Conservative Rewritability of Description Logic TBoxes: First Results

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Abstract. We want to understand when a given TBox $\mathcal{T}$ in a description logic $\mathcal{L}$ can be rewritten into a TBox $\mathcal{T}'$ in a weaker description logic $\mathcal{L}'$. Two notions of rewritability are considered: model-conservative rewritability ($\mathcal{T}'$ entails $\mathcal{T}$ and all models of $\mathcal{T}$ can be expanded to models of $\mathcal{T}'$) and $\mathcal{L}$-conservative rewritability ($\mathcal{T}'$ entails $\mathcal{T}$ and every $\mathcal{L}$-consequence of $\mathcal{T}'$ in the signature of $\mathcal{T}$ is a consequence of $\mathcal{T}$) and investigate rewritability of TBoxes in $\mathcal{ALCI}$ to $\mathcal{ALC}$, $\mathcal{ALCQ}$ to $\mathcal{ALC}$, $\mathcal{ALC}$ to $\mathcal{EL}_\bot$, and $\mathcal{ALCI}$ to *DL-Lite*$_{horn}$. We compare conservative rewritability with equivalent rewritability, give model-theoretic characterizations of conservative rewritability, prove complexity results for deciding rewritability, and provide some rewriting algorithms.

Over the past 30 years, a multitude of different description logics (DLs) have been designed, investigated, and used in practice as ontology languages. The introduction of new DLs has been driven both by the need for additional expressive power (such as transitive roles in the 1990s) and by applications that require efficient reasoning of a novel type (such as ontology-based data access in the late 2000s). While the resulting flexibility in choosing DLs has had the positive effect of making DLs available for a large number of domains and applications, it has also led to the development of ontologies with language constructors that are not really required to axiomatize their knowledge. A constructor to be 'not required' can mean different things here, ranging from the high-level 'this domain can be represented in an adequate way in a weaker DL' to the very concrete 'this ontology is logically equivalent to an ontology in a weaker DL'. In this paper, we take the latter understanding as our starting point. Equivalent rewritability of a given DL ontology (TBox) to a weaker DL has been investigated in [17], where model-theoretic characterizations and the complexity of deciding rewritability were investigated. For example, equivalent rewritability of an $\mathcal{ALC}$ TBox to an $\mathcal{EL}_\bot$ TBox has been characterized in terms of preservation under products and global equisimilations, and a NExpTime upper bound for deciding equivalent rewritability has been established. Equivalent rewritability is a very strong notion, however, that appears to apply to a very small number of real-world TBoxes. A more practically relevant notion we propose in this paper is *conservative* rewritability, which allows one to use new concept and role names when rewriting a given ontology into a weaker DL. In this case, we clearly cannot demand that the new TBox is logically equivalent to the original one, but only that it entails the original TBox. To avoid uncontrolled additional consequences of the new TBox, we can also require that (i) it does not entail any new consequences in the language of the original TBox, or even that (ii) every model of the original TBox can be expanded a model of the new TBox. The latter type of conservative extension is known as *model-conservative extension* [16], and we call a TBox $\mathcal{T}$ model-conservatively $\mathcal{L}$-rewritable if a model-conservative rewriting of $\mathcal{T}$ in the DL $\mathcal{L}$ exists. The former type of conservative extension is known as a *language-conservative extension* or *deductive conservative extension* [12] and, given a DL $\mathcal{L}$ in which $\mathcal{T}$ is formulated and a weaker DL $\mathcal{L}'$, we call $\mathcal{T}$ $\mathcal{L}$-conservatively $\mathcal{L}'$-rewritable if there is a TBox $\mathcal{T}'$ in $\mathcal{L}'$ such that $\mathcal{T}'$ has the same $\mathcal{L}$-consequences as $\mathcal{T}$ in the signature of $\mathcal{T}$. Model-conservative rewritability is the more robust notion as it is language-independent and does not only leave unchanged the entailed concept inclusions of the original TBox but also, for example, certain answers if the ontologies are used to access data.

The main result of this paper is that there are important DLs for which model-conservative and $\mathcal{L}$-conservative rewritability can be transparently characterized, effectively decided, and where rewriting algorithms can be designed. This is in contrast to the undecidability of the problem whether one TBox is a model-conservative extension of another one even for weak DLs such as $\mathcal{EL}$ [18, 16]. In particular, we show that, given an $\mathcal{ALCI}$ TBox, one can compute in polynomial time its model-conservative $\mathcal{ALC}$-rewriting provided if such a rewriting exists, which can be decided in ExpTime. We characterize model-conservative $\mathcal{ALC}$-rewritability in terms of preservation under generated subinterpretations and show that $\mathcal{ALCI}$-conservative $\mathcal{ALC}$-rewritability coincides with model-conservative one. For $\mathcal{ALCQ}$ TBoxes, we show that model-conservative $\mathcal{ALC}$-rewritability coincides with equivalent rewritability, but is different from $\mathcal{ALCQ}$-conservative rewritability. The latter can be characterized using bounded morphisms, and all these rewritability notions are decidable in 2ExpTime. Unlike the $\mathcal{ALCI}$ case, we currently do not have polynomial rewritings for $\mathcal{ALCQ}$ TBoxes. For rewritability from $\mathcal{ALCI}$ to *DL-Lite*$_{horn}$, we observe that all our notions of rewritability coincide and are ExpTime-complete. In contrast, for rewritability from $\mathcal{ALC}$ to $\mathcal{EL}_\bot$ they are all distinct and, in fact, rather intricate and difficult to analyse. We prove decidability of model-conservative rewritability and give necessary semantic conditions for both $\mathcal{ALC}$-conservative and model-conservative $\mathcal{EL}_\bot$-rewritability.

Related work. Conservative rewritings of TBoxes are ubiquitous in DL research. For example, many rewritings of TBoxes into normal forms are model-conservative [14, 4]. Regarding rewritability of TBoxes into weaker DLs, the focus has been on polynomial satisfability preserving rewritings as a pre-processing step to reasoning [11, 9, 8] or to prove complexity results for reasoning [10]. Such rewritings are mostly not conservative. There has been significant work on rewritings of ontology-mediated queries (pairs of ontologies and queries), which preserve their certain answers, into datalog or ontology-mediated queries based on weaker DLs [13, 5]. It seems, however, that this problem is different from TBox conservative rewritability. In [2], the expressive power of DLs and corresponding notions of rewritability are introduced based on a variant of model-conservative extension, and the relationship to $\mathcal{L}$-conservative extensions is discussed.

All the omitted proofs can be found in the Appendix.

1 Conservative Rewritability

We consider the standard description logics $\mathcal{ALC}$, $\mathcal{ALCI}$, $\mathcal{ALCQ}$, $\mathcal{EL}_\bot$, and *DL-Lite*$_{horn}$ [3, 4, 7, 1], where $\mathcal{EL}_\bot$ is $\mathcal{EL}$ extended with the concept $\bot$, and *DL-Lite*$_{horn}$ is *DL-Lite*$_{core}$ extended with conjunctions of basic concepts on the left-hand side of concept inclusions. As usual, the alphabet of DLs consists of countably infinite sets $\mathsf{N_C}$ of *concept names* and $\mathsf{N_R}$ of *role names*. By a *signature*, Σ, we mean any set of concept and role names. The *signature* $\mathsf{sig}(\mathcal{T})$ *of a TBox* $\mathcal{T}$ is the set of concept and role names occurring in $\mathcal{T}$.

Before introducing conservative rewritability, we remind the reader of a simpler notion of TBox rewritability. Suppose $\mathcal{L}$ and $\mathcal{L}'$ are DLs; we typically assume that $\mathcal{L}$ is more expressive than $\mathcal{L}'$.

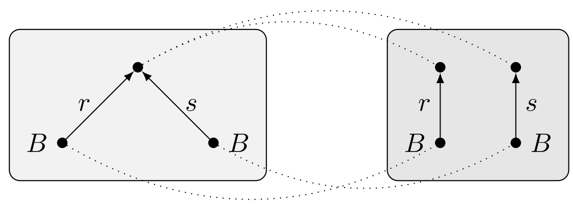
Definition 1 (equivalent $\mathcal{L}$-to-$\mathcal{L}'$ rewritability). An $\mathcal{L}'$ TBox $\mathcal{T}'$ is called an *equivalent $\mathcal{L}'$-rewriting* of an $\mathcal{L}$ TBox $\mathcal{T}$ if $\mathcal{T} \models \mathcal{T}'$ and $\mathcal{T}' \models \mathcal{T}$ (in other words, if $\mathcal{T}$ and $\mathcal{T}'$ have the same models). An $\mathcal{L}$ TBox is called *equivalently $\mathcal{L}'$-rewritable* if it has an equivalent $\mathcal{L}'$-rewriting.

Equivalent $\mathcal{L}$-to-$\mathcal{L}'$ rewritability has been studied in [17], where semantic characterizations are given and complexity results for deciding equivalent rewritability are obtained for various DLs $\mathcal{L}$ and $\mathcal{L}'$. For example, if $\mathcal{L}$ is $\mathcal{ALCI}$ or $\mathcal{ALCQ}$ and $\mathcal{L}'$ is $\mathcal{ALC}$, then an $\mathcal{L}$ TBox $\mathcal{T}$ is equivalently $\mathcal{L}'$-rewritable just in case its class of models is preserved under global bisimulations, which are defined as follows. Given interpretations $\mathcal{I}_i = (\Delta^{\mathcal{I}_i}, \cdot^{\mathcal{I}_i})$, for $i = 1, 2$, and a signature Σ, we call a relation $S \subseteq \Delta^{\mathcal{I}_1} \times \Delta^{\mathcal{I}_2}$ a *Σ-bisimulation between $\mathcal{I}_1$ and $\mathcal{I}_2$* if

- for any $A \in \Sigma$, whenever $(d_1, d_2) \in S$ then $d_1 \in A^{\mathcal{I}_1}$ iff $d_2 \in A^{\mathcal{I}_2}$;
- for any $r \in \Sigma$ and $(d_1, d_2) \in S$,
  if $(d_1, e_1) \in r^{\mathcal{I}_1}$ then there is e_2 such that $(e_1, e_2) \in S$ and $(d_2, e_2) \in r^{\mathcal{I}_2}$,
  if $(d_2, e_2) \in r^{\mathcal{I}_2}$ then there is e_1 such that $(e_1, e_2) \in S$ and $(d_1, e_1) \in r^{\mathcal{I}_1}$.

S is a *global* Σ-bisimulation between $\mathcal{I}_1$ and $\mathcal{I}_2$ if $\Delta^{\mathcal{I}_1}$ is the domain of S and $\Delta^{\mathcal{I}_2}$ its range. $\mathcal{I}_1$ and $\mathcal{I}_2$ are *globally Σ-bisimilar* if there is a global Σ-bisimulation between them, in which case we write $\mathcal{I}_1 \sim^{\Sigma}_{\mathcal{ALC}} \mathcal{I}_2$. For $d_1 \in \Delta^{\mathcal{I}_1}$ and $d_2 \in \Delta^{\mathcal{I}_2}$, we say that $(\mathcal{I}_1, d_1)$ is *Σ-bisimilar to* $(\mathcal{I}_2, d_2)$ if there is a Σ-bisimulation S between $\mathcal{I}_1$ and $\mathcal{I}_2$ such that $(d_1, d_2) \in S$. If $\Sigma = \mathsf{N_C} \cup \mathsf{N_R}$, we omit Σ, write $\mathcal{I}_1 \sim_{\mathcal{ALC}} \mathcal{I}_2$ and say simply '(global) bisimulation.'

Example 1. The $\mathcal{ALCI}$ TBox $\{\exists r^-.B \sqsubseteq A\}$ can be equivalently rewritten to the $\mathcal{ALC}$ TBox $\{B \sqsubseteq \forall r.A\}$. However, the $\mathcal{ALCI}$ TBox $\mathcal{T} = \{\exists r^-.B \sqcap \exists s^-.B \sqsubseteq A\}$ is not equivalently $\mathcal{ALC}$-rewritable. Indeed, the interpretation on the right-hand side in the picture below is a model of $\mathcal{T}$ and globally bisimilar to the interpretation on the left-hand side, which is not a model of $\mathcal{T}$.



We now introduce two subtler notions of TBox rewritability, which allow the use of fresh concept and role names in rewritings. For an interpretation $\mathcal{I}$ and signature Σ, the *Σ-reduct of* $\mathcal{I}$ is the interpretation $\mathcal{I}_{|\Sigma}$ coinciding with $\mathcal{I}$ on the names in Σ and having $X^{\mathcal{I}_{|\Sigma}} = \emptyset$ for all $X \notin \Sigma$. We say that interpretations $\mathcal{I}$ and $\mathcal{J}$ *coincide on* Σ and write $\mathcal{I} =_\Sigma \mathcal{J}$ if the Σ-reducts of $\mathcal{I}$ and $\mathcal{J}$ coincide. A TBox $\mathcal{T}'$ is a *model-conservative extension* of $\mathcal{T}$ if an interpretation $\mathcal{I}$ is a model of $\mathcal{T}$ just in case there is a model $\mathcal{I}'$ of $\mathcal{T}'$ such that $\mathcal{I} =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}'$.

Definition 2 (model-conservative $\mathcal{L}$-to-$\mathcal{L}'$-rewritability). An $\mathcal{L}'$ TBox $\mathcal{T}'$ is called a *model-conservative $\mathcal{L}'$-rewriting* of an $\mathcal{L}$ TBox $\mathcal{T}$ if $\mathcal{T}'$ is a model-conservative extension of $\mathcal{T}$. An $\mathcal{L}$ TBox $\mathcal{T}$ is *model-conservatively $\mathcal{L}'$-rewritable* if a model-conservative $\mathcal{L}'$-rewriting of $\mathcal{T}$ exists.

Clearly, any equivalent $\mathcal{L}'$-rewriting of a TBox $\mathcal{T}$ is also a model-conservative $\mathcal{L}'$-rewriting of $\mathcal{T}$. The next example shows that the converse does not hold.

Example 2. The $\mathcal{ALCI}$ TBox $\mathcal{T} = \{\exists r^-.B \sqcap \exists s^-.B \sqsubseteq A\}$ from Example 1 is model-conservatively $\mathcal{ALC}$-rewritable to

$$\mathcal{T}' = \{B \sqsubseteq \forall r.B_{\exists r^-.B},\ B \sqsubseteq \forall s.B_{\exists s^-.B},\ B_{\exists r^-.B} \sqcap B_{\exists s^-.B} \sqsubseteq A\},$$

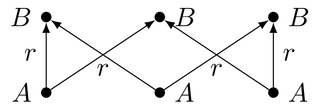
where $B_{\exists r^-.B}$, $B_{\exists s^-.B}$ are *fresh* concept names.

A TBox $\mathcal{T}'$ is called an *$\mathcal{L}$-conservative extension* of $\mathcal{T}$ if $\mathcal{T}' \models \mathcal{T}$ and $\mathcal{T}' \models C \sqsubseteq D$ implies $\mathcal{T} \models C \sqsubseteq D$, for every $\mathcal{L}$-concept inclusion $C \sqsubseteq D$ formulated in $\mathsf{sig}(\mathcal{T})$.

Definition 3 ($\mathcal{L}$-conservative $\mathcal{L}'$-rewritability). An $\mathcal{L}'$ TBox $\mathcal{T}'$ is called an *$\mathcal{L}$-conservative $\mathcal{L}'$-rewriting* of an $\mathcal{L}$ TBox $\mathcal{T}$ if $\mathcal{T}'$ is an $\mathcal{L}$-conservative extension of $\mathcal{T}$. An $\mathcal{L}$ TBox $\mathcal{T}$ is *$\mathcal{L}$-conservatively $\mathcal{L}'$-rewritable* if an $\mathcal{L}$-conservative $\mathcal{L}'$-rewriting of $\mathcal{T}$ exists.

It should be clear that every model-conservative $\mathcal{L}'$-rewriting of an $\mathcal{L}$ TBox $\mathcal{T}$ is also an $\mathcal{L}$-conservative $\mathcal{L}'$-rewriting of $\mathcal{T}$. The next example shows that the converse implication does not hold.

Example 3. The $\mathcal{ALCQ}$ TBox $\mathcal{T} = \{A \sqsubseteq\ \geq 2\, r.B\}$ is $\mathcal{ALCQ}$-conservatively $\mathcal{ALC}$-rewritable to $\mathcal{T}' = \{A \sqsubseteq \exists r.C,\ A \sqsubseteq \exists r.D,\ C \sqsubseteq \neg D,\ C \sqcup D \sqsubseteq B\}$, where C and D are *fresh* concept names. However, $\mathcal{T}'$ is not a model-conservative rewriting of $\mathcal{T}$ because the model of $\mathcal{T}$ shown below is not the $\mathsf{sig}(\mathcal{T})$-reduct of any model of $\mathcal{T}'$. Note that $\mathcal{T}$ is not equivalently $\mathcal{ALC}$-rewritable.



In our examples so far, we have used fresh concept names but no fresh role names. This is no accident: it turns out that, for the DLs considered in this paper, fresh role names in conservative rewritings are not required. More precisely, we call a model-conservative- or $\mathcal{L}$-conservative $\mathcal{L}'$-rewriting $\mathcal{T}'$ of $\mathcal{T}$ a model-conservative- or, respectively, $\mathcal{L}$-conservative $\mathcal{L}'$-*concept rewriting* of $\mathcal{T}$ if $\mathsf{sig}_R(\mathcal{T}) = \mathsf{sig}_R(\mathcal{T}')$, where $\mathsf{sig}_R(\mathcal{T})$ is the set of role names in $\mathcal{T}$.

Say that a DL $\mathcal{L}$ *reflects disjoint unions* if, for any $\mathcal{L}$ TBox $\mathcal{T}$, whenever the *disjoint* union $\bigcup_{i\in I}\mathcal{I}_i$ of interpretations $\mathcal{I}_i$ is a model of $\mathcal{T}$, then each $\mathcal{I}_i$, $i \in I$, is also a model of $\mathcal{T}$. All the DLs considered in this paper reflect disjoint unions.

