Game-Theoretic Semantics for Alternating-Time Temporal Logic

Valentin Goranko[1]
Stockholm University
Sweden

Antti Kuusisto[2]
University of Bremen
Germany

Raine Rönnholm[3]
University of Tampere
Finland

Abstract

We introduce versions of game-theoretic semantics (GTS) for Alternating-Time Temporal Logic (ATL). In GTS, truth is defined in terms of existence of a winning strategy in a semantic evaluation game, and thus the game-theoretic perspective appears in the framework of ATL on two semantic levels: on the object level, in the standard semantics of the strategic operators, and on the meta-level, where game-theoretic logical semantics can be applied to ATL. We unify these two perspectives into semantic evaluation games specially designed for ATL. The novel game-theoretic perspective enables us to identify new variants of the semantics of ATL, based on limiting the time resources available to the verifier and falsifier in the semantic evaluation game; we introduce and analyse an *unbounded* and *bounded* GTS and prove these to be equivalent to the standard (Tarski-style) compositional semantics. We also introduce a non-equivalent *finitely bounded* semantics and argue that it is natural from both logical and game-theoretic perspectives.

1 Introduction

Alternating-Time Temporal Logic ATL was introduced in [3] as a multi-agent extension of the branching-time temporal logic CTL. The semantics of ATL is defined over *multi-agent transition systems*, also known as *concurrent game models*, in which agents take simultaneous actions at the current state and the resulting collective action determines the subsequent state transition. The logic ATL and its extensions such as ATL* have gradually become the most popular logical formalisms for reasoning about strategic abilities of agents in synchronous multi-agent systems.

Game-theoretic semantics (GTS) of logical languages has a complex history going back to Hintikka [7], Lorenzen [10] and others. For an overview of the topic, see [9]. In GTS, truth of a logical formula φ is determined in a formal *debate* between two players, *Eloise* and *Abelard*. Eloise is trying to verify φ, while Abelard is opposing her. Each logical operator is associated with a related rule in the game. The framework of GTS has turned out to be particularly useful for the purpose of defining variants of semantic approaches to different logics. For example, IF-logic of Hintikka and Sandu [8] is an extension of first-order logic which was originally developed using GTS. Also, the game-theoretic approach to semantics has lead to new methods for solving decision problems of logics, e.g., via using parity games for the μ-calculus.

[1] `valentin.goranko@philosophy.su.se`
[2] `kuusisto@uni-bremen.de`
[3] `raine.ronnholm@uta.fi`

In this article we introduce game-theoretic semantics for ATL. In that framework, the rules corresponding to strategic operators involve scenarios where Eloise and Abelard are both controlling (or leading) coalitions of agents with opposing objectives. The perspective offered by GTS enables us to develop novel approaches to ATL based on different time resources available to the players. In *unbounded* GTS, a coalition trying to verify an until-formula is allowed to continue without a time limit, the price of an infinite play being a loss in the game. In *bounded* GTS, the coalition must commit to *finishing in finite time* by submitting an *ordinal number* in the beginning of the game; the ordinal controls available time resources in the game and *guarantees a finite play.* In fact, even safety games (for release-formulae) will be determined in finite time, and thus the bounded and unbounded approaches to GTS are conceptually different.

Despite the differences between the two semantics, we show that they are in fact equivalent to the standard compositional (i.e., Tarski-style) semantics of ATL and therefore to each other. Furthermore, we introduce a restriction of the bounded GTS, called *finitely bounded* GTS, where the ordinals controlling time flow must always be finite. This is a particularly simple system of semantics where the players will always announce the ultimate (always finite) duration of the game before the game begins. We show that the finitely bounded GTS is equivalent to the standard ATL semantics on *image finite models*, and therefore provides an alternative approach to ATL sufficient for most practical purposes.

Since the finitely bounded semantics is new, we also develop an equivalent (over all models) Tarski-style semantics for it. We note that the difference between the finitely bounded and unbounded semantics is conceptually linked to the difference between *for-loops* and *while-loops*.

For all systems of game-theoretic semantics studied in this paper, we establish that positional strategies suffice in the perfect information setting for ATL. In the framework of unbounded semantics, this means that strategies depend on the current state only. In the case of bounded and finitely bounded semantics, strategies may additionally depend on the value of the ordinal guiding the time flow of the game.

The main contributions of this paper are twofold: the conceptual and technical development of game-theoretic semantics for ATL and the introduction of new resource-sensitive versions of logics for multi-agent strategic reasoning. The latter relates conceptually to the study of other resource-bounded versions of ATL, see [2], [11], [1].

The structure of the paper is as follows. After the preliminaries in Section 2 , we develop the bounded and unbounded GTS in Section 3 . We analyse the frameworks in Section 4 , where we show, inter alia, that the two game-theoretic frameworks are equivalent. In Section 5 , we compare the game-theoretic and standard Tarski-style semantics and establish the equivalences between them stated above.

It is worth pointing out that some of our technical results could be derived using more general alternative methods from coalgebraic modal logic. We will discuss this matter in more detail in the concluding section 5.3 .

2 Preliminaries

In this section we define concurrent game models as well as the syntax and standard compositional semantics of ATL.

Definition 2.1. *A* ***concurrent game model*** *(*CGM*)* $\mathcal{M}$ *is a tuple* $(\mathbb{A}\mathrm{gt}, \mathrm{St}, \Pi, \mathrm{Act}, d, o, v)$ *which consists of the following non-empty sets:* ***agents*** $\mathbb{A}\mathrm{gt} = \{1, \dots, k\}$*,* ***states*** St*,* ***proposition symbols*** Π*,* ***actions*** Act*,* ***action function*** $d : \mathbb{A}\mathrm{gt} \times \mathrm{St} \to \mathcal{P}(\mathrm{Act}) \setminus \{\emptyset\}$

assigning a non-empty set of actions available to each agent at each state, and a ***transition function*** o *assigning a unique* ***outcome state*** $o(q, \vec{\alpha})$ *to each state* $q \in \mathrm{St}$ *and* ***action profile*** *(a tuple of actions* $\vec{\alpha} = (\alpha_1, \ldots, \alpha_k)$ *such that* $\alpha_i \in d(i, q)$ *for each* $i \in \mathbb{A}\mathrm{gt}$*), and a* ***valuation function*** $v : \Pi \to \mathcal{P}(\mathrm{St})$*.*

Sets of agents $A \subseteq \mathbb{A}\mathrm{gt}$ are also called **coalitions**. The complement $\overline{A} = \mathbb{A}\mathrm{gt} \setminus A$ of a coalition A is called the **opposing coalition** (of A). We also define the set of action tuples that are *available* to coalition A at a state $q \in \mathrm{St}$: $\mathsf{action}(A, q) := \{(\alpha_i)_{i \in A} \mid \alpha_i \in d(i, q) \text{ for each } i \in A\}$.

Definition 2.2. *Let* $\mathcal{M} = (\mathbb{A}\mathrm{gt}, \mathrm{St}, \Pi, \mathrm{Act}, d, o, v)$ *be a concurrent game model. A* ***strategy***[1] *for an agent* $a \in \mathbb{A}\mathrm{gt}$ *is a function* $s_a : \mathrm{St} \to \mathrm{Act}$ *such that* $s_a(q) \in d(a, q)$ *for each* $q \in \mathrm{St}$*. A* ***collective strategy*** S_A *for* $A \subseteq \mathbb{A}\mathrm{gt}$ *is a tuple of individual strategies, one for each agent in A. A* ***path*** *in* $\mathcal{M}$ *is a sequence of states* Λ *s.t.* $\Lambda[n+1] = o(\Lambda[n], \vec{\alpha})$ *for some admissible action profile* $\vec{\alpha}$*, where* $\Lambda[n]$ *is the n-th state in* Λ $(n \in \mathbb{N})$*. The function* $\mathsf{paths}(q, S_A)$ *returns the set of all paths that can be formed when the agents in A play according to* S_A*, beginning from the state* q*.*

The full **Alternating-time temporal logic** ATL^* introduced in [3], is a logic, suitable for specifying and verifying qualitative objectives of players and coalitions in concurrent game models. The main syntactic construct of ATL^* is a formula of type $\langle\!\langle A \rangle\!\rangle\, \Phi$, intuitively meaning that *the coalition A has a collective strategy to guarantee the satisfaction of the objective* Φ *on every play enabled by that strategy.* Formally, ATL^* is a multi-agent extension of the branching time logic CTL^* with *strategic quantifiers* $\langle\!\langle A \rangle\!\rangle$ indexed with sets (*coalitions*) A of players. ATL^* has two sorts of formulae. *State formulae* are evaluated at states, and *path formulae* are evaluated on plays. The syntax is defined as follows:

State formulae: $\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle\!\langle A \rangle\!\rangle\, \Phi$,
Path formulae: $\Phi ::= \varphi \mid \neg\Phi \mid \Phi \vee \Phi \mid \mathsf{X}\, \Phi \mid \Phi\, \mathsf{U}\, \Phi \mid \Phi\, \mathsf{R}\, \Phi$.

Here $A \subseteq \mathbb{A}\mathrm{gt}$ and $p \in \Pi$. In this paper, we will focus on the semantically simpler and computationally better behaved fragment ATL, which is essentially the state-formulae fragment of ATL^* and can also be viewed as the multi-agent analogue of CTL, only involving state formulae defined as follows, for any $A \subseteq \mathbb{A}\mathrm{gt}$, $p \in \Pi$:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \vee \varphi) \mid \langle\!\langle A \rangle\!\rangle\, \mathsf{X}\, \varphi \mid \langle\!\langle A \rangle\!\rangle\, (\varphi\, \mathsf{U}\, \varphi) \mid \langle\!\langle A \rangle\!\rangle\, (\varphi\, \mathsf{R}\, \varphi)$$

Other Boolean connectives are defined as usual, and the combined operators $\langle\!\langle A \rangle\!\rangle\, \mathsf{F}\, \varphi$ and $\langle\!\langle A \rangle\!\rangle\, \mathsf{G}\, \varphi$ are defined respectively by $\langle\!\langle A \rangle\!\rangle \top \mathsf{U}\, \varphi$ and $\langle\!\langle A \rangle\!\rangle \bot \mathsf{R}\, \varphi$.

Definition 2.3. *Let* $\mathcal{M} = (\mathbb{A}\mathrm{gt}, \mathrm{St}, \Pi, \mathrm{Act}, d, o, v)$ *be a CGM,* $q \in \mathrm{St}$ *a state and* φ *an* ATL*-formula. Truth of* φ *in* $\mathcal{M}$ *and* q*, denoted by* $\mathcal{M}, q \models \varphi$*, is defined as follows:*

- $\mathcal{M}, q \models p$ *iff* $q \in v(p)$ *(for* $p \in \Pi$ *).*
- $\mathcal{M}, q \models \neg\psi$ *iff* $\mathcal{M}, q \not\models \psi$*.*
- $\mathcal{M}, q \models \psi \vee \theta$ *iff* $\mathcal{M}, q \models \psi$ *or* $\mathcal{M}, q \models \theta$*.*
- $\mathcal{M}, q \models \langle\!\langle A \rangle\!\rangle\, \mathsf{X}\, \psi$ *iff there exists* S_A *such that for each* $\Lambda \in \mathsf{paths}(q, S_A)$*, we have* $\mathcal{M}, \Lambda[1] \models \psi$*.*

[1] Unless otherwise specified, a 'strategy' hereafter will mean a positional and deterministic strategy.

- $\mathcal{M}, q \models \langle\!\langle A \rangle\!\rangle\, \psi \,\mathsf{U}\, \theta$ *iff there exists* S_A *such that for each* $\Lambda \in \mathsf{paths}(q, S_A)$, *there is* $i \geq 0$ *such that* $\mathcal{M}, \Lambda[i] \models \theta$ *and* $\mathcal{M}, \Lambda[j] \models \psi$ *for every* $j < i$.

- $\mathcal{M}, q \models \langle\!\langle A \rangle\!\rangle\, \psi \,\mathsf{R}\, \theta$ *iff there exists* S_A *such that for each* $\Lambda \in \mathsf{paths}(q, S_A)$ *and* $i \geq 0$, *we have* $\mathcal{M}, \Lambda[i] \models \theta$ *or there is* $j < i$ *such that* $\mathcal{M}, \Lambda[j] \models \psi$.

3 Game-theoretic semantics

In this section we will introduce unbounded, bounded and finitely bounded evaluation games for ATL. By defining the truth of a formula as the existence of a winning strategy for the verifier in the corresponding evaluation game, these variants of evaluation games lead to three different versions of game-theoretic semantics for ATL.

3.1 Unbounded evaluation games

Given a CGM $\mathcal{M}$, a state q_{in} and a formula φ, the **evaluation game** $\mathcal{G}(\mathcal{M}, q_{in}, \varphi)$ is intuitively an argument between two opponents, **Eloise** (**E**) and **Abelard** (**A**), about whether the formula φ is true at the state q_{in} in the model $\mathcal{M}$. Eloise claims that φ is true, so she adopts (initially) the role of a **verifier** in the game, and Abelard tries to prove the formula false, so he is (initially) the **falsifier**. These roles can swap in the course of the game when negations are encountered in the formula to be evaluated.

We will often use the following notation: if $\mathbf{P} \in \{\mathbf{E}, \mathbf{A}\}$, then $\overline{\mathbf{P}}$ denotes the **opponent** of $\mathbf{P}$, i.e., $\overline{\mathbf{P}} \in \{\mathbf{E}, \mathbf{A}\} \setminus \{\mathbf{P}\}$.

Definition 3.1. *Let* $\mathcal{M} = (\mathbb{A}\mathrm{gt}, \mathrm{St}, \Pi, \mathrm{Act}, d, o, v)$ *be a CGM,* $q_{in} \in \mathrm{St}$ *and* φ *an* ATL*-formula. The* ***unbounded evaluation game*** $\mathcal{G}(\mathcal{M}, q_{in}, \varphi)$ *between the players* $\mathbf{A}$ *and* $\mathbf{E}$ *is defined as follows.*

- *A* ***position*** *of the game is a tuple* $\mathsf{Pos} = (\mathbf{P}, q, \psi)$ *where* $\mathbf{P} \in \{\mathbf{A}, \mathbf{E}\}$, $q \in \mathrm{St}$ *and* ψ *is a subformula of* φ. *The* ***initial position*** *of the game is* $\mathsf{Pos}_0 := (\mathbf{E}, q_{in}, \varphi)$.

- *In every position* $(\mathbf{P}, q, \psi)$, *the player* $\mathbf{P}$ *is called the* ***verifier*** *and* $\overline{\mathbf{P}}$ *the* ***falsifier*** *for that position.*

- *Each position of the game is associated with a rule. The rules for positions where the related formula is either a proposition symbol or has a Boolean connective as its main connective, are defined as follows.*

  1. *If* $\mathsf{Pos}_i = (\mathbf{P}, q, p)$, *where* $p \in \Pi$, *then* Pos_i *is called an* ***ending position*** *of the evaluation game. If* $q \in v(p)$, *then* $\mathbf{P}$ *wins the game. Else* $\overline{\mathbf{P}}$ *wins.*
  2. *Let* $\mathsf{Pos}_i = (\mathbf{P}, q, \neg\psi)$. *The game then moves to the next position,* $\mathsf{Pos}_{i+1} = (\overline{\mathbf{P}}, q, \psi)$.
  3. *Let* $\mathsf{Pos}_i = (\mathbf{P}, q, \psi \vee \theta)$. *Then the player* $\mathbf{P}$ *decides whether* $\mathsf{Pos}_{i+1} = (\mathbf{P}, q, \psi)$ *or* $\mathsf{Pos}_{i+1} = (\mathbf{P}, q, \theta)$.

In order to deal with the strategic operators, we now define a ***one step game****, denoted by* $\mathsf{step}(\mathbf{P}, A, q)$, *where* $A \subseteq \mathbb{A}\mathrm{gt}$. *This game consists of the following two actions.*

i) First $\mathbf{P}$ *chooses an action* $\alpha_i \in d(i, q)$ *for each* $i \in A$.

ii) Then $\overline{\mathbf{P}}$ *chooses an action* $\alpha_i \in d(i, q)$ *for each* $i \in \overline{A}$.

*The **resulting state** of the one step game* $\mathsf{step}(\mathbf{P}, A, q)$ *is the state* $q' := o(q, \alpha_1, \ldots, \alpha_k)$ *arising from the combined action of the agents. We now define how the evaluation game proceeds in positions where the formula is of type* $\langle\langle A\rangle\rangle \mathsf{X}\, \psi$*:*

4. *Let* $\mathsf{Pos}_i = (\mathbf{P}, q, \langle\langle A\rangle\rangle \mathsf{X}\, \psi)$*. The next position* Pos_{i+1} *is* $(\mathbf{P}, q', \psi)$*, where* q' *is the resulting state of* $\mathsf{step}(\mathbf{P}, A, q)$*.*

*The rules for the other strategic operators are obtained by iterating the one step game. For this purpose, we now define the **embedded game*** $\mathbf{G} := \mathbf{g}(\mathbf{V}, \mathbf{C}, A, q_0, \psi_{\mathbf{C}}, \psi_{\overline{\mathbf{C}}})$*, where both* $\mathbf{V}, \mathbf{C} \in \{\mathbf{E}, \mathbf{A}\}$*,* A *is a coalition,* q_0 *a state, and* $\psi_{\mathbf{C}}$ *and* $\psi_{\overline{\mathbf{C}}}$ *are formulae. The player* $\mathbf{V}$ *is called the **verifier** (of the embedded game) and* $\mathbf{C}$ *the **controller**. These may, but need not be, the same player. We let* $\overline{\mathbf{V}}$ *and* $\overline{\mathbf{C}}$ *denote the opponents of* $\mathbf{C}$ *and* $\mathbf{V}$*, respectively.*

The embedded game $\mathbf{G}$ *starts from the **initial state*** q_0 *and proceeds from any state* q *according to the following rules, applied in the order below, until an **exit position** is reached.*

i) $\mathbf{C}$ *may end the game at the exit position* $(\mathbf{V}, q, \psi_{\mathbf{C}})$*.*

ii) $\overline{\mathbf{C}}$ *may end the game at the exit position* $(\mathbf{V}, q, \psi_{\overline{\mathbf{C}}})$*.*

iii) If the game has not ended due to the above rules, the one step game $\mathsf{step}(\mathbf{V}, A, q)$ *is played to produce a resulting state* q'*. The embedded game is continued from* q'*.*

If the embedded game $\mathbf{G}$ *continues an infinite number of rounds, the controller* $\mathbf{C}$ *loses the entire evaluation game* $\mathcal{G}(\mathcal{M}, q_{in}, \varphi)$*. Else the evaluation game resumes from the exit position of the embedded game.*

We now define the rules of the evaluation game for the remaining strategic operators as follows:

5. *Consider a position* $\mathsf{Pos}_i = (\mathbf{P}, q, \langle\langle A\rangle\rangle\, \psi \,\mathsf{U}\, \theta)$*. The next position* Pos_{i+1} *is the exit position of the embedded game* $\mathbf{g}(\mathbf{P}, \mathbf{P}, A, q, \theta, \psi)$*. (Note the order of the formulae.)*

6. *Consider a position* $\mathsf{Pos}_i = (\mathbf{P}, q, \langle\langle A\rangle\rangle\, \psi \,\mathsf{R}\, \theta)$*. The next position* Pos_{i+1} *is the exit position of the embedded game* $\mathbf{g}(\mathbf{P}, \overline{\mathbf{P}}, A, q, \theta, \psi)$*.*

This completes the definition of the evaluation game.

We sometimes say that the embedded game for a formula $\langle\langle A\rangle\rangle\, \psi \,\mathsf{U}\, \theta$ is an **eventuality game** and the embedded game for $\langle\langle A\rangle\rangle\, \psi \,\mathsf{R}\, \theta$ is a **safety game**. The embedded game $\mathbf{g}(\mathbf{V}, \mathbf{C}, A, q, \psi_{\mathbf{C}}, \psi_{\overline{\mathbf{C}}})$ can be seen as a 'simultaneous reachability game' where both players have a goal they are trying to reach before the opponent reaches her/his goal. The verifier $\mathbf{V}$ leads the coalition A and the falsifier $\overline{\mathbf{V}}$ leads the opposing coalition $\overline{A}$. The goal of both $\mathbf{V}$ and $\overline{\mathbf{V}}$ is defined by a formula. When $\mathbf{V} = \mathbf{C}$, the goal of $\mathbf{V}$ is to verify $\psi_{\mathbf{C}}$ and the goal of $\overline{\mathbf{V}}$ is to *falsify* $\psi_{\overline{\mathbf{C}}}$. Note that falsifying $\psi_{\overline{\mathbf{C}}}$, corresponds to reaching the complement of the set of states where $\psi_{\overline{\mathbf{C}}}$ holds. When $\mathbf{V} \neq \mathbf{C}$, the goal of $\mathbf{V}$ is to verify $\psi_{\overline{\mathbf{C}}}$ and that of $\overline{\mathbf{V}}$ is to falsify $\psi_{\mathbf{C}}$. Both players $\mathbf{V}$ and $\overline{\mathbf{V}}$ have the possibility to end the game when they believe that they have reached their goal. However, the controller is responsible for ending the embedded game in finite time, and (s)he will lose if the game continues infinitely long. If both players reach their targets at the same time, the controller $\mathbf{C}$ wins, because $\mathbf{C}$ gets to make the decision to end the embedded game first.

It is worth noting that, even though the coalitions in ATL operate concurrently, in the embedded game the verifier $\mathbf{V}$ has the advantage of making the choices for her/his coalition before $\overline{\mathbf{V}}$ in every round, making the evaluation games fully turn-based.

3.2 Bounded evaluation games

The difference between bounded and unbounded evaluation games is that in the bounded case, the embedded games are associated with a time limit. In a bounded evaluation game, the controller must first announce some possibly infinite ordinal γ which will decrease in each round. This will guarantee that the embedded game, and in fact the entire evaluation game, will end after a finite number of rounds.

