_{\mathsf{f}}^i$, rather than *syntactic containment*. We first establish an auxiliary lemma that helps in bridging this gap.

Since, by assumption, the original TBox $\mathcal{T}$ is in the stricter normal form introduced at the begin of Section 4, all TBoxes $\mathcal{T}_{\mathsf{f}}^i$ contain $\forall$-restrictions only in the form $\forall r.A$. However, due to cycle reversion, functionality restrictions may occur in the form $(\leqslant 1\ r\ L')$ for arbitrary conjunctions L'. Every $\mathcal{T}_{\mathsf{f}}^i$ and every conjunction K that is satisfiable w.r.t. $\mathcal{T}_{\mathsf{f}}^i$ give rise to a TBox $(\mathcal{T}_{\mathsf{f}}^i)_K$ as follows:

1. for all CIs $L \sqsubseteq C \in \mathcal{T}_{\mathsf{f}}^i$ with $\mathcal{T}_{\mathsf{f}}^i \models K \sqsubseteq L$ and C of one of the forms $\exists r.L'$, $\forall r.A$, and $(\leqslant 1\ r\ L')$, include $K \sqsubseteq C$;
2. then exhaustively apply rules **R5** and **R7′**, where **R7′** is obtained from **R7** by replacing *containment* in $\widehat{\mathcal{T}}$ with *entailment* in $\mathcal{T}_{\mathsf{f}}^i$:

$$\textbf{R7}'\ \frac{K \sqsubseteq \exists r.K_1 \qquad K \sqsubseteq \exists r.K_2 \qquad \mathcal{T}_{\mathsf{f}}^i \models K_1 \sqsubseteq K' \\ K \sqsubseteq (\leqslant 1\ r\ K') \qquad\qquad \mathcal{T}_{\mathsf{f}}^i \models K_2 \sqsubseteq K'}{K \sqsubseteq \exists r.(K_1 \sqcap K_2)}$$

It is easy to see that $\mathcal{T}_{\mathsf{f}}^i \models (\mathcal{T}_{\mathsf{f}}^i)_K$. Note that Step 1 above addresses the fact that K need not occur syntactically in $\mathcal{T}_{\mathsf{f}}^i$.

The proof of the following lemma uses $(\mathcal{T}_{\mathsf{f}}^i)_K$ to introduce two variants of the canonical model for $\mathcal{T}_{\mathsf{f}}^i$ and to extract the required witnesses for entailments.

Lemma 23 *For every $i \geq 0$, the following hold.*

1. *If $\mathcal{T}_{\mathsf{f}}^i \models K \sqsubseteq \exists r.K'$ and K is satisfiable w.r.t. $\mathcal{T}_{\mathsf{f}}^i$ then there is some conjunction L' with*
   (a) $\mathcal{T}_{\mathsf{f}}^i \models L' \sqsubseteq K'$ and
   (b) $(\mathcal{T}_{\mathsf{f}}^i)_K \ni K \sqsubseteq \exists r.L'$.
2. *If $\mathcal{T}_{\mathsf{f}}^i \models K \sqsubseteq (\leqslant 1\ r\ K')$ and $\mathcal{T}_{\mathsf{f}}^i \models K' \sqsubseteq \exists r^-.K$ such that K is maximal with this property and K' is satisfiable w.r.t. $\mathcal{T}_{\mathsf{f}}^i$, then there are L, L' with*
   (a) $\mathcal{T}_{\mathsf{f}}^i \models K \sqsubseteq L$, and
   (b) $\mathcal{T}_{\mathsf{f}}^i \models K' \sqsubseteq L'$, and
   (c) $\mathcal{T}_{\mathsf{f}}^i \ni L \sqsubseteq (\leqslant 1\ r\ L')$.

Proof. We begin by constructing a variant of the canonical model for $\mathcal{T}_{\mathsf{f}}^i$ that will be used in the proofs of both points of the lemma. Let K be a conjunction satisfiable w.r.t. $\mathcal{T}_{\mathsf{f}}^i$. The interpretation $\mathcal{I}_K$ is defined as follows. The domain $\Delta^{\mathcal{I}_K}$ consists of words over the alphabet built up of all conjunctions of concept names that occur in $\mathcal{T}_{\mathsf{f}}^i$ and are satisfiable w.r.t. $\mathcal{T}_{\mathsf{f}}^i$. Initially, $\Delta^{\mathcal{I}_K}$ is the singleton set $\{d_0\}$ for $d_0 = K$, and the concept and role names are interpreted such that

$$\mathsf{tp}_{\mathcal{I}_K}(d_0) = \{A \mid \mathcal{T}_{\mathsf{f}}^i \models K \sqsubseteq A\}$$
$$r^{\mathcal{I}_K} = \emptyset$$

Then we add the required successors to the root node d_0. For every $K \sqsubseteq \exists r.L' \in (\mathcal{T}_f^i)_K$ such that L' is maximal with this property,

- add a fresh element $e = KL'$ to $\Delta^{\mathcal{I}_K}$;
- add the pair (d_0, e) to $r^{\mathcal{I}_K}$;
- interpret concept names such that $\mathsf{tp}_{\mathcal{I}_K}(e) = \{A \mid \mathcal{T}_f^i \models L' \sqsubseteq A\}$.

Finally, we exhaustively generate required successors of non-root elements. For every $d = wL \in \Delta^{\mathcal{I}_K}$ with $d \neq d_0$, and every inclusion $L \sqsubseteq \exists r.L'$ such that $\mathcal{T}_f^i \models L \sqsubseteq \exists r.L'$, L' is maximal with this property, and $d \notin (\exists r.L')^{\widehat{\mathcal{I}}}$,

- add a fresh element $e = wLL'$ to $\Delta^{\mathcal{I}_K}$;
- add the pair (d, e) to $r^{\mathcal{I}_K}$;
- interpret concept names such that $\mathsf{tp}_{\mathcal{I}_K}(e) = \{A \mid \mathcal{T}_f^i \models L' \sqsubseteq A\}$.

Note the difference between the treatment of the root node d_0 and all other nodes: for d_0, we consider inclusions $K \sqsubseteq \exists r.L'$ that are syntactically contained in $(\mathcal{T}_f^i)_K$; for all other nodes, we consider inclusions that semantically follow from $\mathcal{T}_f^i$ (equivalently: from $(\mathcal{T}_f^i)_K$).

Claim. $\mathcal{I}_K \models \mathcal{T}_f^i$.

Proof of Claim. Let $L \sqsubseteq C \in \mathcal{T}_f^i$. We distinguish the following cases.

- $C = \bot$. Assume that L has an instance $d = w\widehat{L}$ in $\mathcal{I}_K$. Then $\mathcal{T}_f^i \models \widehat{L} \sqsubseteq L$ due to the construction of $\mathcal{I}_K$; hence $\mathcal{T}_f^i \models \widehat{L} \sqsubseteq \bot$, which is impossible, as the following inductive argument shows. If $d = K$, the claim follows from the assumption that K is satisfiable in $\mathcal{T}_f^i$. If $|d| > 1$ then d was added due to some element $d' = wL$ with $\mathcal{T}_f^i \models L \sqsubseteq \exists r.\widehat{L}$. Then $\mathcal{T}_f^i \models \widehat{L} \sqsubseteq \bot$ implies $\mathcal{T}_f^i \models L \sqsubseteq \bot \in \widehat{\mathcal{T}}$, contradicting the inductive hypothesis.

- $C = A$. Let $d \in L^{\mathcal{I}_K}$ with $d = w\widehat{L}$. Then $\mathcal{T}_f^i \models \widehat{L} \sqsubseteq L$ by construction of $\mathcal{I}_K$. Since $L \sqsubseteq A \in \mathcal{T}_f^i$, we obtain $\mathcal{T}_f^i \models \widehat{L} \sqsubseteq A$; hence $d \in A^{\mathcal{I}_K}$.

- $C = \exists r.L'$. Let $d \in L^{\mathcal{I}_K}$.

  In case $d = d_0 = K$, we have that $\mathcal{T}_f^i \models K \sqsubseteq L$. Together with $L \sqsubseteq \exists r.L' \in \mathcal{T}_f^i$, this implies that $K \sqsubseteq \exists r.L' \in (\mathcal{T}_f^i)_K$ by Step 1 of the construction of $(\mathcal{T}_f^i)_K$. Let K' be maximal with $K \sqsubseteq \exists r.K' \in (\mathcal{T}_f^i)_K$ and $L' \subseteq K'$. In the construction of $\mathcal{I}_K$, we thus create an r-successor e of d with $\mathsf{tp}_{\mathcal{I}_K}(e) \supseteq L'$. Hence $d \in (\exists r.L')^{\mathcal{I}_K}$.

  In case $d \neq d_0$, let $d = wK'$. Then $\mathcal{T}_f^i \models K' \sqsubseteq L$. Together with $L \sqsubseteq \exists r.L' \in \mathcal{T}_f^i$, this implies that $\mathcal{T}_f^i \models K' \sqsubseteq \exists r.L'$. Then the construction of $\mathcal{I}_K$ ensures that there is an r-successor e of d with $\mathsf{tp}_{\mathcal{I}_K}(e) \supseteq L'$. Hence $d \in (\exists r.L')^{\mathcal{I}_K}$.

- $C = \forall r.A$. Let $d \in L^{\mathcal{I}_K}$ and $(d, e) \in r^{\mathcal{I}_K}$.

  In case $d = d_0 = K$ and $e = KK'$, we have that $\mathcal{T}_f^i \models K \sqsubseteq L$; hence $K \sqsubseteq \forall r.A \in (\mathcal{T}_f^i)_K$ as above. Now e was added for some $K \sqsubseteq \exists r.K' \in (\mathcal{T}_f^i)_K$ with K' maximal. Since $(\mathcal{T}_f^i)_K$ is closed under application of **R5**, we have that $K \sqsubseteq \exists r.(K' \sqcap A) \in (\mathcal{T}_f^i)_K$. Maximality of K' implies that $A \in K'$. The construction of $\mathcal{I}_K$ then implies that $A \in \mathsf{tp}_{\mathcal{I}_K}(e)$; i.e., $e \in A^{\mathcal{I}_K}$.

  In case $e = d_0 = K$ and $d = KK'$, we have that $\mathcal{T}_f^i \models K' \sqsubseteq L$; hence $\mathcal{T}_f^i \models K' \sqsubseteq \forall r.A$ (x). Now d was added for some $K \sqsubseteq \exists r^-.K' \in (\mathcal{T}_f^i)_K$ with K'. Since $\mathcal{T}_f^i \models (\mathcal{T}_f^i)_K$, we obtain $\mathcal{T}_f^i \models K \sqsubseteq \exists r^-.K'$. Together with (x), a simple semantic argument implies $\mathcal{T}_f^i \models K \sqsubseteq A$; hence, $e \in A^{\mathcal{I}_K}$.

  In case $d = wK_1 \neq d_0$ and $e = wK_1K_2$, we have that $\mathcal{T}_f^i \models K_1 \sqsubseteq L$; hence $\mathcal{T}_f^i \models K_1 \sqsubseteq \forall r.A$ as above. Now e was added because $\mathcal{T}_f^i \models K_1 \sqsubseteq \exists r.K_2$ with K_2 maximal. A simple semantic argument implies $\mathcal{T}_f^i \models K_1 \sqsubseteq \exists r.(K_2 \sqcap A)$, and maximality of K_2 again yields $A \in K_2$; i.e., $e \in A^{\mathcal{I}_K}$.

  In case $e = wK_1 \neq d_0$ and $d = wK_1K_2$, we have that $\mathcal{T}_f^i \models K_2 \sqsubseteq L$; hence $\mathcal{T}_f^i \models K_2 \sqsubseteq \forall r.A$. Now d was added because $\mathcal{T}_f^i \models K_1 \sqsubseteq \exists r^-.K_2$. A simple semantic argument implies $\mathcal{T}_f^i \models K_1 \sqsubseteq A$, i.e., $e \in A^{\mathcal{I}_K}$.

- $C = (\leqslant 1\ r\ L')$. Let $d \in L^{\mathcal{I}_K}$, and let $(d, e_i) \in r^{\mathcal{I}_K}$ and $e_i \in (L')^{\mathcal{I}_K}$ for $i = 1, 2$.

  In case $d = d_0 = K$ and $e_i = KK_i$, we have that (i) $\mathcal{T}_f^i \models K \sqsubseteq L$ and (ii) $\mathcal{T}_f^i \models K_i \sqsubseteq L'$ for $i = 1, 2$. By construction of $(\mathcal{T}_f^i)_K$, (i) and the assumption imply (iii) $K \sqsubseteq (\leqslant 1\ r\ L') \in (\mathcal{T}_f^i)_K$. Now each e_i was added for some (iv) $K \sqsubseteq \exists r.K_i \in (\mathcal{T}_f^i)_K$ with K_i maximal. Applying **R7′** to (iv), (iii), (ii) yields $K \sqsubseteq \exists r.(K_1 \sqcap K_2) \in (\mathcal{T}_f^i)_K$. Maximality of the K_i implies that $K_1 = K_2$; hence $e_1 = e_2$.

  In case $e_1 = d_0 = K$, $d = KK_1$, and $e_2 = KK_1K_2$, we have that (i) $\mathcal{T}_f^i \models K_1 \sqsubseteq L$ plus (ii) $\mathcal{T}_f^i \models K \sqsubseteq L'$ and (iii) $\mathcal{T}_f^i \models K_2 \sqsubseteq L'$. By construction of $(\mathcal{T}_f^i)_K$, (i) and the assumption imply (iv) $\mathcal{T}_f^i \models K \sqsubseteq (\leqslant 1\ r\ L')$. Now d was added for some (v) $K \sqsubseteq \exists r^-.K_1 \in (\mathcal{T}_f^i)_K$, and e_2 was added for some (vi) $K_1 \sqsubseteq \exists r.K_2 \in (\mathcal{T}_f^i)_K$. A simple semantic argument applied to (v), (vi), (iv), (ii) and (iii) yields $\mathcal{T}_f^i \models K \sqsubseteq K_2$. This contradicts the assumption that e_2 was added for (vi).

  In case $d = wK' \neq d_0$ and $e_i = wK'K_i$, we argue as in the first case, but purely on a semantic basis, i.e., referring to entailment by $\mathcal{T}_f^i$ instead of containment in $(\mathcal{T}_f^i)_K$.