Theorem 1. *Let $\mathcal{L}$ be a DL reflecting disjoint unions, $\mathcal{T}$ an $\mathcal{L}$ TBox, and let $\mathcal{L}' \in \{\mathcal{ALC}, \mathcal{EL}_\bot, \text{DL-Lite}_{horn}\}$. Then $\mathcal{T}$ is model-conservatively* (*or $\mathcal{L}$-conservatively*) *$\mathcal{L}'$-rewritable iff it is model-conservatively* (*or, respectively, $\mathcal{L}$-conservatively*) *$\mathcal{L}'$-concept rewritable.*

2 $\mathcal{ALCI}$-to-$\mathcal{ALC}$ Rewritability

Equivalent $\mathcal{ALCI}$-to-$\mathcal{ALC}$ rewritability was studied in [17], where the characterization in terms of global bisimulations was used to design a 2ExpTime decision procedure. Here, we give a characterization of model-conservative $\mathcal{ALC}$ rewritability of $\mathcal{ALCI}$ TBoxes in terms of generated subinterpretations and use it to show that (i) model-conservative $\mathcal{ALCI}$-to-$\mathcal{ALC}$ rewritings are of polynomial size and can be constructed in polynomial time (if they exist), and that (ii) deciding model-conservative $\mathcal{ALCI}$-to-$\mathcal{ALC}$ rewritability is ExpTime-complete. We also observe that $\mathcal{ALCI}$-conservative $\mathcal{ALC}$-rewritability coincides with model-conservative rewritability.

We remind the reader that an interpretation $\mathcal{I}$ is a *subinterpretation* of an interpretation $\mathcal{J}$ if $\Delta^\mathcal{I} \subseteq \Delta^\mathcal{J}$, $A^\mathcal{I} = A^\mathcal{J} \cap \Delta^\mathcal{I}$ for all concept names A, and $r^\mathcal{I} = r^\mathcal{J} \cap (\Delta^\mathcal{I} \times \Delta^\mathcal{I})$ for all role names r. $\mathcal{I}$ is a *generated subinterpretation* of $\mathcal{J}$ if, in addition, whenever $d \in \Delta^\mathcal{I}$ and $(d, d') \in r^\mathcal{J}$, r a role name, then $d' \in \Delta^\mathcal{I}$. We say that a TBox $\mathcal{T}$ is *preserved under generated subinterpretations* if every generated subinterpretation of a model of $\mathcal{T}$ is also a model of $\mathcal{T}$. As well known, every $\mathcal{ALC}$ TBox is preserved under generated subinterpretations.

Suppose we want to find a model-conservative $\mathcal{ALC}$-rewriting of an $\mathcal{ALCI}$ TBox $\mathcal{T}$. Without loss of generality, we assume that $\mathcal{T} = \{\top \sqsubseteq C_\mathcal{T}\}$ and $C_\mathcal{T}$ is built using $\neg$, $\sqcap$ and $\exists$ only. By $\mathsf{sub}(\mathcal{T})$ we denote the closure under single negation of the set of (subconcepts) of concepts in $\mathcal{T}$. For every role name r in $\mathcal{T}$, we take a fresh role name $\bar{r}$ and, for every $\exists r.C$ in $\mathsf{sub}(\mathcal{T})$ (where r is a role name or its inverse), we take a fresh concept name $B_{\exists r.C}$. Denote by $D^\sharp$ the $\mathcal{ALC}$-concept obtained from any $D \in \mathsf{sub}(\mathcal{T})$ by replacing every top-most occurrence of a subconcept of the form $\exists r.C$ in it with $B_{\exists r.C}$. Now, let $\mathcal{T}^\dagger$ be an $\mathcal{ALC}$ TBox comprised of the following concept inclusions, for $r \in \mathsf{N_R}$: $\top \sqsubseteq C^\sharp_\mathcal{T}$,

$$C^\sharp \sqsubseteq \forall \bar{r}.B_{\exists r.C}, \quad B_{\exists r.C} \equiv \exists r.C^\sharp, \qquad \text{for every } \exists r.C \in \mathsf{sub}(\mathcal{T}),$$

$$C^\sharp \sqsubseteq \forall r.B_{\exists r^-.C}, \quad B_{\exists r^-.C} \equiv \exists \bar{r}.C^\sharp, \qquad \text{for every } \exists r^-.C \in \mathsf{sub}(\mathcal{T}).$$

Clearly, $\mathcal{T}^\dagger$ can be constructed in polynomial time in the size of $\mathcal{T}$.

Theorem 2. *An $\mathcal{ALCI}$ TBox $\mathcal{T}$ is model-conservatively $\mathcal{ALC}$-rewritable iff $\mathcal{T}$ is preserved generated subinterpretations. Moreover, if $\mathcal{T}$ is model-conservatively $\mathcal{ALC}$-rewritable, then $\mathcal{T}^\dagger$ is a model-conservative $\mathcal{ALC}$-rewriting.*

It is now easy to show that model-conservative $\mathcal{ALCI}$-to-$\mathcal{ALC}$ rewritability is decidable in ExpTime. By Theorem 2, this amounts to deciding whether $\mathcal{T}^\dagger$ is a model-conservative extension of $\mathcal{T}$. In general, this is an undecidable problem. It is, however, easy to see that, for every model $\mathcal{I}$ of $\mathcal{T}$, there is a model $\mathcal{I}'$ of $\mathcal{T}^\dagger$ such that $\mathcal{I} =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}'$. It thus remains to decide whether every interpretation $\mathcal{I}$ with $\mathcal{I} =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}'$, for some model $\mathcal{I}'$ of $\mathcal{T}^\dagger$, is a model of $\mathcal{T}$. In other words, this simply means to decide whether $\mathcal{T}^\dagger \models \mathcal{T}$, which can be done in ExpTime. A matching lower bound is easily obtained by reducing satisfiability in $\mathcal{ALC}$.

Corollary 1. *The problem of deciding model-conservative $\mathcal{ALCI}$-to-$\mathcal{ALC}$ rewritability is* ExpTime*-complete.*

$\mathcal{ALCI}$-conservative $\mathcal{ALC}$-rewritability of $\mathcal{ALCI}$ TBoxes coincides with model-conservative $\mathcal{ALC}$-rewritability. This can be proved using the characterization via subinterpretations and robustness under replacement of $\mathcal{ALCI}$ TBoxes, an important property in the context of modular ontology design [15, Theorem 4].

Theorem 3. *An $\mathcal{ALCI}$-TBox $\mathcal{T}$ is $\mathcal{ALCI}$-conservatively $\mathcal{ALC}$-rewritable iff $\mathcal{T}$ is model-conservatively $\mathcal{ALC}$-rewritable.*

3 $\mathcal{ALCQ}$-to-$\mathcal{ALC}$ Rewritability

Equivalent $\mathcal{ALCQ}$-to-$\mathcal{ALC}$ rewritability was characterized in [17] in terms of preservation under global bisimulations. Below, we use this characterization to give a 2ExpTime algorithm for checking equivalent $\mathcal{ALC}$-rewritability.

We first prove a characterization of $\mathcal{ALCQ}$-conservative $\mathcal{ALC}$-rewritability in terms of preservation under inverse bounded morphisms and use it to show that one can (*i*) decide $\mathcal{ALCQ}$-conservative $\mathcal{ALC}$-rewritability in 2ExpTime and (*ii*) construct effectively an $\mathcal{ALCQ}$-conservative rewriting if it exists. We also show that, unlike $\mathcal{ALCI}$-to-$\mathcal{ALC}$-rewritings, model-conservative $\mathcal{ALC}$-rewritability of $\mathcal{ALCQ}$ TBoxes coincides with equivalent rewritability.

A *bounded Σ-morphism* from an interpretation $\mathcal{I}_1$ to an interpretation $\mathcal{I}_2$ is a global Σ-bisimulation S between $\mathcal{I}_1$ and $\mathcal{I}_2$ such that S is a function from $\Delta^{\mathcal{I}_1}$ to $\Delta^{\mathcal{I}_2}$. A class $\mathcal{K}$ of interpretations is *preserved under inverse bounded Σ-morphisms* if whenever there is a bounded Σ-morphism from an interpretation $\mathcal{I}_1$ to some $\mathcal{I}_2 \in \mathcal{K}$, then $\mathcal{I}_1 \in \mathcal{K}$. The following lemma provides the fundamental property of bounded morphisms:

Lemma 1. *Suppose $f\colon \mathcal{I}_1 \to \mathcal{I}_2$ is a bounded Σ-morphism, where $\mathcal{I}_2$ is a model of an $\mathcal{ALC}$ TBox $\mathcal{T}$ and $\mathsf{sig}_R(\mathcal{T}) \subseteq \Sigma$. Then there is $\mathcal{J}_1 \models \mathcal{T}$ such that $\mathcal{J}_1 =_\Sigma \mathcal{I}_1$.*

Proof. We define $\mathcal{J}_1$ in the same way as $\mathcal{I}_1$ except that $B^{\mathcal{J}_1} := f^{-1}(B^{\mathcal{I}_2})$ for all concept names $B \in \mathsf{sig}(\mathcal{T}_1) \setminus \Sigma$. Then f is a bounded $\mathsf{sig}(\mathcal{T})$-morphism from $\mathcal{J}_1$ to $\mathcal{I}_2$. Thus $\mathcal{J}_1$ is a model of $\mathcal{T}$ since $\mathcal{I}_1$ is a model of $\mathcal{T}$. ❑

An interpretation $\mathcal{I}$ is a *directed tree interpretation* if $r^\mathcal{I} \cap s^\mathcal{I} = \emptyset$, for $r \neq s$, and the directed graph with nodes $\Delta^\mathcal{I}$ and edges E defined by setting $(d, d') \in E$ iff $(d, d') \in \bigcup_{r \in \mathsf{N_R}} r^\mathcal{I}$ is a directed tree. We start our investigation with the observation that $\mathcal{ALCQ}$-conservative $\mathcal{ALCQ}$-to-$\mathcal{ALC}$ rewritability can be regarded as a principled approximation of model-conservative rewritability:

Lemma 2. *An $\mathcal{ALC}$ TBox $\mathcal{T}'$ is an $\mathcal{ALCQ}$-conservative rewriting of an $\mathcal{ALCQ}$ TBox $\mathcal{T}$ iff $\mathcal{T}'$ is a model-conservative rewriting of $\mathcal{T}$ over the class of directed tree interpretations of finite outdegree.*

Suppose we want to find an $\mathcal{ALCQ}$-conservative $\mathcal{ALC}$-rewriting of an $\mathcal{ALCQ}$-TBox $\mathcal{T}$. Without loss of generality, we assume that $\mathcal{T}$ is of the form $\{\top \sqsubseteq C_\mathcal{T}\}$ and that $C_\mathcal{T}$ is built using $\neg$, $\sqcap$, $(\geqslant n\ r\ C)$ only. Construct a TBox $\mathcal{T}^\dagger$ as follows. Take fresh concept names $B_D, B_1^D, \dots, B_n^D$ for every $D = (\geqslant n\ r\ C) \in \mathsf{sub}(\mathcal{T})$. We use Σ to denote $\mathsf{sig}(\mathcal{T})$ extended with all fresh concept names of the form B_i^D. For each $C \in \mathsf{sub}(\mathcal{T})$, $C^\sharp$ denotes the $\mathcal{ALC}$-concept that results from C by replacing all top-most occurrences of any $D = (\geqslant n\ r\ C)$ in $\mathcal{T}$ with B_D. Now, define $\mathcal{T}^\dagger$ to be the infinite TBox that consists of the following inclusions:

- $\top \sqsubseteq C^\sharp_\mathcal{T}$,
- $B_D \sqsubseteq \exists r.(C^\sharp \sqcap B_1^D) \sqcap \dots \sqcap \exists r.(C^\sharp \sqcap B_n^D)$,
- $B_i^D \sqsubseteq \neg B_j^D$, for $i \neq j$, and
- for all $\mathcal{ALC}$-concepts $C_1, \dots, C_n$ in Σ and all $D = (\geqslant n\ r\ C) \in \mathsf{sub}(\mathcal{T})$,

$$\bigsqcap_{1 \leq i \leq n} (\exists r.(C^\sharp \sqcap C_i^\sharp \sqcap \bigsqcap_{j \neq i} \neg C_j^\sharp)) \sqsubseteq B_D.$$

The next theorem characterizes $\mathcal{ALCQ}$-conservative $\mathcal{ALC}$-rewritability.

Theorem 4. *An $\mathcal{ALCQ}$ TBox $\mathcal{T}$ is $\mathcal{ALCQ}$-conservatively $\mathcal{ALC}$-rewritable iff $\mathcal{T}$ is preserved under inverse bounded $\mathsf{sig}(\mathcal{T})$-morphisms. Moreover, if $\mathcal{T}$ is $\mathcal{ALCQ}$-conservatively $\mathcal{ALC}$-rewritable, then $\mathcal{T}^\dagger$ is an (infinite) rewriting.*

The semantic characterization of Theorem 4 can be employed to prove the following complexity result using a type elimination argument. We assume that numbers in number restrictions are coded in unary.

Theorem 5. *For $\mathcal{ALCQ}$ TBoxes, $\mathcal{ALCQ}$-conservative $\mathcal{ALC}$-rewritability is decidable in 2*EXPTIME.

It follows that, given an $\mathcal{ALCQ}$ TBox $\mathcal{T}$, one can first decide $\mathcal{ALCQ}$-conservative $\mathcal{ALC}$-rewritability and then, in case of a positive answer, effectively construct a rewriting by going through the finite subsets of $\mathcal{T}^\dagger$ in a systematic way until a finite $\mathcal{T}' \subseteq \mathcal{T}^\dagger$ with $\mathcal{T}' \models \mathcal{T}$ is reached. By compactness, such a set $\mathcal{T}'$ exists.

We finally show that every model-conservatively $\mathcal{ALC}$-rewritable $\mathcal{ALCQ}$ TBox is equivalently $\mathcal{ALC}$-rewritable.

Theorem 6. *An $\mathcal{ALCQ}$ TBox is model-conservatively $\mathcal{ALC}$-rewritable iff it is equivalently $\mathcal{ALC}$-rewritable, which is decidable in 2*ExpTime.

4 $\mathcal{ALCI}$-to-*DL-Lite*$_{horn}$ and $\mathcal{ALC}$-to-$\mathcal{EL}_\bot$ Rewritability

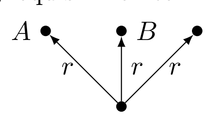
We first observe that all notions of rewritability introduced in this paper coincide in the case of $\mathcal{ALCI}$-to-*DL-Lite*$_{horn}$ rewritability. Deciding rewritability is ExpTime-complete in all cases since deciding equivalent $\mathcal{ALCI}$-to-*DL-Lite*$_{horn}$ rewritability is ExpTime-complete [17]:

Theorem 7. *For $\mathcal{ALCI}$ TBoxes, equivalent DL-Lite$_{horn}$-rewritability, model-conservative DL-Lite$_{horn}$-rewritability, as well as $\mathcal{ALCI}$-conservative DL-Lite$_{horn}$-rewritability coincide and are* ExpTime*-complete.*

We now provide separating examples for $\mathcal{ALC}$-to-$\mathcal{EL}_\bot$ rewritability and then prove decidability of model-conservative $\mathcal{EL}_\bot$-rewritability. In contrast to the previous sections, we have not been able to find a purely model-theoretic characterization of conservative $\mathcal{EL}_\bot$-rewritability. Finally, we give necessary model-theoretic conditions for conservative $\mathcal{EL}_\bot$-rewritability.

Equivalent $\mathcal{ALC}$-to-$\mathcal{EL}_\bot$ rewritability has been characterized in [17] in terms of preservation under products and global equisimulations. A *simulation* between interpretations $\mathcal{I}$ and $\mathcal{J}$ is a relation $S \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{J}$ such that, for any $A \in \mathsf{N_C}$, $r \in \mathsf{N_R}$ and $(d_1, d_2) \in S$, if $d_1 \in A^{\mathcal{I}_1}$ then $d_2 \in A^{\mathcal{I}_2}$, and if $(d_1, e_1) \in r^\mathcal{I}$ then there exists e_2 with $(e_1, e_2) \in S$ and $(d_2, e_2) \in r^\mathcal{J}$. $(\mathcal{I}, d)$ is *simulated* by $(\mathcal{J}, e)$ if there is a simulation S between $\mathcal{I}$ and $\mathcal{J}$ such that $(d, e) \in S$. Interpretations $\mathcal{I}$ and $\mathcal{J}$ are *globally equisimilar* if, for any $d \in \Delta^\mathcal{I}$, there exists $e \in \Delta^\mathcal{J}$ such that $(\mathcal{I}, d)$ is simulated by $(\mathcal{J}, e)$ and $(\mathcal{J}, e)$ is simulated by $(\mathcal{I}, d)$. According to [17, Theorem 17], an $\mathcal{ALC}$ TBox is equivalently $\mathcal{EL}_\bot$-rewritable if its models are preserved under products and global equisimulations.

Example 4. The TBox $\mathcal{T} = \{\exists r.A \sqcap \exists r.B \sqcap \forall r.(A \sqcup B) \sqsubseteq E \sqcup F,\ A \sqcap B \sqsubseteq \bot\}$ is not equivalently $\mathcal{EL}_\bot$-rewritable because its models are not preserved under global equisimulations. Indeed, the interpretation $\mathcal{I}$ shown below is clearly a model of $\mathcal{T}$. However, by removing the rightmost r-arrow from $\mathcal{I}$, we obtain an interpretation which is globally equisimilar to $\mathcal{I}$ but not a model of $\mathcal{T}$.

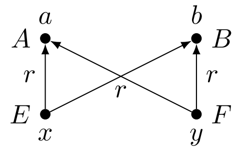


On the other hand, the $\mathcal{EL}_\bot$ TBox

$$\{\exists r.A \sqcap \exists r.B \sqsubseteq \exists r.G,\ \exists r.(G \sqcap A) \sqsubseteq E,\ \exists r.(G \sqcap B) \sqsubseteq F,\ A \sqcap B \sqsubseteq \bot\}$$

is easily seen to be an $\mathcal{ALC}$-conservative $\mathcal{EL}_\bot$ rewriting of $\mathcal{T}$. We now show that $\mathcal{T}$ is not model-conservatively $\mathcal{EL}_\bot$-rewritable. For suppose $\mathcal{T}$ has such a rewriting $\mathcal{T}'$ given in standard normal form (with inclusions of the form $A_1 \sqcap \ldots \sqcap A_n \sqsubseteq B$, $\exists r.B \sqsubseteq A$, or $A \sqsubseteq \exists r.B$ where $A_1, \ldots, A_n, A, B \in \mathsf{N_C} \cup \{\bot\}$). Consider the model

$\mathcal{I}$ of $\mathcal{T}$ depicted below, and let $\mathcal{I}'$ be a model of $\mathcal{T}'$ such that $\mathcal{I} =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}'$.