Bounded evaluation games $\mathcal{G}(\mathcal{M}, q_{in}, \varphi, \Gamma)$ have an additional parameter Γ, which is an ordinal that fixes an upper bound for the ordinals that the players can announce during the related embedded games. Different parameters Γ give rise to different kinds of evaluation games and thus lead to different kinds of game theoretic semantics, as we will see.

Definition 3.2. *Let $\mathcal{M}$ be a CGM, $q_{in} \in \mathrm{St}$, φ an* ATL*-formula and Γ an ordinal. The* ***bounded evaluation game*** *$\mathcal{G}(\mathcal{M}, q_{in}, \varphi, \Gamma)$ is defined as the unbounded evaluation game $\mathcal{G}(\mathcal{M}, q_{in}, \varphi)$, the only difference between the two games being the treatment of until- and release-formulae.*

Let $\mathbf{G} = \mathbf{g}(\mathbf{V}, \mathbf{C}, A, q_0, \psi_{\mathbf{C}}, \psi_{\overline{\mathbf{C}}})$ be an embedded game that arises from a position Pos *in $\mathcal{G}(\mathcal{M}, q_{in}, \varphi)$. In that same position* Pos *in the bounded evaluation game $\mathcal{G}(\mathcal{M}, q_{in}, \varphi, \Gamma)$, the player $\mathbf{C}$ first chooses some ordinal $\gamma_0 < \Gamma$ as the* ***initial time limit*** *for the embedded game $\mathbf{G}$. This choice leads to a* ***bounded embedded game*** *that is denoted by $\mathbf{G}[\gamma_0]$.*

A ***configuration*** *of $\mathbf{G}[\gamma_0]$ is a pair (γ, q), where γ is a (possibly infinite) ordinal called the* ***current time limit*** *and $q \in \mathrm{St}$ a state called the* ***current state****. The bounded embedded game $\mathbf{G}[\gamma_0]$ starts from the* ***initial configuration*** *(γ_0, q_0) and proceeds from any configuration (γ, q) according to the following rules, applied in the given order.*

i) If $\gamma = 0$, the game ends at the exit position $(\mathbf{V}, q, \psi_{\mathbf{C}})$.

ii) $\mathbf{C}$ may end the game at the exit position $(\mathbf{V}, q, \psi_{\mathbf{C}})$.

iii) $\overline{\mathbf{C}}$ may end the game at the exit position $(\mathbf{V}, q, \psi_{\overline{\mathbf{C}}})$.

iv) If the game has not ended due to the previous rules, then $\mathsf{step}(\mathbf{V}, A, q)$ *is played in order to produce a resulting state q'. Then the bounded embedded game continues from the configuration (γ', q'), where $\gamma' = \gamma - 1$ if γ is a successor ordinal, and if γ is a limit ordinal, then γ' is an ordinal smaller than γ and chosen by $\mathbf{C}$.*

We denote the set of configurations in $\mathbf{G}[\gamma_0]$ by $\mathsf{Conf}_{\mathbf{G}[\gamma_0]}$. After the bounded embedded game $\mathbf{G}[\gamma_0]$ has reached an exit position—which it will, because ordinals are well-founded—the evaluation game resumes from the exit position.

It is clear that bounded evaluation games end after a finite number of rounds because bounded embedded games do. Note that if time limits are infinite ordinals, they do not directly refer to the number of rounds left in the game, but instead they are related to the game duration in a more abstract way. Different kinds of ways to use ordinals in game-theoretic considerations go way back. An important and relatively early reference is [12] which contains references to even earlier related articles.

It is possible to analyse embedded games as separate entities independent of evaluation games. An embedded game of the form $\mathbf{G} = \mathbf{g}(\mathbf{V}, \mathbf{C}, A, q_0, \psi_{\mathbf{C}}, \psi_{\overline{\mathbf{C}}})$ can be played without a time limit as in unbounded evaluation games, or it can be given some time limit γ_0 as a parameter, which leads to the related bounded embedded game $\mathbf{G}[\gamma_0]$. When we use the plain notation $\mathbf{G}$ (as opposed to $\mathbf{G}[\gamma_0]$), we always assume that the embedded game $\mathbf{G}$ is not bounded—we may even emphasize this by calling $\mathbf{G}$ an *unbounded* embedded game.

Evaluation games of the form $\mathcal{G}(\mathcal{M}, q_{in}, \varphi, \omega)$ constitute a particularly interesting subclass of bounded evaluation games. We call the games in this class **finitely bounded evaluation games.** In these games, only *finite* time limits are allowed to be announced for bounded embedded games.

Example I. *In the current article, the operators $\langle\!\langle A \rangle\!\rangle \mathsf{F}$ and $\langle\!\langle A \rangle\!\rangle \mathsf{G}$ are syntactic abbreviations, and therefore the above games show explicitly how the associated rules look like. We next define alternative rules that could be directly given to $\langle\!\langle A \rangle\!\rangle \mathsf{F}$ and $\langle\!\langle A \rangle\!\rangle \mathsf{G}$ in the* ***finitely bounded*** *evaluation games without affecting the results of the article. (The fact that this indeed holds will ultimately be straightforward to observe.) Similar (but not identical) rules could be given to $\langle\!\langle A \rangle\!\rangle \mathsf{F}$ and $\langle\!\langle A \rangle\!\rangle \mathsf{G}$ also in the framework based on unbounded and as well as general bounded games.*

- *Let $\mathsf{Pos}_i = (\mathbf{P}, q, \langle\!\langle A \rangle\!\rangle \mathsf{F}\, \psi)$. First the player $\mathbf{P}$ chooses $n \in \mathbb{N}$ and then the players iterate $\mathsf{step}(\mathbf{P}, A, q)$ for at most n times. The player $\mathbf{P}$ may decide to stop at the current state q' after any number $m \leq n$ of iterations and continue the evaluation game from $\mathsf{Pos}_{i+1} = (\mathbf{P}, q', \psi)$.*

- *Let $\mathsf{Pos}_i = (\mathbf{P}, q, \langle\!\langle A \rangle\!\rangle \mathsf{G}\, \psi)$. First the player $\overline{\mathbf{P}}$ chooses $n \in \mathbb{N}$ and then the players iterate $\mathsf{step}(\mathbf{P}, A, q)$ for at most n times. The player $\overline{\mathbf{P}}$ may decide to stop at the current state q' after any number $m \leq n$ of iterations and continue the evaluation game from $\mathsf{Pos}_{i+1} = (\mathbf{P}, q', \psi)$.*

3.3 Game-theoretic semantics

A strategy for a player $\mathbf{P} \in \{\mathbf{A}, \mathbf{E}\}$ will be defined below to be a function on game positions; in positions where $\mathbf{P}$ is not the player required to make a move, the strategy of $\mathbf{P}$ will output a special value "void". We occasionally also give the value void to some other functions when the output is not relevant (e.g., when formulating a winning strategy, we may assign void for losing positions).

Definition 3.3. *Let $\mathbf{G} = \mathbf{g}(\mathbf{V}, \mathbf{C}, A, q_0, \psi_{\mathbf{C}}, \psi_{\overline{\mathbf{C}}})$ be an embedded game and $\mathbf{P} \in \{\mathbf{A}, \mathbf{E}\}$.* ***A strategy for the player $\mathbf{P}$ in $\mathbf{G}$*** *is a function $\sigma_{\mathbf{P}}$ whose domain is St and whose range is specified below. Firstly, for any $q \in \mathrm{St}$, it is possible to define $\sigma_{\mathbf{P}}(q) \in \{\psi_{\mathbf{C}}, \psi_{\overline{\mathbf{C}}}\}$; then $\sigma_{\mathbf{P}}$ instructs $\mathbf{P}$ to end the game at the state q. Here it is required that if $\mathbf{P} = \mathbf{C}$, then $\sigma_{\mathbf{P}}(q) = \psi_{\mathbf{C}}$ and if $\mathbf{P} = \overline{\mathbf{C}}$, then $\sigma_{\mathbf{P}}(q) = \psi_{\overline{\mathbf{C}}}$. If $\sigma_{\mathbf{P}}(q) \notin \{\psi_{\mathbf{C}}, \psi_{\overline{\mathbf{C}}}\}$, then the following conditions hold.*

- *If $\mathbf{P} = \mathbf{V}$, then $\sigma_{\mathbf{P}}(q)$ is a tuple of actions from $\mathsf{action}(A, q)$.*

- *If $\mathbf{P} = \overline{\mathbf{V}}$, then $\sigma_{\mathbf{P}}(q)$ is defined to be a* ***response function*** *$f : \mathsf{action}(A, q) \to \mathsf{action}(\overline{A}, q)$ that assigns a tuple of actions for $\overline{A}$ as a response to any tuple of actions for A.*

Let γ_0 be an ordinal. A strategy $\sigma_{\mathbf{P}}$ for $\mathbf{P}$ in $\mathbf{G}[\gamma_0]$ is defined in the same way as a strategy in $\mathbf{G}$, but the domain of this strategy is the set of all possible configurations $\mathsf{Conf}_{\mathbf{G}[\gamma_0]}$.

Note that strategies in embedded games are positional, i.e., they depend only on the current state in the unbounded case and the current configuration in the bounded case. We will see later on that if strategies were allowed to depend on more information, such as the sequence of states played, the resulting semantic systems would be equivalent to the current ones.

Any strategy $\sigma_{\mathbf{P}}$ for an unbounded embedded game $\mathbf{G}$ can be used also in any bounded embedded game $\mathbf{G}[\gamma_0]$: we simply use the same action $\sigma_{\mathbf{P}}(q)$ for each configuration $(\gamma, q) \in \mathsf{Conf}_{\mathbf{G}[\gamma_0]}$. Also note that if a strategy $\sigma_{\mathbf{P}}$ for a bounded embedded game $\mathbf{G}[\gamma_0]$ is independent of time limits (and thus depends on states only), it can also be used in the unbounded embedded game $\mathbf{G}$.

We next define the notion of strategy for evaluation games, using strategies for embedded games as sub-strategies.

Definition 3.4. *Let $\mathbf{P} \in \{\mathbf{A}, \mathbf{E}\}$. A **strategy for player $\mathbf{P}$ in an unbounded evaluation game** $\mathcal{G} = \mathcal{G}(\mathcal{M}, q_{in}, \varphi)$ is a function $\Sigma_{\mathbf{P}}$ defined on the set of positions* POS *of $\mathcal{G}$ (with the range specified below) satisfying the following conditions.*

1. *If $\mathsf{Pos} = (\mathbf{P}, q, \psi \vee \theta)$, then $\Sigma_{\mathbf{P}}(\mathsf{Pos}) \in \{\psi, \theta\}$.*
2. *If $\mathsf{Pos} = (\mathbf{P}, q, \langle\langle A\rangle\rangle \mathsf{X}\, \psi)$, then $\Sigma_{\mathbf{P}}(\mathsf{Pos})$ is a tuple of actions from $\mathsf{action}(A, q)$ for the one step game $\mathsf{step}(\mathbf{P}, A, q)$.*
3. *If $\mathsf{Pos} = (\overline{\mathbf{P}}, q, \langle\langle A\rangle\rangle \mathsf{X}\, \psi)$, then $\Sigma_{\mathbf{P}}(\mathsf{Pos})$ is a response function $f : \mathsf{action}(A, q) \to \mathsf{action}(\overline{A}, q)$ for $\mathsf{step}(\mathbf{P}, A, q)$.*
4. *Let $\mathsf{Pos} = (\mathbf{P}, q, \langle\langle A\rangle\rangle\, \psi \mathsf{T} \theta)$ or $\mathsf{Pos} = (\overline{\mathbf{P}}, q, \langle\langle A\rangle\rangle\, \psi \mathsf{T} \theta)$, where $\mathsf{T} \in \{\mathsf{U}, \mathsf{R}\}$. Then $\Sigma_{\mathbf{P}}(\mathsf{Pos})$ is a strategy $\sigma_{\mathbf{P}}$ for $\mathbf{P}$ in the respective embedded game $\mathbf{g}(\mathbf{V}, \mathbf{C}, A, q, \theta, \psi)$.*
5. *In all other cases, $\Sigma_{\mathbf{P}}(\mathsf{Pos}) = \mathsf{void}$.*

*We say that the player $\mathbf{P}$ plays according to the strategy $\Sigma_{\mathbf{P}}$ in the evaluation game $\mathcal{G}$ if $\mathbf{P}$ makes her/his choices in $\mathcal{G}$ according to that strategy. We say that $\Sigma_{\mathbf{P}}$ is a **winning strategy** for $\mathbf{P}$ in $\mathcal{G}$ if $\mathbf{P}$ wins all plays of $\mathcal{G}$ where (s)he plays according to that strategy.*

Definition 3.5. *A **strategy for player $\mathbf{P}$ in a bounded evaluation game** $\mathcal{G} = \mathcal{G}(\mathcal{M}, q_{in}, \varphi, \Gamma)$ is defined as in Definition 3.4, with the exception of positions with until- and release-formulae, which are treated as follows.*

4. *Let $\mathsf{Pos} = (\mathbf{P}, q, \langle\langle A\rangle\rangle\, \psi \mathsf{T} \theta)$ or $\mathsf{Pos} = (\overline{\mathbf{P}}, q, \langle\langle A\rangle\rangle\, \psi \mathsf{T} \theta)$, where $\mathsf{T} \in \{\mathsf{U}, \mathsf{R}\}$, and let $\mathbf{G} = \mathbf{g}(\mathbf{V}, \mathbf{C}, A, q, \theta, \psi)$ denote the embedded game related to Pos. If $\mathbf{P} = \mathbf{C}$, then $\Sigma_{\mathbf{P}}(\mathsf{Pos}) = (\gamma_0, t, \sigma_{\mathbf{P}})$ where the following conditions hold.*
   - *$\gamma_0 < \Gamma$ is an ordinal. It is the choice for the initial time limit that leads to the bounded embedded game $\mathbf{G}[\gamma_0]$.*
   - *t is a function, called **timer**, on pairs (γ, q), where $\gamma \leq \gamma_0$ is a limit ordinal and $q \in \mathrm{St}$. The timer t gives an instruction how to lower the time limit γ after a transition to q has been made; the value of $t(\gamma, q)$ must be an ordinal less than γ.*
   - *$\sigma_{\mathbf{P}}$ is a strategy for $\mathbf{P}$ in $\mathbf{G}[\gamma_0]$.*

   Finally, if $\mathbf{P} \neq \mathbf{C}$, then $\Sigma_{\mathbf{P}}(\mathsf{Pos})$ is a function that maps any ordinal $\gamma_0 < \Gamma$ to some strategy $\sigma_{\mathbf{P}}$ for $\mathbf{P}$ in $\mathbf{G}[\gamma_0]$.

In finitely bounded evaluation games, only finite time limits $\gamma_0 < \omega$ may be announced by $\mathbf{C}$. Since no limit ordinal is reached, the timer t can be omitted from the strategy.

Different choices for time limit bounds Γ give rise to different semantic systems, and most results in the next section will be proven for an arbitrary choice of Γ. However, in this paper we mainly focus on the cases $\Gamma = \omega$ (where ω is the smallest infinite ordinal) and $\Gamma = 2^{\kappa}$, where κ is the cardinality of the model. Note that when $\Gamma = \omega$, the embedded games are finitely bounded. We will prove later that time limit bounds greater than 2^{κ} are not needed.

Definition 3.6. *Let $\mathcal{M}$ be a CGM, $q \in \mathrm{St}$ and φ an* ATL*-formula. Let κ be the cardinality of the model $\mathcal{M}$. We define three different notions of truth of φ in $\mathcal{M}$ and q based on three different evaluation games, thereby defining the **unbounded**, **bounded** and **finitely bounded semantics** (denoted, respectively, by $\models^g_u$, $\models^g_b$, and $\models^g_f$) as follows.*

- *$\mathcal{M}, q \models^g_u \varphi$ iff **E** has a winning strategy in $\mathcal{G}(\mathcal{M}, q, \varphi)$.*
- *$\mathcal{M}, q \models^g_b \varphi$ iff **E** has a winning strategy in $\mathcal{G}(\mathcal{M}, q, \varphi, 2^{\kappa})$.*
- *$\mathcal{M}, q \models^g_f \varphi$ iff **E** has a winning strategy in $\mathcal{G}(\mathcal{M}, q, \varphi, \omega)$.*

*We also write more generally that $\mathcal{M}, q \models^g_\Gamma \varphi$ iff **E** has a winning strategy in $\mathcal{G}(\mathcal{M}, q, \varphi, \Gamma)$.*

We will prove that *both* the bounded and unbounded semantics are equivalent to the standard compositional semantics of Definition 2.3. The finitely bounded sematics, on the other hand, is equivalent to a natural variant of the compositional semantics to be introduced in Section 5. The following example shows that the finitely bounded GTS differs from the unbounded and bounded cases. In particular, the fixed point property of the temporal operator F fails:

Example 3.7. *Let $\mathcal{M} = (\{a\}, \{q_0\} \cup \mathbb{N} \times \mathbb{N}, \{p\}, \mathbb{N}, d, o, v)$, where $v(p) = \{(i,i) \mid i \in \mathbb{N}\}$, $d(a, q_0) = \mathbb{N}$, $d(a,(i,j)) = \{0\}$, $o(q_0, i) = (i, 0)$ and $o((i,j), 0) = (i, j+1)$. In this model $\mathcal{M}, q_0 \not\models^g_f \langle\langle\emptyset\rangle\rangle \mathsf{F}\, p$ while $\mathcal{M}, q_0 \models^g_f \langle\langle\emptyset\rangle\rangle \mathsf{X} \langle\langle\emptyset\rangle\rangle \mathsf{F}\, p$. This is because for every time limit $n < \omega$ chosen by Eloise, Abelard may select the action n in the first round for the agent a, so it will take $n+1$ rounds to reach a state where p is true. But after the first step, the game will be at a state $(i, 0)$ for some $i \in \mathbb{N}$, whence Eloise can choose some time limit $n \geq i$ and reach a state where p is true before time runs out.*

However, $\mathcal{M}, q_0 \models^g_b \langle\langle\emptyset\rangle\rangle \mathsf{F}\, p$, since Eloise can choose ω as the time limit in the beginning of the game and then lower it to $i < \omega$ when the next state $(i, 0)$ is reached. Also, $\mathcal{M}, q_0 \models^g_u \langle\langle\emptyset\rangle\rangle \mathsf{F}\, p$ since a state where p is true will always be reached in finite time. Still, we will show that the three semantics become equivalent over image finite models.

4 Analysing embedded games

In this section we will examine the properties of different versions of embedded games that occur as part of evaluation games. We associate each state with a winning time label which describes how good that state is for the players. Using the optimal labels will lead to a canonical strategy which will be a winning strategy whenever there exists one.

With these definitions we can prove positionality and determinacy of the embedded games. We will also show that if players are allowed to announce sufficiently large ordinals as time limits, the bounded embedded games become essentially equivalent with corresponding unbounded embedded games. We will also analyse how the sizes of the needed ordinals depend on the CGM in which the game is played.

4.1 Winning time labels

Different values of the time limit bound Γ correspond to different classes of bounded embedded games $\mathbf{G}[\gamma_0]$ where $\gamma_0 < \Gamma$. In this section—unless otherwise specified—we use a fixed value of Γ and assume that all bounded embedded games are part of some evaluation game $\mathcal{G}(\mathcal{M}, q_{in}, \varphi, \Gamma)$. Since Γ could have any ordinal value, our results will hold for both the bounded and finitely bounded semantics.

Let $\mathbf{G} = \mathbf{g}(\mathbf{V}, \mathbf{C}, A, q_0, \psi_{\mathbf{C}}, \psi_{\overline{\mathbf{C}}})$ be an embedded game and let $q \in \mathrm{St}$ be a state. We define $\mathbf{G}[q] := \mathbf{g}(\mathbf{V}, \mathbf{C}, A, q, \psi_{\mathbf{C}}, \psi_{\overline{\mathbf{C}}})$. We also use the abbreviation $\mathbf{G}[q, \gamma] := (\mathbf{G}[q])[\gamma]$. This notation is useful, since by the recursive nature of bounded embedded games, any configuration (γ, q) of $\mathbf{G}[\gamma_0]$ (where $\gamma_0 < \Gamma$) is the initial configuration of $\mathbf{G}[q, \gamma]$. Note that since the players use positional strategies, they do not see any difference between initial configurations and other configurations.

We next define winning strategies for embedded games. "Winning an embedded game" means for the player $\mathbf{P}$ that (s)he has a winning strategy in the *evaluation game* that continues from the exit position of the embedded game.

Definition 4.1. *Let $\mathbf{G} = \mathbf{g}(\mathbf{V}, \mathbf{C}, A, q_0, \psi_{\mathbf{C}}, \psi_{\overline{\mathbf{C}}})$ be an embedded game and let $\gamma_0 < \Gamma$.*

*1. We say that $\sigma_{\mathbf{P}}$ is a **winning strategy for the player $\mathbf{P}$ in $\mathbf{G}$** if infinite plays are possible with $\sigma_{\mathbf{P}}$ only if $\mathbf{P} \neq \mathbf{C}$ and the equivalence $\mathcal{M}, q \models_u^g \psi \Leftrightarrow \mathbf{P} = \mathbf{V}$ holds for all exit positions $(\mathbf{V}, q, \psi)$ of $\mathbf{G}$ that can be reached with $\sigma_{\mathbf{P}}$.*

*2. If $\mathbf{P} = \mathbf{C}$, we say that the pair $(\sigma_{\mathbf{P}}, t)$ is a **timed winning strategy for $\mathbf{P}$ in $\mathbf{G}[\gamma_0]$** if $\mathcal{M}, q \models_\Gamma^g \psi \Leftrightarrow \mathbf{P} = \mathbf{V}$ holds for all exit positions $(\mathbf{V}, q, \psi)$ that can be encountered when $\mathbf{P}$ plays using the strategy $\sigma_{\mathbf{P}}$ and timer t.*

*If $\mathbf{P} \neq \mathbf{C}$, we say that $\sigma_{\mathbf{P}}$ is a **winning strategy for $\mathbf{P}$ in $\mathbf{G}[\gamma_0]$** if $\mathcal{M}, q \models_\Gamma^g \psi \Leftrightarrow \mathbf{P} = \mathbf{V}$ holds for all exit positions that can occur when $\mathbf{P}$ plays using $\sigma_{\mathbf{P}}$.*

If the unbounded (respectively, bounded) embedded game in the above definition ends in a position where the equivalence $\mathcal{M}, q \models_u^g \psi \Leftrightarrow \mathbf{P} = \mathbf{V}$ (respectively, $\mathcal{M}, q \models_\Gamma^g \psi \Leftrightarrow \mathbf{P} = \mathbf{V}$) holds, we also say that $\mathbf{P}$ **wins** the embedded game. In the unbounded case, $\overline{\mathbf{C}}$ wins also if the play is infinite.

