# A Note on Algebraic Closure and Closure under Constraints

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**Abstract** A-closure is the equivalent of path consistency for qualitative spatio-temporal calculi with weak composition. We revisit existing attempts to characterize the question whether a-closure is a complete method for deciding consistency of CSPs over such calculi. Renz and Ligozat's characterization via closure under constraints has been refuted by Westphal, Hué and Wölfl. However, for many commonly used calculi, completeness of a-closure and closure under constraints coincide. We show that it is unlikely to obtain an effective procedure to decide closure under constraints by an enumeration process – and thus completeness of a-closure if the characterization were true. We further provide a sufficient condition for closure under constraints via properties of the automorphism group of the set of relations in a calculus.

## 1 Introduction

Qualitative spatio-temporal reasoning (QSTR) is concerned with representing spatial and/or temporal knowledge and drawing inferences. The decision to use a *qualitative* representation allows to model relevant relations between spatial or temporal objects, abstracting away from their concrete location in space or time, or from relations irrelevant for the application at hand. Qualitative representations can also capture incomplete data, which are commonly derived from sensor observations or are due to obstructions.

Constraint-based reasoning is a well-understood and frequently used approach to QSTR. It relies on the central notion of a qualitative calculus, which provides symbols for the relevant basic relations and for the standard operations of converse and composition. Using a calculus, it is possible to express the known relations between a set of objects as a set of constraints – called a (constraint) network – and to apply dedicated methods to draw inferences. Inferences of interest include the question whether a given network describes a valid constellation of objects – i.e., whether it is consistent – and the computation of such a constellation (generating a model). Established techniques from the area of constraint satisfaction problems (CSP) have been transferred to qualitative constraint-based reasoning; however, due to the infinite nature of spatio-temporal domains, not all CSP techniques are transferable, and dedicated QSTR methods are applied.

One convenient and frequently used method for deciding consistency is the enforcement of algebraic closure, for short: a-closure. It is the equivalent of the well-established path-consistency algorithm for CSPs, adapted to *weak composition* [4], a commonly adopted weakening of the requirement that the set of relevant relations be closed under composition. The decision procedure a-closure runs in polynomial time and is thus considered efficient. It is sound in the sense that it outputs "consistent" for every consistent network. However, it is in general not complete: depending on the calculus, there are inconsistent networks where a-closure outputs "consistent". Still, for a number of calculi a-closure is known to be complete; these include "classical" calculi such as the Region Connection Calculus (RCC-5, -8) [9,6,10] and Allen's Interval Algebra [1,13], as well as more recently developed calculi such as the connectivity variant of the dipole calculus [14]. For a number of other calculi, a-closure is known to be incomplete. A complete classification of calculi for which a-closure is complete is still open.

In this paper, we are concerned with the search for a characterization that allows to decide whether a-closure is complete for a given calculus. An important step towards such a characterization was made by Renz and Ligozat's attempt in [11] to characterize completeness of a-closure by a property of a calculus called "closure under constraints", which is the inability to refine a relation of the calculus to non-overlapping subrelations via two networks. The most important gain from this characterization is a sufficient condition for when a-closure is *not* complete: it is enough to find a pair of networks that witnesses a violation of closure under constraints. If there were a known bound on such networks, it would yield a decision procedure for closure under constraints.

Renz and Ligozat's equivalence was refuted recently by Westphal et al. in [15], giving examples of calculi violating each of the two implications. These calculi are based on finite domains of size 2 and 3; thus they do not refute Renz and Ligozat's observation that, for existing calculi, completeness of a-closure seems to correlate with closure under constraints [11]. It is therefore possible that the equivalence can be "rescued" by imposing further assumptions that are typically satisfied by (most) existing calculi.

We pursue the question whether the claimed equivalence – provided that it holds under some additional assumptions – provides an effective criterion for deciding whether a-closure is complete for a given calculus. More precisely, we study possible ways to establish (or refute) closure under constraints. The most obvious one is to enumerate all refinements of all relations, and check whether two of them are non-overlapping, which yields a semi-decision procedure. It is tempting to try and obtain a decision procedure by bounding the size of networks that need to be compared, e.g., by some function of the number of relations in the calculus. Indeed, for some calculi it is possible to obtain non-overlapping refinements from very small sets of constraints. Our main contribution is to show that there is no bound that depends only on the number of relations.