Let $\mathcal{J}$ be the same as $\mathcal{I}'$ except that $x, y \in M^{\mathcal{J}}$ iff both $x \in M^{\mathcal{I}'}$ and $y \in M^{\mathcal{I}'}$, for every $M \in \mathsf{N_C}$. Since $x \notin E^{\mathcal{J}}$ and $y \notin F^{\mathcal{J}}$, $\mathcal{J}$ is not a model of $\mathcal{T}'$. Since the restriction of $\mathcal{I}'$ to $\{a, b\}$ is a model of $\mathcal{T}'$, and the restrictions of $\mathcal{I}'$ to $\{a, b, x\}$ and $\{a, b, y\}$ coincide, there is $(C \sqsubseteq D) \in \mathcal{T}'$ such that $x, y \in C^{\mathcal{J}}$ but $x, y \notin D^{\mathcal{J}}$. As $\mathcal{I}'$ is a model of $\mathcal{T}'$, which is in standard normal form, and by the definition of $\mathcal{J}$, D must be a concept name. Since clearly $x, y \in C^{\mathcal{I}'}$, we must also have $x, y \in D^{\mathcal{I}'}$, and so $x, y \in D^{\mathcal{J}}$, which is a contradiction.

The following modified version of $\mathcal{T}$

$$\mathcal{T}_m \;=\; \{\exists r.A \sqcap \exists r.B \sqcap \forall r.(A \sqcup B) \sqsubseteq \exists r.(A \sqcap E) \sqcup \exists r.(B \sqcap F),\; A \sqcap B \sqsubseteq \bot\}$$

is not equivalently $\mathcal{EL}_\bot$-rewritable, but has a model-conservative $\mathcal{EL}_\bot$-rewriting:

$$\mathcal{T}'_m \;=\; \{\exists r.A \sqcap \exists r.B \sqsubseteq \exists r.M,\; \exists r.(M \sqcap A) \sqsubseteq \exists r.(M \sqcap E), \\ \exists r.(M \sqcap B) \sqsubseteq \exists r.(M \sqcap F),\; A \sqcap B \sqsubseteq \bot\}.$$

The difference to the previous example is that if d is an instance of $\exists r.A \sqcap \exists r.B$, then we can place the 'marker' M onto an r-successor of d which is either in $A \sqcap E$ or in $B \sqcap F$, whereas in the previous example the decision on where to put the 'marker' G was not determined by the r-successors of d but by d itself.

We now prove that if there exists an $\mathcal{EL}_\bot$-rewriting of an $\mathcal{ALC}$ TBox $\mathcal{T}$, then there is one without any 'recursion' for the newly introduced symbols. Let $\Sigma = \mathsf{sig}(\mathcal{T})$. We say that an $\mathcal{EL}_\bot$ TBox $\mathcal{T}'$ is in *Σ-layered form of depth n* if there are mutually disjoint sets $\Gamma_0, \dots, \Gamma_n$ of concept names such that $\Gamma_i \cap \Sigma = \emptyset$ $(0 \leq i \leq n)$ and the inclusions of $\mathcal{T}'$ take the following form, where $r \in \Sigma$:

level i atom inclusions: $A_1 \sqcap \cdots \sqcap A_n \sqsubseteq B$, for $A_1, \dots, A_n, B \in \Sigma \cup \Gamma_i \cup \{\bot\}$,
level i right-atom inclusions: $\exists r.A \sqsubseteq B$ for $A \in \Sigma \cup \Gamma_{i+1}$, $B \in \Sigma \cup \Gamma_i \cup \{\bot\}$,
level i left-atom inclusions: $A \sqsubseteq \exists r.B$, for $A \in \Sigma \cup \Gamma_i$, $B \in \Sigma \cup \Gamma_{i+1} \cup \{\bot\}$.

The *depth* of a concept C is the maximal number of nestings of existential restrictions in C. The *depth* of a TBox is the maximal depth of its concepts.

Lemma 3. *If an $\mathcal{ALC}$ TBox $\mathcal{T}$ of depth n is model- (or $\mathcal{ALC}$-) conservatively $\mathcal{EL}_\bot$-rewritable, then there exists a model- (respectively, $\mathcal{ALC}$-) conservative $\mathcal{EL}_\bot$-rewriting $\mathcal{T}'$ of $\mathcal{T}$ in $\mathsf{sig}(\mathcal{T})$-layered form of depth n.*

We use Lemma 7 to prove decidability of model-conservative $\mathcal{EL}_\bot$-rewritability. An $\mathcal{ALC}$ ABox $\mathcal{A}$ is a finite set of assertions of the form $C(a)$ and $r(a, b)$, where C is an $\mathcal{ALC}$ concept and a, b are *individual names*. The set of individual names that occur in an ABox $\mathcal{A}$ is denoted by $\mathsf{ind}(\mathcal{A})$. When interpreting ABoxes, we adopt the *standard name assumption*: $a^{\mathcal{I}} = a$, for all $a \in \mathsf{ind}(\mathcal{A})$.

Let $\mathcal{T}$ be an $\mathcal{ALC}$ TBox of depth $n > 0$ (the case $n = 0$ is trivial). By $\mathsf{sub}^{n-1}(\mathcal{T})$ we denote the closure under single negation of the set of subconcepts of concepts in $\mathcal{T}$ of depth at most $n-1$. By $\Theta^{n-1}(\mathcal{T})$ we denote the set of maximal subsets $\boldsymbol{t}$ of $\mathsf{sub}^{n-1}(\mathcal{T})$ that are satisfiable in a model of $\mathcal{T}$. A $\mathcal{T}$-*ABox* is an ABox such that $\boldsymbol{t}_{\mathcal{A}}(a) = \{D \mid D(a) \in \mathcal{A}\} \in \Theta^{n-1}(\mathcal{T})$ for all $a \in \mathsf{ind}(\mathcal{A})$. Let $\mathcal{A}$ be a directed tree ABox of depth at most n (that is, all nodes in it are at distance $\leq n$ from the root). We say that $\mathcal{A}$ is *n-strongly satisfiable* w.r.t. $\mathcal{T}$ if there is a model $\mathcal{I}$ of $\mathcal{A}$ and $\mathcal{T}$ such that the $r^{\mathcal{I}}$-successors of $a^{\mathcal{I}}$, for every $a \in \mathsf{ind}(\mathcal{A})$ of depth $< n$ in $\mathcal{A}$, coincide with the r-successors of a in $\mathcal{A}$.

We now define inductively *$(\mathcal{T}, i)$-bisimilarity relations* $\sim_{i,\mathcal{T}}$ between pairs $(\mathcal{A}_1, a_1)$ and $(\mathcal{A}_2, a_2)$, where the $\mathcal{A}_i$ are $\mathcal{T}$-ABoxes and $a_i \in \mathsf{ind}(\mathcal{A}_i)$:

- $(\mathcal{A}_1, a_1) \sim_{0,\mathcal{T}} (\mathcal{A}_2, a_2)$ if $\boldsymbol{t}_{\mathcal{A}_1}(a_1) = \boldsymbol{t}_{\mathcal{A}_2}(a_2)$;
- $(\mathcal{A}_1, a_1) \sim_{i+1,\mathcal{T}} (\mathcal{A}_2, a_2)$ if $(\mathcal{A}_1, a_1) \sim_{0,\mathcal{T}} (\mathcal{A}_2, a_2)$ and, for every $r \in \mathsf{sig}(\mathcal{T})$, if $r(d_1, e_1) \in \mathcal{A}_1$ then there is $r(d_2, e_2) \in \mathcal{A}_2$ such that $(\mathcal{A}_1, e_1) \sim_{i,\mathcal{T}} (\mathcal{A}_2, e_2)$, and vice versa.

For every $i \geq 0$, one can determine a finite set AT_i of finite directed tree $\mathcal{T}$-ABoxes $\mathcal{A}$ with root $\rho_{\mathcal{A}}$ and of depth $\leq i$ such that:

- for every $\mathcal{I} \models \mathcal{T}$ and every $d \in \Delta^{\mathcal{I}}$, $(\mathcal{I}, d)$ is $(\mathcal{T}, i)$-bisimilar to exactly one $(\mathcal{A}, \rho_{\mathcal{A}}) \in \mathsf{AT}_i$;[4]
- every $\mathcal{A} \in \mathsf{AT}_i$ is strongly i-satisfiable w.r.t. $\mathcal{T}$.

We assume that all ABoxes in $\mathsf{AT}_0, \dots, \mathsf{AT}_n$ have mutually distinct roots. We define the *canonical ABox* $\mathcal{A}_{\mathcal{T}}$ with individuals $\{\rho_{\mathcal{A}} \mid \mathcal{A} \in \mathsf{AT}_i, i \leq n\}$ as follows:

- for $\mathcal{A}_i \in \mathsf{AT}_i$, $\mathcal{A}_{i+1} \in \mathsf{AT}_{i+1}$ and $r \in \mathsf{sig}(\mathcal{T})$, we have $r(\rho_{\mathcal{A}_{i+1}}, \rho_{\mathcal{A}_i}) \in \mathcal{A}_{\mathcal{T}}$ if there exists $r(\rho_{\mathcal{A}_{i+1}}, b) \in \mathcal{A}_{i+1}$ such that the subtree of $\mathcal{A}_{i+1}$ rooted at b is $(i, \mathcal{T})$-bisimilar to $\mathcal{A}_i$;
- for $\mathcal{A}_i \in \mathsf{AT}_i$ and $A \in \mathsf{sig}(\mathcal{T})$, we have $A(\rho_{\mathcal{A}_i}) \in \mathcal{A}_{\mathcal{T}}$ iff $A(\rho_{\mathcal{A}_i}) \in \mathcal{A}_i$.

Note that $\mathcal{A}_{\mathcal{T}}$ is acyclic (but not a directed tree ABox).

Lemma 4. *Let $\mathcal{T}$ be an $\mathcal{ALC}$ TBox of depth n. An $\mathcal{EL}_\bot$ TBox $\mathcal{T}'$ in $\mathsf{sig}(\mathcal{T})$-layered form of depth n is a model-conservative $\mathcal{EL}_\bot$-rewriting of $\mathcal{T}$ iff*

- *$\mathcal{T}' \models \mathcal{T}$ and*
- *there exists $\mathcal{A}' =_{\mathsf{sig}(\mathcal{T})} \mathcal{A}_{\mathcal{T}}$ such that, for all $i = 0, \dots, n$, $\mathcal{A}'$ satisfies all level i inclusions in $\mathcal{T}'$ at all $\rho_{\mathcal{A}_i}$ with $\mathcal{A}_i \in \mathsf{AT}_{n-i}$.*

Theorem 8. *Model-conservative $\mathcal{EL}_\bot$-rewritability of $\mathcal{ALC}$ TBoxes is decidable.*

Proof. Given an $\mathcal{ALC}$ TBox $\mathcal{T}$, we first construct the canonical ABox $\mathcal{A}_{\mathcal{T}}$. If an $\mathcal{EL}_\bot$ TBox $\mathcal{T}'$ in Σ-layered form of depth n satisfies the conditions of Lemma 8, then there exists such a TBox with at most $2^{|\mathcal{A}_{\mathcal{T}}|}$ distinct fresh concept names. As the number of such $\mathcal{EL}_\bot$ TBoxes is finite one can check for each of them whether the conditions of Lemma 8 are satisfied. ❑

[4] Here we identify $\mathcal{I}$ with the ABox with assertions $r(a, b)$, for $(a, b) \in r^{\mathcal{I}}$, and $D(a)$, for $D \in \mathsf{sub}^{n-1}(\mathcal{T})$ and $a \in D^{\mathcal{I}}$.

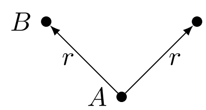
We now give necessary conditions for $\mathcal{ALC}$-conservative $\mathcal{EL}_\bot$-rewritability of $\mathcal{ALC}$ TBoxes. First, we still have the preservation under products:

Theorem 9. *Every $\mathcal{ALC}$-conservatively $\mathcal{EL}_\bot$-rewritable $\mathcal{ALC}$ TBox is preserved under products.*

Theorem 9 can be used to show that TBoxes such as $\{A \sqsubseteq B \sqcup E\}$ are not $\mathcal{ALC}$-conservatively $\mathcal{EL}_\bot$-rewritable. To separate equivalently rewritable TBoxes from $\mathcal{ALC}$-conservatively rewritable TBoxes, we generalize the construction of Example 4. In that case, we removed an r-arrow (d_0, d) from a tree-shaped model $\mathcal{I}$ of $\mathcal{T}$ and obtained a model that is globally equisimilar to the original model but not a model of $\mathcal{T}$. It turns out that $\mathcal{ALC}$-conservatively $\mathcal{EL}_\bot$-rewritable $\mathcal{ALC}$ TBoxes of depth 1 are preserved under the inverse of this operation. We say that $(\mathcal{I}, d)$ is *$\subseteq_1$-simulated by* $(\mathcal{J}, e)$ if (i) $d \in A^\mathcal{I}$ iff $e \in A^\mathcal{J}$, for all $A \in \mathsf{N_C}$; (ii) for all $r \in \mathsf{N_R}$, if $(e, e') \in r^\mathcal{J}$ then there exists d' with $(d, d') \in r^\mathcal{I}$ and, for all $A \in \mathsf{N_C}$, if $e' \in A^\mathcal{J}$ then $d' \in A^\mathcal{I}$; (iii) for all $r \in \mathsf{N_R}$, if $(d, d') \in r^\mathcal{I}$ then there exists e' with $(e, e') \in r^\mathcal{J}$ and, for all $A \in \mathsf{N_C}$, we have $d' \in A^\mathcal{I}$ iff $e' \in A^\mathcal{J}$. Say that $\mathcal{I}$ is *globally $\subseteq_1$-simulated* by $\mathcal{J}$ if, for every $e \in \Delta^\mathcal{J}$, there exists $d \in \Delta^\mathcal{I}$ such that $(\mathcal{I}, d)$ is $\subseteq_1$-simulated by $(\mathcal{J}, e)$. An $\mathcal{ALC}$ TBox is *preserved under $\subseteq_1$-simulations* if every interpretation that globally $\subseteq_1$-simulates a model of $\mathcal{T}$ is a model of $\mathcal{T}$.

Theorem 10. *Every $\mathcal{ALC}$-conservatively $\mathcal{EL}_\bot$-rewritable $\mathcal{ALC}$ TBox of depth 1 is preserved under global $\subseteq_1$-simulations.*

This result can be used to show, for example, that $\mathcal{T} = \{A \sqsubseteq \forall r.B\}$ is not $\mathcal{ALC}$-conservatively $\mathcal{EL}_\bot$-rewritable. For the interpretation below is *not* a model



of $\mathcal{T}$, but by removing from it the rightmost r-arrow, we obtain an interpretation which is globally $\subseteq_1$-simulated by $\mathcal{J}$ and is a model of $\mathcal{T}$. It remains open whether preservation under products and global $\subseteq_1$-simulations is sufficient for an $\mathcal{ALC}$ TBox of depth 1 to be $\mathcal{ALC}$-conservatively $\mathcal{EL}_\bot$-rewritable.

5 Conclusion

Conservative rewritings of ontologies provide more flexibility than equivalent rewritings and are more natural in practice. However, they are also technically much more challenging to analyse. For future work, we are particularly interested in better understanding conservative rewritings to $\mathcal{EL}$ and related logics. For example, can we find transparent model-theoretic characterizations and explicit axiomatizations of the rewritten TBoxes? The results in Section 4 should provide a good starting point. Another challenging problem could be to investigate rewritability to OWL 2 QL—that is, *DL-Lite*$_{core}$ extended with role inclusions—which preserves answers to conjunctive queries over all possible ABoxes. (Recall [6] that conjunctive query inseparability for OWL 2 QL TBoxes is ExpTime-complete.)

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A Proofs for Section 1

We prove Theorem 1. In what follows we denote by $\mathsf{sub}(\mathcal{T})$ closure under single negation of the set of subconcepts of concepts in $\mathcal{T}$. A $\mathcal{T}$-type $\mathbf{t}$ is a maximal subset of $\mathsf{sub}(\mathcal{T})$.

Proof of Theorem 1. We first consider rewritings to $\mathcal{ALC}$. Let $\mathcal{T}'$ be an $\mathcal{ALC}$-rewriting of $\mathcal{T}$ using additional roles (the proof does not depend on whether we consider model-conservative or $\mathcal{L}$-conservative rewritings). Define $\mathcal{T}''$ as follows: introduce for every $\exists R.C \in \mathsf{sub}(\mathcal{T}')$ with $R \notin \mathsf{sig}(\mathcal{T})$ a fresh concept name $A_{\exists R.C}$. Denote by D^* the result of replacing any such $\exists R.C$ in D by $A_{\exists R.C}$. Define $\mathcal{T}''$ by taking

- $\bigsqcap_{C\in\mathbf{t}} C^* \sqsubseteq \bot$ for any $\mathcal{T}'$-type $\mathbf{t}$ not satisfiable w.r.t. $\mathcal{T}'$.