  In case $e_1wK' \neq d_0$, $d = wK'K_1$, and $e_2 = wK'K_1K_2$, we argue "semantically" as in the second case.

◆

To prove claim (1), assume $\mathcal{T}_f^i \models K \sqsubseteq \exists r.K'$ with K satisfiable w.r.t. $\mathcal{T}_f^i$. Since $\mathcal{I}_K \models \mathcal{T}_f^i$ and due to Step 1 of the construction of $\mathcal{I}_K$, there is some L' with $K \sqsubseteq \exists r.L' \in (\mathcal{T}_f^i)_K$ and $\mathcal{T}_f^i \models L' \sqsubseteq K'$.

To prove claim (2), assume $\mathcal{T}_f^i \models K \sqsubseteq (\leqslant 1\ r\ K')$ and $\mathcal{T}_f^i \models K' \sqsubseteq \exists r^-.K$ with K maximal and K' – and thus K –

satisfiable w.r.t. $\mathcal{T}_f^i$. We construct an interpretation $\mathcal{J}$ from the models $\mathcal{I}_K$ and $\mathcal{I}_{K'}$ as follows. Start with two copies of $\mathcal{I}_{K'}$ and one of $\mathcal{I}_K$, pairwise disjoint. Since $\mathcal{T}_f^i \models K' \sqsubseteq \exists r^-.K$ with K maximal, the root d_i of each of the two copies of $\mathcal{I}_{K'}$ has an r^--successor e_i of type K. Delete the subtrees starting at e_i and replace the r-edges (e_i, d_i) with (d_0, d_i), where d_0 is the root of the copy of $\mathcal{I}_K$.

Now the proof of the previous claim that $\mathcal{I}_K$ and $\mathcal{I}_{K'}$ are models of $\mathcal{T}_f^i$ can be easily refined to yield that

- all axioms $L \sqsubseteq C \in \mathcal{T}_f^i$ where C is of the form $A, \bot, \exists s.L', \forall s.A$ are satisfied by $\mathcal{J}$;
- if an axiom $L \sqsubseteq (\leqslant 1\ s\ L') \in \mathcal{T}_f^i$ is violated by $\mathcal{J}$, then it is violated by the root of the copy of $\mathcal{I}_K$, i.e., $s = r$ and $d_0 \in L^{\mathcal{J}}$ but $d_0 \notin (\leqslant 1\ r\ L')^{\mathcal{J}}$.

Since $\mathcal{T}_f^i \models K \sqsubseteq (\leqslant 1\ r\ K')$ but obviously $\mathcal{J} \not\models K \sqsubseteq (\leqslant 1\ r\ K')$, we have that $\mathcal{J} \not\models \mathcal{T}_f^i$. Consequently, there is an axiom $L \sqsubseteq (\leqslant 1\ r\ L') \in \mathcal{T}_f^i$ which is violated by d_0; that is, $d_0 \in L^{\mathcal{J}} \setminus (\leqslant 1\ r\ L')^{\mathcal{J}}$. This establishes (c) directly and implies (a) and (b): first, by construction of the $\mathcal{I}_{K'}$, we get $\mathcal{T}_f^i \models K' \sqsubseteq L'$, which is (b). Second, with $L \sqsubseteq (\leqslant 1\ r\ L') \in \mathcal{T}_f^i$ and (b), we obtain $\mathcal{T}_f^i \models L \sqsubseteq (\leqslant 1\ r\ K')$. Since K is maximal with this property, we have $L \subseteq K$, which implies (a). ❑

A conjunction K of concept names is *active in* $\widehat{\mathcal{T}}$ if there is a $\widehat{K} \in \mathsf{KON}(\widehat{\mathcal{T}})$ with $\widehat{K} \vdash_{\widehat{\mathcal{T}}} K$. Point (1) of the following lemma implies the required statement $\widehat{\mathcal{I}} \models \mathcal{T}_f^{\max}$.

Lemma 24 *For every $i \geq 0$, the following hold.*

1. *If $\mathcal{T}_f^i \ni K \sqsubseteq \exists r.K'$ and K is active in $\widehat{\mathcal{T}}$, then there is a $\widehat{K}' \in \mathsf{KON}(\widehat{\mathcal{T}})$ such that*
   (a) $\widehat{K}' \vdash_{\widehat{\mathcal{T}}} K'$;
   (b) for all $\widehat{K} \in \mathsf{KON}(\widehat{\mathcal{T}})$ with $\widehat{K} \vdash_{\widehat{\mathcal{T}}} K$, we have $\widehat{\mathcal{T}} \ni \widehat{K} \sqsubseteq \exists r.\widehat{K}'$.
2. *If $\mathcal{T}_f^i \ni K \sqsubseteq (\leqslant 1\ r\ K')$ and K is active in $\widehat{\mathcal{T}}$, then there is a concept name A such that*
   (a) for all $\widetilde{K}' \in \mathsf{KON}(\widehat{\mathcal{T}})$: if $\widetilde{K}' \vdash_{\widehat{\mathcal{T}}} K'$ then $\widetilde{K}' \sqsubseteq A \in \widehat{\mathcal{T}}$.
   (b) for all $\widehat{K} \in \mathsf{KON}(\widehat{\mathcal{T}})$ with $\widehat{K} \vdash_{\widehat{\mathcal{T}}} K$, we have $\widehat{\mathcal{T}} \ni \widehat{K} \sqsubseteq (\leqslant 1\ r\ A)$.
3. *$\widehat{\mathcal{I}} \models \mathcal{T}_f^i$*
4. *Point 1 holds when "$\mathcal{T}_f^i \ni K \sqsubseteq \exists r.K'$" is replaced with "$\mathcal{T}_f^i \models K \sqsubseteq \exists r.K'$".*
5. *Point 2 holds when "$\mathcal{T}_f^i \ni K \sqsubseteq (\leqslant 1\ r\ K')$" is replaced with "$\mathcal{T}_f^i \models K \sqsubseteq (\leqslant 1\ r\ K')$, $\mathcal{T}_f^i \models K' \sqsubseteq \exists r^-.K$, K is maximal with this property, and K' is active in $\widehat{\mathcal{T}}$".*

Proof. We simultaneously prove Points 1–5 by induction on i, showing

- the straightforward base case for Points 1–3;
- that Points 1–3 imply Points 4 and 5 for *every* $i \geq 0$;
- the induction step for Points 1–2 simultaneously, and for Point 3.

For Point 1 of the base case, assume that $\mathcal{T}_f^0 = \mathcal{T} \ni K \sqsubseteq \exists r.K'$ with K active in $\widehat{\mathcal{T}}$. Then $K \in \mathsf{KON}(\widehat{\mathcal{T}})$ because $\mathcal{T} \subseteq \widehat{\mathcal{T}}$, and K' is the required $\widehat{K}'$: (a) and (b) are due to Rules **R1** and **R3**, respectively.

For Point 2 of the base case, assume that $\mathcal{T}_f^0 = \mathcal{T} \ni K \sqsubseteq (\leqslant 1\ r\ K')$. Then K' is in fact a concept name A because we are assuming $\mathcal{T}$ to be in the stricter normal form. This A is the required concept name: (a) holds trivially, and (b) is due to Rule **R3**.

Point 3 of the base case follows from $\widehat{\mathcal{I}} \models \widehat{\mathcal{T}}$ (Lemma 10) and $\widehat{\mathcal{T}} \supseteq \mathcal{T} = \mathcal{T}_f^0$.

For Point 4, we will show that, for *every* $i \geq 0$, Point 4 follows from Points 1–3. The following argument thus combines base case and induction step for Point 4.

Assume that $\mathcal{T}_f^i \models K \sqsubseteq \exists r.K'$ with K active in $\widehat{\mathcal{T}}$. We also have that K is satisfiable w.r.t. $\mathcal{T}_f^i$ since $\widehat{\mathcal{I}} \models \mathcal{T}_f^i$ by Point 3 and, by construction, $\widehat{\mathcal{I}}$ has an instance of K. Consider the TBox $(\mathcal{T}_f^i)_K$. By Lemma 23 (1), there is some L' such that

(a′) $\mathcal{T}_f^i \models L' \sqsubseteq K'$ and

(b′) $(\mathcal{T}_f^i)_K \ni K \sqsubseteq \exists r.L'$.

We shall show below that, for every $K \sqsubseteq \exists r.L'$ in $(\mathcal{T}_f^i)_K$, there is some $\widehat{K}' \in \mathsf{KON}(\widehat{\mathcal{T}})$ with

(a″) $\widehat{K}' \vdash_{\widehat{\mathcal{T}}} L'$

(b″) for all $\widehat{K} \in \mathsf{KON}(\widehat{\mathcal{T}})$ with $\widehat{K} \vdash_{\widehat{\mathcal{T}}} K$, we have $\widehat{\mathcal{T}} \ni \widehat{K} \sqsubseteq \exists r.\widehat{K}'$.

This implies (a) and (b). (b) is exactly (b‴), and (a) follows from (a′) and (a″) by inspecting the root element $\widehat{K}'$ of $\Delta^{\widehat{\mathcal{I}}}$: by construction of $\widehat{\mathcal{I}}$ and (a″), this element is an instance of L'; since $\widehat{\mathcal{I}} \models \widehat{\mathcal{T}}_f^i$ (Point 3), it is an instance of K' too; by construction of $\widehat{\mathcal{I}}$, we get $\widehat{K}' \vdash_{\widehat{\mathcal{T}}} K'$.

To prove the above, we use induction on the number of rule applications used to construct $(\mathcal{T}_f^i)_K$. The base case is that $K \sqsubseteq \exists r.L'$ enters $(\mathcal{T}_f^i)_K$ in Step 1 of the construction. Then there is some $L \sqsubseteq \exists r.L' \in \mathcal{T}_f^i$ with $\mathcal{T}_f^i \models K \sqsubseteq L$. Since K is active in $\widehat{\mathcal{T}}$, so is L: for some $\widetilde{K} \in \mathsf{KON}(\widehat{\mathcal{T}})$ with $\widetilde{K} \vdash_{\widehat{\mathcal{T}}} K$, the root element $\widetilde{K}$ of $\Delta^{\widehat{\mathcal{I}}}$ must make L true. By Point 1, there is a $\widehat{K}' \in \mathsf{KON}(\widehat{\mathcal{T}})$ with (a″) and (b″) as required.

In the induction step, $K \sqsubseteq \exists r.L'$ enters $(\mathcal{T}_f^i)_K$ in Step 2 of the construction. In case this happens via an application of **R5**, we have that $L' = L'_1 \sqcap A$ and

(i) $K \sqsubseteq \exists r.L'_1 \in (\mathcal{T}_f^i)_K$

(ii) $K \sqsubseteq \forall r.A \in (\mathcal{T}_f^i)_K$

Applying the induction hypothesis to (i), we obtain

(i′) there is some $\widehat{K}'_1 \in \mathsf{KON}(\widehat{\mathcal{T}})$ with

(a‴) $\widehat{K}'_1 \vdash_{\widehat{\mathcal{T}}} L'_1$

(b‴) for all $\widehat{K} \in \mathsf{KON}(\widehat{\mathcal{T}})$ with $\widehat{K} \vdash_{\widehat{\mathcal{T}}} K$, we have $\widehat{\mathcal{T}} \ni \widehat{K} \sqsubseteq \exists r.\widehat{K}'_1$.

From (ii), we obtain $L \sqsubseteq \forall r.A \in \mathcal{T}_\mathsf{f}^i$ for some L with $\mathcal{T}_\mathsf{f}^i \models K \sqsubseteq L$ because axioms with ∀-restrictions never enter $(\mathcal{T}_\mathsf{f}^i)_K$ in Step 2 of the construction. Since such axioms are not generated by closing cycles either, we even have $L \sqsubseteq \forall r.A \in \mathcal{T}$; hence

(ii′) $L \sqsubseteq \forall r.A \in \widehat{\mathcal{T}}$ with $\mathcal{T}_\mathsf{f}^i \models K \sqsubseteq L$.

We now observe that $\mathcal{T}_\mathsf{f}^i \models K \sqsubseteq L$ and $\widehat{K} \vdash_{\widehat{\mathcal{T}}} K$ imply $\widehat{K} \vdash_{\widehat{\mathcal{T}}} L$ (again by consulting the domain element of $\widehat{\mathcal{I}}$ created for $\widehat{K}$). Hence, application of **R3** to (ii′) yields

(ii″) $\widehat{K} \sqsubseteq \forall r.A \in \widehat{\mathcal{T}}$ for all $\widehat{K} \in \mathsf{KON}(\widehat{\mathcal{T}})$ with $\widehat{K} \vdash_{\widehat{\mathcal{T}}} K$.

Now $\widehat{K}' := \widehat{K}'_1 \sqcap A$ is as required:

- $\widehat{K}' \in \mathsf{KON}(\widehat{\mathcal{T}})$.

  Since K is active in $\widehat{\mathcal{T}}$, there is some $\widehat{K} \in \mathsf{KON}(\widehat{\mathcal{T}})$ with $\widehat{K} \vdash_{\widehat{\mathcal{T}}} K$. By (b‴), we thus have $\widehat{K} \sqsubseteq \exists r.\widehat{K}'_1 \in \widehat{\mathcal{T}}$. Applying **R5** to this and (ii′) yields $\widehat{K} \sqsubseteq \exists r.(\widehat{K}'_1 \sqcap A) \in \widehat{\mathcal{T}}$. Hence $\widehat{K}'_1 \sqcap A \in \mathsf{KON}(\widehat{\mathcal{T}})$.
- (a″) is satisfied, that is, $\widehat{K}'_1 \sqcap A \vdash_{\widehat{\mathcal{T}}} L'_1 \sqcap A$.

  For every $A' \in L'_1$, we have $\widehat{K}'_1 \sqcap A \sqsubseteq A' \in \widehat{\mathcal{T}}$ because of (a‴), **R1**, **R3**. Furthermore, $\widehat{K}'_1 \sqcap A \sqsubseteq A \in \widehat{\mathcal{T}}$ due to **R1**.
- (b″) is satisfied.

  Let $\widehat{K} \in \mathsf{KON}(\widehat{\mathcal{T}})$ such that $\widehat{K} \vdash_{\widehat{\mathcal{T}}} K$. Then **R5** applied to (b‴) and (ii″) implies $\widehat{K} \sqsubseteq \exists r.(\widehat{K}'_1 \sqcap A) \in \widehat{\mathcal{T}}$.