Another possible way to establish closure under constraints is to show that there is no proper refinement at all. We provide a sufficient condition that involves a property of the automorphism group of the set of relations in the calculus, called 2-transitivity. Intuitively, this property means that different instances of a relation are indistinguishable.

The work reported here is work in progress that attempts to explore the possibilities for characterizing completeness of a-closure. Our results show that further attempts to rescue the correspondence with closure under constraints are unlikely to yield an *effective* characterization. We find this insight worth sharing with the QSTR community and hope to ignite the search for different ways of characterizing completeness of a-closure.

## 2 Preliminaries

**Partition schemes.** Commonly a qualitative spatio-temporal calculus is regarded as a set equipped with relations meeting some minimal requirements, and with natural operations on them. These requirements are captured by the notion of a partition scheme [7].

Let $U$ be a non-empty set called a *universe* (or domain). The elements of the universe are called *entities* and we think of spatial or temporal objects e.g. represented by points, intervals or regions in $\mathbb{R}^n$. The set of relations is required to be *JEPD* (jointly exhaustive and pairwise disjoint), that is, for every two elements $x, y \in U$ there is exactly one relation $R$ s.t. $xRy$. So let $\mathcal{B}$ be a finite partition of $U \times U$, that is, $U \times U = \biguplus_{R \in \mathcal{B}} R$. The elements of $\mathcal{B}$ are called *atomic relations*.

The notion of a partition scheme is now derived by imposing two further demands on partitions, involving the *identity relation* $\mathrm{id} = \{(x, x) : x \in U\} \in \mathcal{B}$ and the *converse* of a relation $R$, defined by $R^{\smile} = \{(y, x) : (x, y) \in R\}$.

**Definition 1.** *A* partition scheme *is a pair $(U, \mathcal{B})$ where $\mathcal{B}$ is a partition of $U \times U$, $\mathcal{B}$ contains id and $R \in \mathcal{B}$ implies $R^{\smile} \in \mathcal{B}$.*

As a simple example, consider the point calculus (PC) [12], which is capable of representing the relative location of points on a line. Its underlying partition scheme has the domain $U = \mathbb{R}$ and the atomic relations $\mathcal{B} = \{<, =, >\}$. It can thus express the relations $2^{\mathcal{B}} = \{\emptyset, <, =, >, \leq, \geq, \neq, \lesseqgtr\}$, where $=$ is id, and $\lesseqgtr$ is the universal relation $\mathbb{R} \times \mathbb{R}$.

Typical spatio-temporal calculi, including PC, have infinite domains and *serial* relations; a relation $R$ is serial if, for every $x \in U$, there is some $y \in U$ with $(x, y) \in R$. We call a calculus serial if all its relations are serial. However, the above definition does not rule out finite domains and/or non-serial relations.

**Constraint networks.** Constellations of spatio-temporal objects are given by constraint networks, which are defined as follows.

**Definition 2.** *Let $K = (U, \mathcal{B})$ be a partition scheme. A $K$-*(constraint-)network *is a pair $\Theta = (V, \nu)$ consisting of a finite set $V$, called variables, and a map $\nu : V \times V \to 2^{\mathcal{B}}$ such that $\nu(v_1, v_2) = \nu(v_2, v_1)^{\smile}$ and $\nu(v_1, v_1) = id$ for all $v_1, v_2 \in V$.*

*$\Theta$ is* atomic *if $\nu(v_1, v_2) \in \mathcal{B}$ for all $v_1, v_2 \in V$; the* size *of $\Theta$ is defined by $|\Theta| := |V|$.*

Imagine the range of the variables to be the universe $U$, and for $v_1, v_2 \in V$ the mapping $\nu(v_1, v_2)$ is a constraint between $v_1$ and $v_2$.

A central decision problem in QSTR asks whether a given set of constraints is consistent. This special case of a constraint satisfaction problem is defined as follows.

**Definition 3.** *Let $K = (U, \mathcal{B})$ be a partition scheme and $\Theta = (V, \nu)$ a $K$-network.*

1. *A map $\phi : V \to U$ is a* solution *of $\Theta$ if $(\phi(x), \phi(y)) \in \nu(x, y)$ for all $x, y \in V$.*
2. *$\Theta$ is* consistent *if it has a solution.*

A constraint network $(V, \nu)$ can be visualized as a complete directed graph, whose nodes are from $V$ and whose edges are labeled by the corresponding $\nu(v_1, v_2)$. We can omit loops at a node labeled with *id* and it is sufficient to draw only one of the two directed edged between a pair of nodes.