We show that $\mathcal{T}''$ is a model (respectively, $\mathcal{L}$-) conservative $\mathcal{ALC}$ rewriting of $\mathcal{T}$ without additional role names. Note that for every model $\mathcal{I}$ of $\mathcal{T}'$ there clearly exists a model $\mathcal{J}$ of $\mathcal{T}''$ such that $\mathcal{J} =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}$. Thus, it is sufficient to show that $\mathcal{T}'' \models \mathcal{T}$. Assume this is not the case. Let $\mathcal{I}$ be a model of $\mathcal{T}''$ which is not a model of $\mathcal{T}$. We may assume that $R^{\mathcal{I}} = \emptyset$ for all $R \notin \mathsf{sig}_R(\mathcal{T})$. Now define $\mathcal{J}$ as follows: for every $d \in \Delta^{\mathcal{I}}$ and with $d \in A^{\mathcal{I}}_{\exists R.C}$ take a model $\mathcal{I}_{d,\exists R.C}$ of $\mathcal{T}'$ which satisfies C in $e_{d,\exists R.C}$ and such that whenever $d \notin (\exists R.D)^{\mathcal{I}}$ for some $\exists R.D \in \mathsf{sub}(\mathcal{T}')$, then $e_{d,\exists R.C} \notin D^{\mathcal{I}_{d,\exists R.C}}$. Such an interpretation $\mathcal{I}_{d,\exists R.C}$ exists since the conjunction of all D with $D \in \mathsf{sub}(\mathcal{T}')$ and $d \in (D^*)^{\mathcal{I}}$ is satisfiable w.r.t. $\mathcal{T}'$. Now define $\mathcal{J}$ by taking $\mathcal{I}$ and connecting every d to $e_{d,\exists R.C}$ using R. The resulting interpretation $\mathcal{J}$ is a model of $\mathcal{T}'$ and since $\mathcal{L}_1$ reflects disjoint unions still refutes $\mathcal{T}$. We have derived a contradiction.

We come to rewritings into $\mathcal{EL}$. Let $\mathcal{T}'$ be an $\mathcal{EL}$-rewriting of $\mathcal{T}$ (again, it does not matter whether model or $\mathcal{L}$-conservative) using fresh roles. Assume $\mathcal{T}'$ is in standard normal form (define below). Define $\mathcal{T}''$ as follows: introduce for every $\exists R.B \in \mathsf{sub}(\mathcal{T}')$ with $R \notin \mathsf{sig}(\mathcal{T})$ a fresh concept name $A_{\exists R.B}$. Denote by D^* the result of replacing any such $\exists R.B$ by $A_{\exists R.B}$. Define $\mathcal{T}''$ by taking

- $C^* \sqsubseteq D^*$ for all $C \sqsubseteq D \in \mathcal{T}'$;
- $A_{\exists R.B} \sqsubseteq A_{\exists R.E}$ whenever $\mathcal{T}' \models B \sqsubseteq E$, $\exists R.B, \exists R.E \in \mathsf{sub}(\mathcal{T}')$, and $R \notin \mathsf{sub}_R(\mathcal{T})$.

We show that $\mathcal{T}''$ is a model (respectively, $\mathcal{L}$-) conservative $\mathcal{EL}$ rewriting of $\mathcal{T}$ without additional role names.

Again, it is sufficient to show that $\mathcal{T}'' \models \mathcal{T}$. Assume this is not the case. Let $\mathcal{I}$ be a model of $\mathcal{T}''$ which is not a model of $\mathcal{T}$. We may assume that $R^{\mathcal{I}} = \emptyset$ for all $R \notin \mathsf{sig}_R(\mathcal{T})$. Now define $\mathcal{J}$ as follows: for every $d \in \Delta^{\mathcal{I}}$ and $\exists R.B \in \mathsf{sub}(\mathcal{T}')$ such that $d \in A^{\mathcal{I}}_{\exists R.B}$ take a *canonical* model $\mathcal{I}_{d,\exists R.B}$ of $\mathcal{T}'$ and B satisfying B in its root $e_{d,\exists R.B}$. Now define $\mathcal{J}$ by taking $\mathcal{I}$ and connecting d to $e_{d,\exists R.B}$ using R. We show the following:

For all $d \in \Delta^{\mathcal{I}}$ and all $C \in \mathsf{sub}(\mathcal{T}')$: $d \in (C^*)^{\mathcal{I}}$ iff $d \in C^{\mathcal{J}}$.

The interesting case are concepts of the form $\exists R.B$. Assume first that $d \in A^{\mathcal{I}}_{\exists R.B}$. Then there exists $e_{d,\exists R.B}$ with $(d, e_{d,\exists R.B}) \in R^{\mathcal{J}}$. We have $e_{d,\exists R.B} \in B^{\mathcal{J}}$ and so $d \in (\exists R.B)^{\mathcal{J}}$, as required. Conversely, assume that $d \in (\exists R.B)^{\mathcal{J}}$. Then there exists $\exists R.E \in \mathsf{sub}(\mathcal{T}')$ such that $d \in A^{\mathcal{I}}_{\exists R.E}$ and $e_{d,\exists R.E} \in B^{\mathcal{J}}$. But then $\mathcal{T}' \models E \sqsubseteq B$. Hence $\mathcal{T}'' \models A_{\exists R.E} \sqsubseteq A_{\exists R.B}$. Thus, $d \in A^{\mathcal{J}}_{\exists R.B}$, as required.

It follows that $\mathcal{J}$ is a model of $\mathcal{T}'$. Clearly, $\mathcal{J}$ refutes $\mathcal{T}$ since $\mathcal{I}$ refutes $\mathcal{T}$. We have derived a contradiction to the assumption that $\mathcal{T}' \models \mathcal{T}$. This finishes the proof for rewritings into $\mathcal{EL}$.

We come to rewritings into *DL-Lite*$_{horn}$. Let $\mathcal{T}'$ be a *DL-Lite*$_{horn}$-rewriting (it does not matter whether model or $\mathcal{L}$-conservative) of $\mathcal{T}$ using fresh role names. Define $\mathcal{T}''$ as follows: introduce for every role name R such that R or R^- occur in $\mathcal{T}'$ but not in $\mathcal{T}$ fresh concept name $A_{\exists R.\top}$ and $A_{\exists R^-.\top}$. Denote by D^* the result of replacing any $\exists S.\top$ in D by $A_{\exists S.\top}$. Define $\mathcal{T}''$ by taking

- $C^* \sqsubseteq D^*$ for all $C \sqsubseteq D \in \mathcal{T}'$;
- $A_{\exists S.\top} \sqsubseteq \bot$ whenever $\exists S.\top$ occurs in $\mathcal{T}'$, S does not occur in $\mathcal{T}$, and $\exists S.\top$ is not satisfiable in a model of $\mathcal{T}'$.

We show that $\mathcal{T}''$ is a model-conservative (respectively, $\mathcal{L}$-conservative) *DL-Lite*$_{horn}$-rewriting of $\mathcal{T}$ without additional role names.

Again, it is sufficient to show that $\mathcal{T}'' \models \mathcal{T}$. Assume this is not the case. Let $\mathcal{I}$ be a model of $\mathcal{T}''$ which is not a model of $\mathcal{T}$. We may assume that $R^{\mathcal{I}} = \emptyset$ for all $R \notin \mathsf{sig}_R(\mathcal{T})$. Now define $\mathcal{J}$ as follows: for every $d \in \Delta^{\mathcal{I}}$ and $\exists R.\top \in \mathsf{sub}(\mathcal{T}')$ such that $d \in A^{\mathcal{I}}_{\exists R.\top}$ take a *canonical* model $\mathcal{I}_{d,\exists R.\top}$ of $\mathcal{T}'$ and $\exists R^-.\top$ satisfying $\exists R^-.\top$ in its root $e_{d,\exists R.\top}$. Now define $\mathcal{J}$ by taking $\mathcal{I}$ and connecting d to $e_{d,\exists R.\top}$ using R. One can show the following:

For all $d \in \Delta^{\mathcal{I}}$ and all $C \in \mathsf{sub}(\mathcal{T}')$: $d \in (C^*)^{\mathcal{I}}$ iff $d \in C^{\mathcal{J}}$.

It follows that $\mathcal{J}$ is a model of $\mathcal{T}'$. Clearly, $\mathcal{J}$ refutes $\mathcal{T}$ since $\mathcal{I}$ refutes $\mathcal{T}$. We have derived a contradiction to the assumption that $\mathcal{T}' \models \mathcal{T}$. This finishes the proof of Theorem 1.

B Proofs for Section 2

Theorem 2 An $\mathcal{ALCI}$ TBox $\mathcal{T}$ is model-conservatively $\mathcal{ALC}$-rewritable iff $\mathcal{T}$ is preserved generated subinterpretations. Moreover, if $\mathcal{T}$ is model-conservatively $\mathcal{ALC}$-rewritable, then $\mathcal{T}^\dagger$ is a model-conservative $\mathcal{ALC}$-rewriting.

Proof. We show the following:

1. If an $\mathcal{ALCI}$ TBox $\mathcal{T}$ is model-conservatively $\mathcal{ALC}$-rewritable, then $\mathcal{T}$ is preserved under generated subinterpretations;
2. If an $\mathcal{ALCI}$ TBox $\mathcal{T}$ is preserved under generated subinterpretations, then $\mathcal{T}^\dagger$ is a model-conservative rewriting of $\mathcal{T}$.

1. Assume that $\mathcal{T}'$ is a model-conservative $\mathcal{ALC}$-rewriting of $\mathcal{T}$. By Theorem 1, we can assume that all role names in $\mathcal{T}'$ also occur in $\mathcal{T}$. Assume for a proof by contradiction that $\mathcal{T}$ is not preserved under generated subinterpretations. Then there is a model $\mathcal{J}$ of $\mathcal{T}$ and a generated subinterpretation $\mathcal{I}$ of $\mathcal{J}$ that is not a model of $\mathcal{T}$. We also have a model $\mathcal{J}'$ of $\mathcal{T}'$ such that $\mathcal{J} =_{\mathsf{sig}(\mathcal{T})} \mathcal{J}'$. Let $\mathcal{I}'$ be the restriction of $\mathcal{J}'$ to $\Delta^{\mathcal{I}}$. Since all role names in $\mathcal{T}'$ also occur in $\mathcal{T}$, we may assume that $r^{\mathcal{J}'} = \emptyset$ for all roles r that are not in $\mathcal{T}$. Consequently, $\mathcal{I}'$ is a generated subinterpretation of $\mathcal{J}'$ and, therefore, a model of $\mathcal{T}'$. We have $\mathcal{I}' =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}$, and so $\mathcal{I}$ is a model of $\mathcal{T}$, which is a contradiction.

2. Suppose $\mathcal{T}$ is preserved under generated subinterpretations. We show that $\mathcal{T}^\dagger$ is a model-conservative rewriting of $\mathcal{T}$ (Claim 2 below). We first show an auxiliary claim. An interpretation $\mathcal{I}$ is called *proper* if $\bar{r}^{\mathcal{I}} = \{(d,e) \mid (e,d) \in r^{\mathcal{I}}\}$, for all fresh role names $\bar{r}$, and $B^{\mathcal{I}}_{\exists r.C} = (\exists r.C)^{\mathcal{I}}$, for all fresh concept names $B_{\exists r.C}$.

Claim 1. A proper interpretation is a model of $\mathcal{T}$ iff it is a model of $\mathcal{T}^\dagger$.

Proof sketch. Let $\mathcal{I}$ be proper. It is not hard to show that $C^{\mathcal{I}} = (C^\sharp)^{\mathcal{I}}$ for all $C \in \mathsf{sub}(\mathcal{T})$. This makes both the 'if' and the 'only if' directions easy to verify.

Claim 2. An interpretation $\mathcal{I}$ is a model of $\mathcal{T}$ iff there exists a model $\mathcal{I}'$ of $\mathcal{T}^\dagger$ such that $\mathcal{I} =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}'$.

Proof. ($\Rightarrow$) Let $\mathcal{I}$ be a model of $\mathcal{T}$. Extend $\mathcal{I}$ to an interpretation $\mathcal{I}'$ by setting $B^{\mathcal{I}'}_{\exists r.C} = (\exists r.C)^{\mathcal{I}}$ for every fresh concept name $B_{\exists r.C}$ and $\bar{r}^{\mathcal{I}'} = (r^-)^{\mathcal{I}}$ for every fresh role name $\bar{r}$. Then $\mathcal{I}'$ is proper and, by Claim 1, a model of $\mathcal{T}^\dagger$. Moreover, we clearly have $\mathcal{I} =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}'$.

($\Leftarrow$) Let $\mathcal{I}'$ be a model of $\mathcal{T}^\dagger$ such that $\mathcal{I} =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}'$. Extend $\mathcal{I}'$ by setting $\bar{r}^{\mathcal{I}'} = \bar{r}^{\mathcal{I}'} \cup \{(d,e) \mid (e,d) \in r^{\mathcal{I}'}\}$ for every fresh role name $\bar{r}$, and denote the extended interpretation by $\mathcal{I}''$. It can be verified that $\mathcal{I}''$ is still a model of $\mathcal{T}^\dagger$. As an example, consider the CI $C^\sharp \sqsubseteq \forall \bar{r}.B_{\exists r.C}$. Assume that $(d,e) \in \bar{r}^{\mathcal{I}''} \setminus \bar{r}^{\mathcal{I}'}$ and $d \in (C^\sharp)^{\mathcal{I}''}$. Then $d \in (C^\sharp)^{\mathcal{I}'}$. It suffices to show that $e \in B^{\mathcal{I}'}_{\exists r.C}$, which follows from the facts that $(e,d) \in r^{\mathcal{I}'}$ and $\mathcal{I}' \models \exists r.C^\sharp \sqsubseteq B_{\exists r.C}$. We also note that $\mathcal{I} =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}''$.

We now further modify $\mathcal{I}''$ to an interpretation $\mathcal{J}$. Let $\mathcal{I}_0$ be the disjoint copy of $\mathcal{I}''$ in which every $d \in \Delta^{\mathcal{I}''}$ is renamed to d'. Then $\mathcal{J}$ is constructed by starting with the disjoint union of $\mathcal{I}''$ and $\mathcal{I}_0$ and then

1. replacing each edge $(d,e) \in \bar{r}^{\mathcal{J}}$ such that $d,e \in \Delta^{\mathcal{I}''}$ and $(e,d) \notin r^{\mathcal{J}}$ with the two edges $(e',d) \in r^{\mathcal{J}}$ and $(d,e') \in \bar{r}^{\mathcal{J}}$;
2. for each edge $(d',e') \in \bar{r}^{\mathcal{J}}$ such that $d',e' \in \Delta^{\mathcal{I}_0}$ and $(e',d') \notin r^{\mathcal{J}}$, adding the edge $(e',d') \in r^{\mathcal{J}}$.

It can be verified that $\mathcal{J}$ is still a model of $\mathcal{T}^\dagger$. Consequently, $\mathcal{J}$ is proper and, by Claim 1, a model of $\mathcal{T}$. Now let $\mathcal{J}'$ be obtained from $\mathcal{J}$ by setting $s^{\mathcal{J}'}$ for all role names s that do not occur in $\mathcal{T}$ (including the role names $\bar{r}$). Clearly, $\mathcal{J}'$ is also a model of $\mathcal{T}$. Moreover, $\mathcal{I} =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}'$ and the construction of $\mathcal{J}$ imply that $\mathcal{I}$ is a generated submodel of $\mathcal{J}'$. Since $\mathcal{T}$ is preserved under generated subinterpretations, we have $\mathcal{I} \models \mathcal{T}$ as required. ❑

We now show that $\mathcal{ALCI}$-conservative $\mathcal{ALC}$-rewritability coincides with model-conservative $\mathcal{ALC}$-rewritability for $\mathcal{ALCI}$ TBoxes. We employ robustness under replacement of $\mathcal{ALCI}$, which can be formulated as follows [15, Theorem 4]:

Theorem 11. *Let $\mathcal{T}'$ be an $\mathcal{ALCI}$-conservative $\mathcal{ALC}$-concept rewriting of $\mathcal{T}$. Let $\mathcal{T}''$ and $C \sqsubseteq D$ be in $\mathcal{ALCI}$ with $\mathsf{sig}(\mathcal{T}'', C \sqsubseteq D) \cap (\mathsf{sig}(\mathcal{T}') \setminus \mathsf{sig}(\mathcal{T})) = \emptyset$. Then $\mathcal{T}' \cup \mathcal{T}'' \models C \sqsubseteq D$ iff $\mathcal{T} \cup \mathcal{T}'' \models C \sqsubseteq D$.*

Theorem 12. *An $\mathcal{ALCI}$-TBox $\mathcal{T}$ is $\mathcal{ALCI}$-conservatively $\mathcal{ALC}$-rewritable iff $\mathcal{T}$ is model-conservatively $\mathcal{ALC}$-rewritable.*

Proof. For a concept name A, we define inductively a *relativization* $C_{|A}$ of an $\mathcal{ALCI}$ concept C to A by taking:

$$\begin{aligned}
B_{|A} &= B \sqcap A,\\
(\neg C)_{|A} &= A \sqcap \neg C_{|A},\\
(C \sqcap D)_{|A} &= C_{|A} \sqcap D_{|A},\\
(\exists r.C)_{|A} &= A \sqcap \exists r.(A \sqcap C).
\end{aligned}$$

For an interpretation $\mathcal{I}$ with $A^{\mathcal{I}} \neq \emptyset$, we denote by $\mathcal{I}_{|A}$ the subinterpretation of $\mathcal{I}$ with domain $A^{\mathcal{I}}$. We employ the following easily proved

Claim. For any interpretation $\mathcal{I}$, any $\mathcal{ALCI}$ concept C and any concept name A not in C, the following holds:

- $\mathcal{I}_{|A}$ is a generated subinterpretation of $\mathcal{I}$ iff $\mathcal{I} \models A \sqsubseteq \forall r.A$ for all $r \in \mathsf{N_R}$;
- for all $d \in \Delta^{\mathcal{I}}$, we have $d \in (C_{|A})^{\mathcal{I}}$ iff $d \in C^{\mathcal{I}_{|A}}$.