Consider an embedded game $\mathbf{G} = \mathbf{g}(\mathbf{V}, \mathbf{C}, A, q_0, \psi_{\mathbf{C}}, \psi_{\overline{\mathbf{C}}})$. We next define for $\mathbf{G}$ so called **winning time labels**, $\mathcal{L}_{\mathbf{P}}(q)$, for each $q \in \mathrm{St}$. The labels will indicate how good the state q is for the player $\mathbf{P}$ when different bounded embedded games $\mathbf{G}[q, \gamma_0]$ are played with different time limits $\gamma_0 < \Gamma$. If the label is "win" or "lose", then the state is a winning (respectively, losing) state for $\mathbf{P}$, regardless of the time limit γ_0. If the label is an ordinal $\gamma < \Gamma$, it means that γ is the "critical time limit" for winning or losing the game: if $\mathbf{P} = \mathbf{C}$, then γ is the least time limit needed for $\mathbf{P}$ to win from q, and if $\mathbf{P} \neq \mathbf{C}$, then γ is the least time limit such that $\mathbf{P}$ can no longer guarantee that (s)he will not lose the game from q.

From now on we will often consider separately the cases where the player $\mathbf{P}$ is the controlling player $\mathbf{C}$ and where her/his opponent $\overline{\mathbf{P}}$ is the controlling player. The former case corresponds to the situation where $\mathbf{P}$ is the verifier in an eventuality game and the situation where $\mathbf{P}$ the falsifier in a safety game. The latter case means that either $\mathbf{P}$ is the verifier in a safety game or $\mathbf{P}$ is the falsifier in an eventuality game.

Definition 4.2. *Let $\mathbf{G} = \mathbf{g}(\mathbf{V}, \mathbf{C}, A, q_0, \psi_{\mathbf{C}}, \psi_{\overline{\mathbf{C}}})$ be an embedded game and $\mathbf{P} \in \{\mathbf{E}, \mathbf{A}\}$. The **winning time label $\mathcal{L}_{\mathbf{P}}(q)$ for $\mathbf{P}$ in $\mathbf{G}$ at state** $q \in \mathrm{St}$ is defined as follows.*

***Case 1**. Suppose $\mathbf{P} = \mathbf{C}$. Let $\sigma_{\mathbf{P}}$ be a strategy for $\mathbf{P}$. We first define a **strategy label** $l(q, \sigma_{\mathbf{P}})$ as follows.*

- *Set $l(q, \sigma_{\mathbf{P}}) :=$ lose if $(\sigma_{\mathbf{P}}, t)$ is not a timed winning strategy in $\mathbf{G}[q, \gamma]$ for any timer t and $\gamma < \Gamma$.*

- *Else, set* $l(q, \sigma_{\mathbf{P}}) := \gamma$*, where* $\gamma < \Gamma$ *is the least time limit for which there is a timer* t *such that* $(\sigma_{\mathbf{P}}, t)$ *is a timed winning strategy in* $\mathbf{G}[q, \gamma]$*.*

When there exists at least one $\sigma_{\mathbf{P}}$ *such that* $l(q, \sigma_{\mathbf{P}}) \neq \mathsf{lose}$*, we define*

$$\mathcal{L}_{\mathbf{P}}(q) := min\big\{l(q, \sigma_{\mathbf{P}}) \mid \sigma_{\mathbf{P}} \text{ is a strategy for } \mathbf{P} \text{ s.t. } l(q, \sigma_{\mathbf{P}}) \text{ is an ordinal } \big\}.$$

Else, we define $\mathcal{L}_{\mathbf{P}}(q) := \mathsf{lose}$*.*

Case 2. *Suppose* $\mathbf{P} \neq \mathbf{C}$*. Let* $\sigma_{\mathbf{P}}$ *be a strategy for* $\mathbf{P}$*.*

- *If* $\sigma_{\mathbf{P}}$ *is a winning strategy in* $\mathbf{G}[q, \gamma]$ *for every time limit* $\gamma < \Gamma$*, then set* $l(q, \sigma_{\mathbf{P}}) := \mathsf{win}$*.*
- *Else, set* $l(q, \sigma_{\mathbf{P}}) := \gamma$*, where* $\gamma < \Gamma$ *is the least time limit such that* $\sigma_{\mathbf{P}}$ *is not a winning strategy in* $\mathbf{G}[q, \gamma]$*.*

If $l(q, \sigma_{\mathbf{P}}) = \mathsf{win}$ *for some* $\sigma_{\mathbf{P}}$*, then set* $\mathcal{L}_{\mathbf{P}}(q) := \mathsf{win}$*. Else, set* $\mathcal{L}_{\mathbf{P}}(q)$ *to be the least upper bound for the values* $l(q, \sigma_{\mathbf{P}})$*.*

The following claim shows that if the controller has a timed winning strategy in some embedded game with time limit γ_0, then (s)he has a timed winning strategy with greater time limits as well.

Claim I. *Let* $\mathbf{G} = \mathbf{g}(\mathbf{V}, \mathbf{C}, A, q_0, \psi_{\mathbf{C}}, \psi_{\overline{\mathbf{C}}})$ *be an embedded game. Assume that* $\mathbf{P} = \mathbf{C}$ *and that* $\mathbf{P}$ *has a timed winning strategy* $(\sigma_{\mathbf{P}}, t)$ *in* $\mathbf{G}[\gamma_0]$ *for some* $\gamma_0 < \Gamma$*. Then there is a pair* $(\sigma'_{\mathbf{P}}, t')$ *which is a timed winning strategy in* $\mathbf{G}[\gamma]$ *for any time limit* γ *such that* $\gamma_0 \leq \gamma < \Gamma$*.*

Proof. We define the strategy $\sigma'_{\mathbf{P}}$ for any configuration (γ, q), where $\gamma < \Gamma$ and $q \in \mathrm{St}$, in the following way.

- If $(\sigma_{\mathbf{P}}, t)$ is not a timed winning strategy in $\mathbf{G}[q, \gamma']$ for any $\gamma' \leq \gamma$, set $\sigma'_{\mathbf{P}}(\gamma, q) = \mathsf{void}$.
- Else, set $\sigma'_{\mathbf{P}}(\gamma, q) = \sigma_{\mathbf{P}}(\gamma', q)$, where $\gamma' \leq \gamma$ is the smallest ordinal such that $(\sigma_{\mathbf{P}}, t)$ is a timed winning strategy in $\mathbf{G}[q, \gamma']$.

We define the timer t' for any pair (γ, q), where $\gamma < \Gamma$ is a limit ordinal and $q \in \mathrm{St}$, in the following way.

- If $(\sigma_{\mathbf{P}}, t)$ is not a timed winning strategy in $\mathbf{G}[q, \gamma']$ for any $\gamma' < \gamma$, set $t'(\gamma, q) = \mathsf{void}$.
- Else $t'(\gamma, q) = \gamma'$, where γ' is any ordinal such that $\gamma' < \gamma$ and $(\sigma_{\mathbf{P}}, t)$ is a timed winning strategy in $\mathbf{G}[q, \gamma']$.

We prove by transfinite induction on $\gamma < \Gamma$ for every $q \in \mathrm{St}$ that if $(\sigma_{\mathbf{P}}, t)$ is a timed winning strategy in $\mathbf{G}[q, \delta]$ for some $\delta \leq \gamma$, then $(\sigma'_{\mathbf{P}}, t')$ is a timed winning strategy in $\mathbf{G}[q, \gamma]$. The claim follows from this.

Let the induction hypothesis be that the claim holds for every $\gamma' < \gamma$ and suppose that $(\sigma_{\mathbf{P}}, t)$ is a timed winning strategy in $\mathbf{G}[q, \delta]$ for some $\delta \leq \gamma$. Let $\delta' < \Gamma$ be the smallest ordinal such that $(\sigma_{\mathbf{P}}, t)$ is a timed winning strategy in $\mathbf{G}[q, \delta']$. Now $\delta' \leq \delta \leq \gamma$ and $\sigma'_{\mathbf{P}}(\gamma, q) = \sigma_{\mathbf{P}}(\delta', q)$.

Suppose first that $\delta' = 0$, whence $(\sigma_{\mathbf{P}}, t)$ is a timed winning strategy in $\mathbf{G}[q, 0]$. Now $\sigma'_{\mathbf{P}}(\gamma, q) = \sigma_{\mathbf{P}}(0, q) = \psi_{\mathbf{C}}$ and thus $(\sigma'_{\mathbf{P}}, t')$ is a timed winning strategy in $\mathbf{G}[q, \gamma]$.

Suppose then that $\delta' > 0$, whence we must have $\sigma_{\mathbf{P}}(\delta', q) \neq \psi_{\mathbf{C}}$. Let $q' \in \mathrm{St}$ be any possible successor state of q when $\mathbf{P}$ follows $\sigma_{\mathbf{P}}(\delta', q)$. Now $(\sigma_{\mathbf{P}}, t)$ must be a timed winning strategy in $\mathbf{G}[q', \delta'']$ for some ordinal $\delta'' < \delta'$ (if δ' is a limit ordinal, then $\delta'' = t(\delta', q')$, and if δ' is a successor ordinal, then $\delta'' = \delta' - 1$). Since $\delta' \leq \gamma$, we have $\delta'' < \gamma$.

Suppose first that γ is a successor ordinal. Since we have $\delta'' < \gamma$, we infer by the induction hypothesis that $(\sigma'_{\mathbf{P}}, t')$ is a timed winning strategy in $\mathbf{G}[q', \gamma - 1]$. Hence we see that $(\sigma'_{\mathbf{P}}, t')$ must be a timed winning strategy in $\mathbf{G}[q, \gamma]$. Suppose then that γ is a limit ordinal. Since $(\sigma_{\mathbf{P}}, t)$ is a timed winning strategy in $\mathbf{G}[q', \delta'']$, the value of $t'(\gamma, q')$ is defined such that $t'(\gamma, q') < \gamma$ and $(\sigma_{\mathbf{P}}, t)$ is a timed winning strategy in $\mathbf{G}[q', t'(\gamma, q')]$. Thus, by the induction hypothesis, $(\sigma'_{\mathbf{P}}, t')$ is a timed winning strategy in $\mathbf{G}[q', t'(\gamma, q')]$. Hence we see that $(\sigma'_{\mathbf{P}}, t')$ must be a timed winning strategy in $\mathbf{G}[q, \gamma]$. □

The following proposition relates values of winning time labels to durations of embedded games and existence of winning strategies.

Proposition 4.3. *Let $\mathbf{G} = \mathbf{g}(\mathbf{V}, \mathbf{C}, A, q_0, \psi_{\mathbf{C}}, \psi_{\overline{\mathbf{C}}})$ be an embedded game, $\mathbf{P} \in \{\mathbf{E}, \mathbf{A}\}$ and $q \in \mathrm{St}$.*

1. *Assume $\mathbf{P} = \mathbf{C}$. We have $\mathcal{L}_{\mathbf{P}}(q) = \gamma < \Gamma$ iff there is a pair $(\sigma_{\mathbf{P}}, t)$ that is a timed winning strategy in $\mathbf{G}[q, \gamma']$ for all γ' s.t. $\gamma \leq \gamma' < \Gamma$, but there is no timed winning strategy for $\mathbf{P}$ in $\mathbf{G}[q, \gamma']$ for any $\gamma' < \gamma$.*

   We have $\mathcal{L}_{\mathbf{P}}(q) = \mathsf{lose}$ iff there is no timed winning strategy $(\sigma_{\mathbf{P}}, t)$ for $\mathbf{P}$ in $\mathbf{G}[q, \gamma]$ for any $\gamma < \Gamma$.

2. *Assume $\mathbf{P} \neq \mathbf{C}$. We have $\mathcal{L}_{\mathbf{P}}(q) = \gamma < \Gamma$ iff for every $\gamma' < \gamma$, there is some $\sigma_{\mathbf{P}}$ which is a winning strategy for $\mathbf{P}$ in $\mathbf{G}[q, \gamma']$, but there is no winning strategy for $\mathbf{P}$ in $\mathbf{G}[q, \gamma']$ for any γ' such that $\gamma \leq \gamma' < \Gamma$.*

   We have $\mathcal{L}_{\mathbf{P}}(q) = \mathsf{win}$ iff there is a strategy $\sigma_{\mathbf{P}}$ which is a winning strategy in $\mathbf{G}[q, \gamma]$ for every $\gamma < \Gamma$.

Proof. 1. We first examine the case where $\mathbf{P} = \mathbf{C}$:

Suppose first that $\mathcal{L}_{\mathbf{P}}(q) = \gamma < \Gamma$. By Definition 4.2 there is some strategy $\sigma_{\mathbf{P}}$ for which the strategy label $l(q, \sigma_{\mathbf{P}})$ is γ. Thus there is some timer t such that the pair $(\sigma_{\mathbf{P}}, t)$ is a timed winning strategy for $\mathbf{P}$ in $\mathbf{G}[q, \gamma]$. By Claim I there is a pair $(\sigma'_{\mathbf{P}}, t')$ which is a timed winning strategy in $\mathbf{G}[q, \gamma']$ for any γ' such that $\gamma \leq \gamma' < \Gamma$. If there existed some timed winning strategy $(\sigma''_{\mathbf{P}}, t'')$ for $\mathbf{P}$ in $\mathbf{G}[q, \gamma']$ for some $\gamma' < \gamma$, then we would have $l(q, \sigma''_{\mathbf{P}}) \leq \gamma'$ and thus $\mathcal{L}_{\mathbf{P}}(q) \leq \gamma' < \gamma$, which is a contradiction.

For the other direction, suppose that there is a pair $(\sigma_{\mathbf{P}}, q)$ which is a timed winning strategy in $\mathbf{G}[q, \gamma']$ for any γ' such that $\gamma \leq \gamma' < \Gamma$, but there is no timed winning strategy for $\mathbf{P}$ in $\mathbf{G}[q, \gamma']$ for any $\gamma' < \gamma$. Now $l(q, \sigma_{\mathbf{P}}) = \gamma$, and for any other strategy $\sigma'_{\mathbf{P}}$, we have either $l(q, \sigma'_{\mathbf{P}}) = \mathsf{lose}$ or $l(q, \sigma'_{\mathbf{P}}) \geq \gamma$. Hence the smallest ordinal value for the strategy labels at q is γ, and thus we have $\mathcal{L}_{\mathbf{P}}(q) = \gamma$.

If $\mathcal{L}_{\mathbf{P}}(q) = \mathsf{lose}$, then $l(q, \sigma_{\mathbf{P}}) = \mathsf{lose}$ for every strategy $\sigma_{\mathbf{P}}$ of $\mathbf{P}$. Hence none of the strategy-timer pairs $(\sigma_{\mathbf{P}}, t)$ is a timed winning strategy for $\mathbf{P}$ in $\mathbf{G}[q, \gamma]$ for any $\gamma < \Gamma$. Conversely, if there is no timed winning strategy $(\sigma_{\mathbf{P}}, t)$ in $\mathbf{G}[q, \gamma]$ for any $\gamma < \Gamma$, then we have $l(q, \sigma_{\mathbf{P}}) = \mathsf{lose}$ for every $\sigma_{\mathbf{P}}$ and thus $\mathcal{L}_{\mathbf{P}}(q) = \mathsf{lose}$.

2. We then examine the case where $\mathbf{P} \neq \mathbf{C}$:

If $\mathcal{L}_{\mathbf{P}}(q) = \mathsf{win}$, then $l(q, \sigma_{\mathbf{P}}) = \mathsf{win}$ for some strategy $\sigma_{\mathbf{P}}$, i.e., $\sigma_{\mathbf{P}}$ is a winning strategy in $\mathbf{G}[q, \gamma]$ for any time limit $\gamma < \Gamma$. Conversely, if there is some $\sigma_{\mathbf{P}}$ which is a winning strategy in $\mathbf{G}[q,\gamma]$ for every time limit $\gamma < \Gamma$, then $l(q,\sigma_{\mathbf{P}}) = \mathsf{win}$ and thus $\mathcal{L}_{\mathbf{P}}(q) = \mathsf{win}$.

Suppose that $\mathcal{L}_{\mathbf{P}}(q_0) = \gamma < \Gamma$, i.e., γ is the supremum of the strategy labels $l(q,\sigma_{\mathbf{P}})$. Suppose first that there is some $\sigma_{\mathbf{P}}$ for which $l(q_0,\sigma_{\mathbf{P}}) = \gamma$. Now $\sigma_{\mathbf{P}}$ is a winning strategy in $\mathbf{G}[q,\gamma']$ for any $\gamma' < \gamma$. Suppose then that there is no maximum value for the labels $l(q,\sigma_{\mathbf{P}})$, whence γ must be a limit ordinal. Let $\gamma' < \gamma$. Since γ is the least upper bound for the strategy labels $l(q,\sigma_{\mathbf{P}})$, there must be some strategy $\sigma'_{\mathbf{P}}$ for which $l(q,\sigma'_{\mathbf{P}}) \geq \gamma'+1$, as otherwise $\gamma'+1$ would be a lower upper bound for the strategy labels. We now observe that $\sigma'_{\mathbf{P}}$ is a winning strategy in $\mathbf{G}[q,\gamma']$.

If there existed a winning strategy $\sigma'_{\mathbf{P}}$ for $\mathbf{P}$ in $\mathbf{G}[q,\gamma']$ for some $\gamma' \geq \gamma$, then we would have $l(q,\sigma'_{\mathbf{P}}) > \gamma$, and thus $\mathcal{L}_{\mathbf{P}}(q) > \gamma$. Hence there cannot be any winning strategy for $\mathbf{P}$ in $\mathbf{G}[q,\gamma']$ for any γ' such that $\gamma \leq \gamma' < \Gamma$.

For the other direction, assume that for every $\gamma' < \gamma$, there exists a winning strategy for $\mathbf{P}$ in $\mathbf{G}[q,\gamma']$, but there exists no winning strategy for $\mathbf{P}$ in $\mathbf{G}[q,\gamma']$ for any γ' such that $\gamma \leq \gamma' < \Gamma$. If we had $\gamma < \mathcal{L}_{\mathbf{P}}(q_0) < \Gamma$, then, by the (already proved) other direction of the current claim, there would exist a winning strategy for $\mathbf{P}$ in $\mathbf{G}[q,\gamma]$, which is a contradiction. If we had $\mathcal{L}_{\mathbf{P}}(q) = \mathsf{win}$, we would again end up with a contradiction by the (already proved) result concerning the label win. If we had $\mathcal{L}_{\mathbf{P}}(q) < \gamma$, then, once again by the other direction of the current claim, there would not be any winning strategy for $\mathbf{P}$ in $\mathbf{G}[q,\mathcal{L}_{\mathbf{P}}(q)]$, a contradiction. Hence the only possibility left is that $\mathcal{L}_{\mathbf{P}}(q_0) = \gamma$. □

Winning time labels $\mathcal{L}_{\mathbf{P}}(q)$ of an embedded game are either ordinals less than the time limit bound Γ or labels win, lose. If we increased the value of Γ to some $\Gamma' > \Gamma$ and considered the values of winning time labels of the corresponding embedded game within the evaluation game $\mathcal{G}(\mathcal{M}, q_{in}, \varphi, \Gamma')$, then some of the labels that originally were win or lose, could now obtain ordinal values γ s.t. $\Gamma \leq \gamma < \Gamma'$. Other kinds of changes of labels would also be possible because the truth sets of the goal formulae $\psi_{\mathbf{C}}$ and $\psi_{\overline{\mathbf{C}}}$ could change. However, it is easy to see that if all ordinal valued labels stay strictly below Γ in all embedded games when going from Γ to Γ', then each label in fact remains the same in the transition.

We say that Γ is **stable** for an embedded game $\mathbf{G}$ if the winning time labels of the game cannot be altered by increasing Γ. We say that Γ is **globally stable** for a concurrent game model $\mathcal{M}$ if Γ is stable for *all* bounded embedded games within all evaluation games $\mathcal{G}(\mathcal{M}, q_{in}, \varphi, \Gamma)$. We will see later that there exists a globally stable time limit bound for every concurrent game model. When Γ is globally stable, its role is not so relevant anymore, since players would not benefit from the ability to choose arbitrarily high time limits. However, for technical reasons, we always need some time limit bound to avoid strategies becoming proper classes.

4.2 Canonical strategies for embedded games

Here we define so-called *canonical strategies*. They are guaranteed to be winning strategies whenever a winning strategy exists.