We can furthermore omit edges between nodes $v_1$ and $v_2$ to represent the universal relation, i.e., if $\nu(v_1, v_2) = \mathcal{B}$. This goes along with the idea of the edges being constraints: if there is no edge between to variables, there is no constraint between them.

**Weak composition.** Constraints can be propagated using the *composition* operation $\circ$, defined by $R \circ S = \{(x, y) \mid (x, z) \in R \text{ and } (z, y) \in S \text{ for some } z \in U\}$. For an arbitrary partition scheme $(U, \mathcal{B})$, the set $2^{\mathcal{B}}$ of all possible relations is not necessarily closed under composition. Therefore *weak composition* $\diamond$ has been established as the best approximation that is expressible in the "language" of $(U, \mathcal{B})$ [7]. Weak composition is defined for atomic relations $r, s$ by

$$r \diamond s = \bigcup \{t \in \mathcal{B} \mid (r \circ s) \cap t \neq \emptyset\} \tag{1}$$

and transferred to arbitrary relations $R, S$ via $R \diamond S = \bigcup_{r \in R} \bigcup_{s \in S} r \diamond s$. The $\mathcal{B}$ in (1) indicates that weak composition is relative to the partition scheme $(U, \mathcal{B})$.

A calculus is a partition scheme with weak composition. Weak composition is usually provided in the form of a composition table with $|\mathcal{B}|^2$ entries storing the composition results $r \diamond s$ for all $r, s \in \mathcal{B}$. For our purposes, the particular representation of $\diamond$ is not relevant, and we abstractly define:

**Definition 4.** *A* (qualitative spatio-temporal) calculus *is a triple* $(U, \mathcal{B}, \diamond)$ *where* $(U, \mathcal{B})$ *is a partition scheme and* $\diamond$ *denotes weak composition relative to* $(U, \mathcal{B})$.

**Algebraic Closure.** Let $K = (U, \mathcal{B}, \diamond)$ be a fixed calculus. We consider the *consistency problem* for $K$, which receives as an input a $K$-network $\Theta = (V, \nu)$ and answers the question whether $\Theta$ is consistent. This problem is a constraint satisfaction problem (CSP). However, due to the use of weak composition, standard CSP approaches cannot be applied directly. Instead, for various calculi, dedicated algorithms deciding consistency are known, but these usually cannot (easily) be generalized to other calculi. As a notable exception, the a-closure algorithm (AC) can be applied to every calculus. AC is the result of transferring standard path-consistency algorithms for CSPs [3] to the use of weak composition; its runtime is $O(|V|^3)$ [3].

AC is given in pseudocode in Algorithm 1. For a given network $\Theta = (V, \nu)$, AC checks for every triangle $x, y, z$ whether there are unnecessary atomic relations contained in $\nu(x, z)$, i.e. relations that are not included in $\nu(x, y) \diamond \nu(y, z)$ and thus cannot occur in a solution of $\Theta$. Those will be removed until a fixpoint is reached. If the resulting network contains an edge labelled by $\emptyset$, the algorithm answers *inconsistent*, otherwise *consistent*.

**Algorithm 1:** Algebraic closure

**Input**: Constraint network $\Theta = (V, \nu)$

```
repeat
    foreach x, y, z ∈ V do
        ν(x, z) ← ν(x, z) ∩ (ν(x, y) ⋄ ν(y, z))
until ν(x, z) ⊆ ν(x, y) ⋄ ν(y, z) for all x, y, z ∈ V
if ν(x, y) = ∅ for some x, y ∈ V then return inconsistent else return consistent
```

If AC returns *inconsistent*, then the input network is actually inconsistent. However, the opposite is not generally true, i.e., AC is not necessarily complete. For some calculi, AC is known to be complete; for other calculi there are networks that witness incompleteness. Given the general interest in efficient complete reasoning procedures, attempts have been made to characterize the calculi where AC is complete [11,15]. It has to be noted that choice of relations that are allowed on the labels of $\Theta$ is relevant. For example, AC is complete for atomic networks in Allen's Interval Relations, but for arbitrary networks the problem is NP-hard [13], and the proof yields a minimal example for an a-closed inconsistent network. Current attempts to characterize completeness of a-closure largely consider the restriction to networks with atomic relations plus the universal relation [11,15]. We adopt this restriction.