Now suppose $\mathcal{T}$ has an $\mathcal{ALCI}$-conservative $\mathcal{ALC}$-rewriting $\mathcal{T}'$, but is not preserved under generated subinterpretations. By Theorem 1, we may assume that $\mathcal{T}'$ uses no additional role names. Then we have for $A \in \mathsf{N_C} \setminus \mathsf{sig}(\mathcal{T})$:

$$\mathcal{T} \cup \{A \sqsubseteq \forall r.A \mid r \in \mathsf{sig}_R(\mathcal{T})\} \not\models C|_A \sqsubseteq D|_A,$$

for some $(C \sqsubseteq D) \in \mathcal{T}$. Thus, by Theorem 12,

$$\mathcal{T}' \cup \{A \sqsubseteq \forall r.A \mid r \in \mathsf{sig}_R(\mathcal{T})\} \not\models C|_A \sqsubseteq D|_A.$$

Take a model $\mathcal{I}$ of $\mathcal{T}' \cup \{A \sqsubseteq \forall r.A \mid r \in \mathsf{sig}_R(\mathcal{T})\}$ such that $\mathcal{I} \not\models C|_A \sqsubseteq D|_A$. Then $\mathcal{I}_{|A}$ is a model of $\mathcal{T}'$ such that $C \sqsubseteq D$, which is impossible. ❑

C Proofs for Section 3

Lemma 2 An $\mathcal{ALC}$ TBox $\mathcal{T}'$ is an $\mathcal{ALCQ}$-conservative rewriting of an $\mathcal{ALCQ}$ TBox $\mathcal{T}$ iff $\mathcal{T}'$ is a model-conservative rewriting of $\mathcal{T}$ over the class of directed tree interpretations of finite outdegree.

Proof. ($\Leftarrow$) We have to show that $\mathcal{T} \models C \sqsubseteq D$ iff $\mathcal{T}' \models C \sqsubseteq D$ for all $\mathcal{ALCQ}$-concepts C, D in the signature $\mathsf{sig}(\mathcal{T})$. In the 'if' direction, $\mathcal{T} \not\models C \sqsubseteq D$ implies that there is a model $\mathcal{I}$ of $\mathcal{T}$ with $\mathcal{I} \not\models C \sqsubseteq D$. We can always assume that $\mathcal{I}$ is a directed tree interpretation of finite outdegree. Consequently, there is a directed tree model $\mathcal{J}$ of $\mathcal{T}'$ with $\mathcal{I} =_{\mathsf{sig}(\mathcal{T})} \mathcal{J}$. Thus, $\mathcal{J} \not\models C \sqsubseteq D$, and so $\mathcal{T}' \not\models C \sqsubseteq D$. The converse direction is similar.

($\Rightarrow$) Let $\mathcal{I}$ be a directed tree model of $\mathcal{T}$ of finite outdegree with root d_0. We have to show that there is a model $\mathcal{J}$ of $\mathcal{T}'$ with $\mathcal{I} =_{\mathsf{sig}(\mathcal{T})} \mathcal{J}$. For every $d \in \Delta^{\mathcal{I}}$ and $i \geq 0$, set

$$\begin{aligned} C_d^0 &= \bigsqcap_{A \in \mathsf{sig}(\mathcal{T}), d \in A^{\mathcal{I}}} A \sqcap \bigsqcap_{A \in \mathsf{sig}(\mathcal{T}), d \notin A^{\mathcal{I}}} \neg A, \\ C_d^{i+1} &= C_d^i \sqcap \bigsqcap_{r \in \mathsf{sig}(\mathcal{T}), (d,e) \in r^{\mathcal{I}}} (\leqslant n_{d,r,C_e^i}\ r\ C_e^i) \sqcap (\geqslant n_{d,r,C_e^i}\ r\ C_e^i) \sqcap \\ &\qquad\qquad \bigsqcap_{r \in \mathsf{sig}(\mathcal{T})} \forall r. \bigsqcup_{(d,e) \in r^{\mathcal{I}}} C_e^i, \end{aligned}$$

where $n_{d,r,C}$ is the cardinality of $\{(d,e) \in r^{\mathcal{I}} \mid e \in C^{\mathcal{I}}\}$. Let $\Gamma_{\mathcal{I}} = \{C_{d_0}^i \mid i \geq 0\}$. One can show that a tree interpretation $\mathcal{J}$ satisfies $\Gamma_{\mathcal{I}}$ at the root iff $\mathcal{J} \models_{\mathsf{sig}(\mathcal{T})} \mathcal{I}$. Since $\mathcal{T}'$ is an $\mathcal{ALCQ}$-conservative rewriting of $\mathcal{T}$ and by compactness, Γ_d is satisfiable w.r.t. $\mathcal{T}'$. Clearly, any directed tree model $\mathcal{J}$ of $\mathcal{T}'$ that satisfies Γ_d at the root is as required. The converse direction is similar. ❑

Theorem 13. *An $\mathcal{ALCQ}$ TBox $\mathcal{T}$ is $\mathcal{ALCQ}$-conservatively $\mathcal{ALC}$-rewritable iff $\mathcal{T}$ is preserved under inverse bounded $\mathsf{sig}(\mathcal{T})$-morphisms over the class of directed tree interpretations of finite outdegree. Moreover, if $\mathcal{T}$ is $\mathcal{ALCQ}$-conservatively $\mathcal{ALC}$-rewritable (over this class), then $\mathcal{T}^\dagger$ is an (infinite) rewriting.*

Proof. ($\Rightarrow$) Suppose $\mathcal{T}$ has an $\mathcal{ALCQ}$-conservative $\mathcal{ALC}$-rewriting $\mathcal{T}'$, which only contains fresh concept names, but no fresh role names. Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be directed tree interpretations of finite outdegree such that there is bounded $\mathsf{sig}(\mathcal{T})$-morphism f from $\mathcal{I}_1$ to $\mathcal{I}_2$ and $\mathcal{I}_2$ is a model of $\mathcal{T}$. We have to show that $\mathcal{I}_1$ is a model of $\mathcal{T}$. By Lemma 2, there is a model $\mathcal{J}_2$ of $\mathcal{T}'$ with $\mathcal{J}_2 =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}_2$. Clearly, f is also a bounded $\mathsf{sig}(\mathcal{T})$-morphism f from $\mathcal{I}_1$ to $\mathcal{J}_2$. By Lemma 1 and since $\mathcal{T}'$ contains only fresh concept names, we find a model $\mathcal{J}_1$ of $\mathcal{T}'$ such that $\mathcal{J}_1 =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}_1$. Consequently, $\mathcal{I}_1$ is a model of $\mathcal{T}$.

($\Leftarrow$) Assume that $\mathcal{T}$ is preserved under inverse bounded $\mathsf{sig}(\mathcal{T})$-morphisms on the class of directed tree interpretations of finite outdegree. We show the following, which clearly implies that $\mathcal{T}^\dagger$ is an (infinite) $\mathcal{ALC}$-rewriting of $\mathcal{T}$.

1. if $\mathcal{T}^\dagger \models C \sqsubseteq D$ then $\mathcal{T} \models C \sqsubseteq D$ for all $\mathcal{ALCQ}$ inclusions $C \sqsubseteq D$ in $\mathsf{sig}(\mathcal{T})$;
2. $\mathcal{T}^\dagger \models \mathcal{T}$.

To obtain a finite $\mathcal{ALC}$-rewriting of $\mathcal{T}$, it then remains to invoke compactness: there is a finite subset $\mathcal{T}^\ddagger$ of $\mathcal{T}^\dagger$ such that $\mathcal{T}^\ddagger \models \mathcal{T}$. Clearly, $\mathcal{T}^\ddagger$ is as required.

Proof of Point 1. Assume $\mathcal{T} \not\models C \sqsubseteq D$ for some $\mathcal{ALCQ}$ inclusion $C \sqsubseteq D$ over $\mathsf{sig}(\mathcal{T})$. We find a directed tree interpretation $\mathcal{I}$ that is a model of $\mathcal{T}$ such that $\mathcal{I} \not\models C \sqsubseteq D$. Define $\mathcal{I}'$ in the same way as $\mathcal{I}$ except that $B_D^{\mathcal{I}'} = D^{\mathcal{I}}$ for all $D = (\geqslant n\ r\ C) \in \mathsf{sub}(\mathcal{T})$ and that for $d \in D^{\mathcal{I}}$ we make $B_1^D, \dots, B_n^D$ true in distinct r-successor of d in which C holds. It is readily checked that $\mathcal{I}'$ is a model of $\mathcal{T}^\dagger$. Thus, $\mathcal{T}^\dagger \not\models C \sqsubseteq D$.

Proof of Point 2. Assume that $\mathcal{T}^\dagger \not\models \mathcal{T}$. Take a directed tree interpretation $\mathcal{I}$ satisfying $\mathcal{T}^\dagger$ and refuting $\mathcal{T}$ in its root. First we manipulate $\mathcal{I}$ so that it has finite outdegree.

Clearly, we find a subinterpretation $\mathcal{I}'$ of $\mathcal{I}$ of finite outdegree that refutes $\mathcal{T}$. We have to be careful, however, to ensure that it still satisfies $\mathcal{T}^\dagger$. In particular, we have to ensure that no non-bisimilar successor nodes are introduced when removing nodes from $\mathcal{I}$.

We define $\mathcal{I}'$ as the limit of a sequence $\mathcal{I}_0, \mathcal{I}_1, \dots$ of interpretations:

- Set $\mathcal{I}_0 := \mathcal{I}$;
- Assume $\mathcal{I}_n$ has been defined. Let $\sim_n$ be the Σ-bisimulation relation on points of level n in $\mathcal{I}_n$. Let $[d]_{\sim_n}$ be the equivalence class of d w.r.t. $\sim_n$. Choose for every $D = (\geqslant m\ r\ C) \in \mathsf{sub}(\mathcal{T})$ and $e \in D^{\mathcal{I}_n}$ of level n in $\mathcal{I}_n$, m r-successors $d_1, \dots, d_m \in C^{\mathcal{I}_n}$ and include all $\mathcal{I}_{d_i}$ in $\mathcal{I}_{n+1}$. Also, choose for every $e \in B_D^{\mathcal{I}_n}$ of level n in $\mathcal{I}_n$ r-successors $d_i' \in (B_i^D \sqcap C)^{\mathcal{I}_n}$ for $1 \leq i \leq m$ and include all $\mathcal{I}_{d_i'}$ in $\mathcal{I}_{n+1}$.
  Finally, select for every $e \in [d]_{\sim_n}$ and every selected r-successor f of e for every $e' \in [d]_{\sim_n}$ an r-successor f' of e' that is Σ-bisimilar to f and include $\mathcal{I}_{f'}$ in $\mathcal{I}_{n+1}$ as well. This concludes the definition of $\mathcal{I}_{n+1}$.

Let $\mathcal{I}'$ be the intersection over all $\mathcal{I}_n$. $\mathcal{I}'$ has finite outdegree and clearly refutes $\mathcal{T}$. It remains to show that it is a model of $\mathcal{T}^\dagger$. The interesting inclusions are $\bigsqcap_{1 \leq i \leq n} (\exists r.(C \sqcap C_i \sqcap \bigsqcap_{j \neq i} \neg C_j)) \sqsubseteq B_D$. To show that these are still true in $\mathcal{I}'$ it is sufficient to show that if d, d' are Σ-bisimilar r-successor of d in $\mathcal{I}$ and are included in $\Delta^{\mathcal{I}'}$, then they are Σ-bisimilar in $\mathcal{I}'$. But this is the case by construction.

We now define an interpretation $\mathcal{J}$ as the image of $\mathcal{I}'$ under a bounded $\mathsf{sig}(\mathcal{T})$-morphism. Let $[d]$ denote the set of all nodes of the same level as d that are Σ-bisimilar d. The domain of $\mathcal{J}$ consists of all words $[d_0]r_1[d_1] \cdots r_n[d_n]$, where d_0 is the root of $\mathcal{I}'$ and for all i there exist $e_i \in [d_i]$ and $e_{i+1} \in [d_{i+1}]$ such that $(e_i, e_{i+1}) \in r_{i+1}^{\mathcal{I}'}$. Set $[d_0]r_1[d_1] \cdots r_n[d_n] \in A^{\mathcal{J}}$ iff $d_n \in A^{\mathcal{I}'}$ and set $([d_0]r_1[d_1] \cdots r_n[d_n], [d_0]r_1[d_1] \cdots r_n[d_n]r_{n+1}[d_{n+1}]) \in r^{\mathcal{J}}$ iff $r = r_{n+1}$ and there exists $e_n \in [d_n]$ and $e_{n+1} \in [d_{n+1}]$ such that $(e_n, e_{n+1}) \in r^{\mathcal{I}'}$. This defines $\mathcal{J}$. Now one can show

- $f : d \mapsto [d]$ is a bounded Σ-morphism from $\mathcal{I}'$ to $\mathcal{J}$;
- in $\mathcal{J}$, any Σ-bisimilar r-successors of a node are identical;

– $\mathcal{J} \models \mathcal{T}^\dagger$;
– $\mathcal{J} \models \mathcal{T}$.

The first three points are straightforward. Now for the final Point: we show by induction that $B_D^{\mathcal{J}} = D^{\mathcal{J}}$ for all $D = (\geqslant n\ r\ C) \in \mathsf{sub}(\mathcal{T})$. If $d \in B_D^{\mathcal{J}}$, then $d \in D^{\mathcal{J}}$ holds since $\mathcal{J}$ is a model of $\mathcal{T}^\dagger$ and the the first set of inclusions in $\mathcal{T}^\dagger$. Conversely, assume $d \in D^{\mathcal{J}}$. Then d has n distinct r-successors $d_1, \dots, d_n \in C^{\mathcal{J}}$. None of them is Σ-bisimilar, by Point 2. Thus, there are concepts $C_1, \dots, C_n$ in $\mathcal{ALC}$ and using symbols from Σ such that $d_i \in C_j^{\mathcal{J}'}$ iff $j = i$. By the final set of inclusions in $\mathcal{T}^\dagger$ we have $d \in B_D^{\mathcal{J}}$.

We obtain that $\mathcal{I}'$ is a model of $\mathcal{T}$ since $\mathcal{J}$ is a model of $\mathcal{T}$. But that is a contradiction. ❑

Theorem 4 An $\mathcal{ALCQ}$ TBox $\mathcal{T}$ is $\mathcal{ALCQ}$-conservatively $\mathcal{ALC}$-rewritable iff $\mathcal{T}$ is preserved under inverse bounded $\mathsf{sig}(\mathcal{T})$-morphisms. Moreover, if $\mathcal{T}$ is $\mathcal{ALCQ}$-conservatively $\mathcal{ALC}$-rewritable, then $\mathcal{T}^\dagger$ is an (infinite) rewriting.

Proof. Since ($\Leftarrow$) is an immediate consequence of Theorem 14, we concentrate on ($\Rightarrow$). Suppose an $\mathcal{ALCQ}$ TBox $\mathcal{T}$ is $\mathcal{ALCQ}$-conservatively $\mathcal{ALC}$-rewritable. Consider interpretations $\mathcal{I}_1$ and $\mathcal{I}_2$ such that $\mathcal{I}_2 \models \mathcal{T}$ and there is a bounded $\mathsf{sig}(\mathcal{T})$-morphism f from $\mathcal{I}_1$ to $\mathcal{I}_2$. We have to show that $\mathcal{I}_1 \models \mathcal{T}$. Let $\mathcal{I}_1^\dagger$ and $\mathcal{I}_2^\dagger$ be the unfoldings of $\mathcal{I}_1$ and $\mathcal{I}_2$, respectively. Note that $\mathcal{I}_i \models \mathcal{T}$ iff $\mathcal{I}_i^\dagger \models \mathcal{T}$ and that we can lift f to a bounded Σ-morphism $f^\dagger$ from $\mathcal{I}_1^\dagger$ to $\mathcal{I}_2^\dagger$. It is therefore sufficient to show $\mathcal{I}_1^\dagger \models \mathcal{T}$. By Theorem 14, $\mathcal{T}^\dagger \models \mathcal{T}$, and so $\mathcal{I}_1^\dagger \models \mathcal{T}$ if there exists a model $\mathcal{J}_1$ of $\mathcal{T}^\dagger$ such that $\mathcal{J}_1 =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}_1^\dagger$. By Lemma 1, such a model $\mathcal{J}_1$ exists if there exists a model $\mathcal{J}_2$ of $\mathcal{T}^\dagger$ with $\mathcal{J}_2 =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}_2^\dagger$. But the latter is straightforward using that $\mathcal{I}_2^\dagger$ is a model of $\mathcal{T}$. ❑

C.1 Proof of Theorem 6

We start with a sketch of the proof of Theorem 6 and then give a detailed proof. Assume, for a proof by contradiction, that $\mathcal{T}$ is an $\mathcal{ALCQ}$ TBox which is not preserved under global bisimulations but has a model-conservative $\mathcal{ALC}$-rewriting $\mathcal{T}'$. By Theorem 1, we may also assume that $\mathcal{T}'$ contains no new role names. It follows that there is a model $\mathcal{I}$ of $\mathcal{T}$ and an interpretation $\mathcal{J}$, which is not a model of $\mathcal{T}$, such that $\mathcal{I} \sim_{\mathcal{ALC}} \mathcal{J}$. Then one can show the following:

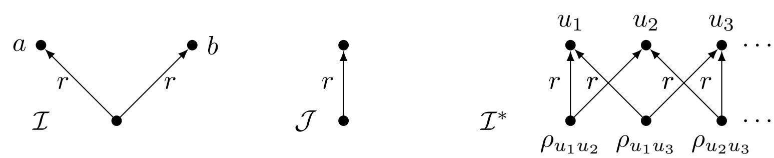
($*$) there exist tree-shaped interpretations $\mathcal{I} \sim_{\mathcal{ALC}} \mathcal{J}$ such that $\mathcal{I}$ is a model of $\mathcal{T}$, $\mathcal{J}$ is not a model of $\mathcal{T}$, and, for any $d \in \Delta^{\mathcal{I}}$, there are no distinct d_1, d_2 with $(d, d_1) \in r^{\mathcal{I}}$, $(d, d_2) \in r^{\mathcal{I}}$ and $(\mathcal{I}, d_1) \sim_{\mathcal{ALC}} (\mathcal{I}, d_2)$.

Observe first that ($*$) leads to a contradiction. It is easy to see that any bisimulation S witnessing ($*$) gives rise to a bounded $\mathsf{sig}(\mathcal{T})$-morphism f from $\mathcal{J}$ to $\mathcal{I}$. Indeed, we may assume that S is a *level-bisimulation* in the sense that if $(d, e) \in S$, then the distance from d to the root of $\mathcal{I}$ is the same as the distance of e to the root of $\mathcal{J}$. The condition for $\mathcal{I}$ in ($*$) implies then that S is a function.