In case $K \sqsubseteq \exists r.L'$ enters $(\mathcal{T}_\mathsf{f}^i)_K$ via an application of **R7′**, we have that $L' = L'_1 \sqcap L'_2$ and

(i) $K \sqsubseteq \exists r.L'_j \in (\mathcal{T}_\mathsf{f}^i)_K$, $j = 1, 2$

(ii) $K \sqsubseteq (\leqslant 1\ r\ L'_3) \in (\mathcal{T}_\mathsf{f}^i)_K$ for some L'_3 with

(iii) $\mathcal{T}_\mathsf{f}^i \models L'_j \sqsubseteq L'_3$, $j = 1, 2$.

Applying the induction hypothesis to (i), we obtain

(i′) there is some $\widehat{K}'_j \in \mathsf{KON}(\widehat{\mathcal{T}})$, $j = 1, 2$, with

(a‴) $\widehat{K}'_j \vdash_{\widehat{\mathcal{T}}} L'_j$

(b‴) for all $\widehat{K} \in \mathsf{KON}(\widehat{\mathcal{T}})$ with $\widehat{K} \vdash_{\widehat{\mathcal{T}}} K$, we have $\widehat{\mathcal{T}} \ni \widehat{K} \sqsubseteq \exists r.\widehat{K}'_j$.

Regarding (ii), we observe that axioms with functionality restrictions never enter $(\mathcal{T}_\mathsf{f}^i)_K$ in Step 2. Hence, there is some $L \sqsubseteq (\leqslant 1\ r\ L'_3) \in \mathcal{T}_\mathsf{f}^i$ with $\mathcal{T}_\mathsf{f}^i \models K \sqsubseteq L$, and the same observation as in the previous case yields $\widehat{K} \vdash_{\widehat{\mathcal{T}}} L$ whenever $\widehat{K} \vdash_{\widehat{\mathcal{T}}} K$. We thus obtain from Point 2 that

(ii′) there is some A with

(a⁴) for all $\widetilde{K}' \in \mathsf{KON}(\widehat{\mathcal{T}})$: if $\widetilde{K}' \vdash_{\widehat{\mathcal{T}}} L'_3$, then $\widetilde{K}' \sqsubseteq A \in \widehat{\mathcal{T}}$.

(b⁴) for all $\widehat{K} \in \mathsf{KON}(\widehat{\mathcal{T}})$ with $\widehat{K} \vdash_{\widehat{\mathcal{T}}} K$, we have $\widehat{\mathcal{T}} \ni \widehat{K} \sqsubseteq (\leqslant 1\ r\ A)$.

Furthermore (iii) and (a‴) yield

(iii′) $\widehat{K}'_j \vdash_{\widehat{\mathcal{T}}} L'_3$ for $i = 1, 2$,

and with (a⁴) we get

(iii″) $\widehat{K}'_j \sqsubseteq A \in \widehat{\mathcal{T}}$ for $i = 1, 2$.

Now $\widehat{K}' = \widehat{K}'_1 \sqcap \widehat{K}'_2$ is as required:

- $\widehat{K}' \in \mathsf{KON}(\widehat{\mathcal{T}})$.

  Since K is active in $\widehat{\mathcal{T}}$, there is some $\widehat{K} \in \mathsf{KON}(\widehat{\mathcal{T}})$ with $\widehat{K} \vdash_{\widehat{\mathcal{T}}} K$. By (b‴) and (b⁴), we thus have $\widehat{K} \sqsubseteq \exists r.\widehat{K}'_j \in \widehat{\mathcal{T}}$ and $\widehat{K} \sqsubseteq (\leqslant 1\ r\ A) \in \widehat{\mathcal{T}}$. Applying **R7** to these and (iii″) yields $\widehat{K} \sqsubseteq \exists r.(\widehat{K}'_1 \sqcap \widehat{K}'_2) \in \widehat{\mathcal{T}}$. Hence $\widehat{K}'_1 \sqcap \widehat{K}'_2 \in \mathsf{KON}(\widehat{\mathcal{T}})$.
- (a″) is satisfied, that is, $\widehat{K}'_1 \sqcap \widehat{K}'_2 \vdash_{\widehat{\mathcal{T}}} L'_1 \sqcap L'_2$.

  For every $i = 1, 2$ and $A' \in L'_i$, we have $\widehat{K}'_1 \sqcap \widehat{K}'_2 \sqsubseteq A \in \widehat{\mathcal{T}}$ because of (a‴), **R1**, and **R3**.
- (b″) is satisfied.

  Let $\widehat{K} \in \mathsf{KON}(\widehat{\mathcal{T}})$ such that $\widehat{K} \vdash_{\widehat{\mathcal{T}}} K$. Then **R7** applied to (b‴), (b⁴), (iii″) implies $\widehat{K} \sqsubseteq \exists r.(\widehat{K}'_1 \sqcap \widehat{K}'_2) \in \widehat{\mathcal{T}}$.

For Point 5, we will show that, for *every* $i \geq 0$, Point 5 follows from Points 1–3.

Assume that $\mathcal{T}_\mathsf{f}^i \models K \sqsubseteq (\leqslant 1\ r\ K')$, $\mathcal{T}_\mathsf{f}^i \models K' \sqsubseteq \exists r^-.K$, K is maximal with this property, and K' is active in $\widehat{\mathcal{T}}$. We have to show that there is an A such that

(a) for all $\widetilde{K}' \in \mathsf{KON}(\widehat{\mathcal{T}})$: if $\widetilde{K}' \vdash_{\widehat{\mathcal{T}}} K'$, then $\widetilde{K}' \sqsubseteq A \in \widehat{\mathcal{T}}$.

(b) for all $\widehat{K} \in \mathsf{KON}(\widehat{\mathcal{T}})$ with $\widehat{K} \vdash_{\widehat{\mathcal{T}}} K$, we have $\widehat{\mathcal{T}} \ni \widehat{K} \sqsubseteq (\leqslant 1\ r\ A)$.

Since K' is active in $\widehat{\mathcal{T}}$ and $\widehat{\mathcal{I}} \models \mathcal{T}_\mathsf{f}^i$ by Point 3, we also have that K' is satisfiable w.r.t. $\mathcal{T}_\mathsf{f}^i$. By Lemma 23 (2), there is some $L \sqsubseteq (\leqslant 1\ r\ L') \in \mathcal{T}_\mathsf{f}^i$ with

(i) $\mathcal{T}_\mathsf{f}^i \models K \sqsubseteq L$ and

(ii) $\mathcal{T}_\mathsf{f}^i \models K' \sqsubseteq L'$.

We apply Point 2 and conclude that there is some A with

(a′) for all $\widetilde{K}' \in \mathsf{KON}(\widehat{\mathcal{T}})$: if $\widetilde{K}' \vdash_{\widehat{\mathcal{T}}} L'$, then $\widetilde{K}' \sqsubseteq A \in \widehat{\mathcal{T}}$;

(b′) for all $\widehat{K} \in \mathsf{KON}(\widehat{\mathcal{T}})$ with $\widehat{K} \vdash_{\widehat{\mathcal{T}}} L$, we have $\widehat{\mathcal{T}} \ni \widehat{K} \sqsubseteq (\leqslant 1\ r\ A)$.

It remains to show that A is the required conjunction. For (a), take some $\widetilde{K}' \in \mathsf{KON}(\widehat{\mathcal{T}})$ and assume that $\widetilde{K}' \vdash_{\widehat{\mathcal{T}}} K'$. Then (ii) and Point 3 ($\widehat{\mathcal{I}} \models \mathcal{T}_\mathsf{f}^i$) imply that $\widetilde{K}' \vdash_{\widehat{\mathcal{T}}} L'$. Now (a′) implies that $\widetilde{K}' \sqsubseteq A \in \widehat{\mathcal{T}}$.

For (b), take some $\widehat{K} \in \mathsf{KON}(\widehat{\mathcal{T}})$ with $\widehat{K} \vdash_{\widehat{\mathcal{T}}} K$. Then (i) and $\widehat{\mathcal{I}} \models \mathcal{T}_\mathsf{f}^i$ imply that $\widehat{K} \vdash_{\widehat{\mathcal{T}}} L$. Now (b′) implies that $\widehat{\mathcal{T}} \ni \widehat{K} \sqsubseteq (\leqslant 1\ r\ A)$.

For Points 1–2 of the induction step, we prove both points simultaneously. If any of the CIs $K \sqsubseteq \exists r.K'$ and $K \sqsubseteq (\leqslant 1\ r\ K')$ is in $\mathcal{T}_\mathsf{f}^{i-1}$, then we can use the induction hypothesis for it. Otherwise, the respective CI has been introduced by closing a cycle $K_1, r_1, K_2, \ldots, r_{n-1}, K_n$ in $\mathcal{T}_\mathsf{f}^{i-1}$ with $K =$

K_j for some $j \in \{1, \ldots, n-1\}$, and thus K_j is active in $\widehat{\mathcal{T}}$. Take some $\widetilde{K} \in \mathsf{KON}(\widehat{\mathcal{T}})$ with $\widetilde{K} \vdash_{\widehat{\mathcal{T}}} K_j$. Applying Point 4 of the induction hypothesis to $\mathcal{T}_{\mathsf{f}}^{i-1} \models K_j \sqsubseteq \exists r_j.K_{j+1}$, we find a $\widehat{K}_{j+1} \in \mathsf{KON}(\widehat{\mathcal{T}})$ such that

(i) $\widehat{K}_{j+1} \vdash_{\widehat{\mathcal{T}}} K_{j+1}$, and

(ii) for all $\widehat{K} \in \mathsf{KON}(\widehat{\mathcal{T}})$ with $\widehat{K} \vdash_{\widehat{\mathcal{T}}} K_j$, we have $\widehat{\mathcal{T}} \ni \widehat{K} \sqsubseteq \exists r_j.\widehat{K}_{j+1}$.

By Point (i), K_{j+1} is active in $\widehat{\mathcal{T}}$ and thus we can iterate the argument to find

$$\widehat{K}_{j+2}, \ldots, \widehat{K}_n = \widehat{K}_1, \widehat{K}_2, \ldots, \widehat{K}_j$$

with the following properties, for $1 \leq j < n$.

(iii) $\widehat{K}_j \vdash_{\widehat{\mathcal{T}}} K_j$ and $\widehat{K}_j \in \mathsf{KON}(\widehat{\mathcal{T}})$;

(iv) $\widehat{\mathcal{T}} \ni \widehat{K}_j \sqsubseteq \exists r_j.\widehat{K}_{j+1}$.

Let $1 < j \leq n$. Applying Point 5 of the induction hypothesis to $\mathcal{T}_{\mathsf{f}}^{i-1} \models K_{j+1} \sqsubseteq (\leqslant 1\ r_j^-\ K_j)$, we find an A_j such that

(v) for all $\widetilde{K}_j \in \mathsf{KON}(\widehat{\mathcal{T}})$: if $\widetilde{K}_j \vdash_{\widehat{\mathcal{T}}} K_j$, then $\widetilde{K}_j \sqsubseteq A_j \in \widehat{\mathcal{T}}$;

(vi) for all $\widehat{K} \in \mathsf{KON}(\widehat{\mathcal{T}})$ with $\widehat{K} \vdash_{\widehat{\mathcal{T}}} K_{j+1}$, we have $\widehat{\mathcal{T}} \ni \widehat{K} \sqsubseteq (\leqslant 1\ r_j^-\ A_j)$.

From (vi) and (i), in particular we obtain:

(vii) $\widehat{\mathcal{T}} \ni \widehat{K}_{j+1} \sqsubseteq (\leqslant 1\ r_j^-\ A_j)$.

From (iii) and (v), we obtain:

(viii) $\widehat{K}_j \sqsubseteq A_j \in \widehat{\mathcal{T}}$.

We can now apply the cycle rule **R9** to the CIs in (iv), (vii) and (viii), obtaining

(ix) $\widehat{\mathcal{T}} \ni \widehat{K}_{j+1} \sqsubseteq \exists r_j^-.\widehat{K}_j$

(x) $\widehat{\mathcal{T}} \ni \widehat{K}_j \sqsubseteq (\leqslant 1\ r_j\ A_{j+1})$

To establish both Points 1 and 2, we set $\widehat{K}' = \widehat{K}_j$ and $A = A_{j+1}$, and we have to show

(1a) $\widehat{K}_j \vdash_{\widehat{\mathcal{T}}} K'$;

(1b) for all $\widehat{K} \in \mathsf{KON}(\widehat{\mathcal{T}})$ with $\widehat{K} \vdash_{\widehat{\mathcal{T}}} K_{j+1}$, we have $\widehat{\mathcal{T}} \ni \widehat{K} \sqsubseteq \exists r_j^-.\widehat{K}_j$;

(2a) for all $\widetilde{K}' \in \mathsf{KON}(\widehat{\mathcal{T}})$: if $\widetilde{K}' \vdash_{\widehat{\mathcal{T}}} K_{j+1}$ then $\widetilde{K}' \sqsubseteq A_{j+1} \in \widehat{\mathcal{T}}$;

(2b) for all $\widehat{K} \in \mathsf{KON}(\widehat{\mathcal{T}})$ with $\widehat{K} \vdash_{\widehat{\mathcal{T}}} K_j$, we have $\widehat{\mathcal{T}} \ni \widehat{K} \sqsubseteq (\leqslant 1\ r_j\ A_{j+1})$.

Now (1a) and (2a) are just (iii) and (v); hence it remains to show (1b) and (2b). We first claim that

(xi) For every $j \geq 1$, we have $\widehat{K} \sqsubseteq A \in \widehat{\mathcal{T}}$ for all $A \in \widehat{K}_j$.

To show (xi), let $\widehat{K} \in \mathsf{KON}(\widehat{\mathcal{T}})$ with $\widehat{K} \vdash_{\widehat{\mathcal{T}}} K_j$. From (ii) and (v), we get that

(xii) $\widehat{K} \sqsubseteq \exists r_j.\widehat{K}_{j+1} \in \widehat{\mathcal{T}}$.

From (viii), and from (v) with $\widehat{K} \vdash_{\widehat{\mathcal{T}}} K_j$, we obtain:

(xiii) $\widehat{K}_j \sqsubseteq A,\ \widehat{K} \sqsubseteq A_j \in \widehat{\mathcal{T}}$.

Applying **R8** to (xii), (ix), (xiii), (vi) yields (xi).