Definition 4.4. *Let $\mathbf{G} = \mathbf{g}(\mathbf{V}, \mathbf{C}, A, q_0, \psi_{\mathbf{C}}, \psi_{\overline{\mathbf{C}}})$ be an embedded game, let $\mathbf{P} \in \{\mathbf{E}, \mathbf{A}\}$ and suppose that $\mathbf{P} = \mathbf{C}$. We define the* ***canonical strategy*** *$\tau_{\mathbf{P}}$ and* ***canonical timer*** *t_{can} for $\mathbf{P}$ in $\mathbf{G}$ as follows.*

If $\mathcal{L}_{\mathbf{P}}(q) = \gamma$, then $\tau_{\mathbf{P}}(q) = \sigma_{\mathbf{P}}(\gamma, q)$ for some strategy $\sigma_{\mathbf{P}}$ for which there is a timer t such that $(\sigma_{\mathbf{P}}, t)$ is a timed winning strategy in $\mathbf{G}[q,\gamma']$ for all γ' s.t. $\gamma \leq \gamma' < \Gamma$. (Note that such a strategy exists by Proposition 4.3). If $\mathcal{L}_{\mathbf{P}}(q) = \mathsf{lose}$, then $\tau_{\mathbf{P}}(q) = \mathsf{void}$.

We define t_{can} for any pair (γ, q) ($\gamma < \Gamma$ is a limit ordinal and $q \in \mathrm{St}$) such that if $\mathcal{L}_{\mathbf{P}}(q) \neq \mathsf{lose}$ and $\mathcal{L}_{\mathbf{P}}(q) < \gamma$, then $t_{can}(\gamma, q) = \mathcal{L}_{\mathbf{P}}(q)$, and otherwise $t_{can}(\gamma, q) = \mathsf{void}$.

*We call the pair $(\tau_{\mathbf{P}}, t_{can})$ the **canonically timed strategy** (for the controller).*

Note that $\tau_{\mathbf{P}}$ is not necessarily unique since we may have to choose one from several strategies. However, these choices are all equally good for our purposes. Note that when $\mathbf{P} = \mathbf{C}$, the canonical strategy depends on states only and can thus be used in both unbounded and bounded embedded games. Also note that $\tau_{\mathbf{P}}$ and t_{can} are defined such that they depend neither on the initial state q_0 nor the initial time limit γ_0. We will see that if $\mathbf{P}$ can win $\mathbf{G}[\gamma_0]$ for some $\gamma_0 < \Gamma$, then $\mathbf{P}$ wins $\mathbf{G}[\gamma_0]$ with $(\tau_{\mathbf{P}}, t_{can})$. Since the canonical strategy follows the strategies with the lowest strategy labels, it can be seen, in some sense, as optimal for winning the game as fast as possible.

Definition 4.5. *Let $\mathbf{G} = \mathbf{g}(\mathbf{V}, \mathbf{C}, A, q_0, \psi_{\mathbf{C}}, \psi_{\overline{\mathbf{C}}})$ be an embedded game, let $\mathbf{P} \in \{\mathbf{E}, \mathbf{A}\}$ and assume that $\mathbf{P} \neq \mathbf{C}$. We define the **canonical strategy** $\tau_{\mathbf{P}}$ for $\mathbf{P}$ in $\mathbf{G}[\gamma_0]$ for all $\gamma_0 < \Gamma$. We define $\tau_{\mathbf{P}}$ at every configuration (γ, q), where $\gamma < \Gamma$ and $q \in \mathrm{St}$, as follows.*

If $\mathcal{L}_{\mathbf{P}}(q) = \mathsf{win}$, then $\tau_{\mathbf{P}}(\gamma, q) = \sigma_{\mathbf{P}}(\gamma, q)$ for some strategy $\sigma_{\mathbf{P}}$ for which the strategy label $l(q, \sigma_{\mathbf{P}})$ is win (see Def 4.2). Else, if $\mathcal{L}_{\mathbf{P}}(q) = \gamma'$ and $\gamma' > \gamma$, then $\tau_{\mathbf{P}}(\gamma, q) = \sigma_{\mathbf{P}}(\gamma, q)$ for some $\sigma_{\mathbf{P}}$ for which $l(q, \sigma_{\mathbf{P}}) > \gamma$. Such a strategy exists by Definition 4.2. Otherwise $\tau_{\mathbf{P}}(\gamma, q) = \mathsf{void}$.

*We also define, for every $n < \omega$, the n-**canonical strategy** $\tau^n_{\mathbf{P}}$ for $\mathbf{P}$ in $\mathbf{G}$ and the ∞-**canonical strategy** $\tau^\infty_{\mathbf{P}}$ for $\mathbf{P}$ in $\mathbf{G}$. These are defined for each $q \in \mathrm{St}$ as follows.*

If $\mathcal{L}_{\mathbf{P}}(q) \geq \omega$ or $\mathcal{L}_{\mathbf{P}}(q) = \mathsf{win}$, then $\tau^n_{\mathbf{P}}(q) = \tau_{\mathbf{P}}(n, q)$. Else, if $\mathcal{L}_{\mathbf{P}}(q) = m > 0$, then $\tau^n_{\mathbf{P}}(q) = \sigma_{\mathbf{P}}(m-1, q)$ for some $\sigma_{\mathbf{P}}$ for which $l(q, \sigma_{\mathbf{P}}) = m$. Otherwise $\tau^n_{\mathbf{P}}(q) = \mathsf{void}$.

If $\mathcal{L}_{\mathbf{P}}(q) = \mathsf{win}$, then $\tau^\infty_{\mathbf{P}}(q) = \tau_{\mathbf{P}}(\Gamma-1, q)$, and otherwise $\tau^\infty_{\mathbf{P}}(q) = \mathsf{void}$. Note that to be able to define $\tau^\infty_{\mathbf{P}}$, we have to assume that Γ is a successor ordinal.

When $\mathbf{P} \neq \mathbf{C}$, the canonical strategy $\tau_{\mathbf{P}}$ depends on time limits, and thus it cannot be used in unbounded embedded games. However, both n-canonical and ∞-canonical strategies depend on states only. We fix the notation such that from now on $\tau_{\mathbf{P}}$, $\tau^n_{\mathbf{P}}$ and $\tau^\infty_{\mathbf{P}}$ will always denote canonical strategies (of the respective type) for the player $\mathbf{P}$.

Definition 4.6. *Let $\mathbf{G} = \mathbf{g}(\mathbf{V}, \mathbf{C}, A, q_0, \psi_{\mathbf{C}}, \psi_{\overline{\mathbf{C}}})$ be an embedded game. Let $\sigma_{\mathbf{P}}$ be a strategy in $\mathbf{G}[\gamma_0]$ ($\gamma_0 < \Gamma$). Suppose that (γ, q) is such a configuration that $\sigma_{\mathbf{P}}(\gamma, q)$ is either a tuple of actions for A or some response function for $\overline{A}$. We say that set $Q \subseteq \mathrm{St}$ is **forced** by $\sigma_{\mathbf{P}}(\gamma, q)$ if for each $q' \in \mathrm{St}$, it holds that $q' \in Q$ if and only if there is some play with $\sigma_{\mathbf{P}}$ from (γ, q) such that the next configuration is (γ', q') for some γ'. We use the same terminology for the set forced by $\sigma_{\mathbf{P}}(q)$ when $\sigma_{\mathbf{P}}$ depends on states only.*

The following lemma shows that the canonical strategy is guaranteed to be a winning strategy always when a winning strategy exists. This is quite easy to see since the canonical strategy follows the optimal winning time labels at each state by its definition.

Lemma 4.7. *Let $\mathbf{G} = \mathbf{g}(\mathbf{V}, \mathbf{C}, A, q_0, \psi_{\mathbf{C}}, \psi_{\overline{\mathbf{C}}})$ be an embedded game. Let $\mathbf{P} \in \{\mathbf{E}, \mathbf{A}\}$ and $\gamma_0 < \Gamma$.*

1. *Suppose that $\mathbf{P} = \mathbf{C}$. If $\mathbf{P}$ has a timed winning strategy $(\sigma_{\mathbf{P}}, t)$ in $\mathbf{G}[\gamma_0]$, then $(\tau_{\mathbf{P}}, t_{can})$ is a timed winning strategy for $\mathbf{P}$ in $\mathbf{G}[\gamma_0]$.*

2. *Suppose that $\mathbf{P} \neq \mathbf{C}$. If $\mathbf{P}$ has a winning strategy $\sigma_{\mathbf{P}}$ in $\mathbf{G}[\gamma_0]$, then $\tau_{\mathbf{P}}$ is a winning strategy for $\mathbf{P}$ in $\mathbf{G}[\gamma_0]$.*

Proof. We first discuss the case where $\mathbf{P} = \mathbf{C}$. We will prove by transfinite induction on $\gamma < \Gamma$ that for every $q \in \mathrm{St}$, if $\mathbf{P}$ has a timed winning strategy in $\mathbf{G}[q, \gamma]$, then $(\tau_{\mathbf{P}}, t_{can})$ is a timed winning strategy in $\mathbf{G}[q, \gamma]$. We let the induction hypothesis be that the claim holds for every $\gamma' < \gamma$ and suppose that $\mathbf{P}$ has a timed winning strategy in $\mathbf{G}[q, \gamma]$.

By Proposition 4.3 , we have $\mathcal{L}_{\mathbf{P}}(q) = \delta$ for some $\delta \leq \gamma$. Let $\sigma'_{\mathbf{P}}$ be a strategy for which $\tau_{\mathbf{P}}(q) = \sigma'_{\mathbf{P}}(\delta, q)$ and there is a timer t' such that $(\sigma'_{\mathbf{P}}, t')$ is a timed winning strategy in $\mathbf{G}[q, \delta']$ for all δ' such that $\delta \leq \delta' < \Gamma$. (Such a strategy exists by the definition of the canonical strategy $\tau_{\mathbf{P}}$.) Suppose first that $\delta = 0$, whence $(\sigma'_{\mathbf{P}}, t')$ is a timed winning strategy in $\mathbf{G}[q, 0]$. Now $\tau_{\mathbf{P}}(q) = \sigma'_{\mathbf{P}}(0, q) = \psi_{\mathbf{C}}$ and thus $(\tau_{\mathbf{P}}, t_{can})$ is a timed winning strategy in $\mathbf{G}[q, \gamma]$.

Suppose then that $\delta > 0$, whence $\sigma'_{\mathbf{P}}(\delta, q)$ must be either some tuple of actions for the coalition A or some response function for the coalition $\overline{A}$. Let $Q \subseteq \mathrm{St}$ be the set of states that is forced by $\sigma'_{\mathbf{P}}(\delta, q)$ and let $q' \in Q$. Since $(\sigma'_{\mathbf{P}}, t')$ is a timed winning strategy in $\mathbf{G}[q, \delta]$, there is $\delta' < \delta$ such that $(\sigma'_{\mathbf{P}}, t')$ is a timed winning strategy in $\mathbf{G}[q', \delta']$ (if δ is a limit ordinal, then $\delta' = t'(\delta, q')$, and else $\delta' = \delta - 1$). Since $\delta \leq \gamma$, we have $\delta' < \gamma$.

Suppose first that γ is a successor ordinal. Since we have $\delta' \leq \gamma - 1$, there is a timed winning strategy $(\sigma''_{\mathbf{P}}, t'')$ in $\mathbf{G}[q', \gamma - 1]$ by Claim I . Thus, by the induction hypothesis, $(\tau_{\mathbf{P}}, t_{can})$ is a timed winning strategy in $\mathbf{G}[q', \gamma - 1]$. Since this holds for every $q' \in Q$, we see that $(\tau_{\mathbf{P}}, t_{can})$ is a timed winning strategy in $\mathbf{G}[q, \gamma]$.

Suppose then that γ is a limit ordinal. Since $(\sigma'_{\mathbf{P}}, t')$ is a timed winning strategy in $\mathbf{G}[q', \delta']$, by Proposition 4.3 we must have $\mathcal{L}_{\mathbf{P}}(q') \leq \delta' < \gamma$. Thus, by the definition of the canonical timer, $t_{can}(\gamma, q') = \mathcal{L}_{\mathbf{P}}(q')$. By proposition 4.3 there is a timed winning strategy $(\sigma''_{\mathbf{P}}, t'')$ in $\mathbf{G}[q', \mathcal{L}_{\mathbf{P}}(q')]$. Thus by the induction hypothesis $(\tau_{\mathbf{P}}, t_{can})$ is a timed winning strategy in $\mathbf{G}[q', \mathcal{L}_{\mathbf{P}}(q')]$. Since this holds for every $q' \in Q$ and $t_{can}(\gamma, q') = \mathcal{L}_{\mathbf{P}}(q')$ for every $q' \in Q$, we see that $(\tau_{\mathbf{P}}, t_{can})$ is a timed winning strategy in $\mathbf{G}[q, \gamma]$.

We then discuss the case where $\mathbf{P} \neq \mathbf{C}$. We will prove by transfinite induction on $\gamma < \Gamma$ that for every $q \in Q$, if $\mathbf{P}$ has a winning strategy in $\mathbf{G}[q, \gamma]$, then $\tau_{\mathbf{P}}$ is a winning strategy in $\mathbf{G}[q, \gamma]$.

Suppose first that $\gamma = 0$ and that $\mathbf{P}$ has a winning strategy $\sigma_{\mathbf{P}}$ in $\mathbf{G}[q, 0]$. Now, since with the time limit 0 the game will end at q immediately, every strategy of $\mathbf{P}$ will be a winning strategy in $\mathbf{G}[q, 0]$. Hence, in particular, $\tau_{\mathbf{P}}$ is a winning strategy in $\mathbf{G}[q, 0]$.

Suppose that the claim holds for every $\gamma' < \gamma$ and that $\mathbf{P}$ has a winning strategy $\sigma_{\mathbf{P}}$ in $\mathbf{G}[q, \gamma]$. By Proposition 4.3, we have either $\mathcal{L}_{\mathbf{P}}(q) = \mathsf{win}$ or $\mathcal{L}_{\mathbf{P}}(q) > \gamma$. Assume first that $\mathcal{L}_{\mathbf{P}}(q) = \mathsf{win}$. Let $\sigma_{\mathbf{P}}$ be a strategy for which $l(q, \sigma_{\mathbf{P}}) = \mathsf{win}$ and $\tau_{\mathbf{P}}(\gamma, q) = \sigma_{\mathbf{P}}(\gamma, q)$. (Such a strategy exists by the definition of $\tau_{\mathbf{P}}$.) Let $Q \subseteq \mathrm{St}$ be the set of states forced by $\sigma_{\mathbf{P}}(\gamma, q)$ and let $q' \in Q$. Since $l(q, \sigma_{\mathbf{P}}) = \mathsf{win}$, the strategy $\sigma_{\mathbf{P}}$ is a winning strategy in $\mathbf{G}[q, \delta]$ for every $\delta < \Gamma$, and therefore, as we have $\gamma < \Gamma$, the strategy $\sigma_{\mathbf{P}}$ must also be a winning strategy in $\mathbf{G}[q', \gamma']$ for every $\gamma' < \gamma$. Thus there is a winning strategy in $\mathbf{G}[q', \gamma']$ for every $\gamma' < \gamma$ and every $q' \in Q$. Hence, by the induction hypothesis, $\tau_{\mathbf{P}}$ is a winning strategy in $\mathbf{G}[q', \gamma']$ for every $\gamma' < \gamma$ and $q' \in Q$. Therefore we observe that $\tau_{\mathbf{P}}$ is also a winning strategy in $\mathbf{G}[q, \gamma]$.

Suppose then that $\mathcal{L}_{\mathbf{P}}(q) = \delta > \gamma$. Let $\sigma_{\mathbf{P}}$ be a strategy for which $l(q, \sigma_{\mathbf{P}}) > \gamma$ and $\tau_{\mathbf{P}}(\gamma, q) = \sigma_{\mathbf{P}}(\gamma, q)$. (Such a strategy exists by the definition of $\tau_{\mathbf{P}}$.) Let $Q \subseteq \mathrm{St}$ be the set of states that is forced by $\sigma_{\mathbf{P}}(\gamma, q)$ and let $q' \in Q$. Since $\sigma_{\mathbf{P}}$ is a winning strategy in $\mathbf{G}[q, \delta']$ for every $\delta' < l(q, \sigma_{\mathbf{P}})$ and since we have $\gamma < l(q, \sigma_{\mathbf{P}})$, the strategy $\sigma_{\mathbf{P}}$ must also be a winning strategy in $\mathbf{G}[q', \gamma']$ for every $\gamma' < \gamma$. Hence we can deduce, as before, that $\tau_{\mathbf{P}}$ is a winning strategy in the games for the configurations over Q that follow the configuration (γ, q), and thus $\tau_{\mathbf{P}}$ is also a winning strategy in $\mathbf{G}[q, \gamma]$. $\square$

By the first claim of the previous proposition, we see that it suffices to consider those strategies of player $\mathbf{C}$ which are independent of time limits. The following lemma shows that the same holds for the player $\overline{\mathbf{C}}$ in bounded embedded games with a finite time limit. The key here will be the use of n-canonical strategies.

Lemma 4.8. *Let $\mathbf{G} = \mathbf{g}(\mathbf{V}, \mathbf{C}, A, q_0, \psi_{\mathbf{C}}, \psi_{\overline{\mathbf{C}}})$ be an embedded game, let $\mathbf{P} \in \{\mathbf{E}, \mathbf{A}\}$ and assume that $\mathbf{P} \neq \mathbf{C}$.*

Let $n < \omega$. Now, if $\mathbf{P}$ has a winning strategy $\sigma_{\mathbf{P}}$ in $\mathbf{G}[m]$ for some $m \leq n$, then $\tau^n_{\mathbf{P}}$ is a winning strategy in $\mathbf{G}[m]$.

Proof. We will prove by induction on $m \leq n$ that for any $q \in \mathrm{St}$, if $\mathbf{P}$ has winning strategy in $\mathbf{G}[q, m]$, then $\tau^n_{\mathbf{P}}$ is a winning strategy in $\mathbf{G}[q, m]$. If $m = 0$ and $\mathbf{P}$ has a winning strategy $\sigma_{\mathbf{P}}$ in $\mathbf{G}[q, 0]$, then every strategy of $\mathbf{P}$ will be a winning strategy in $\mathbf{G}[q, 0]$. Hence, in particular, $\tau^n_{\mathbf{P}}$ is a winning strategy in $\mathbf{G}[q, 0]$.

Suppose then that the claim holds for $m-1$ and that $\mathbf{P}$ has a winning strategy in $\mathbf{G}[q, m]$. Thus we have $\mathcal{L}_{\mathbf{P}}(q) > m$ or $\mathcal{L}_{\mathbf{P}}(q) = \mathsf{win}$. Suppose first that $\mathcal{L}_{\mathbf{P}}(q) = m' < \omega$, and let $\sigma_{\mathbf{P}}$ be a strategy such that $l(q, \sigma_{\mathbf{P}}) = m'$ and $\tau^n_{\mathbf{P}}(q) = \sigma_{\mathbf{P}}(m'-1, q)$. (Such a strategy $\sigma_{\mathbf{P}}$ exists by the definition of the n-canonical strategy $\tau^n_{\mathbf{P}}$.) Let $Q \subseteq \mathrm{St}$ be the set of states forced by $\sigma_{\mathbf{P}}(m'-1, q)$. Since $m' > m$, the strategy $\sigma_{\mathbf{P}}$ must be a winning strategy in $\mathbf{G}[q, m]$, and thus it will also be a winning strategy in $\mathbf{G}[q', m-1]$ for every $q' \in Q$. Thus, by the induction hypothesis, $\tau^n_{\mathbf{P}}$ is a winning strategy in $\mathbf{G}[q', m-1]$ for every $q' \in Q$. Therefore we observe that $\tau^n_{\mathbf{P}}$ wil also be a winning strategy in $\mathbf{G}[q', m]$.

Suppose then that $\mathcal{L}_{\mathbf{P}}(q) \geq \omega$ or $\mathcal{L}_{\mathbf{P}}(q) = \mathsf{win}$, and let $\sigma_{\mathbf{P}}$ be a strategy such that $l(q, \sigma_{\mathbf{P}}) \in \{\mathsf{win}\} \cup \{\gamma < \Gamma \mid \gamma > n\}$ and $\tau^n_{\mathbf{P}}(q) = \tau_{\mathbf{P}}(n, q) = \sigma_{\mathbf{P}}(n, q)$. (Recall Definition 4.5; the strategy $\sigma_{\mathbf{P}}$ exists by the definitions of $\tau^n_{\mathbf{P}}$ and $\tau_{\mathbf{P}}$.) Let $Q \subseteq \mathrm{St}$ be the set of states that is forced by $\sigma_{\mathbf{P}}(n, q)$. Since $m \leq n$, the strategy $\sigma_{\mathbf{P}}$ is a winning strategy in $\mathbf{G}[q, m]$, and thus it is also a winning strategy in $\mathbf{G}[q', m-1]$ for every $q' \in Q$. Hence, by the induction hypothesis, $\tau^n_{\mathbf{P}}$ is a winning strategy in $\mathbf{G}[q', m-1]$ for every $q' \in Q$. Thus we observe that $\tau^n_{\mathbf{P}}$ is a winning strategy in $\mathbf{G}[q, m]$. ☐

Example 4.9. *In the cases where $\mathcal{L}_{\overline{\mathbf{C}}}(q) = \omega$, the player $\overline{\mathbf{C}}$ can win the game with any time limit $n < \omega$, but there is no single strategy that would win for every n. But if $\overline{\mathbf{C}}$ knows that the initial time limit is (at most) m, then (s)he knows that the m-canonical strategy will be her/his winning strategy. Therefore $\overline{\mathbf{C}}$ needs to know the time limit when* selecting *the strategy, but not when using it (since n-canonical strategies are independent of time limits).*

We will see later that if the time limit bound Γ is sufficiently large and there exists a winning strategy for each $\gamma < \Gamma$, then $\tau^{\infty}_{\mathbf{P}}$ will be a winning strategy for each $\gamma < \Gamma$.

4.3 Determinacy of embedded games

The following correspondence between the winning time labels of $\mathbf{C}$ and $\overline{\mathbf{C}}$ will be the key for proving determinacy of bounded embedded games.