We require the following fundamental property of bounded Σ-morphisms:

We now apply Lemma 1 to the bounded $\mathsf{sig}(\mathcal{T})$-morphism f from $\mathcal{J}$ to $\mathcal{I}$. Since $\mathcal{I}$ is a model of $\mathcal{T}$, we find $\mathcal{I}' =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}$ such that $\mathcal{I}'$ is a model of $\mathcal{T}'$. Clearly, f is still a bounded $\mathsf{sig}(\mathcal{T})$-morphism from $\mathcal{J}$ to $\mathcal{I}'$. By Lemma 1, there exists a model $\mathcal{J}'$ of $\mathcal{T}'$ such that $\mathcal{J}' =_{\mathsf{sig}(\mathcal{T})} \mathcal{J}$. But then $\mathcal{J}$ is a model of $\mathcal{T}$, and we have obtained a contradiction.

To illustrate the idea behind property $(*)$, consider the interpretations $\mathcal{I}$ and $\mathcal{J}$ in the picture below, assuming that $\mathcal{I}$ is a model of an $\mathcal{ALCQ}$ TBox $\mathcal{T}$.



The model $\mathcal{I}$ does not satisfy $(*)$ because $(\mathcal{I}, a) \sim_{\mathcal{ALC}} (\mathcal{I}, b)$. To find a model of $\mathcal{T}$ for which $(*)$ holds, we construct the interpretation $\mathcal{I}^*$ shown on the right-hand side of the picture, where each pair of distinct u_i, u_j from an infinite set $U = \{u_1, u_2, \dots\}$ has its own 'r-root' $\rho_{u_i u_j}$. Clearly, $\mathcal{I}^*$ is a model of $\mathcal{T}$, and so there exists $\mathcal{J}^* =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}^*$ such that $\mathcal{J}^*$ is a model of the $\mathcal{ALC}$ TBox $\mathcal{T}'$. Now, since U is infinite, there exist two $u_i, u_j \in U$ that are instances in $\mathcal{J}^*$ of the same concept names in $\mathcal{T}'$. The restriction of $\mathcal{J}^*$ to $\{\rho_{u_i,u_j}, u_i, u_j\}$ is a model of $\mathcal{T}'$, which is $\mathsf{sig}(\mathcal{T}')$-bisimilar to the restriction $\mathcal{I}'$ of $\mathcal{J}^*$ to $\{\rho_{u_i,u_j}, u_i\}$. Thus $\mathcal{I}'$ is a model of $\mathcal{T}$ satisfying $(*)$.

We now come to the detailed proof of Theorem 6. A *path* p in an interpretation $\mathcal{I}$ is a word $d_0 r_0 \cdots r_{n-1} d_n$ such that $d_i \in \Delta^{\mathcal{I}}$, $r_i \in \mathsf{N_R}$, and $(d_i, d_{i+1}) \in r^{\mathcal{I}}$ for all $i < n$. By $\mathsf{tail}(p)$ we denote the final element of p. If $\mathcal{I}$ is a directed tree interpretation, then for every $d \in \Delta^{\mathcal{I}}$ there exists a unique path p starting from the root $\rho_{\mathcal{I}}$ of p such that $\mathsf{tail}(p) = d$. We denote this path by $p^{\mathcal{I}}(d)$. Let $\mathcal{I}$ and $\mathcal{J}$ be directed tree interpretations. A global bisimulation S between $\mathcal{I}$ and $\mathcal{J}$ is a *level bisimulation* if $(d, d') \in S$ implies that the length of $p^{\mathcal{I}}(d)$ equals the length of $p^{\mathcal{I}}(d')$. For $d \in \Delta^{\mathcal{I}}$ we denote by $\mathcal{I}_d$ the interpretation rooted at d.

Lemma 5. *Let $\mathcal{I}$ and $\mathcal{J}$ be globally bisimilar interpretations such that $\mathcal{I}$ is a model of an $\mathcal{ALCQ}$ TBox $\mathcal{T}$ and $\mathcal{J}$ is not a model of $\mathcal{T}$. Then there are directed tree interpretations $\mathcal{I}'$ and $\mathcal{J}'$ such that $\mathcal{I}'$ is a model of $\mathcal{T}$, $\mathcal{J}'$ is not a model of $\mathcal{T}$ and there is a level bisimulation S between $\mathcal{I}'$ and $\mathcal{J}'$. Moreover, we can assume that the outdegrees of $\mathcal{I}^*$ and $\mathcal{J}^*$ are finite.*

Proof. Assume $(\mathcal{I}, d)$ and $(\mathcal{J}, e)$ are globally bisimilar and $e \in C^{\mathcal{J}} \setminus D^{\mathcal{J}}$ for some $C \sqsubseteq D \in \mathcal{T}$. We unfold $(\mathcal{I}, d)$ and $(\mathcal{J}, e)$ to $\mathcal{I}^*$ and $\mathcal{J}^*$ as follows:

- $\Delta^{\mathcal{I}^*}$ is the set of all paths in $\mathcal{I}$ starting at d;
- $p \in A^{\mathcal{I}^*}$ if $\mathsf{tail}(p) \in A^{\mathcal{I}}$;
- $(p, p \cdot r \cdot f) \in r^{\mathcal{I}^*}$ if $(\mathsf{tail}(p), f) \in r^{\mathcal{I}}$.

$\mathcal{J}^*$ is defined analogously with paths in $\mathcal{J}$ starting at e. It is readily checked that $\mathcal{I}^*$ and $\mathcal{J}^*$ satisfy the conditions of the lemma except the bound on the outdegree. Let S be the level bisimulation.

We now define subinterpretations of $\mathcal{I}^*$ and $\mathcal{J}^*$ that have finite outdegree. The construction is by selective filtrations. We construct pairs (X, Y), where $X \subseteq \Delta^{\mathcal{I}^*}$ and $Y \subseteq \Delta^{\mathcal{J}^*}$.

- We start with $X = \{\rho_{\mathcal{I}^*}\}$ and $Y = \{\rho_{\mathcal{J}^*}\}$;
- Assume (X, Y) has been defined. Let $(d, e) \in S$ with $d \in X$ and $e \in Y$ such that no successor of d is in X. We find a subseteq X' of the set of successors of d with $|X'| \leq m_r$ such that whenever $d \in (\geqslant n\ r\ C)^{\mathcal{I}^*}$ and $(\geqslant n\ r\ C) \in \mathsf{sub}(\mathcal{T})$, then there are at least n r successors of d in $C^{\mathcal{I}^*}$. Similarly we find such a set Y' of successors of e. Choose for every $d' \in X'$ with $(d, d') \in r^{\mathcal{I}^*}$ an e' with $(e, e') \in r^{\mathcal{J}^*}$ such that $(d', e') \in S$ and insert it into Y''. Also, choose for every $e' \in Y'$ with $(e, e') \in r^{\mathcal{J}^*}$ a d' with $(d, d') \in r^{\mathcal{I}^*}$ such that $(d', e') \in S$ and insert it into X''. Now set $X := X \cup X' \cup X''$ and $Y = Y \cup Y' \cup Y''$.

Let $\mathcal{I}'$ be the restriction of $\mathcal{I}^*$ to X and $\mathcal{J}'$ be the restriction of $\mathcal{J}^*$. It is readily checked that $\mathcal{I}', \mathcal{J}'$ are as required. ❑

Theorem 14. *An $\mathcal{ALCQ}$-TBox $\mathcal{T}$ is model-projectively $\mathcal{ALC}$-rewritable iff it is preserved under global bisimulations (and thus iff it is equivalent to an $\mathcal{ALC}$ TBox).*

Proof. Assume $\mathcal{T}$ is not preserved under global bisimulations and $\mathcal{T}'$ is a model-projective $\mathcal{ALC}$ rewriting of $\mathcal{T}$. By Lemma 3, there exist directed tree interpretations $\mathcal{I}$ and $\mathcal{J}$, both of finite outdegree, such that $\mathcal{I}$ is a model of $\mathcal{T}$, $\mathcal{J}$ is not a model of $\mathcal{T}$, and there is a level bisimulation S between $\mathcal{I}$ and $\mathcal{J}$. We first show the following

Claim 1. There exists a directed tree interpretation $\mathcal{I}'$ that is a model of $\mathcal{T}$ and is globally bisimilar to $\mathcal{I}$ such that for any two bisimilar $(\mathcal{I}', d)$ and $(\mathcal{I}', d')$ with d, d' points at the same level in $\mathcal{I}'$ the interpretations $\mathcal{I}'_d$ and $\mathcal{I}'_{d'}$ are isomorphic.

Proof of Claim 1. We define a sequence $\mathcal{I}_0, \mathcal{I}_1, \ldots$ of directed tree interpretations as follows:

- $\mathcal{I}_0 := \mathcal{I}$;
- Assume $\mathcal{I}_n$ has been defined. Let $\sim_n$ be the minimal bisimulation relation on points of level n in $\mathcal{I}_n$. For any equivalence class $[d]_{\sim_n} = \{d_1, \ldots, d_m\}$ with respect to $\sim_n$ and any role name r in $\mathsf{sig}_R(\mathcal{T})$, take m disjoint copies $\mathcal{I}_e^1, \ldots, \mathcal{I}_e^m$ of every $\mathcal{I}_e$ with e an r-successor of some $d_j \in [d]_{\sim_n}$ and attach $\mathcal{I}_e^i$ to d_i, for all $1 \leq i \leq m$, by connecting d_i and the the root of $\mathcal{I}_e^i$ using r. We assume $\mathcal{I}_e^j = \mathcal{I}_e$. Let $\mathcal{I}_{n+1}$ be the resulting interpretation.

Define $\mathcal{I}'$ as the union of all $\mathcal{I}_n$ (note that $\mathcal{I}_n$ is a subintererpretation of $\mathcal{I}_{n+1}$ since we assume $\mathcal{I}_e^j = \mathcal{I}_e$). It is readily checked that that for any two bisimilar $(\mathcal{I}', d)$ and $(\mathcal{I}', d')$ with d, d' points at the same level in $\mathcal{I}'$ the interpretations $\mathcal{I}'_d$ and $\mathcal{I}'_{d'}$ are isomorphic.

To prove that $\mathcal{I}'$ is a model of $\mathcal{T}$ observe that there is a bounded $\mathsf{sig}(\mathcal{T})$-morphism f from $\mathcal{I}'$ to $\mathcal{I}$. Since $\mathcal{T}'$ is a model conservative $\mathcal{ALC}$-rewriting of $\mathcal{T}$, there exists a model $\mathcal{J}$ of $\mathcal{T}'$ with $\mathcal{J} =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}$. Thus, by Lemma 1, there exists a model $\mathcal{J}'$ of $\mathcal{T}'$ with $\mathcal{J}' =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}'$. Thus $\mathcal{I}'$ is a model of $\mathcal{T}$. This finishes the proof of Claim 1.

Claim 2. There exists a directed tree interpretation $\mathcal{I}''$ which is a model of $\mathcal{T}$ and is globally bisimilar to $\mathcal{I}'$ such that there do not exist two distinct globally bisimilar $\mathcal{I}_{d_1}$ and $\mathcal{I}_{d_2}$ with d_1, d_2 r-successors of some d in $\mathcal{I}''$.

We construct $\mathcal{I}''$ as the limit of a sequence $\mathcal{I}'_0, \mathcal{I}'_1, \ldots$ defined as follows:

- $\mathcal{I}'_0 := \mathcal{I}'$;
- Assume $\mathcal{I}'_n$ has been defined. Consider a lowest level occurrence of *distinct* globally bisimilar $\mathcal{I}_{d_1}$ and $\mathcal{I}_{d_2}$ with d_1, d_2 r-successors of some d in $\mathcal{I}'_n$. (If this situation does not occur, set $\mathcal{I}'_{n+1} := \mathcal{I}'_n$.) Take such a d with r-successors $d_1, \ldots, d_m$, $m > 1$, such that $\mathcal{I}_{d_1}, \ldots, \mathcal{I}_{d_m}$ are globally bisimilar. By Claim 1 $\mathcal{I}_{d_1}, \ldots, \mathcal{I}_{d_m}$ are isomorphic. We define $\mathcal{I}'_{n+1}$ as the result of removing $\mathcal{I}_{d_2}, \ldots, \mathcal{I}_{d_m}$ from $\mathcal{I}'_n$.

We show that if $\mathcal{I}'_n$ is a model of $\mathcal{T}$, then $\mathcal{I}'_{n+1}$ is a model of $\mathcal{T}$.

Let U be a set of cardinality $\kappa > 2^{\aleph_0}$ and take for every $u \in U$ a copy $\mathcal{I}_u$ of the interpretation $\mathcal{I}_{d_1}$. We assume that the $\mathcal{I}_u$, $u \in U$, are mutually disjoint. For any m-element subset $W = \{w_1, \ldots, w_m\}$ of U define an interpretation $\mathcal{I}_W$ that is obtained from $\mathcal{I}'_n$ by replacing the subinterpretations $\mathcal{I}_{d_1}, \ldots, \mathcal{I}_{d_m}$ by $\mathcal{I}_{w_1}, \ldots, \mathcal{I}_{w_m}$, respectively. We assume that the $\mathcal{I}_W$ are mutually disjoint except for the nodes in $\Delta^{\mathcal{I}_u}$, $u \in U$. Note that all $\mathcal{I}_W$ are isomorphic to $\mathcal{I}'_n$. Let $\mathcal{J}$ be the union of all $\mathcal{I}_W$. The point generated subinterpretations of $\mathcal{J}$ are all isomorphic to generated subinterpretations of $\mathcal{I}'_n$. Thus $\mathcal{J}$ is a model of $\mathcal{T}$ since $\mathcal{I}'_n$ is a model of $\mathcal{T}$. Hence there exists a model $\mathcal{J}'$ of $\mathcal{T}'$ such that $\mathcal{J}' =_{\mathsf{sig}(\mathcal{T})} \mathcal{J}$. As U has cardinality $> 2^{\aleph_0}$, there is a set $W_0 = \{w_1, \ldots, w_m\} \subseteq U$ of cardinality m such that the restrictions of $\mathcal{J}'$ to $\Delta^{\mathcal{I}_{w_i}}$ are isomorphic, for all $w_i \in W_0$. Let $\mathcal{I}'_{W_0}$ be the restriction of $\mathcal{J}'$ to $\Delta^{\mathcal{I}_{W_0}}$. The resulting interpretation $\mathcal{I}''_{W_0}$ after removing all points in $\mathcal{I}_{w_2}, \ldots, \mathcal{I}_{w_m}$ from $\mathcal{I}'_{W_0}$ is clearly again a model of $\mathcal{T}'$ and $\mathcal{I}''_{W_0} =_{\mathsf{sig}(\mathcal{T})} \mathcal{I}''_n$. Thus $\mathcal{I}''_n$ is a model of $\mathcal{T}$.

Define $\mathcal{I}''$ as the limit of the sequence $\mathcal{I}'_0, \mathcal{I}'_1, \ldots$. It is readily checked that $\mathcal{I}''$ is as required. This finishes the proof of Claim 2.

The interpretation $\mathcal{I}''$ obtained in Claim 2 is globally bisimilar to $\mathcal{J}$. So we have a level bisimulation S between $\mathcal{J}$ and $\mathcal{I}''$. From the condition for $\mathcal{I}''$ that no node has distinct bisimilar r-successors we obtain that S is a function, thus a bounded $\mathsf{sig}(\mathcal{T})$-morphism. By Lemma 1 there exists a model $\mathcal{J}'$ of $\mathcal{T}'$ such that $\mathcal{J}' =_{\Sigma} \mathcal{J}$. Thus, $\mathcal{J}$ is a model of $\mathcal{T}$ and we have derived a contradiction. ❑

C.2 Proofs of 2ExpTime upper bounds for rewritability

Theorem 15. *For $\mathcal{ALCQ}$ TBoxes the following holds: equivalent $\mathcal{ALC}$-rewritability, model-conservative $\mathcal{ALC}$-rewritability, and $\mathcal{ALCQ}$-conservative $\mathcal{ALC}$-rewritability are decidable in 2ExpTime.*

Proof. We employ the model-theoretic criteria and use type elimination procedures.

First we show that it is decidable in 2ExpTime whether a $\mathcal{ALCQ}$ TBox is preserved under global bisimulations. Assume a $\mathcal{ALCQ}$ TBox $\mathcal{T}$ is given. The 2ExpTime algorithm deciding preservation under global bisimulations is as follows. Consider the set tp of all types over $\mathsf{sub}(\mathcal{T})$ and its subset $\mathsf{tp}(\mathcal{T})$ of all types in tp that are satisfiable w.r.t. $\mathcal{T}$. The following rules are applied recursively to the set $\mathcal{E}$ of elements of $2^{\mathsf{tp}} \times 2^{\mathsf{tp}(\mathcal{T})}$:

(A) Remove (T, T') from $\mathcal{E}$ if not all $t \in T \cup T'$ contain the same concept names.

(EX) Remove (T, T') from $\mathcal{E}$ if there is a role name r such that there are no interpretations $\mathcal{I}_t, t \in T \cup T'$, and $d_t \in \Delta^{\mathcal{I}_t}$ such that
- all $\mathcal{I}_t, t \in T'$, are models of $\mathcal{T}$;
- d_t satisfies t, for all $t \in T \cup T'$;
- for each $t_0 \in T \cup T'$ and $(d_{t_0}, e_{t_0}) \in r^{\mathcal{I}_{t_0}}$ there exist $(d_t, e_t) \in r^{\mathcal{I}_t}$ for $t \in (T \cup T') \setminus \{t_0\}$, such that there exists $(S, S') \in \mathcal{E}$ with S the set of types realized by the nodes e_t, $t \in T$, and S' the set of types realized by the nodels e_t, $t \in T'$.

Denote by $\mathcal{E}_0$ the remaining set. One can show that $\mathcal{E}_0$ is the set of all (T, T') such that there exist models $\mathcal{I}_t$, $t \in T \cup T'$, and $d_t \in \Delta^{\mathcal{I}_t}$ such that

- all $\mathcal{I}_t$, $t \in T'$, are models of $\mathcal{T}$;
- d_t satisfies t, for all $t \in T \cup T'$;
- all $(\mathcal{I}_t, d_t)$, $t \in T \cup T'$, are bisimilar.