For showing (1b), take a $\widehat{K} \in \mathsf{KON}(\widehat{\mathcal{T}})$ with $\widehat{K} \vdash_{\widehat{\mathcal{T}}} K_{j+1}$. With (xi), we get $\widehat{K} \sqsubseteq A \in \widehat{\mathcal{T}}$ for every $A \in \widehat{K}_{j+1}$. Hence, **R3** and (ix) give us that $\widehat{K} \sqsubseteq \exists r_j^-.\widehat{K}_j \in \widehat{\mathcal{T}}$. For showing (2b), take $\widehat{K} \in \mathsf{KON}(\widehat{\mathcal{T}})$ with $\widehat{K} \vdash_{\widehat{\mathcal{T}}} K_j$. Again with (xi), we get that $\widehat{K} \sqsubseteq A \in \widehat{\mathcal{T}}$ for every $A \in \widehat{K}_j$ which, by **R3** and (x), yields $\widehat{K} \sqsubseteq (\leqslant 1\ r_j\ A_{j+1}) \in \widehat{\mathcal{T}}$ as required.

For Point 3 of the induction step. For every $K \sqsubseteq C \in \mathcal{T}_{\mathsf{f}}^i$, if K is realized in $\widehat{\mathcal{I}}$, then in the form of a supertype from $\mathsf{KON}(\widehat{\mathcal{T}})$. Thus it is easy to show that $\widehat{\mathcal{I}}$ is a model of every $K \sqsubseteq C \in \mathcal{T}_{\mathsf{f}}^i$, using Points 1 and 2 for the cases $C = \exists r.K'$ and $C = (\leqslant 1\ r\ K')$, and deriving the remaining cases from $\widehat{\mathcal{I}} \models \widehat{\mathcal{T}}$.

❑

C Proofs for Section 5

Recall that, to establish Proposition 15 we have to show the following.

Proposition 25 *For every $n_0 > 0$, there is a finite model $\mathcal{J}_{n_0}$ of $\mathcal{A}$ and $\mathcal{T}$ such that there is a homomorphism from any n_0-substructure of $\mathcal{J}_{n_0}$ to $\mathcal{U}$.*

As explained in the main paper, the first step towards proving Proposition 25 is to show Proposition 16, repeated here for convenience.

Proposition 16 For every $n_0 > 0$, there is a finite model $\mathcal{I}_{n_0}$ of $\mathcal{A}$ and $\mathcal{T}$ such that $\mathcal{I}_{n_0} \preceq_{n_0} \mathcal{U}$.

For the rest of Section C, fix a concrete n_0 (and, as in the main paper, an ABox $\mathcal{A}$ and TBox $\mathcal{T}$). To prove Proposition 16, we modify the finite model construction of Section 3. In particular, we modify the initial interpretation, use modified versions of the completion rules (**c2**) and (**c3**), and change the strategy of rule application. Here is a more detailed summary of the implemented changes:

- The elements d_t introduced in the initial version of $\mathcal{I}$ in the original construction and used as targets for applications of (**c3**) are not present.
- Instead of including only $\mathcal{A}$ in the initial version of $\mathcal{I}$, we include an initial piece of the canonical model $\mathcal{U}$ of $\mathcal{A}$ and $\mathcal{T}$, truncated to depth n_0.
- We then apply only (**c1**) and (**c2**), where the latter is modified in a way so that all new paths introduced between elements that existed already before a rule application exceed length n_0.
- To provide targets for applications of (**c3**), we determine all relevant n_0-simulation types and add them disjointly to the constructed interpretation.
- The previous two steps are iterated until no new n_0-simulation types are added.
- We then apply a modified version of (**c3**) that respects n_0-simulation types.

Details are provided in what follows.

We introduce some missing bits of notation. An (unbounded) *simulation of* $\mathcal{I}_1$ *in* $\mathcal{I}_2$ is a relation $\rho \subseteq \Delta^{\mathcal{I}_1} \times \Delta^{\mathcal{I}_2}$ such that for all $(d, e) \in \rho$, Condition 1 of bounded simulations is satisfied, as well as the following variation of Condition 2:

2′. if $(d, d') \in r^{\mathcal{I}_1}$ for some (possibly inverse) role r with $\mathsf{sig}(r) \subseteq \mathsf{sig}(\mathcal{A}) \cup \mathsf{sig}(\mathcal{T})$, then there is an $e' \in \Delta^{\mathcal{I}_2}$ with $(e, e') \in r^{\mathcal{I}_2}$ and $(d', e') \in \rho$.

We write $(\mathcal{I}_1, d) \preceq (\mathcal{I}_2, d)$ if there is a simulation ρ of $\mathcal{I}_1$ in $\mathcal{I}_2$ such that $(d, e) \in \rho$ and for all $a \in \mathsf{Ind}(\mathcal{A}) \cap \Delta^{\mathcal{I}}$, we have $(a, a) \in \rho$.

Proof of Proposition 16: Applying (c1) and (c2)

We define a finite interpretation $\mathcal{I}_0$ by starting with the subinterpretation $\mathcal{U}_0$ of $\mathcal{U}$ to those elements that can be reached from an ABox individual by traveling at most n_0 role edges. Note that the ABox $\mathcal{A}$ is a substructure of $\mathcal{U}_0$, but that the elements d_t from the initial interpretation $\mathcal{I}$ in Section 3 are not present. Next, we exhaustively apply the two completion rules $(\mathbf{c1})$ and $(\mathbf{c2'})$ to the initial $\mathcal{I}_0$ just defined, where $(\mathbf{c1})$ is as in Section 3 and $(\mathbf{c2'})$ is a modified version of $(\mathbf{c2})$. Note that, also in the original construction of the finite model $\mathcal{I}$ in Section 3, it is safe to apply $(\mathbf{c3})$ only after no further applications of $(\mathbf{c1})$ and $(\mathbf{c2})$ are possible since $(\mathbf{c3})$ cannot trigger an application of $(\mathbf{c1})$ or $(\mathbf{c2})$.

$(\mathbf{c2'})$ Choose a type class P that is minimal w.r.t. the order $\prec^+$ and such that there is a $\lambda = s\ {}^1\!\!\leftrightarrow^1_r s'$ with $s \in P$, and an element $d \in \Delta^{\mathcal{I}_0}$ with $d \in s^{\mathcal{I}_0} \setminus (\exists r.s')^{\mathcal{I}_0}$.

For each $\lambda = s\ {}^1\!\!\leftrightarrow^1_r s'$ with $s \in P$, set

$$X^{\mathcal{I}_0}_{\lambda,1} = s^{\mathcal{I}_0} \setminus (\exists r.s')^{\mathcal{I}_0} \qquad X^{\mathcal{I}_0}_{\lambda,2} = s'^{\mathcal{I}_0} \setminus (\exists r^-.s)^{\mathcal{I}_0}.$$

Take (i) a fresh set Δ_s for each $s \in P$ and (ii) a bijection π_λ between $X^{\mathcal{I}_0}_{\lambda,1} \cup \Delta_s$ and $X^{\mathcal{I}_0}_{\lambda,2} \cup \Delta_{s'}$ for each $\lambda = s\ {}^1\!\!\leftrightarrow^1_r s'$ with $s, s' \in P$ and r a role name, and extend $\mathcal{I}_0$ as follows:

- add all domain elements in $\biguplus_{s\in P} \Delta_s$;
- extend $r^{\mathcal{I}_0}$ with π_λ, for each $\lambda = s\ {}^1\!\!\leftrightarrow^1_r s'$ with $s, s' \in P$ and r a role name;
- interpret concept names so that $\mathsf{tp}_{\mathcal{I}_0}(d) = s$ for all $d \in \Delta_s$, $s \in P$.

An element d in the extended $\mathcal{I}_0$ is called *old* if it existed already before the extension (that is, $d \notin \biguplus_s \Delta_s$) and *new* otherwise. A path is a sequence $d_1 r_1 d_2 \cdots d_k r_k d_{k+1}$, with $d_1, \ldots, d_{k+1} \in \Delta^{\mathcal{I}_0}$, $r_1, \ldots, r_k$ (potentially inverse) roles and $(d_i, d_{i+1}) \in r_i^{\mathcal{I}_0}$ for $1 \le i \le k$. A path is *simple* if there are no multiple occurrences of the same node. We will show below that we can choose the sets Δ_s and bijections π_s such that:

1. $|\biguplus_{s\in P} \Delta_s| \le \mathcal{O}(2^{|\mathcal{T}|} \cdot |\mathcal{T}|^{n_0} \cdot |\Delta^{\mathcal{I}_0}|)$;
2. no edge $(d_1, d_2) \in r^{\mathcal{I}_0}$ is introduced with both d_1, d_2 old;
3. for each new element $d_1 \in \biguplus_s \Delta_s$, there is at most one simple path $d_1 r_1 d_2 \cdots d_k r_k d_{k+1}$ of length at most n_0 such that $d_1, \ldots, d_k$ are new and d_{k+1} is old.

Rules are applied in the same preference order as in Section 3.

We now argue that, in rule $(\mathbf{c2'})$ above, the sets Δ_s and π_λ indeed exist also with these modified conditions. We first extend $\mathcal{I}_0$ to a new interpretation $\mathcal{I}_0'$ by adding required successors up to depth n_0, which results in tree-shaped substructures of depth n_0 to be attached to elements of $\mathcal{I}_0$. During the process, we assign to each newly generated element a level and a type. In detail:

- all elements of $\mathcal{I}_0$ are assigned *level 0*;
- whenever there is an element d on level $\ell < n_0$ and a $\lambda = s\ {}^1\!\!\leftrightarrow^1_r s'$ with $s, s' \in P$ and r a (potentially inverse) role such that $\mathsf{tp}_{\mathcal{I}_0}(d) = s$ and there is no $(d, d') \in r^{\mathcal{I}_0}$ with $d' \in s'^{\mathcal{I}_0}$, then add a new element d', put $\mathsf{tp}_{\mathcal{I}_0}(d') = s'$, add (d, d') to $r^{\mathcal{I}_0}$, and assign to d' level $\ell + 1$.

Call the resulting interpretation $\mathcal{I}_0'$. We can now apply the original $(\mathbf{c2})$ operation to $\mathcal{I}_0'$ instead of to $\mathcal{I}_0$. This gives us a set Δ_s' for each $s \in P$ and a bijection π_λ' from $X^{\mathcal{I}_0'}_{\lambda,1} \cup \Delta_s'$ to $X^{\mathcal{I}_0'}_{\lambda,2} \cup \Delta_{s'}'$ for each $\lambda = s\ {}^1\!\!\leftrightarrow^1_r s'$ with $s, s' \in P$ and r a role name.

Set $\Delta_s = \Delta_s' \cup \{d \in \Delta^{\mathcal{I}_0'} \setminus \Delta^{\mathcal{I}_0} \mid \mathsf{tp}_{\mathcal{I}_0'}(d) = s\}$, and define π_λ as the extension of π_λ' by all pairs $(d, d') \in r^{\mathcal{I}_0'}$ such that $d, d' \in \Delta^{\mathcal{I}_0'} \setminus \Delta^{\mathcal{I}_0}$. It can be verified that π_λ is a bijection from $X^{\mathcal{I}_0}_{\lambda,1} \cup \Delta_s$ to $X^{\mathcal{I}_0}_{\lambda,2} \cup \Delta_{s'}$. Moreover, we have $|\Delta^{\mathcal{I}_0'}| \le |\Delta^{\mathcal{I}_0}| + (|\Delta^{\mathcal{I}_0}| \cdot |\mathcal{T}|^{n_0})$, and thus the size bound in Point 1 above is a consequence of the fact that $|\biguplus_{s\in P} \Delta_s'| \le 2^{|\mathcal{T}|} \cdot |\Delta^{\mathcal{I}_0'}|$.

The invariants $(\mathbf{i1})$–$(\mathbf{i3})$ from Section 3 are satisfied also with the modified initial interpretation and the modified version of $(\mathbf{c2'})$:

- The modified initial interpretation satisfies the invariants: $(\mathbf{i1})$, $(\mathbf{i2})$ are satisfied by construction of $\mathcal{U}$, and $(\mathbf{i3})$ holds because $\mathcal{U} \models \mathcal{T}_\mathsf{f}$ (Lemma 14).
- To show that the invariants are preserved by applications of $(\mathbf{c1})$ and the *old* $(\mathbf{c2})$ starting from the modified initial interpretation, the same arguments as for Theorem 3 (1) go through.
- The invariants are preserved by the new $(\mathbf{c2'})$, too, because the model extension described in $(\mathbf{c2'})$ is identical to that in $(\mathbf{c2})$ except for the potentially larger number of elements added, and the arguments for why the old $(\mathbf{c2})$ preserves all invariants do not depend on that exact number.

Furthermore, termination can be proved in exactly the same way as before.

In the following three lemmas, we will show that bounded simulations from $\mathcal{I}_0$ and extensions thereof in $\mathcal{U}$ exist. It suffices to consider n_0-*neighborhoods*: for a given model $\mathcal{J}$, we denote by $\mathcal{J}|^{n_0}_{d^*}$ the restrictions of $\mathcal{I}$ to the elements reachable from a given element d^* by travelling along at most n_0 role edges.

Lemma 26 $\mathcal{I}_0 \preceq_{n_0} \mathcal{U}$.

Proof. Let $d^* \in \Delta^{\mathcal{I}_0}$. We show that $(\mathcal{I}_0|^{n_0}_{d^*}, d^*) \preceq (\mathcal{U}, e)$ for some $e \in \Delta^{\mathcal{U}}$. For brevity, we use Δ to denote the domain of $\mathcal{I}_0|^{n_0}_{d^*}$. An element $d \in \Delta$ is an *initial element* if it is present in the initial version of (the modified) $\mathcal{I}$. A *forward path* is a sequence $d_0 r_0 d_1 \cdots d_{k-1} r_{k-1} d_k$ such that

(a) $(d_i, d_{i+1}) \in r_i^{\mathcal{I}_0|^{n_0}_{d^*}}$ for all $i < k$.

(b) $\mathsf{tp}_{\mathcal{I}_0}(d_i) \rightarrow_{r_i} \mathsf{tp}_{\mathcal{I}_0}(d_{i+1})$ for all $i < k$.

(c) $(r_i^-, d_{i+1}) \neq (r_{i-1}, d_{i-1})$ for $0 < i < k$;

(d) $d_1, \ldots, d_k$ are not initial.