Proposition 4.10. *Let $\mathbf{G} = \mathbf{g}(\mathbf{V}, \mathbf{C}, A, q_0, \psi_{\mathbf{C}}, \psi_{\overline{\mathbf{C}}})$ be an embedded game. The equivalence $\mathcal{L}_{\mathbf{C}}(q) = \gamma$ iff $\mathcal{L}_{\overline{\mathbf{C}}}(q) = \gamma$ holds for each state $q \in \mathrm{St}$ and each ordinal $\gamma < \Gamma$.*

Proof. We prove this claim by transfinite induction on the ordinal $\gamma < \Gamma$. We first prove the special case where $\gamma = 0$. Let $q \in \mathrm{St}$ and suppose first that $\mathcal{L}_{\mathbf{C}}(q) = 0$, whence by Proposition 4.3 the player $\mathbf{C}$ has a timed winning strategy $(\sigma_{\mathbf{C}}, t)$ in $\mathbf{G}[q, 0]$. This is possible only if the exit position $(\mathbf{C}, q, \psi_{\mathbf{C}})$ is a winning position for $\mathbf{C}$. In that case $\overline{\mathbf{C}}$ loses $\mathbf{G}[q, \gamma']$ with any time limit γ' and thus $\mathcal{L}_{\overline{\mathbf{C}}}(q) = 0$. Suppose then that $\mathcal{L}_{\overline{\mathbf{C}}}(q) = 0$, whence there no winning strategy for $\overline{\mathbf{C}}$ in $\mathbf{G}[q,0]$. This is possible only if $\mathbf{C}$ wins the game at the exit position $(\mathbf{C}, q, \psi_{\mathbf{C}})$, whence $\mathcal{L}_{\mathbf{C}}(q) = 0$.

We then suppose that $\gamma > 0$ and let the induction hypothesis be that the claim holds for every ordinal $\gamma' < \gamma$. Suppose first that $\mathcal{L}_{\mathbf{C}}(q) = \gamma$. By Proposition 4.7 there is a timed winning strategy $(\sigma_{\mathbf{P}}, t)$ in $\mathbf{G}[q,\gamma]$. Thus $\overline{\mathbf{C}}$ cannot have a winning strategy in $\mathbf{G}[q,\gamma]$ and thus by Proposition 4.3 we must have $\mathcal{L}_{\overline{\mathbf{C}}}(q) \leq \gamma$. If we had $\mathcal{L}_{\overline{\mathbf{C}}}(q) < \gamma$, then, by the induction hypothesis, we would have $\mathcal{L}_{\mathbf{C}}(q) = \mathcal{L}_{\overline{\mathbf{C}}}(q) < \gamma$. This is a contradiction. Hence $\mathcal{L}_{\overline{\mathbf{C}}}(q) = \gamma$.

Suppose then that $\mathcal{L}_{\overline{\mathbf{C}}}(q) = \gamma$. We will next show that for any strategy $\sigma_{\overline{\mathbf{C}}}$ and any set $Q \subseteq \mathrm{St}$ forced by $\sigma_{\overline{\mathbf{C}}}(\gamma, q)$, there is a state $q' \in Q$ for which $\mathcal{L}_{\overline{\mathbf{C}}}(q') < \gamma$. For the sake of contradiction, suppose that there is a strategy $\sigma_{\overline{\mathbf{C}}}$ such that for the set $Q \subseteq \mathrm{St}$ forced by $\sigma_{\mathbf{C}}(\gamma, q)$, we have $\mathcal{L}_{\overline{\mathbf{C}}}(q') \geq \gamma$ or $\mathcal{L}_{\overline{\mathbf{C}}}(q') = \mathsf{win}$ for every $q' \in Q$. We formulate the following strategy $\sigma'_{\overline{\mathbf{C}}}$ for $\overline{\mathbf{C}}$ in the embedded game $\mathbf{G}[q,\gamma]$:

$$\sigma'_{\overline{\mathbf{C}}}(\delta, q) = \sigma_{\overline{\mathbf{C}}}(\gamma, q) \text{ for every } \delta \leq \gamma,$$
$$\sigma'_{\overline{\mathbf{C}}}(\delta, q') = \tau_{\overline{\mathbf{C}}}(\delta, q') \text{ for every } \delta \leq \gamma \text{ and } q' \in \mathrm{St} \setminus \{q\}.$$

Since $\mathcal{L}_{\overline{\mathbf{C}}}(q') \geq \gamma$ or $\mathcal{L}_{\overline{\mathbf{C}}}(q') = \mathsf{win}$ for every $q' \in Q$, by Proposition 4.3, the canonical strategy $\tau_{\overline{\mathbf{C}}}$ is a winning strategy in $\mathbf{G}[q', \delta]$ for any $q' \in Q$ and $\delta < \gamma$. Thus it is easy to see that $\sigma'_{\overline{\mathbf{C}}}$ is a winning strategy in $\mathbf{G}[q,\gamma]$. Hence by Proposition 4.3 we must have $\mathcal{L}_{\overline{\mathbf{C}}}(q) > \gamma$ or $\mathcal{L}_{\overline{\mathbf{C}}}(q) = \mathsf{win}$, which is a contradiction. Therefore, we infer that

$$\text{For any strategy } \sigma_{\overline{\mathbf{C}}} \text{ and } Q \subseteq \mathrm{St} \text{ forced by } \sigma_{\overline{\mathbf{C}}}(\gamma, q),$$
$$\text{there is some } q' \in Q \text{ such that } \mathcal{L}_{\overline{\mathbf{C}}}(q') < \gamma. \qquad (\star)$$

Let $Q' := \{q' \in \mathrm{St} \mid \mathcal{L}_{\overline{\mathbf{C}}}(q') < \gamma\}$. By the induction hypothesis, we have $\mathcal{L}_{\mathbf{C}}(q') < \gamma$ for every $q' \in Q'$. We will show that $\mathbf{C}$ can play in such a way at q that all possible successor states will be in Q'. Suppose first that $\mathbf{C} = \overline{\mathbf{V}}$. Since for every $\vec{\alpha} \in \mathsf{action}(q, A)$ there is some strategy $\sigma_{\overline{\mathbf{C}}}$ s.t. $\sigma_{\overline{\mathbf{C}}}(\gamma, q) = \vec{\alpha}$, we infer by $(\star)$ that there is some response function for $\overline{A}$ which forces the next state to be in Q'.

Suppose then that $\mathbf{C} = \mathbf{V}$. If for every $\vec{\alpha} \in \mathsf{action}(q, A)$ there existed some $\vec{\beta} \in \mathsf{action}(q, \overline{A})$ such that the outcome state of these actions was not in Q', then there would be some strategy $\sigma_{\overline{\mathbf{C}}}$ such that the set forced by $\sigma_{\overline{\mathbf{C}}}(\gamma, q)$ would not intersect Q'. This is a contradiction by $(\star)$, and thus there is some tuple of actions for A at q such that all possible successor states will be in Q'.

We next formulate a strategy $\sigma_{\mathbf{C}}$ for $\mathbf{C}$ in $\mathbf{G}[q,\gamma]$. By the description above, we can define $\sigma_{\mathbf{C}}$ for every configuration (γ', q), where $\gamma' \leq \gamma$, in such a way that the set forced by $\sigma_{\mathbf{C}}(\gamma', q)$ is a subset of Q'. For all configurations (γ', q'), where $\gamma' \leq \gamma$ and $q' \in \mathrm{St} \setminus \{q\}$, we define $\sigma_{\mathbf{C}}(\gamma', q') = \tau_{\mathbf{C}}(q')$.

Since $\mathcal{L}_{\mathbf{C}}(q') < \gamma$ for every $q' \in Q'$, we infer that we have $t_{can}(\gamma, q') = \mathcal{L}_{\mathbf{C}}(q')$ for every $q' \in Q$. By Propositions 4.3 and 4.7, $(\tau_{\mathbf{C}}, t_{can})$ is a timed winning strategy in $\mathbf{G}[q', t_{can}(\gamma, q')]$ for any $q' \in Q'$. Thus it is easy to see that $(\sigma_{\mathbf{C}}, t_{can})$ is a timed winning strategy in $\mathbf{G}[q,\gamma]$. However, since $\mathcal{L}_{\overline{\mathbf{C}}}(q) = \gamma$, we conclude, using the induction hypothesis, that there cannot be a timed winning strategy for $\mathbf{C}$ in $\mathbf{G}[q,\gamma']$ for any $\gamma' < \gamma$. Hence by Proposition 4.3 we must have $\mathcal{L}_{\mathbf{C}}(q) = \gamma$. ☐

Apart from ordinal values that are less than the bound Γ, the only possible winning time label for $\mathbf{C}$ is the label lose. For $\overline{\mathbf{C}}$, the only non-ordinal value is win. Hence by the previous proposition, we also have $\mathcal{L}_{\mathbf{C}}(q) = \mathsf{lose}$ if and only if $\mathcal{L}_{\overline{\mathbf{C}}}(q) = \mathsf{win}$. We are now ready to prove that all bounded embedded games are determined.

Proposition 4.11. *The controller* $\mathbf{C}$ *has a timed winning strategy in a bounded embedded game* $\mathbf{g}(\mathbf{V}, \mathbf{C}, A, q_0, \psi_{\mathbf{C}}, \psi_{\overline{\mathbf{C}}})[\gamma_0]$ *iff* $\overline{\mathbf{C}}$ *does not have a winning strategy in that game.*

Proof. If $\mathcal{L}_{\mathbf{C}}(q_0) = \mathsf{lose}$, then $\mathcal{L}_{\overline{\mathbf{C}}}(q_0) = \mathsf{win}$, whence by Proposition 4.3, the player $\overline{\mathbf{C}}$ has a winning strategy and $\mathbf{C}$ does not have a timed winning strategy. Else $\mathcal{L}_{\mathbf{C}}(q_0) = \gamma$ for some $\gamma < \Gamma$. Now, by Proposition 4.10, also $\mathcal{L}_{\overline{\mathbf{C}}}(q_0) = \gamma$. If $\gamma \leq \gamma_0$, then by Proposition 4.3 the player $\mathbf{C}$ has a timed winning strategy, while $\overline{\mathbf{C}}$ does not have a winning strategy. Analogously, if $\gamma > \gamma_0$, then $\overline{\mathbf{C}}$ has a winning strategy, while $\mathbf{C}$ does not have a timed winning strategy. □

4.4 Finding stable time limit bounds

Definition 4.12. *Let* $\mathcal{M}$ *be a CGM and let* $q \in \mathrm{St}$. *We define the* ***branching degree of*** q, $\mathsf{BD}(q)$, *as the cardinality of the set of states accessible from* q *with a single transition:*

$$\mathsf{BD}(q) := \mathsf{card}(\{o(q, \vec{\alpha}) \mid \vec{\alpha} \in \mathsf{action}(\mathbb{A}\mathrm{gt}, q)\}).$$

We define the ***infinite branching bound of*** $\mathcal{M}$, $\mathsf{IBB}(\mathcal{M})$, *as the smallest infinite cardinal* κ *such that* $\kappa > \mathsf{BD}(q)$ *for every* $q \in \mathrm{St}$.

With this definition $\mathsf{IBB}(\mathcal{M}) = \omega$ iff $\mathcal{M}$ is image finite. Also note that if for the of available actions $\mathsf{card}(\mathrm{Act}) < \kappa$, then $\mathsf{IBB}(\mathcal{M}) \leq \kappa$. We will see that the value of $\mathsf{IBB}(\mathcal{M})$ is closely related to the sizes of a globally stable time limit bounds for $\mathcal{M}$. The following lemma shows an important correspondence between the canonical strategy and the winning time labels of the controller.

Lemma 4.13. *Let* $\mathbf{G} = \mathbf{g}(\mathbf{V}, \mathbf{C}, A, q_0, \psi_{\mathbf{C}}, \psi_{\overline{\mathbf{C}}})$ *be an embedded game and* $\mathbf{P} = \mathbf{C}$. *Now the following holds for every* $q \in \mathrm{St}$: *If* $\mathcal{L}_{\mathbf{P}}(q) = \gamma > 0$ *and* $Q \subseteq \mathrm{St}$ *is forced by* $\tau_{\mathbf{P}}(q)$, *we have* $\mathcal{L}_{\mathbf{P}}(q') < \gamma$ *for every* $q' \in Q$, *and*

- $\max\{\mathcal{L}_{\mathbf{P}}(q') \mid q' \in Q\} = \gamma - 1$ *if* γ *is a successor ordinal,*
- $\sup\{\mathcal{L}_{\mathbf{P}}(q') \mid q' \in Q\} = \gamma$ *if* γ *is a limit ordinal.*

Proof. Suppose that $\mathcal{L}_{\mathbf{P}}(q) = \gamma > 0$. Since $\mathcal{L}_{\mathbf{P}}(q) \neq 0$, canonically timed strategy $(\tau_{\mathbf{P}}, t_{can})$ is not a timed winning strategy for $\mathbf{P}$ in $\mathbf{G}[q, 0]$. Therefore $\tau_{\mathbf{P}}(q)$ is either some tuple of actions for A or some response function for $\overline{A}$. Let $Q \subseteq \mathrm{St}$ be the set of states that is forced by $\tau_{\mathbf{P}}(q)$. We first show that $\mathcal{L}_{\mathbf{P}}(q') < \gamma$ for every $q' \in Q$. Since $(\tau_{\mathbf{P}}, t_{can})$ is a timed winning strategy in $\mathbf{G}[q, \gamma]$, it must also be a timed winning strategy in $\mathbf{G}[q', t_{can}(\gamma, q')]$ for every $q' \in Q$. Hence by the definition of canonical timer $\gamma > t_{can}(\gamma, q') = \mathcal{L}_{\mathbf{P}}(q')$ for every $q' \in Q$.

Suppose first that γ is a successor ordinal. If we would have $\mathcal{L}_{\mathbf{P}}(q') < \gamma-1$ for every $q' \in Q$, then $(\tau_{\mathbf{P}}, t_{can})$ would be a winning strategy in $\mathbf{G}[q, \gamma-1]$, and thus we would have $\mathcal{L}_{\mathbf{P}}(q) \leq \gamma-1$. Hence $\gamma > \mathcal{L}_{\mathbf{P}}(q') \geq \gamma-1$ for some $q' \in Q$ and thus $\max\{\mathcal{L}_{\mathbf{P}}(q') \mid q' \in Q\} = \gamma-1$. Suppose then that γ is a limit ordinal. If $\gamma' < \gamma$ would be an upper bound for the winning time labels in Q, then we would have $\mathcal{L}_{\mathbf{P}}(q') < \gamma'+1$ for every $q' \in Q$. Hence $(\tau_{\mathbf{P}}, t_{can})$ would be a timed winning strategy in $\mathbf{G}[q, \gamma'+1]$, and thus we would have $\mathcal{L}_{\mathbf{P}}(q) \leq \gamma'+1 < \gamma$. This is impossible and thus $\sup\{\mathcal{L}_{\mathbf{P}}(q') \mid q' \in Q\} = \gamma$. □

The following lemma shows that if a certain ordinal valued winning time label exist for an embedded game, then all the smaller winning time labels must exist for that game as well.

Lemma 4.14. *Let* $\mathbf{G} = \mathbf{g}(\mathbf{V}, \mathbf{C}, A, q_0, \psi_{\mathbf{C}}, \psi_{\overline{\mathbf{C}}})$ *be an embedded game and* $\gamma < \Gamma$ *an ordinal. Assume that* $\mathcal{L}_{\mathbf{P}}(q) = \gamma$ *for some* $q \in \mathrm{St}$ *and* $\mathbf{P} \in \{\mathbf{E}, \mathbf{A}\}$*. Now for every* $\delta \leq \gamma$ *there is a state* q_δ *for which* $\mathcal{L}_{\mathbf{C}}(q_\delta) = \delta$*.*

Proof. We prove the claim by transfinite induction on $\gamma < \Gamma$ for every $q \in \mathrm{St}$. We let the induction hypothesis be that the claim holds for every $\gamma' < \gamma$ and suppose that either of the players has winning time label γ at some state q. By Proposition 4.10 , we have $\mathcal{L}_{\mathbf{C}}(q) = \gamma$.

Assume that $\delta < \gamma$. If γ is a successor ordinal, then by Lemma 4.13 there is a state q' s.t. $\mathcal{L}_{\mathbf{C}}(q') = \gamma - 1$. Since $\delta \leq \gamma - 1$, by the induction hypothesis there is a state q_δ for which $\mathcal{L}_{\mathbf{C}}(q_\delta) = \delta$. Suppose then that γ is a limit ordinal. By Lemma 4.13 , there must be a state $q' \in \mathrm{St}$ such that $\mathcal{L}_{\mathbf{C}}(q') = \gamma'$ for some ordinal γ' such that $\delta < \gamma' < \gamma$. Hence by the induction hypothesis there is a state q_δ for which $\mathcal{L}_{\mathbf{C}}(q_\delta) = \delta$. □

Example 4.15. *In finite models all winning time labels are strictly smaller than the cardinality of the model, i.e. if* $\mathsf{card}(\mathcal{M}) = n < \omega$*, then* n *is a globally stable time limit bound for* $\mathcal{M}$*. This can be seen by the following reasoning.*

If there was some state with a winning time label $\gamma \geq n$*, then by Lemma 4.14, there would be a state* $q \in \mathrm{St}$ *for which* $\mathcal{L}_{\mathbf{C}}(q) = n$*. Further, by Lemma 4.14 we would now find states with winning time labels* $n-1, n-2, \ldots, 0$*. But since winning time labels are unique for each state, this would mean that* $\mathsf{card}(\mathcal{M}) > n$*, a contradiction.*

This result is quite obvious by the observation that the controller can only win the embedded game by reaching a state in the truth set of the formula $\psi_{\mathbf{C}}$*. Hence it would not be beneficial for the controller to go in cycles.*

The following proposition shows how we can find an upper bound for the values of possible winning time labels by just looking at the infinite branching bound of a model.

Proposition 4.16. *Let* $\mathcal{M}$ *be a CGM such that we have* $\mathsf{IBB}(\mathcal{M}) = \kappa$*. We define an ordinal* Γ *as follows:*

$$\begin{cases} \Gamma := \kappa \text{ if } \kappa \text{ is a regular cardinal,} \\ \Gamma := \kappa^+ \text{(the successor } \textbf{cardinal} \text{ of } \kappa\text{) otherwise.} \end{cases}$$

Now Γ *is a globally stable time limit bound for* $\mathcal{M}$*.*

Proof. For the sake of contradiction, suppose that there is $\Gamma' > \Gamma$ and embedded game $\mathbf{G}$ within a bounded evaluation game $\mathcal{G}(\mathcal{M}, q_{in}, \varphi, \Gamma')$ such that in $\mathbf{G}$ either of the players has winning time labels that are greater or equal to Γ. By Lemma 4.14 , there is a state $q \in \mathrm{St}$ for which $\mathcal{L}_{\mathbf{C}}(q) = \Gamma$. Let $Q \subseteq \mathrm{St}$ be the set of states that is forced by $\tau_{\mathbf{C}}(q)$. Since $\mathsf{IBB}(\mathcal{M}) = \kappa$, we have $\mathsf{card}(Q) < \kappa \leq \Gamma$. By Lemma 4.13 , $\mathcal{L}_{\mathbf{C}}(q') < \Gamma$ for every state $q' \in Q$, and furthermore, since Γ is a limit ordinal, Γ must be the supremum of the winning time labels of the states in Q.