**Subatomic refinements.** These underlie the notion of closure under constraints, which has been linked with completeness of a-closure [11], and are defined as follows.

**Definition 5.** *Let $K = (U, \mathcal{B}, \diamond)$ be a calculus and $R \in \mathcal{B}$ a atomic relation. A subset $R' \subseteq R$ is a* subatomic refinement (SAR) *of $R$ (or of $K$), if there exists a constraint network $\Theta = (V, \nu)$ and a constraint $\nu(x,y) = \{R\}$, s.t. a map $\phi : \{x,y\} \to U$ can be extended to a solution of $\Theta$ if and only if $\phi(x) R' \phi(y)$.*

Unlike in [11] we don't define SARs via atomic, but via arbitrary networks. It will become apparent further below that this doesn't make a difference for the resulting definition of closure under constraints.

Every atomic relation $R$ has at least the SARs $R$ and $\emptyset$. We call SARs other than $R$ and $\emptyset$ *proper*. If a calculus has proper SARs, then its actual expressiveness is stronger than its atomic relations pretend. It is easily seen that SARs are closed under finite intersections: Let $R_1 \subseteq R$ and $R_2 \subseteq R$ be SARs, then $R_1 \cap R_2$ is generated by the union of the two generating networks for $R_1$ and $R_2$, where the refined edges are merged and all other nodes are kept distinct.

**Definition 6.** *A calculus $K$ is called* closed under constraints (CUC) *if, for every atomic relation $R$, every two proper SARs $R_1 \subseteq R$ and $R_2 \subseteq R$ have a non-empty intersection.*

The properties of closure under constraints and completeness of AC often appear together. Their equivalence, as claimed in [11, Theorem 1], has been refuted, see [15, Proposition 2]. However, the given counterexamples are unusual calculi, they lack properties like seriality or infiniteness of the domain, which the more common calculi retain. So for many calculi the equivalence still holds and the relation between these two properties should be examined. Therefore, we define the CUC-size of a calculus.

**Definition 7.** *Let $K = (U, \mathcal{B}, \diamond)$ be a calculus. If $K$ is closed under constraints, we set the CUC-size $c(K) = \infty$. Otherwise we define:*

$$\begin{aligned} c(K,R) &= \min\{\max\{|\Theta_1|, |\Theta_2|\} : \Theta_1 \text{ and } \Theta_2 \text{ generate two disjoint proper SARs of } R\} \\ c(K) &= \min\{c(K,R) : R \in \mathcal{B}\} \end{aligned}$$

In other words: $c(K)$ is the smallest number $n$ s.t. there exist two networks, each of size at most $n$, which generate two disjoint proper SARs of a relation in $\mathcal{B}$. Example 3 of [11] shows that, for the interval-duration calculus $\mathcal{INDU}$ [8], we have $c(\mathcal{INDU}) = 3$, i.e., two triangles suffice to generate disjoint proper substomic refinements. Such low CUC-sizes are common among existing calculi.

We will now see that CUC-size and closure under constraints would be untouched if we had defined SARs via atomic instead of arbitrary networks.

**Lemma 8.** *Let $K$ be a calculus, that is not closed under constraints. Then there are* atomic *networks $\Theta_1$ and $\Theta_2$, that refine an atomic relation $R$ into two disjoint proper SARs and* $\max(|\Theta_1|, |\Theta_2|) = c(K)$.

**Proof.** By definition there exist $\Theta'$ and $\Theta''$, that refine $R$ into two disjoint proper SARs $R'$ and $R''$, while the larger network has size $c(K)$. Especially, $\Theta'$ and $\Theta''$ are consistent.

Then there are consistent, atomic networks $\Theta_1$ and $\Theta_2$ that result from $\Theta'$ and $\Theta''$ by choosing an appropriate atomic relation from every labeled edge, guided by the solutions leading to the refinements. Now $\Theta_1$ and $\Theta_2$ generate SARs $R_1 \subseteq R'$ and $R_2 \subseteq R''$. Since $R' \cap R'' = \emptyset$ we also have $R_1 \cap R_2 = \emptyset$. So $\Theta_1$ and $\Theta_2$ are the desired networks. ❏

The lemma allows us to restrict ourselves to atomic networks when investigating closure under constraints of a calculus.