An element $d \in \Delta$ is a *root* if the following two conditions are satisfied:

(i) if $(d, e) \in r^{\mathcal{I}_0|^{n_0}_{d^*}}$, then both d and e are initial or $\mathsf{tp}_{\mathcal{I}_0}(d) \rightarrow_r \mathsf{tp}_{\mathcal{I}_0}(e)$;

(ii) for every forward path $d = d_0 r_0 d_1 \cdots r_{k-1} d_k$ and each $(d_k, e) \in r^{\mathcal{I}_0|^{n_0}_{d^*}}$, we have that $\mathsf{tp}_{\mathcal{I}_0}(d_k) \rightarrow_r \mathsf{tp}_{\mathcal{I}_0}(e)$ or $e = d_{k-1}$ and $r = r_{k-1}^-$.

We first show the following:

(1) every initial element is a root;

(2) Δ contains at least one root d_r such that d^* is reachable from d_r on a forward path.

To show (1) and (2), we write $d \prec e$ if the rule application that created d happened before the one that created e. We write $d \preceq e$ if $d \preceq e$ or d and e are initial elements or d and e have been created in the same application (thus of $(\mathbf{c2'})$).

Take some element $d_r \in \Delta$ that is minimal w.r.t. $\preceq$, i.e., whenever $d \preceq d_r$, we also have $d_r \preceq d$. We will now show that

(α) d_r is a root, and

(β) d^* is reachable from d_r on a forward path.

Since every initial element is $\preceq$-minimal, (α) establishes (1). Furthermore, (α) and (β) establish (2).

For (α), we have to show Conditions (i) and (ii) of roots. For (i), take some element e with $(d_r, e) \in r^{\mathcal{I}_0|^{n_0}_{d^*}}$. If both d_r, e are initial, we are done. If d_r is initial and e is not, we have $\mathsf{tp}_{\mathcal{I}_0}(d_r) \rightarrow_r \mathsf{tp}_{\mathcal{I}_0}(e)$ due to the construction of $\mathcal{I}_0$. If e is initial and d_r is not, then this contradicts d_r being minimal. Finally, if none of d_r, e is initial, then we consider the rule application that created the edge $(d_r, e) \in r^{\mathcal{I}_0|^{n_0}_{d^*}}$: if it was $(\mathbf{c1})$, then $\preceq$-minimality of d_r ensures $\mathsf{tp}_{\mathcal{I}_0}(d_r) \rightarrow_r \mathsf{tp}_{\mathcal{I}_0}(e)$. If it was $(\mathbf{c2'})$, we have $\mathsf{tp}_{\mathcal{I}_0}(d_r)\ {}^1{\leftrightarrow}^1_r\ \mathsf{tp}_{\mathcal{I}_0}(e)$.

To show (ii), let $d_r = d_0 r_0 d_1 \cdots r_{k-1} d_k$ be a forward path and $(d_k, e) \in r^{\mathcal{I}_0|^{n_0}_{d^*}}$. In case the edge $(d_k, e) \in r^{\mathcal{I}_0|^{n_0}_{d^*}}$ was created in an application of $(\mathbf{c2'})$, we have $\mathsf{tp}_{\mathcal{I}_0}(d_k)\ {}^1{\leftrightarrow}^1_r\ \mathsf{tp}_{\mathcal{I}_0}(e)$ and we are done. Otherwise, this edge was created in an application of $(\mathbf{c1})$, and either $\mathsf{tp}_{\mathcal{I}_0}(d_k) \rightarrow_r \mathsf{tp}_{\mathcal{I}_0}(e)$ (in which case we are done) or $\mathsf{tp}_{\mathcal{I}_0}(e) \rightarrow_{r^-} \mathsf{tp}_{\mathcal{I}_0}(d_k)$. Then the edge $(d_{k-1}, d_k) \in r_{k-1}^{\mathcal{I}_0|^{n_0}_{d^*}}$ was created in an application of $(\mathbf{c1})$ (in which case we must have $e = d_{k-1}$ and $r = r_{k-1}^-$ and are done) or of $(\mathbf{c2'})$.

In the latter case, we trace that $(\mathbf{c2'})$ application backwards on the path. If all elements $d_r = d_0, \ldots, d_k$ have been created by the same application, we have that $e \prec d_0$, which contradicts d_r being $\preceq$-minimal. Otherwise, d_0 is an old element of that application, and we consider a simple path $d_0 = d'_0 r'_0 d'_1 \cdots r'_{\ell-1} d'_\ell = d_k$ of length $\leq 2n_0$ from $d_r = d_0 = d'_0$ to $d_k = d'_\ell$. Due to its construction, $\mathcal{I}_0|^{n_0}_{d^*}$ has to contain such a path. Then some edge on this new path must have been created in an application of $(\mathbf{c2'})$: otherwise, we would have $d'_0 \prec d'_1 \prec \cdots \prec d'_j \succ \cdots \succ d'_{\ell-1} \succ d'_\ell$, that is, d'_j would have been created in two different $(\mathbf{c1})$ rule applications.

Consider the latest such $(\mathbf{c2'})$ application in the construction of $\mathcal{I}_0$ and observe that $d'_\ell = d_k$ is old for it (we reuse the argument from above). If $d'_0 = d_0$ is old for it as well, then we get a contradiction as follows. Take the maximal index f such that $d'_0, \ldots, d'_f$ are all old and the minimal index g such that $d'_g, \ldots, d'_\ell$ are all old. Since $f < g$ due to Condition 2 of $(\mathbf{c2'})$, there is a middle element $d'_{j'}$ with $j' = \lfloor \frac{f+g}{2} \rfloor$, which has simple paths of length $\leq n_0$ to both old elements d'_f and d'_g. These paths coincide due to Condition 3 of $(\mathbf{c2'})$, which contradicts the assumption that $d'_0 r'_0 d'_1 \cdots r'_{\ell-1} d'_\ell$ is simple.

Otherwise, if d'_0 had been created in the same $(\mathbf{c2'})$ application, then we would get $d_k \prec d'_0$ which contradicts d_r being $\preceq$-minimal. Finally, if d'_0 had been created after that $(\mathbf{c2'})$ application, we would get $d'_1 \prec d'_0$, again contradicting $\preceq$-minimality of d_r.

For (β), we observe that, by construction of $\mathcal{I}_0|^{n_0}_{d^*}$, there is a simple path $d_r = d_0 r_0 d_1 \ldots r_{k-1} d_k = d^*$ in $\mathcal{I}_0|^{n_0}_{d^*}$ with $k \leq n_0$ such that no element other than possibly d_0 is initial. We pick such a path, and our choice ensures Conditions (a), (c) and (d) of forward paths. This leaves us with showing Condition (b).

Since $(d_i, d_{i+1}) \in r_i^{\mathcal{I}_0|^{n_0}_{d^*}}$ for all $i < k$, we have that either $\mathsf{tp}_{\mathcal{I}_0}(d_i) \rightarrow_{r_i} \mathsf{tp}_{\mathcal{I}_0}(d_{i+1})$ or $\mathsf{tp}_{\mathcal{I}_0}(d_{i+1}) \rightarrow_{r_i^-} \mathsf{tp}_{\mathcal{I}_0}(d_i)$ for all $i < k$. Assume that there is some i with $\mathsf{tp}_{\mathcal{I}_0}(d_{i+1}) \rightarrow_{r_i^-} \mathsf{tp}_{\mathcal{I}_0}(d_i)$ but $\mathsf{tp}_{\mathcal{I}_0}(d_i) \not\rightarrow_{r_i} \mathsf{tp}_{\mathcal{I}_0}(d_{i+1})$, and take the smallest such i. Then the edge $(d_i, d_{i+1}) \in r_i^{\mathcal{I}_0}$ has been introduced in some application of $(\mathbf{c1})$, and the previous edge $(d_{i-1}, d_i) \in r_{i-1}^{\mathcal{I}_0}$ has been introduced in some application of $(\mathbf{c2'})$ (otherwise, we would have $d_{i+1} = d_{i-1}$ and $r_i = r_{i-1}^-$, contradicting Condition (d) of forward paths). We can now reuse the above argument, tracing back that application of $(\mathbf{c2'})$, and derive a contradiction.

We say that a relation $\rho \subseteq \Delta \times \Delta^{\mathcal{U}}$ is *rooted* if for every $(d, e) \in \rho$, there is a forward path $d_0 r_0 d_1 \cdots r_{k-1} d_k$ and elements $e_0, \ldots, e_k \in \Delta^{\mathcal{U}}$ such that d_0 is a root, $d_k = d$, $e_k = e$, $(d_i, e_i) \in \rho$ for all $i \leq k$ and $(e_i, e_{i+1}) \in r_i^{\mathcal{U}}$ for all $i < k$. We define a sequence of relations

$$\rho_0 \subseteq \rho_1 \subseteq \cdots \subseteq \Delta \times \Delta^{\mathcal{U}}$$

such that

(†) if $(d, e) \in \rho_i$, then $\mathsf{tp}_{\mathcal{I}_0}(d) = \mathsf{tp}_{\mathcal{U}}(e)$;

(‡) ρ_i is rooted.

Choose a root d_r such that d^* is reachable from d_r by a forward path, whose existence is guaranteed by Point 2 above. Also choose an $e_r \in \Delta^{\mathcal{U}}$ with $\mathsf{tp}_{\mathcal{U}}(e_r) = \mathsf{tp}_{\mathcal{I}_0}(d_r)$, which exists by invariant (**i1**). Then

- set
$$\rho_0 = \{(d,d) \mid d \in \Delta \text{ is initial}\} \cup \{(d_r, e_r)\}$$
Note that (†) and (‡) are trivially satisfied.
- ρ_{i+1} is obained from ρ_i by doing the following for each $(d,e) \in \rho_i$ and $(d,d') \in r^{\mathcal{I}_0|^{n_0}_{d^*}}$. By (‡), there is a forward path $d_0 r_0 d_1 \cdots r_{k-1} d_k$ and elements $e_0, \ldots, e_k \in \Delta^{\mathcal{U}}$ such that d_0 is a root, $d_k = d$, $e_k = e$, $(d_i, e_i) \in \rho$ for all $i \leq k$ and $(e_i, e_{i+1}) \in r_i^{\mathcal{U}}$ for all $i < k$. By Point 2 above, we can distinguish two cases:
  - $\mathsf{tp}_{\mathcal{I}_0}(d_k) \rightarrow_r \mathsf{tp}_{\mathcal{I}_0}(d')$ and it is not true that $d' = d_{k-1}$ and $r = r^-_{k-1}$.
  Then $\mathsf{tp}_{\mathcal{I}_0}(d) \rightarrow_r \mathsf{tp}_{\mathcal{I}_0}(d')$. By (†), we have $\mathsf{tp}_{\mathcal{U}}(e) \rightarrow_r \mathsf{tp}_{\mathcal{I}_0}(d')$ and thus we find an $e' \in \Delta^{\mathcal{U}}$ with $(e,e') \in r^{\mathcal{U}}$ and $\mathsf{tp}_{\mathcal{U}}(e') = \mathsf{tp}_{\mathcal{I}_0}(d')$. Include (d', e') in ρ_{i+1}. Clearly, (†) is still satisfied. Using that it is not true that $d' = d_{k-1}$ and $r = r^-_{k-1}$, it is also straightforward to show that (‡) is still satisfied.
  - $d' = d_{k-1}$ and $r = r^-_{k-1}$.
  Then we do not need to add an extra tuple to ρ_i since there already is a $(d', e') \in \rho_i$ such that $(e, e') \in r^{\mathcal{U}}$. To see this, recall that e_{k-1} and e_k are such that $(d_{k-1}, e_{k-1}) \in \rho_i$, $(d_k, e_k) \in \rho_i$, and $(e_{k-1}, e_k) \in r^{\mathcal{U}}_{k-1}$. Since $d' = d_{k-1}$ and $r = r^-_{k-1}$, e_{k-1} can serve as the required e'.

Set $\rho = \bigcup_{i \geq 0} \rho_i$. By construction, ρ is a simulation. Since there is a forward path from d_r to d^*, by construction of ρ there must be some $(d^*, e) \in \rho$. Thus we have shown $(\mathcal{I}_0|^{n_0}_{d^*}, d^*) \preceq (\mathcal{U}, e)$. Note that $\mathsf{tp}_{\mathcal{I}_0}(d^*) = \mathsf{tp}_{\mathcal{U}}(e)$ as required. ❑

Proof of Proposition 16: Generating Witnesses for (c3)

To prepare for the application of (a modified version) of the completion rule (**c3**), we need to generate elements that can be used as ‘targets’ for edges in place of the elements d_t from the original finite interpretation $\mathcal{I}$. To prepare for this, we first extend the finite model $\mathcal{I}_0$ constructed so far to an infinite interpretation $\mathcal{I}_0^+$. While $\mathcal{I}_0^+$ will of course not be part of the finite model that we aim to construct, it will guide the further construction.

We obtain $\mathcal{I}_0^+$ from $\mathcal{I}_0$ by starting with $\mathcal{I}_0^+ = \mathcal{I}_0$ and then exhaustively applying the completion rule from the construction of the canonical model $\mathcal{U}$, repeated here for convenience: for all $d \in \Delta^{\mathcal{I}_0^+}$ such that $\mathcal{T}_{\mathsf{f}} \models \mathsf{tp}_{\mathcal{I}_0^+}(d) \sqsubseteq \exists r.t'$, where t' is maximal with this property and $d \notin (\exists r.t')^{\mathcal{I}_0^+}$, add a fresh element d' to $\Delta^{\mathcal{I}_0^+}$, the edge (d,d') to $r^{\mathcal{I}_0^+}$, and d' to the interpretation $A^{\mathcal{I}_0^+}$ of all concept names $A \in t'$.