Now every winning time label in Q is smaller than Γ and the cardinality of Q is less than Γ. Because successor cardinals are regular, Γ is necessarily a regular cardinal, and thus it is equal to its own cofinality. Hence we have $\sup\{\mathcal{L}_{\mathbf{C}}(q') \mid q' \in Q\} < \Gamma$. This is a contradiction and thus Γ must be a globally stable time limit bound. □

The result of the previous proposition cannot be improved, since we can show that no lower time limit bound is guaranteed to be globally stable. That is, if κ is an infinite cardinal, $\Gamma \in \{\kappa, \kappa^+\}$ as defined above and $\gamma < \Gamma$ is an ordinal, then there is some model $\mathcal{M}$ for which $\mathsf{IBB}(\mathcal{M}) \leq \kappa$ and in which the winning time label γ is realized. This is demonstrated by the following example:

Example I. *Let* κ *be an infinite cardinal. If* κ *is regular, let* $\Gamma := \kappa$ *and else let* $\Gamma := \kappa^+$*. Let* $\gamma < \Gamma$ *be an ordinal. We show by transfinite induction on* γ *that there exist a concurrent game model* $\mathcal{M}$ *such that* $\mathsf{IBB}(\mathcal{M}) \leq \kappa$ *and a state of* $\mathcal{M}$ *that realizes the winning time label* γ *for some embedded game in* $\mathcal{M}$*.*

All models that we shall construct in this example will be of the form

$$\mathcal{M} = (\mathbb{A}\mathrm{gt}, \mathrm{St}, \Pi, \mathrm{Act}, d, o, v),$$

where $\mathbb{A}\mathrm{gt} = \{a\}$*,* $q_0 \in \mathrm{St}$*,* $\Pi = \{p\}$ *and* $\mathrm{Act} = \{\delta \mid \delta < \Gamma \text{ is an ordinal}\}$*. We will always consider the winning time label for Eloise at the state* q_0 *in the embedded game* $\mathbf{G} = (\mathbf{E}, \mathbf{E}, \emptyset, q_0, p, \top)$ *(which arises when verifying the formula* $\varphi = \langle\langle\emptyset\rangle\rangle \,\mathsf{F}\, p$ *at* q_0*).*

If $\gamma = 0$*, we define* $\mathcal{M}_0 = (\mathbb{A}\mathrm{gt}, \mathrm{St}, \Pi, \mathrm{Act}, d, o, v)$ *where* $\mathrm{St} = \{q_0\}$*,* $d(q_0, a) = \{0\}$*,* $o(q_0, 0) = q_0$ *and* $v(p) = q_0$*. Since* $\mathsf{BD}(q_0) = 1$*, we have* $\mathsf{IBB}(\mathcal{M}_0) = \omega \leq \kappa$*. Also, clearly* $\mathcal{L}_{\mathbf{E}}(q_0) = 0$*.*

Suppose then that γ *is a successor ordinal. By the induction hypothesis, there is a model* $\mathcal{M}_{\gamma-1}$ *s.t.* $\mathsf{IBB}(\mathcal{M}_{\gamma-1}) \leq \kappa$ *and* $\mathcal{L}_{\mathbf{E}}(q_0) = \gamma-1$*. Let* $\mathcal{M}'_{\gamma-1} = (\mathbb{A}\mathrm{gt}, \mathrm{St}', \Pi, \mathrm{Act}, d', o', v')$ *be an isomorphic copy of* $\mathcal{M}_{\gamma-1}$ *in which the state* q_0 *is replaced by a new state* q'*. Let* $\mathcal{M}_\gamma = (\mathbb{A}\mathrm{gt}, \mathrm{St}, \Pi, \mathrm{Act}, d, o, v)$*, where we define* $\mathrm{St} := \mathrm{St}' \cup \{q_0\}$*,* $d := d' \cup \{((q_0, a), \{0\})\}$*,* $o := o' \cup \{((q_0, 0), q')\}$ *and* $v := v'$*. Since* $\mathsf{BD}(q_0) = 1$*,* $\mathsf{IBB}(\mathcal{M}_\gamma) = \mathsf{IBB}(\mathcal{M}'_{\gamma-1}) \leq \kappa$*. Also, clearly* $\mathcal{L}_{\mathbf{E}}(q_0) = \gamma$*.*

Suppose then that γ *is a limit ordinal. We next construct a set of ordinals* $\Psi \subseteq \{\delta \mid \delta < \gamma\}$ *such that* $\mathsf{card}(\Psi) < \kappa$ *and* $\sup(\Psi) = \gamma$*. If* κ *is regular, then we have* $\Gamma = \kappa$*, and thus we can define* $\Psi := \{\delta \mid \delta < \gamma\}$*. Since* $\gamma < \kappa$*, clearly* $\mathsf{card}(\Psi) < \kappa$*. If* κ *is not regular, then* $\Gamma = \kappa^+$*. Let* μ *be the cofinality of* γ*, whence there exists some set* Ψ *of ordinals less than* γ *such that* $\mathsf{card}(\Psi) = \mu$ *and* $\sup(\Psi) = \gamma$*. Since* $\gamma < \kappa^+$ *and since the cofinality of any ordinal is always a regural cardinal (which is smaller than the ordinal itself), we must have* $\mu < \kappa$*.*

By the induction hypothesis, for every ordinal $\delta \in \Psi$*, there is a model* $\mathcal{M}_\delta$ *for which* $\mathsf{IBB}(\mathcal{M}_\delta) \leq \kappa$ *and* $\mathcal{L}_{\mathbf{E}}(q_0) = \delta$*. We build an isomorphic copy*

$$\mathcal{M}'_\delta = (\mathbb{A}\mathrm{gt}, \mathrm{St}_\delta, \Pi, \mathrm{Act}, d_\delta, o_\delta, v_\delta)$$

of every model $\mathcal{M}_\delta$ *such that* $\bigcap_{\delta\in\Psi} \mathrm{St}_\delta = \emptyset$ *and such that the state* q_0 *is replaced with a new state* q_δ *in every model* $\mathcal{M}'_\delta$*. Let* $\mathcal{M}_\gamma = (\mathbb{A}\mathrm{gt}, \mathrm{St}, \Pi, \mathrm{Act}, d, o, v)$*, where we have* $\mathrm{St} = \bigcup_{\delta\in\Psi} \mathrm{St}_\delta \cup \{q_0\}$*,* $d = \bigcup_{\delta\in\Psi} d_\delta \cup \{((q_0, a), \{\delta \mid \delta \in \Psi\})\}$ *and* $o = \bigcup_{\delta\in\Psi} o_\delta \cup \bigcup_{\delta\in\Psi} \{((q_0, \delta), q_\delta)\}$ *and* $v := \bigcup_{\delta\in\Psi} v_\delta$*. Because* $\mathsf{BD}(q_0) = \mathsf{card}(\Psi) < \kappa$ *and* $\mathsf{IBB}(\mathcal{M}_\delta) \leq \kappa$ *for every* $\delta \in \Psi$*, we have* $\mathsf{IBB}(\mathcal{M}_\gamma) \leq \kappa$*. Since* $\sup(\Psi) = \gamma$*, it is also easy to see that* $\mathcal{L}_{\mathbf{E}}(q_0) = \gamma$*.*

From this proposition we obtain the following corollary which tells when the time limit bounds for the bounded and finitely bounded semantics are guaranteed to be stable.

Corollary 4.17. *Let* $\mathcal{M}$ *be a CGM. If* $\mathsf{card}(\mathcal{M}) = \kappa$*, then* 2^κ *is a globally stable time limit bound for* $\mathcal{M}$*. If* $\mathcal{M}$ *is image finite,* ω *is a globally stable time limit bound for* $\mathcal{M}$*.*

Proof. Suppose that $\mathsf{card}(\mathcal{M}) = \kappa$. If $\kappa < \omega$, then by Example 4.15, κ is globally stable for $\mathcal{M}$, whence also 2^κ is globally stable for $\mathcal{M}$. Suppose then that $\kappa \geq \omega$ and let $\mathsf{IBB}(\mathcal{M}) = \mu$. Since $\mathsf{card}(\mathcal{M}) = \kappa$, we have $\mathsf{BD}(q) \leq \kappa$ for every $q \in \mathrm{St}$ and thus $\mu \leq \kappa^+$. If $\mu = \kappa^+$, then μ is regular and thus by Proposition 4.16 μ is globally stable for $\mathcal{M}$. Suppose then $\mu < \kappa^+$, whence we have $\mu^+ \leq \kappa^+$. By Proposition 4.16 μ^+ is globally stable for $\mathcal{M}$, regardless of whether μ is a regular cardinal or not. Therefore κ^+ and thus also 2^κ are globally stable for $\mathcal{M}$.

Suppose then that $\mathcal{M}$ is image finite, whence we have $\mathsf{IBB}(\mathcal{M}) = \omega$. Since ω is a regular cardinal, by Proposition 4.16, ω is globally stable for $\mathcal{M}$. ☐

4.5 Relationship between the unbounded and bounded embedded games

The following lemma shows that if $\mathbf{P}$ uses $\tau_{\mathbf{P}}^{\infty}$ and begins from a state with label win, then $\mathbf{P}$ will always stay in states with label win. We must make some extra assumptions on the time limit bound Γ in order to prove this claim.

Lemma 4.18. *Let $\mathbf{G} = \mathbf{g}(\mathbf{V}, \mathbf{C}, A, q_0, \psi_{\mathbf{C}}, \psi_{\overline{\mathbf{C}}})$ be an embedded game, $\mathbf{P} \in \{\mathbf{E}, \mathbf{A}\}$ and $\mathbf{P} \neq \mathbf{C}$. Assume that the time limit bound Γ is a successor ordinal and $\Gamma{-}1$ is stable for $\mathbf{G}$. Now for every $q \in \mathrm{St}$, if $\mathcal{L}_{\mathbf{P}}(q) = \mathsf{win}$ and $Q \subseteq \mathrm{St}$ is forced by $\tau_{\mathbf{P}}^{\infty}$, then $\mathcal{L}_{\mathbf{P}}(q') = \mathsf{win}$ for every $q' \in Q$.*

Proof. Suppose that $\mathcal{L}_{\mathbf{P}}(q) = \mathsf{win}$ and $Q \subseteq \mathrm{St}$ is forced by $\tau_{\mathbf{P}}^{\infty}(q)$. Let $\sigma_{\mathbf{P}}$ be a strategy for which $l(q, \sigma_{\mathbf{P}}) = \mathsf{win}$ and $\tau_{\mathbf{P}}^{\infty}(q) = \sigma_{\mathbf{P}}(\Gamma{-}1, q)$. For the sake of contradiction, suppose that there is some $q' \in Q$ for which $\mathcal{L}_{\mathbf{P}}(q') \neq \mathsf{win}$ and thus $\mathcal{L}_{\mathbf{P}}(q') = \gamma$ for some $\gamma < \Gamma$. Since $\Gamma - 1$ is stable, we must also have $\gamma < \Gamma - 1$. Now there is a play of $\mathbf{G}[q, \Gamma - 1]$ in which $\mathbf{P}$ is using $\sigma_{\mathbf{P}}$ and the configuration (γ, q') follows $(\Gamma{-}1, q)$. But since $\mathcal{L}_{\mathbf{P}}(q') = \gamma$, the strategy $\sigma_{\mathbf{P}}$ cannot be a winning strategy in $\mathbf{G}[q', \gamma]$. Hence $\sigma_{\mathbf{P}}$ is not a winning strategy in $\mathbf{G}[q, \Gamma - 1]$, which is a contradiction since $l(q, \sigma_{\mathbf{P}}) = \mathsf{win}$. $\square$

The following proposition shows that when the time limit bound is stable, then bounded embedded games become essentially equivalent with unbounded embedded games.

Proposition 4.19. *Let $\mathbf{G} = \mathbf{g}(\mathbf{V}, \mathbf{C}, A, q_0, \psi_{\mathbf{C}}, \psi_{\overline{\mathbf{C}}},)$ be an embedded game and suppose that Γ is stable for $\mathbf{G}$. Now the following equivalences hold:*

- *If $\mathbf{P} = \mathbf{C}$, there is a winning strategy $\sigma_{\mathbf{P}}$ in $\mathbf{G}$ iff there is $\gamma_0 < \Gamma$ and a timed winning strategy $(\sigma'_{\mathbf{P}}, t)$ in $\mathbf{G}[\gamma_0]$.*
- *If $\mathbf{P} \neq \mathbf{C}$, there is a winning strategy $\sigma_{\mathbf{P}}$ in $\mathbf{G}$ iff there is $\sigma'_{\mathbf{P}}$ which is a winning strategy in $\mathbf{G}[\gamma_0]$ for every $\gamma_0 < \Gamma$.*

Proof. Suppose first that $\mathbf{P} = \mathbf{C}$. If $(\sigma'_{\mathbf{P}}, t)$ is a timed winning strategy in $\mathbf{G}[\gamma_0]$ for some time limit $\gamma_0 < \Gamma$ then then by Proposition 4.7 $(\tau_{\mathbf{P}}, t_{can})$ is a winning strategy in $\mathbf{G}[\gamma_0]$. Now the strategy $\tau_{\mathbf{P}}$ will also be a winning strategy in $\mathbf{G}$ (note that infinite plays are impossible with $\tau_{\mathbf{P}}$ and that $\tau_{\mathbf{P}}$ depends on states only). For the other direction, suppose that there is a winning strategy $\sigma_{\mathbf{P}}$ for $\mathbf{P}$ in $\mathbf{G}$. If $\mathcal{L}_{\mathbf{P}}(q_0) = \gamma < \Gamma$ for some ordinal γ, then by Proposition 4.3 there is a timed winning strategy for $\mathbf{P}$ in $\mathbf{G}[\gamma]$. Else we have $\mathcal{L}_{\mathbf{P}}(q_0) = \mathsf{lose}$, i.e. there no timed winning strategy in $\mathbf{G}[\gamma]$ with any time limit γ (which does not have to be lower than Γ since Γ is stable). We show that this leads to a contradiction.

Let $Q \subseteq \mathrm{St}$ be the set of states that is forced by $\sigma_{\mathbf{P}}(q_0)$. Suppose first that $\mathcal{L}_{\mathbf{P}}(q)$ has some ordinal value less than Γ for every $q \in Q$. Hence we can formulate a timed winning strategy $(\sigma'_{\mathbf{P}}, t_{can})$ for $\mathbf{P}$ in $\mathbf{G}[\Gamma]$ by defining $\sigma'_{\mathbf{P}}(\gamma, q_0) = \sigma_{\mathbf{P}}(q_0)$ for every $\gamma < \Gamma$ and using $\tau_{\mathbf{P}}$ for all other cases. This is impossible and thus there must be some $q' \in Q$ for which $\mathcal{L}_{\mathbf{P}}(q') = \mathsf{lose}$. We can repeat this same argumentation for the set forced by $\sigma_{\mathbf{P}}(q')$ to find again a state with winning time label lose. Hence it follows by induction that after any finite number of transitions by using $\sigma_{\mathbf{P}}$, it is possible end up at a state that has a winning time label lose. But for $\sigma_{\mathbf{P}}$ to be a winning strategy in $\mathbf{G}$, it should always reach a state with the winning time label 0 in a finite number of rounds.

Suppose then that $\mathbf{P} \neq \mathbf{C}$. If there is a winning strategy $\sigma_{\mathbf{P}}$ for $\mathbf{P}$ in $\mathbf{G}$, then we can define $\sigma'_{\mathbf{P}}(\gamma, q) = \sigma_{\mathbf{P}}(q)$ for every $\gamma < \Gamma$, whence $\sigma'_{\mathbf{P}}$ will be a winning strategy in $\mathbf{G}[\gamma_0]$ for every time limit $\gamma_0 < \Gamma$. For the other direction, suppose that there is a strategy $\sigma'_{\mathbf{P}}$ which is a winning strategy in $\mathbf{G}[\gamma_0]$ for every time limit $\gamma_0 < \Gamma$, whence by Proposition 4.3 we have $\mathcal{L}_{\mathbf{P}}(q_0) = \mathsf{win}$. We will show that now we can formulate a winning strategy for $\mathbf{P}$ in $\mathbf{G}$.

Let $\Gamma' := \Gamma+1$. We formulate the ∞-canonical strategy $\tau_{\mathbf{P}}^{\infty}$ by using the strategies that correspond to embedded games with the time limit bound Γ'. Since Γ is stable, the winning time label of the states will not change, and in particular, the state q will still have the value win. Now the assumptions of Lemma 4.18 hold, and thus we can use it to deduce that all (finite) plays of $\mathbf{G}$ with $\tau_{\mathbf{P}}^{\infty}$ will have the label win at every state that is reached. But to lose the embedded game at some state, it should have the winning time label 0. Hence $\tau_{\mathbf{P}}^{\infty}$ is a winning strategy for $\mathbf{P}$ in $\mathbf{G}$ (also note that $\tau_{\mathbf{P}}^{\infty}$ depends on states only). ☐

As bounded embedded games are determined, the previous proposition implies that also unbounded embedded games are determined. By this result, we see that even if we defined memory based strategies for bounded or unbounded embedded games, the semantics so obtained would remain equivalent to the current one. We can now prove the equivalence of unbounded and bounded game theoretic semantics.

Theorem 4.20. *Let $\mathcal{M}$ be a CGM, $q \in \mathrm{St}$ and φ an* ATL*-formula. We have $\mathcal{M}, q \models_u^g \varphi$ iff $\mathcal{M}, q \models_b^g \varphi$.*

Proof. Assume that $\mathsf{card}(\mathcal{M}) = \kappa$. By Corollary 4.17, 2^κ is globally stable for $\mathcal{M}$. Consider an embedded game $\mathbf{G}$. If Eloise is the controller in $\mathbf{G}$, then by Proposition 4.19 she has a winning strategy in $\mathbf{G}$ iff there is some $\gamma < 2^\kappa$ s.t. she has a winning strategy in $\mathbf{G}[\gamma]$. If Eloise is not the controller in $\mathbf{G}$, then by Proposition 4.19, she has a winning strategy in $\mathbf{G}$ iff she has a winning strategy in $\mathbf{G}[\gamma]$ for every $\gamma < 2^\kappa$. Hence we can prove by straightforward induction on φ that Eloise has a winning strategy in $\mathcal{G}(\mathcal{M}, q, \varphi)$ iff she has a winning strategy in $\mathcal{G}(\mathcal{M}, q, \varphi, 2^\kappa)$. ☐

Even though the finitely bounded semantics is not equivalent to bounded semantics, the two systems become equivalent on a natural class of concurrent game models:

Theorem 4.21. *Let $\mathcal{M}$ be an image finite CGM, $q \in \mathrm{St}$ and φ an* ATL*-formula. Now $\mathcal{M}, q \models_f^g \varphi$ iff $\mathcal{M}, q \models_b^g \varphi$.*

Proof. By Corollary 4.17, in image finite models all ordinal valued winning time labels are finite. Thus the controller would gain nothing from being able to use infinite ordinals in embedded games. Hence we can prove the claim by a straightforward induction on the formula φ. ☐

5 Comparing game-theoretic and compositional semantics

5.1 Equivalence between unbounded GTS and compositional semantics

We next establish that the unbounded GTS is equivalent to the standard compositional semantics of ATL.

Theorem 5.1. *Let $\mathcal{M}$ be a CGM, $q_{in} \in \mathrm{St}$ and φ an* ATL*-formula. Now $\mathcal{M}, q_{in} \models \varphi$ iff $\mathcal{M}, q_{in} \models_u^g \varphi$.*

Proof. Suppose first that $\mathcal{M} \models_u^g \varphi$, i.e., Eloise has a winning strategy $\Sigma_{\mathbf{E}}$ in $\mathcal{G}(\mathcal{M}, q_{in}, \varphi)$. We will show that if a position $\mathsf{Pos} = (\mathbf{P}, q, \psi)$ can be reached in a game when Eloise plays $\Sigma_{\mathbf{E}}$, then the following condition holds for Pos:

$$\mathcal{M}, q \models \psi \quad \text{iff} \quad \mathbf{P} = \mathbf{E}. \qquad (\star)$$

We can prove this by structural induction on φ, since all positions of the game are of the form $(\mathbf{P}, q, \psi)$, where ψ is a subformula of φ.

Suppose that a position $\mathsf{Pos} = (\mathbf{P}, q, p)$ $(p \in \Pi)$ can be reached in the game. Since $\Sigma_{\mathbf{E}}$ is a winning strategy, we have $q \in v(p)$ if and only if $\mathbf{P} = \mathbf{E}$. Hence the condition $(\star)$ holds for Pos.

Suppose that a position $\mathsf{Pos} = (\mathbf{P}, q, \neg\psi)$ can be reached in the game. If $\mathbf{P} = \mathbf{E}$, the next position is $(\mathbf{A}, q, \psi)$, whence by the induction hypothesis $\mathcal{M}, q \not\models \psi$. Hence $\mathcal{M}, q \models \neg\psi$, and thus $(\star)$ holds for Pos. If $\mathbf{P} = \mathbf{A}$, the next position is $(\mathbf{E}, q, \psi)$, whence by the induction hypothesis $\mathcal{M}, q \models \psi$. Hence $\mathcal{M}, q \not\models \neg\psi$, and thus $(\star)$ holds for Pos.

Suppose that a position $\mathsf{Pos} = (\mathbf{P}, q, \psi \vee \theta)$ can be reached in the game. If $\mathbf{P} = \mathbf{E}$, then the strategy $\Sigma_{\mathbf{E}}$ picks either ψ or θ. If $\Sigma_{\mathbf{E}}(\mathsf{Pos}) = \psi$, then the position $(\mathbf{E}, q, \psi)$ can be reached with $\Sigma_{\mathbf{E}}$, and thus by the induction hypothesis, we have $\mathcal{M}, q \models \psi$. If $\Sigma_{\mathbf{E}}(\mathsf{Pos}) = \theta$, then we analogously have $\mathcal{M}, q \models \theta$. Hence $\mathcal{M}, q \models \psi \vee \theta$, and thus $(\star)$ holds for Pos.

If $\mathbf{P} = \mathbf{A}$, then both of the positions $(\mathbf{A}, q, \psi)$ and $(\mathbf{A}, q, \theta)$ can be reached with $\Sigma_{\mathbf{E}}$ (since Abelard makes the choice). Thus, by the induction hypothesis, we must have $\mathcal{M}, q \not\models \psi$ and $\mathcal{M}, q \not\models \theta$, i.e. $\mathcal{M}, q \not\models \psi \vee \theta$. Hence $(\star)$ holds for Pos.

Suppose that a position $\mathsf{Pos} = (\mathbf{P}, q, \langle\!\langle A \rangle\!\rangle \mathsf{X}\, \psi)$ can be reached in the game. If $\mathbf{P} = \mathbf{E}$, then $\Sigma_{\mathbf{E}}$ assigns some tuple of choices for the agents in A. We can now formulate a related collective stategy S_A (recall the definition of the compositional semantics) by using those choices at q; the choices at other states may be arbitrary. Let $\Lambda \in \mathsf{paths}(q, S_A)$. Now Abelard can choose such actions for $\overline{A}$ that the resulting state is $\Lambda[1]$. Thus the position $(\mathbf{E}, \Lambda[1], \psi)$ can be reached with the winning strategy $\Sigma_{\mathbf{E}}$, and thus, using the induction hypothesis, we infer that $\mathcal{M}, \Lambda[1] \models \psi$. Hence $\mathcal{M}, q \models \langle\!\langle A \rangle\!\rangle \mathsf{X}\, \psi$, and thus $(\star)$ holds for Pos.

If $\mathbf{P} = \mathbf{A}$, then $\Sigma_{\mathbf{E}}$ assigns some some tuple of actions for $\overline{A}$ as a response to any tuple of actions chosen by Abelard. Let S_A be any collective strategy for the coalition A. Now, if Abelard chooses the actions for the agents in A according to S_A and Eloise responds using $\Sigma_{\mathbf{E}}(\mathsf{Pos})$, then the resulting state must be $\Lambda[1]$ for some $\Lambda \in \mathsf{paths}(q, S_A)$. Thus the position $(\mathbf{A}, \Lambda[1], \psi)$ can be reached using $\Sigma_{\mathbf{E}}$, and thus, by the induction hypothesis, we have $\mathcal{M}, \Lambda[1] \not\models \psi$. Hence $\mathcal{M}, q \not\models \langle\!\langle A \rangle\!\rangle \mathsf{X}\, \psi$, and thus $(\star)$ holds for Pos.