## 3 The Difficulty of Verifying Closure under Constraints

If there is a causal relation between closure under constraints and completeness of AC, it would be useful to have a way to check a calculus for being closed under constraints. One might hope that it is enough to enumerate only finitely many pairs of networks, e.g., up to a size depending on the number of atomic relations, and check whether they refine some relation into two disjoint proper SARs. Unfortunately, this hope is easy to dash by a counterexample involving variants of the point calculus restricted to finite domains: it can be shown that the relations $\{<, =, >\}$ over the domain $\{1, \ldots, 3n + 2\}$ cannot be refined via networks of size $n$ but via networks of size $3n + 2$ – i.e., this calculus has a finite CUC-size $\geq n$. We do not present the detailed proof here because it is arguable whether this artificial example makes a statement about "real-world" calculi. Featuring non-serial relations over finite domains, the calculi in our example violate two properties shared by the majority of existing calculi (and even a third, which generalizes seriality and is addressed in the following section). We are therefore concerned, in this section, with establishing a stronger negative statement that is supported by more realistic counterexamples. We proceed in two steps: we first argue about serial calculi with finite domains (9), and then consider a general construction that turns these calculi into serial calculi with infinite domains and an arbitrary finite number of relations (11).

**Theorem 9.** *For every $n \in \mathbb{N}$ there exists a serial calculus $K$ with 3 atomic relations and $n \leq c(K) < \infty$.*

**Proof.** We consider the calculus $K_k = (U_k, \mathcal{B}_k, \diamond)$ with $U_k = \{0, 1, \ldots, 2k\}$. Besides the identity $\mathcal{B}_k$ contains the relation $R_< = \{(a, b) \mid \exists 0 < c \leq k \text{ with } b - a \equiv c \pmod{2k + 1}\}$ and its converse $R_> = \{(a, b) \mid \exists k + 1 \leq c \leq 2k \text{ with } b - a \equiv c \pmod{2k + 1}\}$. Imagine the points of the universe arranged in a circle, clockwise increasing. Then $(a, b) \in R_<$ holds if and only if the path from $a$ to $b$ in clockwise direction is at most half of the circle. Of course, $K_k$ is always serial.

We notice that every SAR of $K_k$ is a union of relations $R_j = \{(a, b) \mid b - a \equiv j \pmod{2k + 1}\}$: Let $(a, b)$ be contained in an SAR $R' \subseteq R$. There exists $j \in \{0, \ldots, 2k\}$ s.t. $a - b \equiv j \pmod{2k + 1}$. Then $R_j \subseteq R$, since from the solution that instantiates the refining edge with $(a, b)$ we can increment all variables by the same number $\pmod{2k+1}$ and obtain solutions where the same edge is instantiated with all pairs in $R_j$. We have $R_= = R_0$, $R_< = \bigcup_{j=1}^{k} R_j$ and $R_> = \bigcup_{j=k+1}^{2k} R_j$. In particular, $R_=$ doesn't have any proper SARs and we can restrict ourselves to constraint networks that only use $R_<$ and $R_>$.

We consider the canonical network $\Theta = (V, \nu)$ for $K_k$, whose variables are just the elements of $U_k$ and where $\nu(x, y) = \{R\}$, where $R$ the unique atomic relation for which $xRy$ holds. $\Theta$ generates all $R_j$ as SARs. Hence, $K_k$ is not closed under constraints for $k \geq 2$, since e.g. $R_1$ and $R_2$ are disjoint SARs of $R_<$.

Further we notice that a consistent atomic constraint network $\Theta = (V, \nu)$ determines a unique successor for every variable: We start with any variable $x_0 \in V$ and consider the set $N(x_0) = \{y \in V \mid \nu(x, y) = \{R_<\}\}$. (If this is empty, argue analoguosly with $R_>$.) Consider the sub-network of $\Theta$ restricted to $N(x_0)$, there exists a total order since any two points are related by exactly one of the relations $R_<, R_=, R_>$. (Transitivity follows from the fact that all these points are inside a semicircle and $\Theta$ is consistent und atomic, i.e. a-closed). We call the minimal element $x_1$ and inductively define $x_{i+1}$ as the minimum of $N(x_i)$, up to $x_{|V|-1}$.