Lemma 27 *$\mathcal{I}_0^+ \preceq_{n_0} \mathcal{U}$.*

Proof. (sketch) Let $d^* \in \Delta^{\mathcal{I}_0^+}$. We show that $(\mathcal{I}_0^+|^{n_0}_{d^*}, d^*) \preceq (\mathcal{U}, e)$ for some $e \in \Delta^{\mathcal{U}}$. For brevity, we use Δ to denote the domain of $\mathcal{I}_0^+|^{n_0}_{d^*}$. We distinguish three cases:

1. d^* is from $\Delta^{\mathcal{I}_0}$.
Then Lemma 26 gives us an n_0-bounded simulation ρ of $(\mathcal{I}_0, d^*)$ in $(\mathcal{U}, e)$ for some e. We can extend ρ to the desired n_0-bounded simulation of $(\mathcal{I}_0^+, d^*)$ in $(\mathcal{U}, e)$ by following the applications of the completion rule applied to construct $\mathcal{I}_0^+$ from $\mathcal{I}_0$, and exploiting that $\mathcal{U}$ is constructed by applying the same rule.
2. d^* is not from $\Delta^{\mathcal{I}_0}$ and Δ contains elements from $\Delta^{\mathcal{I}_0}$.
Let d_0 be the unique element from Δ that is in $\Delta^{\mathcal{I}_0}$ and can be reached from d^* in $\mathcal{I}_0^+|^{n_0}_{d^*}$ on a path of minimal length.[4] Start with a n_0-bounded simulation ρ of $(\mathcal{I}_0, d_0)$ in $(\mathcal{U}, e)$ for some e (given by Lemma 26), restricted to the elements of Δ. Then proceed as in Case 1.
3. Δ contains no elements from $\Delta^{\mathcal{I}_0}$.
Exploiting Invariant (**i1**), it is easy to show by induction on the number of rule applications used to construct $\mathcal{I}_0^+$ that for every $d \in \Delta^{\mathcal{I}_0^+}$, there is an $e \in \Delta^{\mathcal{U}}$ with $\mathsf{tp}_{\mathcal{I}_0^+}(d) = \mathsf{tp}_{\mathcal{U}}(e)$. For $d, d' \in \Delta$, we write $d \prec d'$ if d' was created by a later rule appliction than d during the construction of $\mathcal{I}_0^+$ from $\mathcal{I}_0$. Let d_0 be the unique element of Δ that is minimal w.r.t. $\prec$. We start with the initial bounded simulation $\rho = \{(d_0, n_0, e)\}$ for some e with $\mathsf{tp}_{\mathcal{I}_0^+}(d_0) = \mathsf{tp}_{\mathcal{U}}(e)$ and then proceed as in Case 1 above.

❑

We now choose one representative $(\mathcal{J}, d) \in S$ of each n_0-simulation type S realized in $\mathcal{I}_0^+$, i.e., such that there is some $d \in \Delta^{\mathcal{I}_0^+}$ with $(\mathcal{I}_0^+, d) \in S$. Then extend $\mathcal{I}_0$ with pairwise disjoint copies of all the chosen representatives. By Lemma 27, the resulting interpretation $\mathcal{I}_1$ satisfies $\mathcal{I}_1 \preceq_{n_0} \mathcal{U}$. We treat $\mathcal{I}_1$ as an initial interpretation in the same way as we have treated $\mathcal{U}_0$ as an initial intepretation for constructing $\mathcal{I}_0$ and repeat the application of (**c1**) and (**c2′**) as described above, which results in a completed version of the interpretation $\mathcal{I}_1$. Lemmas 26 and 27 apply also to $\mathcal{I}_1$ in place of $\mathcal{I}_0$, with the proofs going through without modification. The same is true for the invariants (**i1**) to (**i3**). Since $\mathcal{I}_1^+$ might realize n_0-simulation types that are not realized in $\mathcal{I}_0^+$, we then disjointly add copies of the new simulation types. Repeating this process leads to a sequence of finite interpretations $\mathcal{I}_0, \mathcal{I}_1, \ldots$. Since there are only finitely many n_0-simulation types and since the simulation type of added representatives does not change by applying the rules (**c1**) and (**c2′**), this process eventually stabilizes. Call the resulting finite interpretation $\mathcal{I}_\omega$. By what was said above, we have the following.

Lemma 28 *$\mathcal{I}_\omega \preceq_{n_0} \mathcal{U}$ and $\mathcal{I}_\omega^+ \preceq_{n_0} \mathcal{U}$.*

The disjoint copies just added will serve as the desired ‘targets’ for applying (a modified version) of the completion rule (**c3**), described in the next section.

[4]This element d_0 is unique since $\mathcal{I}_0^+$ extends $\mathcal{I}_0$ by attaching tree-shaped structures to existing elements.

Proof of Proposition 16: Applying (c3)

To construct the desired finite interpretation $\mathcal{I}$, it remains to start with the infinite model $\mathcal{I} = \mathcal{I}_\omega^+$ and exhaustively apply a modified version $(\mathbf{c3}')$ of the completion rule $(\mathbf{c3})$.

$(\mathbf{c3}')$ If $d \in \Delta^{\mathcal{I}}$, $\mathsf{tp}_{\mathcal{I}}(d) \to_r t$, and $d \notin (\exists r.t)^{\mathcal{I}}$, then by construction of $\mathcal{I}_\omega^+$, we find an element $e \in \Delta^{\mathcal{I}_\omega^+}$ such that $(d,e) \in r^{\mathcal{I}_\omega^+}$ and $\mathsf{tp}_{\mathcal{I}_\omega^+}(e) = t$. By construction of $\mathcal{I}_\omega$, there is an element $e' \in \Delta^{\mathcal{I}_\omega}$ such that $(\mathcal{I}_\omega^+|_e^{n_0}, e)$ and $(\mathcal{I}_\omega|_{e'}^{n_0}, e')$ have the same simulation type. Include in $r^{\mathcal{I}}$ the edge (d, e').

The modified version of $(\mathbf{c3})$ preserves all invariants because the same arguments as for the old $(\mathbf{c3})$ go through.

Lemma 29 *$\mathcal{I}$ is a model of $\mathcal{A}$ and $\mathcal{T}_\mathsf{f}$.*

Proof. It is easy to see that the proof of Proposition 22 goes through also for the modified version of $\mathcal{I}$: the essential ingredients of that proof are the invariants $(\mathbf{i1})$–$(\mathbf{i3})$, which hold for $\mathcal{I}$ as argued above, plus the argument after the case distinction (1)–(3) for axioms of the form $K \sqsubseteq \exists r.K'$, which is unaffected by our modification of the rules. ❏

We can now establish the main property that is satisfied by the finite model $\mathcal{I}$ just constructed, but not by the finite model $\mathcal{I}$ built in Section 3.

Lemma 30 *For every n_0-substructure $\mathcal{I}'$ of $\mathcal{I}$, $\mathcal{I}' \preceq \mathcal{U}$.*

Proof. By Lemma 28, it suffices to show that $\mathcal{I} \preceq_{n_0} \mathcal{I}_\omega^+$. We call an edge $(d,e) \in r^{\mathcal{I}}$ *special* if it was added in the construction of $\mathcal{I}$ from $\mathcal{I}_\omega$, that is, by applying $(\mathbf{c3}')$. The *source* of the special edge $(d,e) \in r^{\mathcal{I}}$ is the element from $\{d,e\}$ that plays the role of d in the formulation of $(\mathbf{c3}')$.

Let $d^* \in \Delta^{\mathcal{I}}$. In the following, we construct an n_0-bounded simulation ρ of $(\mathcal{I}, d^*)$ in $(\mathcal{I}_\omega^+, d^*)$. To assist with the construction of ρ, we associate with every tuple (d,i,e) in the partially constructed ρ an i-bounded simulation $\rho_{d,i,e}$ of $(\mathcal{I}_\omega^+, d)$ in $(\mathcal{I}_\omega^+, e)$ whose purpose is to guide the further construction.

We start with setting $\rho = \{(d^*, n_0, d^*)\}$. As the required n_0-bounded simulation ρ_{d^*,n_0,d^*} of $(\mathcal{I}_\omega^+, d^*)$ in $(\mathcal{I}_\omega^+, d^*)$, we use the identity, that is, the set of all triples (d,i,d) with $d \in \Delta^{\mathcal{I}_\omega^+}$ and $i \leq n_0$. To extend the initial ρ just defined, we distingush three cases.

Assume that $(d,i,e) \in \rho$ with $i > 0$ and $(d,d') \in r^{\mathcal{I}}$ is non-special. Then $(d,d') \in r^{\mathcal{I}_\omega} \subseteq r^{\mathcal{I}_\omega^+}$ and thus we find a triple $(d', i-1, e') \in \rho_{d,i,e}$ with $(e,e') \in r^{\mathcal{I}_\omega^+}$. Add $(d', i-1, e')$ to ρ and set $\rho_{d',i-1,e'} = \rho_{d,i,e}$.

Now assume that $(d,i,e) \in \rho$ with $i > 0$, $(d,d') \in r^{\mathcal{I}}$ is special, and d is the source of this edge. Then there is a $d'' \in \Delta^{\mathcal{I}_\omega^+}$ such that $(d,d'') \in r^{\mathcal{I}_\omega^+}$ and $(\mathcal{I}_\omega^+, d'')$ has the same n_0-simulation type as $(\mathcal{I}_\omega, d')$. Then $(\mathcal{I}_\omega^+, d'')$ must also have the same n_0-simulation type as $(\mathcal{I}_\omega^+, d')$. We can thus find an $(i-1)$-bounded simulation ν of $(\mathcal{I}_\omega^+, d')$ in $(\mathcal{I}_\omega^+, d'')$. Since $(d,i,e) \in \rho_{d,i,e}$, there must be an $e'' \in \Delta^{\mathcal{I}_\omega^+}$ with $(d'', i-1, e'') \in \rho_{d,i,e}$ and $(e,e'') \in r^{\mathcal{I}_\omega^+}$. We add $(d', i-1, e'')$ to ρ. The required $(i-1)$-bounded simulation $\rho_{d',i-1,e''}$ of $(\mathcal{I}_\omega^+, d')$ in $(\mathcal{I}_\omega^+, e'')$ is obtained by composing ν with $\rho_{d,i,e}$.

Finally assume that $(d,i,e) \in \rho$ with $i > 0$, $(d,d') \in r^{\mathcal{I}}$ is special, and e is the source of this edge. Then there is a $\widehat{d} \in \Delta^{\mathcal{I}_\omega^+}$ such that $(\widehat{d}, d') \in r^{\mathcal{I}_\omega^+}$ and $(\mathcal{I}_\omega^+, \widehat{d})$ has the same n_0-simulation type as $(\mathcal{I}_\omega, d)$. Then $(\mathcal{I}_\omega^+, \widehat{d})$ must also have the same n_0-simulation type as $(\mathcal{I}_\omega^+, d)$. We can thus find an i-bounded simulation ν of $(\mathcal{I}_\omega^+, \widehat{d})$ in $(\mathcal{I}_\omega^+, d)$. Composing ν with $\rho_{d,i,e}$, we find an i-bounded simulation η of $(\mathcal{I}_\omega^+, \widehat{d})$ in $(\mathcal{I}_\omega^+, e)$. Since $(\widehat{d}, d') \in r^{\mathcal{I}_\omega^+}$, there must be some $\widehat{e} \in \Delta^{\mathcal{I}_\omega^+}$ such that $(d', i-1, \widehat{e}) \in \eta$ and $(e, \widehat{e}) \in r^{\mathcal{I}_\omega^+}$. Add $(d', i-1, \widehat{e})$ to ρ. The required $(i-1)$-bounded simulation $\rho_{d',i-1,\widehat{e}}$ of $(\mathcal{I}_\omega^+, d')$ in $(\mathcal{I}_\omega^+, \widehat{e})$ is provided by η. ❏

Proof of Proposition 25: Products

Recall that we are looking for a finite model of $\mathcal{A}$ and $\mathcal{T}_\mathsf{f}$ such that every n_0-substructure of this model homomorphically embeds into $\mathcal{U}$. The model $\mathcal{I}$ from the previous section is still not as required since it may contain cycles that are not present in $\mathcal{U}$. While such cycles cannot be completely avoided, they can be made large enough so that they are not 'visible' in n_0-substructures. To achieve this, we take the product of $\mathcal{I}$ with a finite group of high girth, see (Otto 2004). We start with recalling some basic notions of group theory.

Let $(G, \circ)$ be a finite group generated by a (finite) set $\{g_i \mid i \geq 0\}$ of *involutive* generators, i.e., $g_i = g_i^{-1}$. The *Cayley graph* of G is the undirected graph that has as vertices the group elements $h \in G$ and where $\{h, h'\}$ is an edge if $h' = h \circ g_i$.

We are interested in groups whose Cayley graph satisfies two properties. First, it should have high girth, where the *girth* of a graph is the length of a shortest cycle contained in that graph. Second, for all group elements h and generators g_1, g_2, we want to have $h \circ g_1 \neq h \circ g_2$; in other words, the outdegree of every node in the Cayley graph should be exactly k, with k the number of generators. Such a graph is called *k-regular*.

Explicit constructions of k-regular graphs with girth greater than m, for any k and m, have been studied in the literature. The following is a known result, see e.g. (Alon 1995) for a full discussion of the construction.

Theorem 31 (Margulis 1982; Imrich 1984) *For every $k, m > 0$ there exists a finite group G which is generated by a set of k involutive generators, and whose Cayley graph has* regular degree *k and girth at least m.*

Let $\mathcal{J}$ be an interpretation. We use $E_{\mathcal{J}}$ to denote the set of all edges of $\mathcal{J}$, that is, all sets $\{d,e\}$ such that $(d,e) \in r^{\mathcal{I}}$ for some role r. Let $(G, \circ)$ be a finite group with involutive generators g_S, $S \in E_{\mathcal{J}}$: that is, the set of edges $E_{\mathcal{J}}$ can be embedded via an injection into a set generating G. The existence of such a group G is granted by Theorem 31. We use $\mathcal{J} \otimes G$ to denote the interpretation with domain $\Delta^{\mathcal{J}} \times G$ defined as follows:

$$A^{\mathcal{J} \otimes G} = \{\langle d, h\rangle \in \Delta^{\mathcal{J}} \times G \mid d \in A^{\mathcal{J}}\}$$
$$r^{\mathcal{J} \otimes G} = \{(\langle d, h\rangle, \langle d', h \circ g_{\{d,d'\}}\rangle) \mid (d,d') \in r^{\mathcal{J}}\}.$$

Reduction to loop free models. As a preliminary, we first transform $\mathcal{I}$ to rule out cycles of length 1 (reflexive loops) or 2. More precisely, an interpretation $\mathcal{J}$ is called *loop free* if it satisfies the following three conditions for all individuals d, e and (possibly inverse) roles r, s:

1. If $d \notin \mathsf{Ind}(\mathcal{A})$, then $(d, d) \notin r^{\mathcal{J}}$.
2. If $(d, e) \notin \mathsf{Ind}(\mathcal{A}) \times \mathsf{Ind}(\mathcal{A})$ and $(d, e) \in r^{\mathcal{J}} \cap s^{\mathcal{J}}$, then $r = s$.