It follows that $\mathcal{T}$ is not preserved under global bisimulations iff there exists $(\{t\}, \{t'\}) \in \mathcal{E}_0$ such that $t \not\in \mathsf{tp}(\mathcal{T})$.

Now we show that it is decidable in 2ExpTime whether an $\mathcal{ALCQ}$ TBox $\mathcal{T}$ is preserved under inverse $\mathsf{sig}(\mathcal{T})$-morphisms. Assume an $\mathcal{ALCQ}$ TBox $\mathcal{T}$ is given. The 2ExpTime algorithm is as follows. The following rules are applied recursively to the set $\mathcal{E}$ of all elements of $2^{\mathsf{tp}} \times \mathsf{tp}(\mathcal{T})$:

(A) Remove (T, s) from $\mathcal{E}$ if not all $t \in T \cup \{s\}$ contain the same concept names.

(EX) Remove (T, s) from $\mathcal{E}$ if there is a role name r such that there are no interpretations $\mathcal{I}_t, t \in T \cup \{s\}$, and $d_t \in \Delta^{\mathcal{I}_t}$ such that
- $\mathcal{I}_s$ is a model of $\mathcal{T}$;
- d_t satisfies t, for all $t \in T \cup \{s\}$;
- for each $t \in T$ there is a function f_t from the set of $r^{\mathcal{I}_t}$-successors of d_t onto the set of $r^{\mathcal{I}_s}$-successor of d_s such that for each $r^{\mathcal{I}_s}$-successor e_s of d_s there exists $(S, s') \in \mathcal{E}$ such that s' is the type of e_s and S is the set of types realized in $\bigcup_{t \in T} f_t^{-1}(e_s)$.

Denote by $\mathcal{E}_0$ the remaining set. One can show that $\mathcal{E}_0$ is the set of all (T, s) such that there exist models $\mathcal{I}_t$, $t \in T \cup \{s\}$, and $d_t \in \Delta^{\mathcal{I}_t}$ such that

- $\mathcal{I}_s$ is a model of $\mathcal{T}$;
- d_t satisfies t, for all $t \in T \cup \{s\}$;

– there are $\mathsf{sig}(\mathcal{T})$-bounded morphisms f_t from each $\mathcal{I}_t$ onto $\mathcal{I}_s$ with $f_t(d_t) = d_s$).

It follows that $\mathcal{T}$ is not preserved under inverse $\mathsf{sig}(\mathcal{T})$-bounded morphisms iff there exists $(\{t\}, s) \in \mathcal{E}_0$ such that $t \notin \mathsf{tp}(\mathcal{T})$. ❏

D Proofs for Section 4

Theorem 11 follows from the following result.

Theorem 16. *An any $\mathcal{ALCI}$-TBox $\mathcal{T}$ the following conditions are equivalent:*

- *$\mathcal{T}$ is equivalently DL-Lite$_{horn}$-rewritable;*
- *$\mathcal{T}$ is $\mathcal{ALCI}$-conservatively DL-Lite$_{horn}$-rewritable;*
- *$\mathcal{T}$ is model-conservatively DL-Lite$_{horn}$-rewritable.*

Proof. Assume $\mathcal{T}'$ is an $\mathcal{ALCI}$-conservative *DL-Lite$_{horn}$*-rewriting of an $\mathcal{ALCI}$ TBox $\mathcal{T}$. By Theorem 1, we may assume that $\mathcal{T}'$ does not contain additional role names. Let $\mathcal{T}''$ be the set of *DL-Lite$_{horn}$*-inclusions $C \sqsubseteq D$ in $\mathsf{sig}(\mathcal{T})$ such that C, D do not contain redundant conjuncts (and so $\mathcal{T}''$ is finite) and $\mathcal{T}' \models C \sqsubseteq D$. It is sufficient to show that $\mathcal{T}''$ is an equivalent *DL-Lite$_{horn}$*-rewriting of $\mathcal{T}$. Clearly $\mathcal{T} \models \mathcal{T}''$. Thus, assume $\mathcal{T}'' \not\models \mathcal{T}$. Let $\mathcal{I}$ be a model of $\mathcal{T}''$ that is not a model of $\mathcal{T}$. We expand $\mathcal{I}$ to an interpretation $\mathcal{I}'$ by setting for any concept name $M \notin \mathsf{sig}(\mathcal{T})$ and $d \in \Delta^{\mathcal{I}}$, $d \in M^{\mathcal{I}'}$ iff $\mathcal{T}' \models D \sqsubseteq M$, where D is the conjunction of all basic *DL-Lite$_{horn}$* concepts B with $\mathsf{sig}(B) \subseteq \mathsf{sig}(\mathcal{T})$ and $d \in B^{\mathcal{I}}$. Then, since $\mathcal{T}'$ does not contain any additional role names, $\mathcal{I}'$ is a model of $\mathcal{T}'$. Thus $\mathcal{T}' \not\models \mathcal{T}$ and we have derived a contradiction. ❏

The *product* $\mathcal{I} = \prod_{i \in I} \mathcal{I}_i$ of a family $\mathcal{I}_i$, $i \in I$, of interpretations is the interpretation with domain

$$\{f \colon I \to \bigcup_{i \in I} \Delta^{\mathcal{I}_i} \mid f(i) \in \Delta^{\mathcal{I}_i} \text{ for } i \in I\}$$

and $f \in A^{\mathcal{I}}$ iff $f(i) \in A^{\mathcal{I}_i}$ for all $i \in I$, and $(f, g) \in r^{\mathcal{I}}$ iff $(f(i), g(i)) \in r^{\mathcal{I}_i}$ for all $i \in I$. We first show

Lemma 7 If $\mathcal{T}$ is language or model-conservatively $\mathcal{EL}$-rewritable $\mathcal{ALC}$ TBox of depth n, then there exists a language or, respectively, model-conservative $\mathcal{EL}$ rewriting $\mathcal{T}'$ of $\mathcal{T}$ in Σ layered normal form of depth n, where $\Sigma = \mathsf{sig}(\mathcal{T})$.

Proof. We use the notation introduced in Section 4 for ABoxes. Let $\mathcal{T}$ be an $\mathcal{ALC}$ TBox of depth n and $\Sigma = \mathsf{sig}(\mathcal{T})$. Assume $\mathcal{T}'$ is a model-conservative $\mathcal{EL}_{\bot}$-rewriting of $\mathcal{T}$ using, in addition to Σ, concept names from Γ. We may assume that $\mathcal{T}'$ has depth 1 and is in *standard normal form*, that is, its inclusions take the form

(a) $A_1 \sqcap A_1 \sqsubseteq B$,
(b) $\exists r.B \sqsubseteq A$, or

(c) $\exists r.A \sqsubseteq B$,

where $A, A_1, A_2, B \in \mathsf{N_C} \cup \{\bot\}$. Introduce for every $M \in \Gamma$, fresh concept names $M_0, \dots, M_n$ and set $\Gamma_i = \{M_i \mid M \in \Gamma\}$ for $i \leq n$. For any concept C over $\Sigma \cup \Gamma$ denote by C_i its translation into $\Sigma \cup \Gamma_i$ defined by replacing each $M \in \Gamma$ by M_i. Conversely, for a concept C in $\Sigma \cup \Gamma_i$ denote by C^{-i} the concept in $\Sigma \cup \Gamma$ such that $(C^{-i})_i = C$. Now define $\mathcal{T}''$ as follows:

(a') for each inclusion $C \sqsubseteq D$ of the form (a) with $\mathcal{T}' \models C \sqsubseteq D$, include in $\mathcal{T}''$ the inclusion $C_i \sqsubseteq D_i$, for all $0 \leq i \leq n$;
(b') for each inclusion $\exists r.A \sqsubseteq B$ of the form (b) with $\mathcal{T}' \models \exists r.A \sqsubseteq B$, include in $\mathcal{T}''$ the inclusion $\exists r.A_{i+1} \sqsubseteq B_i$, for all $1 \leq i < n$;
(c') for each inclusion $A \sqsubseteq \exists r.B$ of the form (c) with $\mathcal{T}' \models A \sqsubseteq \exists r.B$, include in $\mathcal{T}''$ the inclusion $A_i \sqsubseteq \exists r.B_{i+1}$, for all $i < n$.

Clearly $\mathcal{T}''$ is in Σ-layered form of depth n. We show that $\mathcal{T}''$ is as required.

We first show that for every model $\mathcal{I}$ of $\mathcal{T}'$ there exists a model $\mathcal{I}'$ of $\mathcal{T}''$ such that $\mathcal{I}' =_\Sigma \mathcal{I}$. Assume $\mathcal{I}$ is given. Define $\mathcal{I}'$ in the same way as $\mathcal{I}$ except that $M_i^{\mathcal{I}'} := M^{\mathcal{I}}$ for all $0 \leq i \leq n$ and all $M \in \Gamma$. It is readily checked that $\mathcal{I}'$ is a model of $\mathcal{T}''$ and clearly $\mathcal{I}' =_\Sigma \mathcal{I}$. It follows that if $\mathcal{T}'$ is a model-conservative $\mathcal{EL}_\bot$-rewriting of $\mathcal{T}$, then for every model $\mathcal{I}$ of $\mathcal{T}$ there exists a model $\mathcal{I}'$ of $\mathcal{T}''$ such that $\mathcal{I} =_\Sigma \mathcal{I}'$. If $\mathcal{T}'$ is an $\mathcal{ALC}$-conservative rewriting of $\mathcal{T}$, then $\mathcal{T}'' \models C \sqsubseteq D$ for some $\mathcal{ALC}$-concepts C, D in Σ implies then $\mathcal{T} \models C \sqsubseteq D$. It therefore remains to to prove that $\mathcal{T}'' \models \mathcal{T}$. Assume $\mathcal{T}'' \not\models \mathcal{T}$. Take a directed tree interpretation $\mathcal{I}$ that is a model of $\mathcal{T}''$ but refutes an inclusion $C \sqsubseteq D \in \mathcal{T}$ in its root ρ. Since $\mathcal{T}$ has depth n, the refutation of $C \sqsubseteq D$ in $\mathcal{I}$ depends only on the nodes with distance $\leq n$ from ρ in $\mathcal{I}$.

Define an ABox $\mathcal{A}$ based on the subinterpretation of $\mathcal{I}$ with nodes of distance $\leq n$ from ρ by setting $A(a) \in \mathcal{A}$ if $a \in A^{\mathcal{I}}$, $\neg A(a) \in \mathcal{A}$ if $a \notin A^{\mathcal{I}}$, and $r(a,b) \in \mathcal{A}$ if $(a,b) \in r^{\mathcal{I}}$ for all $a, b \in \Delta^{\mathcal{I}}$ of distance $\leq n$ from ρ. The following claim contradicts the assumption that $\mathcal{I}$ refutes $\mathcal{T}$ in ρ:

Claim. $\mathcal{A}$ is strongly n-satisfiable w.r.t. $\mathcal{T}$.

Define $\mathcal{A}'$ as the modification of $\mathcal{A}$ obtained by replacing for every a in $\mathcal{A}$:

– every $M_i(a) \in \mathcal{A}$ with $M \in \Gamma$ by $M(a)$;
– every $\neg M_i(a) \in \mathcal{A}$ with $M \in \Gamma$ by $\neg M(a)$.

To prove this claim it is sufficient to show that $\mathcal{A}'$ is strongly n-satisfiable w.r.t. $\mathcal{T}'$. We first show that to prove this, it is sufficient to show:

– for any a in $\mathcal{A}'$, $\{A(a) \mid A(a) \in \mathcal{A}'\} \cup \{\neg A(a) \mid \neg A(a) \in \mathcal{A}'\}$ is satisfiable w.r.t. $\mathcal{T}'$;
– if $\exists r.A \sqsubseteq B \in \mathcal{T}'$, $r(a,b) \in \mathcal{A}'$, and $A(b) \in \mathcal{A}'$, then $B(a) \in \mathcal{A}'$;
– if $A \sqsubseteq \exists r.B \in \mathcal{T}'$ and $A(a) \in \mathcal{A}'$ of co-depth $< n$, then there exists $r(a,b) \in \mathcal{A}'$ with $B(b) \in \mathcal{A}'$.

Assume Point 1 to 3 hold. Attach to each leaf $a \in \mathsf{Ind}(\mathcal{A})$ a tree-shaped interpretation $\mathcal{I}_a$ of $\mathcal{T}'$ satisfying $\{A(a) \mid A(a) \in \mathcal{A}'\} \cup \{\neg A(a) \mid \neg A(a) \in \mathcal{A}'\}$ in its root a. Let $\mathcal{I}$ be the union of $\mathcal{I}_{\mathcal{A}'}$ and the $\mathcal{I}_a$, a a leaf of $\mathcal{A}'$. Clearly $\mathcal{I}'$ is a model of $\mathcal{T}'$.

Now, Point 1 follows from the inclusions of type (a) in $\mathcal{T}''$, Point 2 follows from the inclusions of type (b) in $\mathcal{T}''$. Point 3 follows from the inclusions of type (c) in $\mathcal{T}''$. ❑

To prove Lemma 8 we first show the following.

Lemma 6. *There exists a model $\mathcal{I}$ of $\mathcal{T}$ satisfying $\mathcal{A}_\mathcal{T}$ such that the only nodes with r-successors in $\mathcal{I}$ outside $\mathcal{A}_\mathcal{T}$ are the $\rho_\mathcal{A}$ with $\mathcal{A} \in \mathsf{AT}_0$.*

Proof. Take for any $\rho_\mathcal{A}$ with $\mathcal{A} \in \mathsf{AT}_0$ a directed tree model $\mathcal{I}_\mathcal{A}$ of $\mathcal{T}$ satisfying $\mathcal{A}$ in its root. Hook the $\mathcal{I}_\mathcal{A}$ to the ABox $\mathcal{A}_\mathcal{T}$ at $\rho_\mathcal{A}$. This defines an interpretation $\mathcal{I}$. We show it is a model of $\mathcal{T}$.

Take for each $\mathcal{A} \in \mathsf{AT}_i$, $1 \leq i \leq n$, an interpretation $\mathcal{I}_\mathcal{A}$ witnessing strong i-satisfiability of $\mathcal{A}$ w.r.t. $\mathcal{T}$. We show

Claim 1. For all $C \in \mathsf{sub}(\mathcal{T})$ and all $\rho_\mathcal{A}$ in $\mathcal{A}_\mathcal{T}$ with $\mathcal{A} \in \mathsf{AT}_i$, $1 \leq i \leq n$, $\rho_\mathcal{A} \in C^\mathcal{I}$ iff $\rho_\mathcal{A} \in C^{\mathcal{I}_\mathcal{A}}$.

The proof is by induction over the structure of C. The interesting case is $C = \exists r.D$. Assume $\rho_\mathcal{A} \in C^\mathcal{I}$ and $\mathcal{A} \in \mathsf{AT}_i$ for some $1 \leq i \leq n$. Then there exists $\rho_{\mathcal{A}'}$ with $r(\rho_\mathcal{A}, \rho_{\mathcal{A}'}) \in \mathcal{A}_\mathcal{T}$ such that $\rho_{\mathcal{A}'} \in D^\mathcal{I}$. By induction hypothesis, $\rho_{\mathcal{A}'} \in D^{\mathcal{I}_{\mathcal{A}'}}$. The depth of D does not exceed $n-1$, hence we obtain $D(\rho_{\mathcal{A}'}) \in \mathcal{A}'$. By definition of $\mathcal{A}_\mathcal{T}$, there exists $r(\rho_\mathcal{A}, b) \in \mathcal{A}$ such that the subtree of $\mathcal{A}$ rooted at b is $(i-1, \mathcal{T})$-bisimilar to $\mathcal{A}'$. But then $D(b) \in \mathcal{A}$ and so $\rho_\mathcal{A} \in C^{\mathcal{I}_\mathcal{A}}$.

Assume $\rho_\mathcal{A} \in C^{\mathcal{I}_\mathcal{A}}$ and $\mathcal{A} \in \mathsf{AT}_i$ for some $1 \leq i \leq n$. There exists $r(\rho_\mathcal{A}, b) \in \mathcal{A}$ such that $b \in D^{\mathcal{I}_\mathcal{A}}$. The depth of D does not exceed $n-1$, so we obtain $D(b) \in \mathcal{A}$. There exists $\mathcal{A}' \in \mathsf{AT}_{i-1}$ such that the subtree of $\mathcal{A}$ rooted at b is $(i-1, \mathcal{T})$-bisimilar to $(\mathcal{A}', \rho_{\mathcal{A}'})$. Hence $r(\rho_\mathcal{A}, \rho_{\mathcal{A}'}) \in \mathcal{A}_\mathcal{T}$ and $D(\rho_{\mathcal{A}'}) \in \mathcal{A}'$. Then $\rho_{\mathcal{A}'} \in D^{\mathcal{I}_{\mathcal{A}'}}$ and by induction hypothesis $\rho_{\mathcal{A}'} \in D^\mathcal{I}$. But then $\rho_\mathcal{A} \in C^\mathcal{I}$.

Claim 1 together with the choice of the $\mathcal{I}_\mathcal{A}$ with $\mathcal{A} \in \mathsf{AT}_0$ implies that $\mathcal{I}$ is a model of $\mathcal{T}$, as required. ❑

Lemma 8 Assume $\mathcal{T}$ is an $\mathcal{ALC}$ TBox of depth n and $\Sigma = \mathsf{sig}(\mathcal{T})$. An $\mathcal{EL}_\bot$ TBox $\mathcal{T}'$ in Σ-layered form of depth n is a model-conservative $\mathcal{EL}_\bot$-rewriting of $\mathcal{T}$ iff

- $\mathcal{T}' \models \mathcal{T}$;
- There exists $\mathcal{A}' =_\Sigma \mathcal{A}_\mathcal{T}$ such that, for all $0 \leq i \leq n$, $\mathcal{A}'$ satisfies all level i inclusions in $\mathcal{T}'$ in all $\rho_{\mathcal{A}_i}$ with $\mathcal{A}_i \in \mathsf{AT}_{n-i}$.

Proof. The "only if" direction is straighforward: $\mathcal{T}' \models \mathcal{T}$ follows from the definition of model-conservative $\mathcal{EL}_\bot$-rewritings and the second condition follows from Lemma 4.