Suppose that a position $\mathsf{Pos} = (\mathbf{P}, q, \langle\!\langle A \rangle\!\rangle\, \psi \,\mathsf{U}\, \theta)$ can be reached in the game. If $\mathbf{P} = \mathbf{E}$, then $\Sigma_{\mathbf{E}}(\mathsf{Pos})$ is some strategy $\sigma_{\mathbf{E}}$ for the corresponding embedded game. We can now formulate a collective strategy S_A that is related to the strategy $\sigma_{\mathbf{E}}$: For any state where $\sigma_{\mathbf{E}}$ assigns some tuple of actions for agents in A, we define the same actions for S_A. For states where $\sigma_{\mathbf{E}}$ instructs to end the game, we may define arbitrary actions for S_A. We will use this same method from now on, when we define collective strategies S_A related to the strategies of $\mathbf{V}$ in an embedded game.

Let $\Lambda \in \mathsf{paths}(q, S_A)$. Now, when Eloise uses $\sigma_{\mathbf{E}}$, there will be actions of Abelard such that the states of the embedded game are on Λ until some configuration $\Lambda[i]$ at which Eloise ends the game at the exit position $(\mathbf{E}, \Lambda[i], \theta.)$ (Note that since $\Sigma_{\mathbf{E}}$ is a winning strategy, Eloise must always end the embedded game after finitely many steps.) Thus, by the induction hypothesis, $\mathcal{M}, \Lambda[i] \models \theta$. Let then $j < i$. Since Abelard can end the game after j rounds at a position $(\mathbf{E}, \Lambda[j], \psi)$, we conclude by the induction hypothesis that $\mathcal{M}, \Lambda[j] \models \psi$. Hence $\mathcal{M}, q \models \langle\!\langle A \rangle\!\rangle\, \psi \,\mathsf{U}\, \theta$, and thus $(\star)$ holds for Pos.

If $\mathbf{P} = \mathbf{A}$, then $\Sigma_{\mathbf{E}}(\mathsf{Pos})$ is some a strategy $\sigma_{\mathbf{E}}$. For the sake of contradiction, suppose that $\mathcal{M}, q \models \langle\!\langle A \rangle\!\rangle\, \psi \,\mathsf{U}\, \theta$, i.e., there exists some S_A such that for each $\Lambda \in \mathsf{paths}(q, S_A)$, there is $i \geq 0$ such that $\mathcal{M}, \Lambda[i] \models \theta$ and $\mathcal{M}, \Lambda[j] \models \psi$ for every $j < i$. We define $\sigma_{\mathbf{A}}(q') = \theta$ for each $q' \in \mathrm{St}$ for which $\mathcal{M}, q' \models \theta$. For all other states $q' \in \mathrm{St}$, let

$\sigma_{\mathbf{A}}(q')$ be the actions for agents in A determined by S_A. Now, if Abelard uses $\sigma_{\mathbf{A}}$ and Eloise $\sigma_{\mathbf{E}}$, then the exit position of the game must be $(\mathbf{A}, \Lambda[i], \mu)$, where $\mu \in \{\psi, \theta\}$, $\Lambda \in \mathsf{paths}(q, S_A)$ and $i \geq 0$. But for this exit position, the condition $(\star)$ does not hold, even thought it is reached by the winning strategy $\Sigma_{\mathbf{E}}$. This contradicts the induction hypothesis. Hence we have $\mathcal{M}, q \not\models \langle\!\langle A \rangle\!\rangle\, \psi \,\mathsf{U}\, \theta$, and thus $(\star)$ holds for Pos.

Suppose that a position $\mathsf{Pos} = (\mathbf{P}, q, \langle\!\langle A \rangle\!\rangle\, \psi \,\mathsf{R}\, \theta)$ can be reached in the game. If $\mathbf{P} = \mathbf{E}$, then $\Sigma_{\mathbf{E}}(\mathsf{Pos})$ is a strategy $\sigma_{\mathbf{E}}$. Let S_A be the collective strategy that is related to $\sigma_{\mathbf{E}}$ and let $\Lambda \in \mathsf{paths}(q, S_A)$. Now, when Eloise uses $\sigma_{\mathbf{E}}$, there exist actions of Abelard such that the states of the embedded game are on Λ (until a state is reached where Eloise ends the game – if such a state exists). We need to show that for every $i \geq 0$ either $\mathcal{M}, \Lambda[i] \models \theta$ or there is $j < i$ s.t. $\mathcal{M}, \Lambda[j] \models \psi$. Let $i \geq 0$. If Eloise ends the game before i rounds, the game ends at a exit position $(\mathbf{E}, \Lambda[j], \psi)$ for some $j < i$, and we can conclude that $\mathcal{M}, \Lambda[j] \models \psi$ by the induction hypothesis. If Eloise does not end the game before i rounds have been played, then Abelard can end it the position position $(\mathbf{E}, \Lambda[i], \theta)$. We can then conclude that $\mathcal{M}, \Lambda[i] \models \theta$ by the induction hypothesis. Hence we have $\mathcal{M}, q \models \langle\!\langle A \rangle\!\rangle\, \psi \,\mathsf{R}\, \theta$, and thus $(\star)$ holds for Pos.

If $\mathbf{P} = \mathbf{A}$, then $\Sigma_{\mathbf{E}}(\mathsf{Pos})$ is some strategy $\sigma_{\mathbf{E}}$. For the sake of contradiction, suppose that $\mathcal{M}, q \models \langle\!\langle A \rangle\!\rangle\, \psi \,\mathsf{R}\, \theta$, i.e., there exists a strategy S_A such that for each $\Lambda \in \mathsf{paths}(q, S_A)$ and $i \geq 0$, we have either $\mathcal{M}, \Lambda[i] \models \theta$ or there is some $j < i$ such that $\mathcal{M}, \Lambda[j] \models \psi$. Let $\sigma_{\mathbf{A}}(q') = \psi$ for each $q' \in \mathrm{St}$ where $\mathcal{M}, q' \models \psi$, and for all other states $q' \in \mathrm{St}$, let $\sigma_{\mathbf{A}}(q')$ be the tuple of actions for agents in A determined by S_A. Now, if Abelard uses $\sigma_{\mathbf{A}}$ and Eloise $\sigma_{\mathbf{E}}$, the exit position of the game must be $(\mathbf{A}, \Lambda[i], \mu)$, where $\mu \in \{\psi, \theta\}$, $\Lambda \in \mathsf{paths}(q, S_A)$ and $i \geq 0$. But for this exit position the condition $(\star)$ does not hold, even thought it is reached using the winning strategy $\Sigma_{\mathbf{E}}$. This contradicts the induction hypothesis. Hence we have $\mathcal{M}, q \not\models \langle\!\langle A \rangle\!\rangle\, \psi \,\mathsf{R}\, \theta$, and thus $(\star)$ holds for Pos.

We now conclude that the condition $(\star)$ must hold for the initial position $\mathsf{Pos}_0 = (\mathbf{E}, q_{in}, \varphi)$. Therefore we have $\mathcal{M}, q_{in} \models \varphi$. This concludes the first direction of the proof of the current theorem.

Suppose then that $\mathcal{M}, q_{in} \models \varphi$. We will formulate such a strategy $\Sigma_{\mathbf{E}}$ for Eloise that the condition $(\star)$ will hold at each position $\mathsf{Pos}_i = (\mathbf{P}, q, \psi)$ of the game. The condition holds in the initial position $\mathsf{Pos}_0 = (\mathbf{E}, q_{in}, \varphi)$ by the assumption. We let the induction hypothesis be that the condition $(\star)$ holds in the a position Pos_i and show that we can define $\Sigma_{\mathbf{E}}(\mathsf{Pos}_i)$ in such a way that $(\star)$ holds in the next position Pos_{i+1}:

Let $\mathsf{Pos}_i = (\mathbf{P}, q, \neg\psi)$. If $\mathbf{P} = \mathbf{E}$, by the induction hypothesis $\mathcal{M}, q \models \neg\psi$, i.e., $\mathcal{M}, q \not\models \psi$. Thus $(\star)$ holds in the next position $\mathsf{Pos}_{i+1} = (\mathbf{A}, q, \psi)$. If $\mathbf{P} = \mathbf{A}$, then by the induction hypothesis $\mathcal{M}, q \not\models \neg\psi$, i.e., $\mathcal{M}, q \models \psi$. Thus $(\star)$ holds in the next position $\mathsf{Pos}_{i+1} = (\mathbf{E}, q, \psi)$.

Let $\mathsf{Pos}_i = (\mathbf{P}, q, \psi \vee \theta)$. If $\mathbf{P} = \mathbf{E}$, then by the induction hypothesis $\mathcal{M}, q \models \psi \vee \theta$, i.e., $\mathcal{M}, q \models \psi$ or $\mathcal{M}, q \models \psi$. If the former holds, we define $\Sigma_{\mathbf{E}}(\mathsf{Pos}_i) = \psi$ and else we define $\Sigma_{\mathbf{E}}(\mathsf{Pos}_i) = \theta$. Now $(\star)$ will hold in the next position Pos_{i+1}. If $\mathbf{P} = \mathbf{A}$, by the induction hypothesis $\mathcal{M}, q \not\models \psi \vee \theta$, i.e., $\mathcal{M}, q \not\models \psi$ and $\mathcal{M}, q \not\models \theta$. Thus $(\star)$ holds in the next position Pos_{i+1} regardless of the choice made by Abelard.

Let $\mathsf{Pos}_i = (\mathbf{P}, q, \langle\!\langle A \rangle\!\rangle\, \mathsf{X}\, \psi)$. If $\mathbf{P} = \mathbf{E}$, by the induction hypothesis $\mathcal{M}, q \models \langle\!\langle A \rangle\!\rangle\, \mathsf{X}\, \psi$, i.e., there exists a strategy S_A such that for each $\Lambda \in \mathsf{paths}(q, S_A)$, we have $\mathcal{M}, \Lambda[1] \models \psi$. Let $\Sigma_{\mathbf{E}}(\mathsf{Pos}_i)$ be the tuple in $\mathsf{action}(A, q)$ which is determined by S_A at q. Now, regardless of the actions chosen by Abelard for the agents in $\overline{A}$, the resulting state q' must be $\Lambda[1]$ for some $\Lambda \in \mathsf{paths}(q, S_A)$. Hence $\mathcal{M}, q' \models \psi$. Thus $(\star)$ holds in the next position $\mathsf{Pos}_{i+1} = (\mathbf{E}, q', \psi)$.

If $\mathbf{P} = \mathbf{A}$, by the induction hypothesis $\mathcal{M}, q \not\models \langle\!\langle A \rangle\!\rangle \mathsf{X}\, \psi$, i.e., for each S_A, there exists a path $\Lambda \in \mathsf{paths}(q, S_A)$ such that $\mathcal{M}, \Lambda[1] \not\models \psi$. Let $\vec{\alpha} \in \mathsf{action}(A, q)$, whence there is some strategy S_A that coincides with $\vec{\alpha}$ at the state q. Now there exists some $\Lambda \in \mathsf{paths}(q, S_A)$ such that $\mathcal{M}, \Lambda[1] \not\models \psi$. Hence there is some $\vec{\beta} \in \mathsf{action}(\overline{A}, q)$ such that when $\vec{\alpha}$ and $\vec{\beta}$ are chosen, the resulting state q' is $\Lambda[1]$. Thus we can define $\Sigma_{\mathbf{E}}(\mathsf{Pos}_i)$ to be a response function in such a way that $(\star)$ will hold in the next position $\mathsf{Pos}_{i+1} = (\mathbf{A}, q', \psi)$.

Let $\mathsf{Pos}_i = (\mathbf{P}, q, \langle\!\langle A \rangle\!\rangle \psi \,\mathsf{U}\, \theta)$. If $\mathbf{P} = \mathbf{E}$, by the induction hypothesis $\mathcal{M}, q \models \langle\!\langle A \rangle\!\rangle \psi \,\mathsf{U}\, \theta$, i.e., there exists S_A such that for each $\Lambda \in \mathsf{paths}(q, S_A)$, there is some $i \geq 0$ such that $\mathcal{M}, \Lambda[i] \models \theta$ and $\mathcal{M}, \Lambda[j] \models \psi$ for every $j < i$. Let $\Sigma_{\mathbf{E}}(\mathsf{Pos}_i)$ be the strategy $\sigma_{\mathbf{E}}$ that is defined as follows: let $\sigma_{\mathbf{E}}(q') = \theta$ for each $q' \in \mathrm{St}$ where $\mathcal{M}, q' \models \theta$, and for all other states $q' \in \mathrm{St}$, let $\sigma_{\mathbf{E}}(q')$ be the tuple of actions for the agents in A determined by S_A. Now, regardless of the actions of Abelard, all of the states that are reached in the embedded game must be states $\Lambda[i]$ for some $\Lambda \in \mathsf{paths}(q, S_A)$ and $i \geq 0$. Thus, when Eloise uses $\sigma_{\mathbf{E}}$, a state q' where $\mathcal{M}, q' \models \theta$ is reached in a finite number of rounds. If Abelard ends the game before that at some state q', then we have $\mathcal{M}, q' \models \psi$. Hence $(\star)$ holds for every possible exit position, and thus it holds in the next position Pos_{i+1}.

If $\mathbf{P} = \mathbf{A}$, by the induction hypothesis $\mathcal{M}, q \not\models \langle\!\langle A \rangle\!\rangle \psi \,\mathsf{U}\, \theta$, i.e., for every S_A, there exists $\Lambda \in \mathsf{paths}(q, S_A)$ such that for each $i \geq 0$ we have $\mathcal{M}, \Lambda[i] \not\models \theta$ or $\mathcal{M}, \Lambda[j] \not\models \psi$ for some $j < i$. For the sake of contradiction, suppose that Eloise does not have a winning strategy in the embedded game $\mathbf{g}(\mathbf{A}, \mathbf{A}, A, q, \theta, \psi)$. Since embedded games are determined, there must be a winning strategy $\sigma_{\mathbf{A}}$ for Abelard.

Since $\sigma_{\mathbf{A}}$ is a winning strategy for Abelard, for every possible exit position $(\mathbf{A}, q', \mu)$ (where $\mu \in \{\psi, \theta\}$) of the embedded game we have $\mathcal{M}, q' \models^{g}_{u} \mu$. Hence by the other direction of the claim of the current theorem (which we have proved above), $\mathcal{M}, q' \models \mu$ for every possible exit position $(\mathbf{A}, q', \mu)$. Hence we see that the condition $(\star)$ cannot hold for any exit position that is reached with $\sigma_{\mathbf{A}}$. Also note that Abelard must end the game in finite time because he is the controlling player.

Let S_A be the collective strategy related to $\sigma_{\mathbf{A}}$. By our earlier observations, there exists some $\Lambda \in \mathsf{paths}(q, S_A)$ such that for each $i \geq 0$, we have $\mathcal{M}, \Lambda[i] \not\models \theta$ or $\mathcal{M}, \Lambda[j] \not\models \psi$ for some $j < i$. We see that Eloise can force Abelard on this path when he is using $\sigma_{\mathbf{A}}$. But now Eloise can play in such a way that the condition $(\star)$ will hold for all possible exit positions of the embedded game. This is a contradiction and thus Eloise must have a winning strategy $\sigma_{\mathbf{E}}$ in the embedded game, and we can define $\Sigma_{\mathbf{E}}(\mathsf{Pos}_i) = \sigma_{\mathbf{E}}$. When she uses $\sigma_{\mathbf{E}}$, the game will either be infinitely long whence she wins the whole evaluation game, or alternatively $\mathcal{M}, q' \not\models^{g}_{u} \mu$ for every possible exit position $(\mathbf{A}, q', \mu)$, whence, by the other direction of the claim of the current theorem, the condition $(\star)$ will hold for the next position Pos_{i+1}.

Let $\mathsf{Pos}_i = (\mathbf{P}, q, \langle\!\langle A \rangle\!\rangle \psi \,\mathsf{R}\, \theta)$. If $\mathbf{P} = \mathbf{E}$, by the induction hypothesis $\mathcal{M}, q \models \langle\!\langle A \rangle\!\rangle \psi \,\mathsf{R}\, \theta$, i.e., there exists a strategy S_A such that for each $\Lambda \in \mathsf{paths}(q, S_A)$ and $i \geq 0$ either $\mathcal{M}, \Lambda[i] \models \theta$ or there is $j < i$ such that $\mathcal{M}, \Lambda[j] \models \psi$. Let $\Sigma_{\mathbf{E}}(\mathsf{Pos}_i)$ be the strategy $\sigma_{\mathbf{E}}$ that is defined such that $\sigma_{\mathbf{E}}(q') = \psi$ for each $q' \in \mathrm{St}$ where $\mathcal{M}, q' \models \psi$ and for all other states $q' \in \mathrm{St}$, let $\sigma_{\mathbf{E}}(q')$ be the tuple of actions for the agents in A determined by S_A. Now all states that are reached in the embedded game must be states $\Lambda[i]$ for some $\Lambda \in \mathsf{paths}(q, S_A)$ and $i \geq 0$. Thus, when Eloise uses $\sigma_{\mathbf{E}}$, she will either stay at states q' where $\mathcal{M}, q' \models \theta$ for infinitely long or reach a state q' where $\mathcal{M}, q' \models \psi$ at some point. Hence either the embedded game continues infinitely long, whence Eloise wins the whole evaluation game, or $(\star)$ holds for the exit position which is the next position Pos_{i+1}.

If $\mathbf{P} = \mathbf{A}$, by the induction hypothesis $\mathcal{M}, q \not\models \langle\!\langle A \rangle\!\rangle \psi \,\mathsf{R}\, \theta$, i.e., for every S_A, there exists $\Lambda \in \mathsf{paths}(q, S_A)$ and $i \geq 0$ such that we have $\mathcal{M}, \Lambda[i] \not\models \theta$ and $\mathcal{M}, \Lambda[j] \not\models \psi$ for every $j < i$. For the sake of contradiction, suppose that Eloise does not have a winning strategy in the embedded game $\mathbf{g}(\mathbf{A}, \mathbf{E}, A, q, \theta, \psi)$. Since embedded games are determined, there must be a winning strategy $\sigma_{\mathbf{A}}$ for Abelard. Since $\sigma_{\mathbf{A}}$ is a winning strategy for Abelard, by the other direction of the current theorem, we conclude that the condition $(\star)$ cannot hold for any exit position that is reached with $\sigma_{\mathbf{A}}$.

Let S_A be the collective strategy related to $\sigma_{\mathbf{A}}$. Now there is $\Lambda \in \mathsf{paths}(q, S_A)$ and $i \geq 0$ such that we have $\mathcal{M}, \Lambda[i] \not\models \theta$ and $\mathcal{M}, \Lambda[j] \not\models \psi$ for every $j < i$. Now Eloise can force Abelard on this path, when he is playing $\sigma_{\mathbf{A}}$. But now Eloise can play in such a way that that the the condition $(\star)$ will hold for all possible exit positions of the game and she can reach such an exit position in a finite time. This is contradiction and thus Eloise must have a winning strategy $\sigma_{\mathbf{E}}$ in the embedded game and we can define $\Sigma_{\mathbf{E}}(\mathsf{Pos}_i) = \sigma_{\mathbf{E}}$. If she uses $\sigma_{\mathbf{E}}$, then the embedded game cannot be infinitely long and, by the other direction of the current theorem, the condition $(\star)$ will hold for the next position Pos_{i+1}.

Since the strategy $\Sigma_{\mathbf{E}}$ can maintain the condition $(\star)$ on every position of the game, it also holds in every ending position $\mathsf{Pos}_i = (\mathbf{P}, q, p)$ $(p \in \Pi)$. If $\mathbf{P} = \mathbf{E}$, we have $\mathcal{M}, q \models p$, i.e., $q \in v(p)$, whence Eloise wins. If $\mathbf{P} = \mathbf{A}$, then $\mathcal{M}, q \not\models p$, i.e., $q \notin v(p)$, whence again Eloise wins. Hence the strategy $\Sigma_{\mathbf{E}}$ is indeed a winning strategy for Eloise in $\mathcal{G}(\mathcal{M}, q_{in}, \varphi)$, and thus $\mathcal{M}, q_{in} \models_u^g \varphi$. □

5.2 Finitely bounded compositional semantics

By our earlier observations, the finitely bounded game-theoretic semantics is not equivalent to the standard compositional semantics of ATL. However, it can be shown equivalent to a natural semantics, to be defined next, which we call **finitely bounded compositional semantics**.

Definition 5.2. *Let $\mathcal{M} = (\mathbb{A}\text{gt}, \text{St}, \Pi, \text{Act}, d, o, v)$ be a CGM, $q \in \text{St}$ and φ an* ATL*-formula. The truth of φ in $\mathcal{M}$ at q according to **finitely bounded semantics**, denoted by $\mathcal{M}, q \models_f \varphi$, is defined recursively as follows:*

- *The semantics for $p \in \Pi$, $\neg\psi$, $\psi \vee \theta$ and $\langle\!\langle A \rangle\!\rangle \mathsf{X}\, \psi$ are as in the standard compositional semantics of* ATL *(Def 2.3).*
- *$\mathcal{M}, q \models_f \langle\!\langle A \rangle\!\rangle \psi \,\mathsf{U}\, \theta$ iff there exists $n < \omega$ and S_A such that for each $\Lambda \in \mathsf{paths}(q, S_A)$, there is $i \leq n$ such that $\mathcal{M}, \Lambda[i] \models_f \theta$ and $\mathcal{M}, \Lambda[j] \models_f \psi$ for every $j < i$.*
- *$\mathcal{M}, q \models_f \langle\!\langle A \rangle\!\rangle \psi \,\mathsf{R}\, \theta$ iff for every $n < \omega$, there exists $S_{A,n}$ such that for each $\Lambda \in \mathsf{paths}(q, S_{A,n})$ and $i \leq n$, either $\mathcal{M}, \Lambda[i] \models_f \theta$ or there is $j < i$ such that $\mathcal{M}, \Lambda[j] \models_f \psi$.*

For $\langle\!\langle A \rangle\!\rangle \mathsf{F}$ and $\langle\!\langle A \rangle\!\rangle \mathsf{G}$ we obtain the following semantics:

- $\mathcal{M}, q \models_b \langle\!\langle A \rangle\!\rangle \mathsf{F}\, \psi$ iff there exists $n < \omega$ and S_A such that for each $\Lambda \in \mathsf{paths}(q, S_A)$ there is $i \leq n$ such that $\mathcal{M}, \Lambda[i] \models_f \psi$.
- $\mathcal{M}, q \models_f \langle\!\langle A \rangle\!\rangle \mathsf{G}\, \psi$ iff for every $n \geq 0$ there exists $S_{A,n}$ such that for each $\Lambda \in \mathsf{paths}(q, S_{A,n})$ and $i \leq n$ we have $\mathcal{M}, \Lambda[i] \models_f \psi$.