The following construction shows how to transform $\mathcal{I}$ into a loop free model $\mathcal{I}'$. Let $r_1, \ldots, r_R$ be the role names occurring in $\mathcal{A}$ and $\mathcal{T}_\mathsf{f}$. We take $2R + 2$ disjoint copies of $\Delta^{\mathcal{I}}$, and interpret ABox elements in the last copy and concept names in every copy the same way as in $\Delta^{\mathcal{I}}$. In the model to be constructed, r_k-edges between non-ABox elements jump over k copies (modulo $2R + 2$), and r_k-edges between ABox elements remain in the last copy if they originate there, or otherwise jump over k copies (modulo $2R+1$ this time). This way, we leave the ABox structure intact in the last copy, and break up all other cycles of length 1 and 2. More precisely,

$$\begin{aligned}
\Delta^{\mathcal{I}'} &= \Delta^{\mathcal{I}} \times \{0, \ldots, 2R + 1\} \\
A^{\mathcal{I}'} &= \{\langle d, i\rangle \mid d \in A^{\mathcal{I}},\ 0 \le i \le 2R + 1\} \\
r_k^{\mathcal{I}'} &= \{(\langle d, i\rangle, \langle e, i \oplus_{2R+2} k\rangle) \mid (d, e) \in r_k^{\mathcal{I}} \setminus \mathsf{Ind}(\mathcal{A})^2\} \\
&\quad \cup \{(\langle d, i\rangle, \langle e, i \oplus_{2R+1} k\rangle) \mid (d, e) \in r_k^{\mathcal{I}} \cap \mathsf{Ind}(\mathcal{A})^2, \\
&\qquad\qquad i \le 2R\} \\
&\quad \cup \{(\langle d, 2R + 1\rangle, \langle e, 2R + 1\rangle) \mid (d, e) \in r_k^{\mathcal{I}} \cap \mathsf{Ind}(\mathcal{A})^2\}
\end{aligned}$$

ABox individuals are interpreted in the first copy, that is, we identify a with $\langle a, 2R + 1\rangle$. Note that the last line preserves the structure of the ABox, and the preceding lines ensure loop freeness (but do not generally rule out cycles of length 3).

Indeed, $\mathcal{I}'$ is loop free:

1. Whenever $(\langle d, i\rangle, \langle d, i\rangle) \in r_k^{\mathcal{I}'}$, the construction ensures that $i = 2R + 1$ and d is an ABox element.
2. Let $(\langle d, i\rangle, \langle e, j\rangle) \in (r^{\mathcal{I}'} \cap s^{\mathcal{I}'}) \setminus \mathsf{Ind}(\mathcal{A})^2$. We distinguish three cases.

   Both r, s are role names: $r = r_k$, $s = r_\ell$. Then the above pair has been added in the first or second line of the constructions of both $r_k^{\mathcal{I}'}$ and $r_\ell^{\mathcal{I}'}$. If it was added in the second line, then we have $j = i \oplus_{2R+1} k = i \oplus_{2R+1} \ell$ which, due to $0 < k, \ell \le R$, implies $k = \ell$ and hence $r_k = r_\ell$. The case for the first line is analogous.

   One of r, s is a role name; the other is not: $r = r_k$, $s = r_\ell^-$. As in the previous case, the above pair must have been added in the first or second line of the constructions of both role interpretations. If it was added in the second line, then we have $j = i \oplus_{2R+1} k$ and $i = j \oplus_{2R+1} \ell$. Inserting the first equation into the second, we get $i = i \oplus_{2R+1} k \oplus_{2R+1} \ell$, which is impossible because $0 < k + \ell \le 2R$. The case for the first line is analogous.

   None of r, s are role names: $r = r_k^-$, $s = r_\ell^-$. This case reduces to the first case if we swap $\langle d, i\rangle$ and $\langle e, j\rangle$.

It is an easy exercise to show that $\mathcal{I}'$ is a model of $\mathcal{A}$ and $\mathcal{T}_\mathsf{f}$, and that $\mathcal{I}' \preceq \mathcal{I}$. From Lemma 30, we thus get $\mathcal{I}' \preceq_{n_0} \mathcal{U}$.

Products of loop free models. Now consider the finite model $\mathcal{I}$ of $\mathcal{A}$ and $\mathcal{T}_\mathsf{f}$ constructed in the previous section. By the reduction just shown, we may assume that $\mathcal{I}$ is loop free. By Theorem 31, we can take a finite group G with $|E_{\mathcal{I}}|$ involutive generators $\{g_S \mid S \in E_{\mathcal{I}}\}$, such that the Cayley graph of G has girth higher than n_0 and is $|E_{\mathcal{I}}|$-regular. We then form the product $\mathcal{I} \otimes G$. This interpretation is almost as required, but does not necessarily satisfy the ABox. To fix this, we consider the interpretation $\widehat{\mathcal{J}}$ that can be obtained as follows:

- start with $\mathcal{I}^- \otimes G$, where $\mathcal{I}^-$ is obtained from $\mathcal{I}$ by removing, for each $r(a, b) \in \mathcal{A}$, the pair (a, b) from $r^{\mathcal{I}}$;
- then take an arbitrary but fixed $h_{\mathcal{A}} \in G$, for every $a \in \mathsf{Ind}(\mathcal{A})$ and identify each ABox element a with $(a, h_{\mathcal{A}})$;
- finally, for each $r(a, b) \in \mathcal{A}$, add $(\langle a, h\rangle, \langle b, h\rangle)$ to $r^{\widehat{\mathcal{J}}}$, for every pair $\langle a, h\rangle, \langle b, h\rangle \in \Delta^{\mathcal{J}}$.

Note that all copies of the ABox in $\widehat{\mathcal{J}}$, not just the 'main' one identified by $h_{\mathcal{A}}$, inherit the relational structure of the ABox. We first observe that $\widehat{\mathcal{J}}$ is still a (finite!) model of $\mathcal{A}$ and $\mathcal{T}_\mathsf{f}$. This essentially follows from the observations in (Otto 2004).

Lemma 32 *$\widehat{\mathcal{J}}$ is a model of $\mathcal{A}$ and $\mathcal{T}_\mathsf{f}$.*

Proof. To show that $\widehat{\mathcal{J}}$ is a model of $\mathcal{T}_\mathsf{f}$, we use the fact that $\mathcal{I}$ is a model of $\mathcal{T}_\mathsf{f}$ and the construction of $\widehat{\mathcal{J}}$. CIs of the form $K \sqsubseteq A$, $K \sqsubseteq \bot$, $K \sqsubseteq \exists r.K'$, and $K \sqsubseteq \forall r.K'$ are easy to deal with. We thus concentrate on CIs $K \sqsubseteq (\leqslant 1\ r\ K')$. Assume that $\langle d, h\rangle \in K^{\widehat{\mathcal{I}}}$. By construction of $\widehat{\mathcal{J}}$, this means $d \in K^{\mathcal{I}}$. Assume to the contrary of what is to be shown that there are $(\langle d, h\rangle, \langle e_i, h_i\rangle) \in r^{\widehat{\mathcal{J}}}$ for $i = 1, 2$ such that $\langle e_i, h_i\rangle \in K'^{\widehat{\mathcal{J}}}$ and $\langle e_1, h_1\rangle \neq \langle e_2, h_2\rangle$. Then $(d, e_i) \in r^{\mathcal{I}}$ and, since $\mathcal{I}$ is a model of $\mathcal{T}_\mathsf{f}$, we obtain $e_1 = e_2 =: e$. Now the construction of $\widehat{\mathcal{J}}$ yields that, if both d, e interpret ABox elements in $\mathcal{I}$, then $h_1 = h_2 = h$, and that otherwise $h_i = h \circ g_{\{d,e\}}$ for both $i = 1, 2$. Finally, using the construction of $\widehat{\mathcal{J}}$, it is easy to observe that $\widehat{\mathcal{J}}$ is a model of $\mathcal{A}$. ❑

A *cycle in $\widehat{\mathcal{J}}$* (of length n) is a path $p_1, r_1, \ldots, r_n, p_{n+1}$, where $n > 2$, $p_i \in \Delta^{\widehat{\mathcal{J}}}$, each r_i is a (possibly inverse) role such that $(p_i, p_{i+1}) \in r_i^{\widehat{\mathcal{J}}}$, and $p_1 = p_{n+1}$. Further, a cycle is *simple* if, for $1 \le i < j \le n$, we have $p_i \neq p_j$. An element $p = \langle d, h\rangle \in \Delta^{\widehat{\mathcal{J}}}$ is an *ABox element* if $d \in \mathsf{Ind}(\mathcal{A})$. Note that this definition includes all "copies" of ABox elements from $\mathcal{I}$, not just those that interpret ABox individuals. We say that $\widehat{\mathcal{J}}$ is *k-acyclic relative to $\mathcal{A}$* if every simple cycle in $\widehat{\mathcal{J}}$ of length at most k contains exclusively ABox elements.

Lemma 33 *$\widehat{\mathcal{J}}$ is n_0-acyclic relative to $\mathcal{A}$.*

Proof. We start with the following observation:

Claim. If $(p_1, p_2) \in r^{\widehat{\mathcal{J}}}$ with $p_i = \langle d_i, h_i\rangle$, $i \in \{1, 2\}$, and at least one of d, e is not an ABox element, then $h_2 = h_1 \circ g_{\{d_1, d_2\}}$.

In fact, this is immediate if r is a role name. If $r = s^-$, then $h_1 = h_2 \circ g_{\{d_1, d_2\}}$, which by multiplication with $g_{\{d_1, d_2\}}$ and due to the generators being involutive yields $h_1 \circ g_{\{d_1,d_2\}} = h_2$.

Let $\alpha = p_1, r_1, \dots, r_n, p_{n+1}$ be a simple cycle in $\widehat{\mathcal{J}}$, with $p_i = \langle d_i, h_i \rangle$ for $1 \leq i \leq n$, such that for some i, p_i is not an ABox element. Assume to the contrary of what is to be shown that $n \leq n_0$. We show that h_{i-1}, h_i, and h_{i+1} are all different. Consequently, α gives rise to a cycle of length between three and n_0 in the Cayley graph of G (even if some of the other elements on α should coincide), which contradicts the non-existence of such cycles. We have $h_{i-1} \neq h_i$ since otherwise $h_{i-1} = h_{i-1} \circ g_{i-1}$, which is not possible due to n_0-regularity of the Cayley graph of G; for the same reason, $h_i \neq h_{i+1}$. Finally, assume to the contrary of what we want to show that $h_{i-1} = h_{i+1}$. Then $h_{i-1} \circ g_{i-1} \circ g_i = h_{i-1}$. Multiplying with g_i yields $h_{i-1} \circ g_{i-1} = h_{i-1} \circ g_i$, which gives $g_{i-1} = g_i$ by k-regularity of G. Since $g_{i-1} = g_{\{d_{i-1},d_i\}}$ and $g_i = g_{\{d_i,d_{i+1}\}}$, this yields $d_{i-1} = d_{i+1}$, thus $p_{i-1} = p_i$ in contrast to α being simple. ❏

Lemma 34 *Every n_0-substructure $\mathcal{J}'$ of $\widehat{\mathcal{J}}$ homomorphically embeds into $\mathcal{U}$, the canonical model of $\mathcal{A}$ and $\mathcal{T}_{\mathsf{f}}$.*

Proof. We may assume w.l.o.g. that $\mathcal{J}'$ is connected. We start with making a useful observation.

Claim 1. For each $p_1 \in \Delta^{\widehat{\mathcal{J}}}$ that is not an ABox element, there is at most one simple path $p_1 r_1 p_2 \cdots p_k r_k p_{k+1}$ in $\mathcal{J}'$ such that $p_1, \dots, p_k$ are not ABox elements and p_{k+1} is an ABox element.

Proof. Assume there is a $p_1 \in \Delta^{\widehat{\mathcal{J}}}$ that is not an ABox element and such that there are two simple paths in $\mathcal{J}'$ of the described form. Each such path $p_1 r_1 p_2 \cdots p_k r_k p_{k+1}$ gives rise to a corresponding path $d_1 r_1 d_2 \cdots d_k r_k d_{k+1}$ in $\mathcal{I}$ such that $d_1, \dots, d_k$ are not ABox individuals, but d_{k+1} is. Note that the initial version of the modified finite interpretation $\mathcal{I}$ contains $\mathcal{U}_0$, which takes the form of the ABox $\mathcal{A}$ extended with a tree of depth n_0 below each ABox individual, and that later steps in the construction of $\mathcal{I}$ only add successors to leaves in these trees. Therefore and since the length of all mentioned paths is clearly bounded by n_0, both paths in $\mathcal{I}$ must be inside the same tree of $\mathcal{U}_0$. But then, since they start and end at the same element and are simple, they must be identical.

Choose an arbitrary $p_0 = (d_0, h_0) \in \Delta^{\mathcal{J}'}$. By Lemma 30, we know that $(\mathcal{I}, d_0) \preceq_{n_0} \mathcal{U}$, witnessed by an n_0-bounded simulation ρ. We use ρ to construct the desired homomorphism η from $\mathcal{J}'$ to $\mathcal{U}$.

In what follows, let π be the projection on the first component of elements in $\Delta^{\widehat{\mathcal{J}}}$. We define a sequence of partial homomorphisms η_i, $i \geq 0$, that is, partial functions $\eta_i : \Delta^{\mathcal{J}'} \times \Delta^{\mathcal{U}}$ that satisfy Conditions 1 to 3 of homomorphisms. The desired homomorphism η is then obtained in the limit. We will make sure that all η_i satisfy the following properties:

(a) $(\pi(p), n_0 - i, \eta_i(p)) \in \rho$ for all $p \in \Delta^{\mathcal{J}'}$;

(b) if $\mathcal{J}'$ contains a path of length $\leq i$ from an initial element to p, then $\eta_i(p)$ is defined, where an element q is *initial* if $\eta_0(q)$ is defined.

We start by defining η_0 as follows.