Now we show that for every given $n \in \mathbb{N}$ in the calculus $K_k$ with $k = 2n^2$ the networks of size at most $n$ don't generate disjoint SARs. Let $\Theta = (V, \nu)$ be a consistent, atomic constraint network with $|\Theta| \leq n$ and let $xR_<y$ be an edge. We construct a solution $\phi$ with $\phi(x) = 0$ and $\phi(y) = n^2$, so the distance between $x$ and $y$ is a quarter of the circle. Then $(0, n^2)$ will be contained in every SAR of $R_<$.

For every $v \in V$ the relations to $x$ and $y$ determine in which quarter $v$ has to be put.

| Set of variables | We need to achieve |
|---|---|
| $V_1 = \{v \in V \mid vR_>x, vR_<y\}$ | $\phi(V_1) \subseteq [1, n^2 - 1]$ |
| $V_2 = \{v \in V \mid vR_>x, vR_>y\}$ | $\phi(V_2) \subseteq [n^2 + 1, 2n^2]$ |
| $V_3 = \{v \in V \mid vR_<x, vR_>y\}$ | $\phi(V_3) \subseteq [2n^2 + 1, 3n^2]$ |
| $V_4 = \{v \in V \mid vR_<x, vR_<y\}$ | $\phi(V_4) \subseteq [3n^2 + 1, 4n^2 + 1]$ |

Let $V_i = \{x_{i,1}, \ldots, x_{i,m_i}\}$, with the variables in increasing order. We set $\phi(x_{1,j}) = n \cdot j$ and $\phi(x_{2,j}) = n^2 + n \cdot j$. The variables in $V_3$ depend on $V_1$: For every $x_{3,j}$ the possible values are an interval of size $n - 1$ since the difference of any two variables in $V_1$ is at least $n$. Since $|V_3| \leq n - 2$, there is always such a solution, even if all $x_{3,j}$ fall into the same interval. In the same way we instantiate the variables of $V_4$, depending on $V_2$. ❑

To lift the statement of Theorem 9 to calculi with infinite domains, we use a construction that transforms any calculus into an infinite one, preserving seriality and the CUC-size. The construction works as follows: Let $K = (U, \mathcal{B}, \diamond)$ be a calculus. We define $K_{\mathbb{N}} := (U \times \mathbb{N}, \mathcal{B}', \diamond')$ with $\mathcal{B}'$ consisting of id, a relation $R'$ for every $R \in \mathcal{B} \setminus \{id\}$ with $(x, n)R'(y, m)$ if $xRy$, and an additional relation $\sim$ with $(x, n) \sim (y, m)$ if $x = y$ and $n \neq m$.

**Lemma 10.** *Let $K$ be a calculus. Then $c(K) = c(K_{\mathbb{N}})$.*

**Proof.** We show this by translating atomic $K$-networks into atomic $K_{\mathbb{N}}$ networks of the same size and vice versa, preserving disjointness of SARs.

We first show $c(K) \geq c(K_{\mathbb{N}})$. Let $\Theta = (V, \nu)$ be a $K$-network. We define the $K_{\mathbb{N}}$-network $\Theta' = (V, \nu')$ by $\nu'(x, y) = \{R'\}$, if $\nu(x, y) = \{R\}$ and $\nu'(x, y) = \{id\}$, if $\nu(x, y) = \{id\}$. If two $K$-networks $\Theta_1, \Theta_2$ generate two disjoint proper SARs $R_1, R_2$ of an atomic relation $R$, then either $R \neq$ id and $\Theta'_1, \Theta'_2$ generate the SARs $R'_i = \{((x, n), (y, m)) \mid (x, y) \in R_i \text{ and } n, m \in \mathbb{N}\}$, which are disjoint again, or $R =$ id and $\Theta'_1, \Theta'_2$ generate $R'_i = \{((x, n), (x, n)) \mid (x, x) \in R_i \text{ and } n \in \mathbb{N}\}$, which are disjoint too.