Conversely, we have to show that for every model $\mathcal{I}$ of $\mathcal{T}$ there exists a model $\mathcal{J}$ of $\mathcal{T}'$ such that $\mathcal{J} =_{\Sigma} \mathcal{I}$. Assume $\mathcal{I}$ is given. Take $\mathcal{A}' =_{\mathsf{sig}(\mathcal{T})} \mathcal{A}_{\mathcal{T}}$ satisfying the properties of the lemma.

We denote by $\mathcal{I}_d^{\leq m}$ the subinterpretation on $\mathcal{I}$ consisting of all points of distance $\leq m$ from d.

For $d \in \Delta^{\mathcal{I}}$ we define $f_i(d) = \rho_{\mathcal{A}}$ for the unique $\mathcal{A} \in \mathsf{AT}_{n-i}$ such that $(\mathcal{I}_d^{\leq n-i}, d) \sim_{n-i,\mathcal{T}} (\mathcal{A}, \rho_{\mathcal{A}})$.

Now we define $\mathcal{J}$ in the same way as $\mathcal{I}$ except that for $0 \leq i \leq n$ and $M_i \in \Gamma_i$:

$$M_i^{\mathcal{J}} = \{d \in \Delta^{\mathcal{I}} \mid M_i(f_i(d)) \in \mathcal{A}_{\mathcal{T}}\}.$$

We show that $\mathcal{J}$ is a model of $\mathcal{T}'$. We consider the different types of inclusions in $\mathcal{T}'$:

(a) All level i atom inclusions inclusions in $\mathcal{T}'$ are true in any d in $\mathcal{J}$ since they are true in $\rho_{\mathcal{A}}$ for $\mathcal{A} \in \mathsf{AT}_{n-i}$.

(b) Level i left-atom inclusions come in four different forms:

(1) $A \sqsubseteq \exists r.B$ with $A, B \in \Sigma$. This case follows from the condition that $\mathcal{I}$ is a model of $\mathcal{T}$.

(2) $A_i \sqsubseteq \exists r.B$ with $B \in \Sigma$ and $A_i \in \Gamma_i$: assume $d \in A_i^{\mathcal{J}}$. Then $A_i(f_i(d)) \in \mathcal{A}_{\mathcal{T}}$, where $f_i(d) = \rho_{\mathcal{A}}$ for the unique $\mathcal{A} \in \mathsf{AT}_{n-i}$ such that $(\mathcal{I}_d^{\leq n-i}, d) \sim_{n-i,\mathcal{T}} (\mathcal{A}, \rho_{\mathcal{A}})$. We have $\exists r.B(\rho_{\mathcal{A}}) \in \mathcal{A}'$. Thus $\exists r.B(\rho_{\mathcal{A}}) \in \mathcal{A}_{\mathcal{T}}$ and so $d \in (\exists r.B)^{\mathcal{I}}$, as required.

(3) $A_i \sqsubseteq \exists r.B_{i+1}$ with $A_i \in \Gamma_i$ and $B_{i+1} \in \Gamma_{i+1}$: assume $d \in A_i^{\mathcal{J}}$. Then $A_i(f_i(d)) \in \mathcal{A}_{\mathcal{T}}$, where $f_i(d) = \rho_{\mathcal{A}}$ for the unique $\mathcal{A} \in \mathsf{AT}_{n-i}$ such that $(\mathcal{I}_d^{\leq n-i}, d) \sim_{n-i,\mathcal{T}} (\mathcal{A}, \rho_{\mathcal{A}})$. We have $\exists r.B_{i+1}(\rho_{\mathcal{A}}) \in \mathcal{A}$. Take $\rho_{\mathcal{A}'}$ with $r(\rho_{\mathcal{A}}, \rho_{\mathcal{A}'}) \in \mathcal{A}_{\mathcal{T}}$ such that $B_{i+1}(\rho_{\mathcal{A}'}) \in \mathcal{A}'$. There exists d' with $(d, d') \in r^{\mathcal{I}}$ such that $(\mathcal{I}_{d'}^{\leq n-(i+1)}, d) \sim_{n-(i+1),\mathcal{T}} (\mathcal{A}', \rho_{\mathcal{A}'})$ Thus $d' \in B_{i+1}^{\mathcal{J}}$, as required.

(4) $A \sqsubseteq \exists r.B_{i+1}$ with $A \in \Sigma$ and $B_{i+1} \in \Gamma_{i+1}$ is considered analogously.

(c) Level i right-atom inclusions come in four different forms:

(1) $\exists r.A \sqsubseteq B$ with $A, B \in \Sigma$. This case follows from the condition that $\mathcal{I}$ is a model of $\mathcal{T}$.

(2) $\exists r.A_{i+1} \sqsubseteq B$ with $B \in \Sigma$ and $A_{i+1} \in \Gamma_{i+1}$: assume $d \in (\exists r.A_{i+1})^{\mathcal{J}}$. There exists d' with $(d, d') \in r^{\mathcal{I}}$ such that $d' \in A_{i+1}^{\mathcal{J}}$. Then $A_{i+1}(f_{i+1}(d')) \in \mathcal{A}_{\mathcal{T}}$, where $f_{i+1}(d') = \rho_{\mathcal{A}'}$ for the unique $\mathcal{A}' \in \mathsf{AT}_{n-(i+1)}$ such that $(\mathcal{I}_{d'}^{\leq n-(i+1)}, d') \sim_{n-(i+1),\mathcal{T}} (\mathcal{A}', \rho_{\mathcal{A}'})$. Let $f_i(d)) \in \mathcal{A}_{\mathcal{T}}$, where $f_i(d) = \rho_{\mathcal{A}}$ for the unique $\mathcal{A} \in \mathsf{AT}_{n-i}$ such that $(\mathcal{I}_d^{\leq n-i}, d) \sim_{n-i,\mathcal{T}} (\mathcal{A}, \rho_{\mathcal{A}})$. We have $r(\rho_{\mathcal{A}}, \rho_{\mathcal{A}'}) \in \mathcal{A}_{\mathcal{T}}$ and so $B(\rho_{\mathcal{A}}) \in \mathcal{A}_{\mathcal{T}}$. But then $d \in B^{\mathcal{I}}$, as required.

(3) $\exists r.A_{i+1} \sqsubseteq B_i$ with $B_i \in \Gamma_i$ and $A_{i+1} \in \Gamma_{i+1}$: this case is considered analogously.

(4) $\exists r.A \sqsubseteq B_i$ with $A \in \Sigma$ and $B_i \in \Gamma_i$: this case is considered analogously.

❏

We prove preservation under products of $\mathcal{ALC}$-conservatively $\mathcal{EL}_\bot$-rewritable $\mathcal{ALC}$-TBoxes. We require the following characterization of conservative extensions from [19].

Theorem 17. *An $\mathcal{ALC}$ TBox $\mathcal{T}'$ is a $\mathcal{ALC}$-conservative extension of an $\mathcal{ALC}$ TBox if the following are equivalent for all interpretations $\mathcal{I}$:*

- *$\mathcal{I}$ is a model of $\mathcal{T}$;*
- *$\mathcal{I}$ is globally Σ-bisimilar to a model of $\mathcal{T}'$, for $\Sigma = \mathsf{sig}(\mathcal{T})$.*

Now we observe:

Lemma 7. *Let $\mathcal{I}_i$ and $\mathcal{J}_i$, $i \in I$, be families of interpretations such that $\mathcal{I}_i$ is globally Σ-bisimilar to $\mathcal{J}_i$. Then $\prod_{i\in I} \mathcal{I}_i$ is globally Σ-bisimilar to $\prod_{i\in I} \mathcal{J}_i$.*

Proof. Let S_i be global Σ-bisimilations between $\mathcal{I}_i$ and $\mathcal{J}_i$ for $i \in I$. Define a relation S between $\Delta^{\prod_{i\in I} \mathcal{I}_i}$ and $\Delta^{\prod_{i\in I} \mathcal{J}_i}$ by set

$$S = \{((a_i)_{i\in I}, (b_i)_{i\in I}) \mid \forall i \in I : (a_i, b_i) \in S_i\}$$

It is straightforward to show that S is a global Σ-bisimulation. ❏

Theorem 9 $\mathcal{ALC}$ TBoxes that are $\mathcal{ALC}$-conservatively $\mathcal{EL}_\bot$-rewritable are preserved under products.

Proof. Assume that $\mathcal{T}'$ is a $\mathcal{ALC}$-conservative $\mathcal{EL}_\bot$-rewriting of $\mathcal{T}$ and that $\mathcal{I}_i$, $i \in I$, are models of $\mathcal{T}$. By Theorem 17, there exist models $\mathcal{J}_i$ of $\mathcal{T}'$ such that $\mathcal{J}_i$ is globally Σ-bisimilar to $\mathcal{I}_i$, for $\Sigma = \mathsf{sig}(\mathcal{T})$. By preservation of $\mathcal{EL}_\bot$ TBoxes under products, we obtain that $\prod_{i\in I} \mathcal{J}_i$ is a model of $\mathcal{T}'$. By Lemma 5 we have that $\prod_{i\in I} \mathcal{J}_i$ is globally Σ-bisimilar to $\prod_{i\in I} \mathcal{I}_i$. Thus, by Theorem 17, $\prod_{i\in I} \mathcal{I}_i$ is a model of $\mathcal{T}$, as required. ❏

For $i \geq 0$ we define the notion of Σ-i-bisimilarity on interpretations similarly to $(\mathcal{T}, i)$-bisimilarity on ABoxes.

- $(\mathcal{I}, d) \sim_0 (\mathcal{J}, e)$ whenever $d \in A^{\mathcal{I}}$ if, and only if, $e \in A^{\mathcal{J}}$ for every concept name $A \in \Sigma$;
- $(\mathcal{I}, d) \sim_{i+1} (\mathcal{J}, e)$ if $(\mathcal{I}, e) \sim_0 (\mathcal{J}, e)$ and, for every $r \in \Sigma$, if $(d, d') \in r^{\mathcal{I}}$ then there is e' such that $(e, e') \in r^{\mathcal{J}}$ with $(\mathcal{I}, d') \sim_i (\mathcal{J}, e')$, and vice versa.

It can be readily seen for every $\mathcal{ALC}$ Σ-concept C of depth i that if $(\mathcal{I}, d)$ is i-bisimilar to $(\mathcal{J}, e)$ then $d \in C^{\mathcal{I}}$ if, and only if, $e \in C^{\mathcal{J}}$.

Theorem 10 Every $\mathcal{ALC}$-conservatively $\mathcal{EL}_\bot$-rewritable $\mathcal{ALC}$ TBox of depth 1 is preserved under global $\subseteq_1$-simulations.

Proof. Let $\mathcal{T}$ be a $\mathcal{ALC}$-conservatively $\mathcal{EL}_\bot$-rewritable $\mathcal{ALC}$ TBox of depth 1. Suppose for the proof by contradiction that there exist interpretations $\mathcal{I}$ and $\mathcal{J}$ such that $\mathcal{I} \models \mathcal{T}$, $\mathcal{I}$ is globally $\subseteq_1$-simulated by $\mathcal{J}$, but $\mathcal{J} \not\models \mathcal{T}$. Let $\mathcal{T}'$ be an $\mathcal{EL}$ rewriting of $\mathcal{T}$. By Theorem 1, we may assume that $\mathcal{T}'$ does not use additional role names. Let $\Sigma = \mathsf{sig}(\mathcal{T})$ and $\Sigma' = \mathsf{sig}(\mathcal{T}')$. As $\mathcal{T}'$ is an $\mathcal{EL}$ rewriting of $\mathcal{T}$, there exists an interpretation $\mathcal{I}'$ globally Σ-bisimilar to $\mathcal{I}$ such that $\mathcal{I}' \models \mathcal{T}'$.

Our goal is to construct an interpretation $\mathcal{J}'$ such that

- $\mathcal{J}'$ is globally Σ'-equisimilar to $\mathcal{I}'$; and
- For every $e \in \Delta^{\mathcal{J}}$ there exists $e' \in \Delta^{\mathcal{J}'}$ such that $(\mathcal{J}, e)$ is Σ-1-bisimilar to $(\mathcal{J}', e')$.

If such an interpretatoin $\mathcal{J}'$ exists, as $\mathcal{J} \not\models \mathcal{T}$ for some $C \sqsubseteq D \in \mathcal{T}$, there is $e_* \in \Delta^{\mathcal{J}}$ such that $e_* \notin (\neg C \sqcup D)^{\mathcal{J}}$. Let $(\mathcal{J}', e'_*)$ be Σ-1-bisimilar to $(\mathcal{J}, e_*)$. Then on the one hand $e'_* \notin (\neg C \sqcup D)^{\mathcal{J}'}$ as 1-bisimulation preserves satisfiability of $\mathcal{ALC}$ concepts of depth 1, and on the other $e'_* \in (\neg C \sqcup D)^{\mathcal{J}'}$ as global equisimulations preserve $\mathcal{EL}$ TBoxes and $\mathcal{T}' \models \mathcal{T}$. A contradiction.

We proceed constructing $\mathcal{J}'$ as follows. Let e be an arbitrary element of $\Delta^{\mathcal{J}}$ and $d \in \Delta^{\mathcal{I}}$ be such that $(\mathcal{I}, d)$ is $\subseteq_1$-simulated by $(\mathcal{J}, e)$. Let $\mathcal{S}_e$ be the set of those $e_1 \in \Delta^{\mathcal{J}}$ that satisfy condition (ii) but not condition (iii) of the definition of a $\subseteq_1$ simulation, that is, $(e, e_1) \in s^{\mathcal{J}}$ for some $s \in \mathsf{N_R} \cap \Sigma$ is such that for every $d_1 \in \Delta^{\mathcal{I}}$ with $(\mathcal{I}, d_1)$ being $\subseteq_1$ simulated by $(\mathcal{J}, e_1)$ if $(d, d_1) \in s^{\mathcal{I}}$ then for some concept name $A \in \Sigma$ we have $d_1 \in A^{\mathcal{I}}$ but $e_1 \notin A^{\mathcal{J}}$. For every $e_1 \in \mathcal{S}_e$ let $\mathcal{I}_{(\mathcal{T}', e_1)}$ be the canonical $\mathcal{EL}$-model of $\left(\bigsqcap_{A \in \Sigma, e_1 \in A^{\mathcal{J}}} A\right)$ and $\mathcal{T}'$ rooted at d_{e_1}. Notice that, by item (ii) of the definition of a $\subseteq_1$ simulation, $\left(\mathcal{T}, \bigsqcap_{A \in \Sigma, e_1 \in A^{\mathcal{J}}} A\right)$ is consistent so such $\mathcal{I}_{(\mathcal{T}', e_1)}$ exists. We assume that the domains of $\mathcal{I}'$ and of all $\mathcal{I}_{(\mathcal{T}', e_1)}$, for $e_1 \in \mathcal{S}_e$, are pairwise disjoint. We define $\mathcal{J}'$ as follows:

- $\Delta^{\mathcal{J}'} = \Delta^{\mathcal{I}'} \cup \bigcup_{e \in \Delta^{\mathcal{J}}} \bigcup_{e_1 \in \mathcal{S}_e} \Delta^{\mathcal{I}_{(\mathcal{T}', e_1)}}$;
- for every $A \in \mathsf{N_C}$, $A^{\mathcal{J}'} = A^{\mathcal{I}'} \cup \bigcup_{e \in \Delta^{\mathcal{J}}} \bigcup_{e_1 \in \mathcal{S}_e} A^{\mathcal{I}_{(\mathcal{T}', e_1)}}$;
- for every $r \in \mathsf{N_R}$, $r^{\mathcal{J}'} = r^{\mathcal{I}'} \cup \bigcup_{e \in \Delta^{\mathcal{J}}} \bigcup_{e_1 \in \mathcal{S}_e} r^{\mathcal{I}_{(\mathcal{T}', e_1)}} \cup \{(d', d_{e_1}) \mid e_1 \in \mathcal{S}_e, (e, e_1) \in r^{\mathcal{J}}\}$,

where $d' \in \Delta^{\mathcal{I}'}$ be such that $(\mathcal{I}, d)$ is Σ-bisimilar to $(\mathcal{I}', d')$.

Since $(\mathcal{I}, d)$ is $\subseteq_1$ simulated by $(\mathcal{J}, e)$ and $(\mathcal{I}, d)$ is Σ-bisimilar to $(\mathcal{I}', d')$, for every $e_1 \in \mathcal{S}_e$ and $s \in \mathsf{N_R}$ such that $(e, e_1) \in s^{\mathcal{J}}$, there exists $d'_1 \in \Delta^{\mathcal{I}'}$ such that $(d', d'_1) \in s^{\mathcal{I}'}$ and if $e_1 \in A^{\mathcal{J}}$ then $d'_1 \in A^{\mathcal{I}'}$, for a concept name $A \in \Sigma$. As $\mathcal{I}_{(\mathcal{T}', e_1)}$ is the canonical $\mathcal{EL}$-model of $\left(\bigsqcap_{A \in \Sigma, e_1 \in A^{\mathcal{J}}} A\right)$ and $\mathcal{T}'$ the interpretation $(\mathcal{I}_{\mathcal{T}', e_1}, d_{e_1})$ is Σ'-simulated by $(\mathcal{I}', d'_1)$. But then $(\mathcal{J}', d')$ is Σ'-simulated by $(\mathcal{I}', d')$. It can be easily seen that this property holds for every $e' \in \Delta^{\mathcal{J}'}$, that is there exists $d' \in \Delta^{\mathcal{I}'}$ such that $(\mathcal{J}', e')$ is simulated by $(\mathcal{I}', d')$. Conversely, as $\mathcal{I}' \subseteq \mathcal{J}'$, obviously, $(\mathcal{I}', d')$ is Σ'-simulated by $(\mathcal{J}', d')$ for every $d' \in \Delta^{\mathcal{I}'}$. Thus $\mathcal{J}'$ is globally Σ'-equisimilar to $\mathcal{I}'$. Notice also that $(\mathcal{J}, e)$ is Σ-1-bisimilar to $(\mathcal{J}', e')$ by construction. So we obtain a contradiction as outlined above. ❏