- If $\Delta^{\mathcal{J}'}$ contains ABox elements, then set $\eta_0(p) = a$ for all $p = \langle a, h \rangle \in \Delta^{\mathcal{J}'}$ with $a \in \mathsf{Ind}(\mathcal{A})$.
- If $\Delta^{\mathcal{J}'}$ does not contain ABox elements, then choose an $e \in \Delta^{\mathcal{U}}$ with $(\pi(p_0), n_0, e) \in \rho$ and set $\eta_0(p_0) = e$.

Clearly, η_0 satisfies (a) and (b) and Condition 1 of homomorphisms. Satisfaction of Condition 2 follows from $(*)$. Finally, satisfaction of Condition 3 follows from the existence of ρ and the fact that, by definition of bounded simulations, ρ preserves all edges between ABox elements.

In the induction step, η_{i+1} is obtained from η_i by defining a value for all $p_2 \in \Delta^{\mathcal{J}'}$ such that there is some edge $(p_1, p_2) \in r^{\widehat{\mathcal{J}}}$ with $\eta_i(p_1)$ defined and $\eta_i(p_2)$ undefined. To define $\eta_{i+1}(p_2)$, we observe that $(\pi(p_1), \pi(p_2)) \in r^{\mathcal{I}}$ follows from $(p_1, p_2) \in r^{\widehat{\mathcal{J}}}$ and $(\pi(p_1), n_0 - i, \eta_i(p_1)) \in \rho$ holds by (a). Moreover, we have $i < n_0$ by (b) and since any two elements in $\mathcal{J}'$ reach each other by a path of length $\leq n_0$, thus $i = n_0$ contradicts $\eta_i(p_2)$ being undefined. Consequently there must be some e such that $(\pi(p_2), n_i - i - 1, e) \in \rho$ and $(\eta_i(p_1), e) \in r^{\mathcal{U}}$. Set $\eta_{i+1}(p_2) = e$.

We next show that η_{i+1} is well-defined, that is, if $(p_1, p) \in r^{\widehat{\mathcal{J}}}$ and $(p_2, p) \in s^{\widehat{\mathcal{J}}}$ with $\eta_i(p_1)$ and $\eta_i(p_2)$ defined and $\eta_i(p)$ undefined, then $(p_1, r) = (p_2, s)$. Assume to the contrary that this is not the case. We distinguish three cases:

- $p \in \{p_1, p_2\}$. Then $(p_1, p_1) \in r^{\widehat{\mathcal{J}}}$ or $(p_2, p) \in s^{\widehat{\mathcal{J}}}$. We address the fomer case, the latter is analogous. Let $p_1 = \langle d_1, h_1 \rangle$. Then $(p_1, p_1) \in r^{\widehat{\mathcal{J}}}$ yields $(d, d) \in r^{\mathcal{I}}$. Since $\mathcal{I}$ is loop free, $d \in \mathsf{Ind}(\mathcal{A})$, thus p is an an ABox element. This is a contradiction to $\eta_i(p)$ being undefined
- $p \notin \{p_1, p_2\}$, $p_1 = p_2$, and $r \neq s$. Let $p_1 = \langle d_1, h_1 \rangle$ and $p = \langle d, h \rangle$. Then we have $(d_1, d) \in r^{\mathcal{I}}$ and $(d_1, d) \in s^{\mathcal{I}}$. Since $\mathcal{I}$ is loop free and $d \notin \mathsf{Ind}(\mathcal{A})$ (because p cannot be an ABox element), this yields $r = s$ as required.
- $p \notin \{p_1, p_2\}$ and $p_1 \neq p_2$. Since $\eta_i(p_1)$ and $\eta_i(p_2)$ are defined and $\eta_{i+1}(p)$ is not, p_j is reachable from some initial element $\widehat{p}_j$ on a path $\mathcal{P}_j$ of length i and this is the shortest path from any initial element to p_j, for $j \in \{1, 2\}$. If $\Delta^{\mathcal{J}'}$ contains no ABox elements, then $\widehat{p}_1 = \widehat{p}_2 = p_0$. Otherwise, $\widehat{p}_1$ and $\widehat{p}_2$ are ABox elements and we obtain from Claim 1 and the fact that both $\widehat{p}_1$ and $\widehat{p}_2$ are reachable from p that $\widehat{p}_1 = \widehat{p}_2$. For readability, we from now on use $\widehat{p}$ to denote $\widehat{p}_1 (= \widehat{p}_2)$.

  Let p' be the element on the path $\mathcal{P}_1$ that also occurs on the path $\mathcal{P}_2$ and is furthest away from $\widehat{p}$ (such an element always exists since $\widehat{p}$ is on both paths). Consider the following cycle in $\widehat{\mathcal{J}}$:

  1. from p' to p_1 along $\mathcal{P}_1$;
  2. from p_1 to p along r;
  3. from p to p_2 along s^-;
  4. from p_2 to p' backwards along $\mathcal{P}_2$.

  Since $p \notin \{p_1, p_2\}$ and $p_1 \neq p_2$, this cycle has length > 2 as required. Moreover, the cycle is simple: by choice, the two travelled subpaths of $\mathcal{P}_1$ and $\mathcal{P}_2$ do not share any elements, including p_1 and p_2. Moreover, p does not occur on these subpaths because η_i must be defined for all elements on the subpaths whereas it is not defined for p. Since p occurs on a simple cycle, p must be an ABox element. This yields a contradiction to $\eta_i(p)$ being undefined.

To finish the proof, we note that it is clear that η_{i+1} satisfies (a), (b), and all three conditions of homomorphisms. ❑

D Proofs for Section 6

Proposition 18 *$\mathcal{T}$ is finitely satisfiable iff $\mathcal{T}'$ is finitely satisfiable.*

Proof. The "if" direction is trivial since every model of $\mathcal{T}'$ is also a model of $\mathcal{T}$. For the "only if" direction, let $\mathcal{I}$ be a finite model of $\mathcal{T}$. We construct a finite model $\mathcal{J}$ of $\mathcal{T}'$ by taking n copies of $\mathcal{I}$ and 'rewiring' all role edges across the different copies such that the concept names B_i can be interpreted in a non-conflicting way.

Specifically, since $\mathcal{I}$ satisfies $K \sqsubseteq (\geqslant n\ r\ K')$ we can choose a function $\mathsf{succ} : K^\mathcal{I} \times \{0, \dots, n-1\} \to \Delta^\mathcal{I}$ such that the following conditions are satisfied:

- for all $d \in K^\mathcal{I}$ and $i < n$: $(d, \mathsf{succ}(d, i)) \in r^\mathcal{I}$ and $\mathsf{succ}(d, i) \in (K')^\mathcal{I}$;
- for all $d \in K^\mathcal{I}$ and $i < j < n$: $\mathsf{succ}(d, i) \neq \mathsf{succ}(d, j)$.

Then define the desired interpretation $\mathcal{J}$ by setting

$$
\begin{aligned}
\Delta^\mathcal{J} &= \{d_i \mid d \in \Delta^\mathcal{I} \text{ and } i < n\}\\
E^\mathcal{J} &= \{d_i \mid d \in E^\mathcal{I} \text{ and } i < n\}\\
&\quad \text{for all } E \in \mathsf{N_C} \setminus \{B_0, \dots, B_{n-1}\}\\
B_i^\mathcal{J} &= \{d_i \mid d \in \Delta^\mathcal{I}\} \text{ for all } i < n\\
s^\mathcal{J} &= \{(d_i, e_i) \mid (d, e) \in s^\mathcal{I} \text{ and } i < n\}\\
&\quad \text{for all } s \in \mathsf{N_R} \setminus \{r\}\\
r^\mathcal{J} &= \{(d_i, e_i) \mid (d, e) \in r^\mathcal{I},\ i < n,\\
&\qquad \text{and } d \notin K^\mathcal{I} \text{ or } e \neq \mathsf{succ}(d, j) \text{ for any } j\}\\
&\quad \cup \{(d_i, e_{(i+j) \bmod n}) \mid (d, e) \in r^\mathcal{I},\ i, j < n,\\
&\qquad \text{and } e = \mathsf{succ}(d, j)\}
\end{aligned}
$$

It remains to verify that $\mathcal{J}$ is indeed a model of $\mathcal{T}'$. Clearly, the CIs in $(*)$ on page 9 are satisfied. To verify that all concept inclusions in $\mathcal{T}$ are satisfied by $\mathcal{J}$, we observe that the construction ensures that the number of r-successors (and -predecessors) in any $A \in \mathsf{CN}$ of every (x, i) is the same as that for x.

We first claim that, for every $d \in \Delta^\mathcal{I}$ and every s-successor e of d in $\mathcal{I}$, the i-th copy of d in $\mathcal{J}$ has exactly one copy of e as an s-successor:

Claim. *Let s be a role, $d_i \in \Delta^\mathcal{J}$, and let $\{e \in \Delta^\mathcal{I} \mid (d, e) \in s^\mathcal{I}\} = \{e_1, \dots, e_\ell\}$ for some $\ell \geq 0$. Then $\{e^j \in \Delta^\mathcal{I} \mid (d^i, e^j) \in s^\mathcal{I}\} = \{e_1^{j_1}, \dots, e_\ell^{j_\ell}\}$, for some $j_1, \dots, j_\ell \in \{0, \dots, n-1\}$.*

This claim is implied by the construction of $s^\mathcal{J}$: consider a given $d_i \in \Delta^\mathcal{J}$ and (possibly inverse) role s. If s is neither r nor r^-, then every e_k contributes exactly one s-successor e_k^i of d^i. The same holds if $s = r$ and $d \notin K^\mathcal{I}$. If $s = r$ and $d \in K^\mathcal{I}$, then each $e_k = \mathsf{succ}(d, j)$ for some j contributes exactly one s-successor $e_k^{(i+j) \bmod n}$ of d^i, and every other e_k contributes e_k^i. For $s = r^-$, then every $e_k \in K^\mathcal{I}$ with $d = \mathsf{succ}(e_k, j)$ for some j contributes $e_k^{(i-j) \bmod n}$, and every other e_k contributes e_k^i.

As an immediate consequence, we obtain that all qualified and unqualified number restrictions in $d \in \Delta^\mathcal{I}$ are preserved in every $d^i \in \Delta^\mathcal{J}$:

Fact. *Let $d^i \in \Delta^\mathcal{J}$ and $D = (\bowtie s\ n\ C)$ where $\bowtie \in \{\leqslant, \geqslant\}$, s is a role or inverse role, and C is either a conjunction of concept names, or the negation of such a conjunction, or $\top$, or $\bot$. Then $d \in D^\mathcal{I}$ iff $d^i \in D^\mathcal{J}$.*

This can be concluded from the previous claim and the observation that e and e^{j_i} satisfy the same concept names. The fact includes the cases $s = r$ and $s = r^-$, and it implies that existential, and universal restrictions are preserved – for the latter it is necessary to allow that C is a negated conjunction.

We are now ready to prove that $\mathcal{J}$ is a model of $\mathcal{T}'$, proceeding by type of CI. We distinguish the following cases.

- $L \sqsubseteq A$ and $L \sqsubseteq \bot$, both in $\mathcal{T}$. These are satisfied because they are satisfied by $\mathcal{I}$ and due to the construction: every d in $\mathcal{I}$ and every d^i in $\mathcal{J}$ are instances of the same non-B_i concept names.
- $L \sqsubseteq \exists s.L'$ in $\mathcal{T}$. Let $d^i \in L^\mathcal{J}$. Then $d \in L^\mathcal{I}$ due to the construction. Since $\mathcal{I}$ satisfies the axiom, $d \in (\geqslant 1\ s\ L')^\mathcal{I}$. With the previous fact, we conclude $d^i \in (\geqslant 1\ s\ L')^\mathcal{J}$, hence $d^i \in (\exists s.L')^\mathcal{J}$. This argument includes the cases $s = r$ and $s = r^-$.
- $L \sqsubseteq \forall s.L'$ in $\mathcal{T}$. In the argument above, replace "$\in (\geqslant 1\ s\ L')^{\cdots}$" with "$\notin (\geqslant 1\ s\ \neg L')^{\cdots}$".
- $L \sqsubseteq (\leqslant 1\ s\ L')$ in $\mathcal{T}$. Then $d^i \in L^\mathcal{J}$ implies $d \in L^\mathcal{I}$, hence $d \in (\leqslant 1\ s\ L')^\mathcal{I}$ and, due to the previous fact, $d^i \in (\leqslant 1\ s\ L')^\mathcal{I}$.
- $L \sqsubseteq (\geqslant m\ s\ L')$ in $\mathcal{T}$. Apply the same argument as above.
- $B_i \sqsubseteq K'$ and $B_i \sqcap B_j \sqsubseteq \bot$. Follows from the construction.
- $K \sqsubseteq \exists r.B_i$. Let $d^j \in K^\mathcal{J}$, which implies $d \in K^\mathcal{I}$. Let $e = \mathsf{succ}(d, (i-j) \bmod n)$. Then the construction yields that $(d^j, e^i) \in r^\mathcal{J}$ — because $i = (j + (i-j) \bmod n) \bmod n$ — and $e^i \in B_i^\mathcal{J}$. Hence $d^j \in (\exists r.B_i)^\mathcal{J}$.

❑

E Examples

Example 1

Consider a TBox $\mathcal{T}$ containing the following axioms:

$$
\begin{array}{ll}
A \sqsubseteq \exists r_1.B & B \sqsubseteq (\leqslant 1\ r_1^-\ A)\\
B \sqsubseteq \exists r_2.A & A \sqsubseteq (\leqslant 1\ r_2^-\ B)\\
C_1 \sqsubseteq A & \\
C_1 \sqsubseteq \forall r_1.C_2 & C_2 \sqsubseteq \forall r_2.C_1
\end{array}
$$

After closing, we get the following relations on types:

$$A \; {}^1{\leftrightarrow}^1_{r_1} \; B \tag{21}$$

$$B \; {}^1{\leftrightarrow}^1_{r_2} \; A \tag{22}$$

$$A, C_1 \; {}^1{\leftrightarrow}^1_{r_1} \; B, C_2 \tag{23}$$

$$B, C_2 \; {}^1{\leftrightarrow}^1_{r_2} \; A, C_1 \tag{24}$$

Hence two type partitions $P_1 = \{\{A\}, \{B\}\}$ and $P_2 = \{\{A, C_1\}, \{B, C_2\}\}$ with $P_2 \prec^+ P_1$.

Now consider the ABox $\mathcal{A} = \{C_1(a), X(a), A(b), X(b)\}$. Where X is just any concept name that "activates" the types A, C_1, X, A, X; but does not affect cycles.