Next we show $c(K) \leq c(K_{\mathbb{N}})$. For every $K_{\mathbb{N}}$-network $\Theta = (V, \nu)$ we define the $K$-network $\Theta' = (V, \nu')$ by $\nu'(x, y) = \{R\}$ if $\nu(x, y) = \{R'\}$, and $\nu'(x, y) = \{id\}$ if $\nu(x, y) = \{id\}$ or $\nu(x, y) = \{\sim\}$. Then every solution of $\Theta$ gives a solution of $\Theta'$ by forgetting the second component. If two $K_{\mathbb{N}}$-networks $\Theta_1, \Theta_2$ generate disjoint proper SARs $R_1, R_2$ of an atomic relation $R \neq \sim$, then $\Theta'_1, \Theta'_2$ generate disjoint proper SARs too. The relation $\sim$ doesn't have proper SARs: in a consistent $K_{\mathbb{N}}$-network $\Theta = (V, \nu)$ with an edge

$\nu(x, y) = \{\sim\}$ and a solution $\phi$ any permutation of $\mathbb{N}$ applied to the second component of $\phi$ still yields a solution of $\Theta$. ❑

By iteratively applying Lemma 10 to the calculus from Theorem 9, we can conclude:

**Theorem 11.** *For every $n \in \mathbb{N}$ and every $k \geq 4$ there exists a serial calculus $K$ with an infinite domain, $k$ atomic relations, and $n \leq c(K) < \infty$.*

## 4 Sufficient Conditions for Closure under Constraints

In the previous chapter we gathered some evidence for the difficulty of the problem of deciding whether a given calculus is closed under constraints. However, some of the calculi that are closed under constraints don't have any proper SARs at all and for those there exists another approach to establish closure under constraints.

**Definition 12.** *A calculus is called* strongly closed under constraints*, if no atomic relation $R$ has other SARs than $\emptyset$ and $R$.*

Obviously, strong closure under constraints implies closure under constraints. The idea for establishing strong closure under constraints is to show that for every atomic relation $R$ the calculus cannot differentiate (in a certain manner) between two pairs of points $wRx$ and $yRz$. For this purpose we adapt the notion of 2-transitivity from algebraic group theory [2] to our calculi.

**Definition 13.** *Let $K = (U, \mathcal{B}, \diamond)$ be a calculus. A bijection $h : U \to U$ is called an* automorphism *of $K$ if for all $R \in \mathcal{B}$ and $x, y \in U$ we have $(x, y) \in R \Longrightarrow (h(x), h(y)) \in R$.*

Let $\mathrm{Aut}(K)$ the set of all automorphisms of $K$. It is easily seen that $\mathrm{Aut}(K)$ forms a group. Furthermore, this group acts on the universe in the following sense.

**Definition 14.** *Let $G$ be a group and $M$ a set. A* group action *is a map $\varphi : G \times M \to M$ s.t.:*

1. *$\varphi(e, x) = x$ for the neutral element $e$ and all $x \in M$*
2. *$\varphi(gh, x) = \varphi(g, \varphi(h, x))$ for all $g, h \in G$ and all $x \in M$*

We follow the convention and interpret the group elements as maps: Instead of $\varphi(g, x) = y$ we write $g(x) = y$.

**Definition 15.**
1. *A group action $\varphi : G \times M \to M$ is called* transitive*, if for every $x, y \in M$ there exists a $g \in G$ with $g(x) = y$.*
2. *A group action $\varphi : G \times M \to M$ is called* 2-transitive*, if for every $x_1, x_2, y_1, y_2 \in M$ with $x_1 \neq x_2$ and $y_1 \neq y_2$ there exists a $g \in G$ with $g(x_1) = y_1$ and $g(x_2) = y_2$.*

We carry these notions over to calculi, i.e. consider the transitivity of the group action of $\mathrm{Aut}(K)$ on the universe of $K$. However we have to weaken 2-transitivity.

**Definition 16.** *Let $K = (U, \mathcal{B}, \diamond)$ be a calculus.*

1. *$K$ is called* transitive *if $\mathrm{Aut}(K)$ acts transitively on $U$.*
2. *$K$ is called* 2-transitive *if for every $R \in \mathcal{B}$ and all $(x_1, x_2), (y_1, y_2) \in R$ there exists an automorphism $h$ s.t. $h(x_1) = y_1$ and $h(x_2) = y_2$.*

Transitivity of a calculus can be thought of as a kind of self-similarity of the universe: The "neighborhoods" of two entities are indistinguishable. We expect that many of the commonly used calculi are transitive. The group action of $\mathrm{Aut}(K)$ on $U$ cannot be 2-transitive in the algebraic sense, since our automorphisms have to preserve atomic relations. We can think of 2-transitivity for calculi like this: A calculus $K$ is 2-transitive, if it is "as 2-transitive as possible" in the algebraic sense.

**Lemma 17.** *Every 2-transitive calculus is strongly closed under constraints.*

**Proof.** Let $K = (U, \mathcal{B}, \diamond)$ be a 2-transitive calculus. Let $\Theta = (V, \nu)$ be a consistent, atomic $K$-network. We show that $\Theta$ does not refine any relation. Let $(v_1, v_2) \in V^2$ and $R := \nu(v_1, v_2) \in \mathcal{B}$ an atomic relation. Since $\Theta$ is consistent, there is a solution $\phi : V \to U$. Let $x_1 := \phi(v_1)$ and $x_2 := \phi(v_2)$. Since $\phi$ is a solution, we have $x_1 R x_2$. Let $(y_1, y_2) \in R$ be any other pair. Since $K$ is 2-transitive, there exists an automorphism $h$ with $h(x_1) = y_1$ and $h(x_2) = y_2$. Then $h \circ \phi$ is also a solution for $\Theta$, but this time the considered edge $(v_1, v_2)$ is instantiated by $(y_1, y_2)$. Since the pair $(y_1, y_2) \in R$ was arbitrary, there is a solution for every such pair. Hence, $R$ doesn't have any proper SARs. Since $R$ was arbitrary, $K$ is strongly closed under constraints. ❏

Lemma 17 is indeed a viable method for establishing strong closure under constraints, since for the commonly used calculi that are strongly closed under constraints, it is usually not too difficult to explicitly give the set $\mathrm{Aut}(K)$ and show that it is 2-transitive. Such calculi include Allen's Interval Relations and the point calculus on $\mathbb{R}$.

We exemplarily show that the point calculus on $\mathbb{R}$ is 2-transitive. It suffices to argue for $R_<$ and $R_=$ because $R_>$ has the same witness automorphisms as $R_<$. For $R_=$, given two pairs $(y, y), (z, z)$, we find the automorphism $h(x) = x - y + z$ that maps $y$ to $z$. For $R_<$ let $x_1 < x_2$ and $y_1 < y_2$. We define:

$$h(x) = \begin{cases} x - x_1 + y_1 & \text{if } x \leq x_1 \\ y_1 + \frac{x - x_1}{x_2 - x_1} \cdot (y_2 - y_1) & \text{if } x_1 < x \leq x_2 \\ x - x_2 + y_2 & \text{if } x > x_2 \end{cases}$$

$h$ is bijective and strictly increasing, so it preserves atomic relations. Also, $h(x_1) = y_1$ and $h(x_2) = y_2$. Hence $h$ is the desired automorphism which can be constructed for every two pairs $(x_1, x_2), (y_1, y_2) \in R_<$. Thus, the point calculus is 2-transitive (and by Lemma 17 it is strongly closed under constraints).

## 5 Discussion

We have seen that, although existing calculi suggest a correlation between completeness of a-closure and closure under constraints, it is unlikely to obtain an effective procedure that decides whether a given calculus is closed under constraints by simply enumerating pairs of networks. Our counterexample resulting from Theorems 9 and 11 is designed to be close to existing calculi; in particular, it is infinite and serial, and even transitive (though not 2-transitive). Of course our negative insight does not rule out the possibility that there are other, effective, ways to decide closure under constraints, but we believe that it is worthwhile to look for alternative characterizations of completeness of a-closure. In this connection, Jeavons et al. [5] provide two characterizations for completeness of the more general *r*-consistency in the context of (finite- and infinite-domain) CSPs, which might turn out to be useful when applied to qualitative spatio-temporal calculi.

The construction used in Section 3 to make a calculus infinite preserves not only CUC-size but also completeness of a-closure. Hence it can also be applied to turn Westphal et al.'s counterexamples for Renz and Ligozat's conjecture (which our counterexamples do not address!) into serial, infinite ones. Generalizing this construction and relating it with existing methods for combining calculi [16] remains for future work.

We have furthermore seen that 2-transitivity is a sufficient condition for strong closure under constraints. It is an open question whether 2-transitivity, possibly in disjunction with other properties, is a necessary condition too.

Finally, since closure under constraints is an intuitive and appealing property, it is promising to try and "rescue" the attempts to prove the equivalence with completeness of a-closure by imposing additional assumptions typically satisfied by existing calculi. We conjecture that these assumptions should include infinite domains, transitivity (which implies seriality), and 2-transitivity.

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