5.3 Equivalence between finitely bounded compositional and game-theoretic semantics

To prove the equivalence between the finitely bounded compositional semantics and finitely bounded GTS, we need to show that it is sufficient to consider only such strategies in the embedded games that depend on states only. This property will be needed because the collective strategies for coalitions in the compositional semantics are of this form.

Lemma 5.3. *If Eloise has a winning strategy $\Sigma_{\mathbf{E}}$ in a finitely bounded evaluation game $\mathcal{G}(\mathcal{M}, q_{in}, \varphi, \omega)$, then she has a winning strategy $\Sigma'_{\mathbf{E}}$ which uses in the embedded games exclusively strategies $\sigma_{\mathbf{E}}$ that depend only on states.*

Proof. Suppose that $\Sigma_{\mathbf{E}}$ is a winning strategy in a finitely bounded evaluation game $\mathcal{G}(\mathcal{M}, q_{in}, \varphi, \omega)$. We first observe that since $\Sigma_{\mathbf{E}}$ is a winning strategy, all strategies $\sigma_{\mathbf{E}}$ that are assigned by $\Sigma_{\mathbf{E}}$ must be winning strategies in the corresponding embedded games.

Let $\mathbf{G} = \mathbf{g}(\mathbf{V}, \mathbf{C}, A, q_0, \psi_{\mathbf{C}}, \psi_{\overline{\mathbf{C}}})$ be an embedded game that is related to some position Pos in the evaluation game $\mathcal{G}(\mathcal{M}, q_{in}, \varphi, \omega)$. Suppose first $\mathbf{E} = \mathbf{C}$, whence $\Sigma_{\mathbf{E}}(\mathsf{Pos}) = (n, \sigma_{\mathbf{E}})$ for some $n < \omega$ (the timer t is not used in finitely bounded case, and thus it can be omitted here). Since $\Sigma_{\mathbf{E}}$ is a winning strategy in the evaluation game, $\sigma_{\mathbf{E}}$ must be winning strategy in the embedded game $\mathbf{G}[n]$. By Proposition 4.7, the canonical strategy $\tau_{\mathbf{E}}$ is a winning strategy in $\mathbf{G}[n]$ (the canonical timer is not needed here).

Suppose then that $\mathbf{E} \neq \mathbf{C}$, whence $\Sigma_{\mathbf{E}}(\mathsf{Pos})$ maps every $n < \omega$ to some strategy $\sigma_{\mathbf{E},n}$. Since $\Sigma_{\mathbf{E}}$ is a winning strategy in the evaluation game, $\sigma_{\mathbf{E},n}$ must be winning strategy in the corresponding bounded embedded game $\mathbf{G}[n]$. By Lemma 4.8 , for every $n < \omega$, the n-canonical strategy $\tau^n_{\mathbf{E}}$ is a winning strategy in $\mathbf{G}[n]$.

By these observations, it is easy to see that we can form a strategy $\Sigma'_{\mathbf{E}}$ for Eloise in such a way that it only uses canonical strategies when $\mathbf{E} = \mathbf{C}$ and maps all $n < \omega$ to n-canonical strategies when $\mathbf{E} \neq \mathbf{C}$. Since these strategies depend on states only, $\Sigma'_{\mathbf{E}}$ satisfies the conditions of this claim. ☐

With the help of the previous lemma, we can now prove the equivalence between the finitely bounded compositional and game-theoretic semantics using a similar induction as the one in the proof of Theorem 5.1.

Theorem 5.4. *Let $\mathcal{M}$ be a CGM, $q_{in} \in \mathrm{St}$ and φ an* ATL*-formula. We have $\mathcal{M}, q_{in} \models_f \varphi$ iff $\mathcal{M}, q_{in} \models^g_f \varphi$.*

Proof. Suppose first that $\mathcal{M} \models^g_f \varphi$, i.e., Eloise has a winning strategy $\Sigma_{\mathbf{E}}$ in $\mathcal{G}(\mathcal{M}, q_{in}, \varphi)$. By Lemma 5.3, we may assume that all of the strategies that $\Sigma_{\mathbf{E}}$ assigns for embedded games depend on states only. This amounts to assuming that their domain is the set of states instead of configurations. We also recall that timers are not needed in the finitely bounded case.

We will show that if a position $\mathsf{Pos} = (\mathbf{P}, q, \psi)$ can be reached in the evaluation game when Eloise uses $\Sigma_{\mathbf{E}}$, then the following condition holds for Pos:

$$\mathcal{M}, q \models_f \psi \quad \text{iff} \quad \mathbf{P} = \mathbf{E}. \tag{$\star$}$$

We will prove this claim by induction on φ.

The cases where $\mathsf{Pos} = (\mathbf{P}, q, p)$ $(p \in \Pi)$, $\mathsf{Pos} = (\mathbf{P}, q, \neg\psi)$, $\mathsf{Pos} = (\mathbf{P}, q, \psi \vee \theta)$ or $\mathsf{Pos} = (\mathbf{P}, q, \langle\!\langle A \rangle\!\rangle \mathsf{X}\, \psi)$ are treated exactly as in the proof of Theorem 5.1.

Suppose that $\mathsf{Pos} = (\mathbf{P}, q, \langle\!\langle A \rangle\!\rangle\, \psi \,\mathsf{U}\, \theta)$ can be reached in the game. If $\mathbf{P} = \mathbf{E}$, then $\Sigma_{\mathbf{E}}(\mathsf{Pos}) = (n, \sigma_{\mathbf{E}})$ where $n < \omega$. Let S_A be the collective strategy that is related to the strategy $\sigma_{\mathbf{E}}$ (see the corresponding part in the proof of Theorem 5.1). Let $\Lambda \in \mathsf{paths}(q, S_A)$. Now, when Eloise uses $\sigma_{\mathbf{E}}$, there exist actions of Abelard such that the states of the embedded game will be on Λ until some configuration $(n-i, \Lambda[i])$ (with

$i \leq n$) at which Eloise ends the game at the position $(\mathbf{E}, \Lambda[i], \theta)$ (If she does not stop the game, then the game will automatically end at the exit position $(\mathbf{E}, \Lambda[n], \theta)$). Thus, by the induction hypothesis, $\mathcal{M}, \Lambda[i] \models_f \theta$. Let then $j < i$. Since Abelard can end the game after j rounds at the position $(\mathbf{E}, \Lambda[j], \psi)$, by the induction hypothesis, we have $\mathcal{M}, \Lambda[j] \models_f \psi$. Hence $\mathcal{M}, q \models_f \langle\langle A \rangle\rangle\, \psi \,\mathsf{U}\, \theta$, and thus $(\star)$ holds for Pos.

If $\mathbf{P} = \mathbf{A}$, then $\Sigma_{\mathbf{E}}$ assigns some strategy $\sigma_{\mathbf{E},n}$ to every $n < \omega$. For the sake of contradiction, suppose that we have $\mathcal{M}, q \models_f \langle\langle A \rangle\rangle\, \psi \,\mathsf{U}\, \theta$, i.e., there exist some $n' < \omega$ and S_A such that for each $\Lambda \in \mathsf{paths}(q, S_A)$, there is some $i \leq n'$ such that $\mathcal{M}, \Lambda[i] \models_f \theta$ and $\mathcal{M}, \Lambda[j] \models_f \psi$ for every $j < i$. Let $\sigma_{\mathbf{A}}(q') = \theta$ for each $q' \in \mathrm{St}$ where $\mathcal{M}, q' \models_f \theta$. For all other states $q' \in \mathrm{St}$, let $\sigma_{\mathbf{A}}(q')$ be the tuple of actions for the agents in A chosen according to S_A. Now, if Abelard uses $\sigma_{\mathbf{A}}$ and Eloise $\sigma_{\mathbf{E},n'}$, then the exit position of the game must be $(\mathbf{A}, \Lambda[i], \mu)$ (where $\mu \in \{\psi, \theta\}$) for some $\Lambda \in \mathsf{paths}(q, S_A)$ and $i \leq n'$. But for this exit position, the condition $(\star)$ does not hold even though the position is reached using the strategy $\Sigma_{\mathbf{E}}$. This contradicts the induction hypothesis. Hence $\mathcal{M}, q \not\models_f \langle\langle A \rangle\rangle\, \psi \,\mathsf{U}\, \theta$, and thus $(\star)$ holds for Pos.

Suppose that $\mathsf{Pos} = (\mathbf{P}, q, \langle\langle A \rangle\rangle\, \psi \,\mathsf{R}\, \theta)$ can be reached in the game. If $\mathbf{P} = \mathbf{E}$, then $\Sigma_{\mathbf{E}}(\mathsf{Pos})$ assigns some strategy $\sigma_{\mathbf{E},n}$ for every $n < \omega$. Let $n < \omega$ and let $S_{A,n}$ be the collective strategy that is related to $\sigma_{\mathbf{E},n}$. Let $\Lambda \in \mathsf{paths}(q, S_{A,n})$. Now, when Eloise plays using $\sigma_{\mathbf{E},n}$, there are some actions of Abelard such that the states of the embedded game are on Λ until Eloise ends the game or n rounds have lapsed. We need to show that for every $i \leq n$ either $\mathcal{M}, \Lambda[i] \models_f \theta$ or there is some $j < i$ s.t. $\mathcal{M}, \Lambda[j] \models_f \psi$. Let $i \leq n$. If Eloise ends the game before i rounds have gone, the game ends at the position $(\mathbf{E}, \Lambda[j], \psi)$ for some $j < i$, whence by the induction hypothesis, $\mathcal{M}, \Lambda[j] \models_f \psi$. If Eloise does not end the game before i rounds have lapsed, then Abelard may end it at the position $(\mathbf{E}, \Lambda[i], \theta)$, whence by the induction hypothesis, $\mathcal{M}, \Lambda[i] \models_f \theta$. Hence $\mathcal{M}, q \models_f \langle\langle A \rangle\rangle\, \psi \,\mathsf{R}\, \theta$, and thus the condition $(\star)$ holds for Pos.

If $\mathbf{P} = \mathbf{A}$, then $\Sigma_{\mathbf{E}}(\mathsf{Pos}) = (n', \sigma_{\mathbf{E}})$ for some $n' < \omega$. For the sake of contradiction, suppose that $\mathcal{M}, q \models_f \langle\langle A \rangle\rangle\, \psi \,\mathsf{R}\, \theta$, i.e., for every $n < \omega$, there exists $S_{A,n}$ such that for each $\Lambda \in \mathsf{paths}(q, S_{A,n})$ and $i \leq n$, we have either $\mathcal{M}, \Lambda[i] \models_f \theta$ or there is some $j < i$ such that $\mathcal{M}, \Lambda[j] \models_f \psi$. Let $\sigma_{\mathbf{A}}(q') = \theta$ for each $q' \in \mathrm{St}$ where $\mathcal{M}, q' \models_f \theta$. For all other states $q' \in \mathrm{St}$, let $\sigma_{\mathbf{A}}(q')$ be the tuple of actions for the agents in A according to $S_{A,n'}$. Now, if Abelard uses $\sigma_{\mathbf{A}}$ and Eloise $\sigma_{\mathbf{E}}$, the exit position of the game must be $(\mathbf{A}, \Lambda[i], \mu)$ (where $\mu \in \{\psi, \theta\}$) for some $\Lambda \in \mathsf{paths}(q, S_A)$ and $i \leq n'$. But for this exit position, the condition $(\star)$ does not hold even thought it is reached by strategy $\Sigma_{\mathbf{E}}$. This is a contradiction. Hence $\mathcal{M}, q \not\models_f \langle\langle A \rangle\rangle\, \psi \,\mathsf{R}\, \theta$, and thus the condition $(\star)$ holds for the position Pos.

We now conclude that the condition $(\star)$ must hold for the initial position $\mathsf{Pos}_0 = (\mathbf{E}, q_{in}, \varphi)$. Therefore we have $\mathcal{M}, q_{in} \models_f \varphi$. This concludes the first direction of the proof of the current theorem.

Suppose then that $\mathcal{M}, q_{in} \models_f \varphi$. We will formulate such a strategy $\Sigma_{\mathbf{E}}$ for Eloise that the condition $(\star)$ holds at each position $\mathsf{Pos}_i = (\mathbf{P}, q, \psi)$ of the game. It holds in the initial position $\mathsf{Pos}_0 = (\mathbf{E}, q_{in}, \varphi)$ by the assumption that $\mathcal{M}, q_{in} \models_f \varphi$. We let the induction hypothesis be that the condition $(\star)$ holds in the a position Pos_i, and show that we can define $\Sigma_{\mathbf{E}}(\mathsf{Pos}_i)$ in such a way that $(\star)$ will hold in the next position Pos_{i+1}.

The cases $\mathsf{Pos}_i = (\mathbf{P}, q, \neg\psi)$, $\mathsf{Pos}_i = (\mathbf{P}, q, \psi \vee \theta)$ and $\mathsf{Pos}_i = (\mathbf{P}, q, \langle\langle A \rangle\rangle\, X\psi)$ are treated exactly as in the proof of Theorem 5.1.

Let $\mathsf{Pos}_i = (\mathbf{P}, q, \langle\langle A \rangle\rangle\, \psi \,\mathsf{U}\, \theta)$. If $\mathbf{P} = \mathbf{E}$, then, by the induction hypothesis, $\mathcal{M}, q \models_f \langle\langle A \rangle\rangle\, \psi \,\mathsf{U}\, \theta$, i.e., there exist some $n < \omega$ and S_A such that for each $\Lambda \in \mathsf{paths}(q, S_A)$, there is some $i \leq n$ s.t. $\mathcal{M}, \Lambda[i] \models_f \theta$ and $\mathcal{M}, \Lambda[j] \models_f \psi$ for every $j < i$. Let

$\Sigma_{\mathbf{E}}(\mathsf{Pos}_i) = (n, \sigma_{\mathbf{E}})$, where $\sigma_{\mathbf{E}}(q') = \theta$ for each $q' \in \mathrm{St}$ where $\mathcal{M}, q' \models_f \theta$ and for all other states $q' \in \mathrm{St}$ let $\sigma_{\mathbf{E}}(q')$ be the tuple of actions for the agents in A chosen according to S_A. Now, when Eloise chooses the time limit to be n and uses $\sigma_{\mathbf{E}}$, then, regardless of the actions of Abelard, all states that are reached in the game must be states $\Lambda[i]$ for some $\Lambda \in \mathsf{paths}(q, S_A)$ and $i \leq n$. Thus Eloise can reach a state q' where $\mathcal{M}, q' \models_f \theta$ in n rounds; if Abelard ends the game before that at some state q', then $\mathcal{M}, q \models_f \psi$. Hence the condition $(\star)$ holds for the next position Pos_{i+1} in any case.

If $\mathbf{P} = \mathbf{A}$, by the induction hypothesis, $\mathcal{M}, q \not\models_f \langle\!\langle A \rangle\!\rangle\, \psi \,\mathsf{U}\, \theta$, i.e., for every $n < \omega$ and collective strategy S_A, there exists a path $\Lambda \in \mathsf{paths}(q, S_A)$ such that for each $i \leq n$, we have either $\mathcal{M}, \Lambda[i] \not\models_f \theta$ or $\mathcal{M}, \Lambda[j] \not\models_f \psi$ for some $j < i$. For the sake of contradiction, suppose that there is some $n' < \omega$ (that Abelard can choose) such that Eloise does not have a winning strategy in the bounded embedded game $\mathbf{g}(\mathbf{A}, \mathbf{A}, A, q, \theta, \psi)[n']$. Since bounded embedded games are determined, there must be a winning strategy $\sigma_{\mathbf{A}}$ for Abelard in that game. When Abelard is using $\sigma_{\mathbf{A}}$, we have $\mathcal{M}, q' \models^g_f \mu$ for every possible exit position $(\mathbf{A}, q', \mu)$. Hence, by the other (already proved) direction of the current theorem, the condition $(\star)$ cannot hold for any exit position that is reached using $\sigma_{\mathbf{A}}$.

Let S_A be the collective strategy related to $\sigma_{\mathbf{A}}$. By earlier observations, we know that there is some $\Lambda \in \mathsf{paths}(q, S_A)$ s.t. for each $i \leq n'$, we have $\mathcal{M}, \Lambda[i] \not\models_f \theta$ or $\mathcal{M}, \Lambda[j] \not\models_f \psi$ for some $j < i$. Now Eloise can force Abelard on this path when he is using $\sigma_{\mathbf{A}}$. But thus Eloise can play in such a way that the the condition $(\star)$ will hold for all possible ending positions of the game. This is contradiction and thus Eloise must have a winning strategy $\sigma_{\mathbf{E},n}$ in the embedded game with any time limit $n < \omega$. Thus we can define $\Sigma_{\mathbf{E}}(\mathsf{Pos}_i)$ as a function that maps every $n < \omega$ to $\sigma_{\mathbf{E},n}$. Now, if Eloise plays using to $\Sigma_{\mathbf{E}}$, then $\mathcal{M}, q' \not\models^g_f \mu$ for every possible exit position $(\mathbf{A}, q', \mu)$. Hence, by the other direction of the current theorem, the condition $(\star)$ will hold in any exit position which will be the next position Pos_{i+1}.

Let $\mathsf{Pos}_i = (\mathbf{P}, q, \langle\!\langle A \rangle\!\rangle\, \psi \,\mathsf{R}\, \theta)$. If $\mathbf{P} = \mathbf{E}$, then, by the induction hypothesis, $\mathcal{M}, q \models_f \langle\!\langle A \rangle\!\rangle\, \psi \,\mathsf{R}\, \theta$, i.e., for all $n < \omega$, there exists a collective strategy $S_{A,n}$ such that for each $\Lambda \in \mathsf{paths}(q, S_{A,n})$ and $i \leq n$, we have either $\mathcal{M}, \Lambda[i] \models_f \theta$ or there is some $j < i$ such that $\mathcal{M}, \Lambda[j] \models_f \psi$. Let $\Sigma_{\mathbf{E}}(\mathsf{Pos}_i)$ be a function that maps every $n < \omega$ to $\sigma_{\mathbf{E},n}$, where $\sigma_{\mathbf{E},n}$ is defined as follows. Let $\sigma_{\mathbf{E},n}(q') = \psi$ for each $q' \in \mathrm{St}$ where $\mathcal{M}, q' \models_f \psi$. For all other states $q' \in \mathrm{St}$, let $\sigma_{\mathbf{E},n}(q')$ be the tuple of actions for the agents in A chosen according to $S_{A,n}$. Now, when Eloise uses $\sigma_{\mathbf{E},n}$, all states that can be reached must be states $\Lambda[i]$ for some $\Lambda \in \mathsf{paths}(q, S_A)$ and $i \leq n$. Thus Eloise will either stay at states q' where $\mathcal{M}, q' \models_f \theta$ for n rounds or reach a state q' where $\mathcal{M}, q' \models_f \psi$ while maintaining the truth of θ. Hence the condition $(\star)$ will hold for the exit position which will be the next position Pos_{i+1}.

If $\mathbf{P} = \mathbf{A}$, then, by the induction hypothesis, we have $\mathcal{M}, q \not\models_f \langle\!\langle A \rangle\!\rangle\, \psi \,\mathsf{R}\, \theta$, i.e., there is some $n \geq 0$ such that for every S_A, there exists a path $\Lambda \in \mathsf{paths}(q, S_A)$ and some $i \leq n$ such that we have $\mathcal{M}, \Lambda[i] \not\models_f \theta$ and $\mathcal{M}, \Lambda[j] \not\models_f \psi$ for every $j < i$. For the sake of contradiction, suppose that Eloise does not have a winning strategy in the embedded game $\mathbf{g}(\mathbf{A}, \mathbf{E}, A, q, \theta, \psi)[n]$. Since embedded games are determined, there must be a winning strategy $\sigma_{\mathbf{A}}$ for Abelard in the game. Hence, by the other direction of the current theorem, the condition $(\star)$ cannot hold for any exit position that is reached when Abelard is using $\sigma_{\mathbf{A}}$.

Let S_A be the collective strategy related to $\sigma_{\mathbf{A}}$. Based on our earlier observations, there is some $\Lambda \in \mathsf{paths}(q, S_A)$ and $i \leq n$ such that we have $\mathcal{M}, \Lambda[i] \not\models_f \theta$ and $\mathcal{M}, \Lambda[j] \not\models_f \psi$ for every $j < i$. Now Eloise can force Abelard on this path when he is using $\sigma_{\mathbf{A}}$. Hence Eloise can play in such a way that that the the condition $(\star)$ will hold for all possible exit positions of the game and she can reach such an exit position in n rounds. This is contradiction and thus Eloise has a winning strategy $\sigma_{\mathbf{E}}$ in the corresponding embedded game. We can thus define $\Sigma_{\mathbf{E}}(\mathsf{Pos}_i) = (n, \sigma_{\mathbf{E}})$, whence, by the other direction of this theorem, the condition $(\star)$ holds for the next position Pos_{i+1}.

Since Eloise can maintain the condition $(\star)$ in every position of the game, it also holds in every ending position $\mathsf{Pos}_i = (\mathbf{P}, q, p)$. If $\mathbf{P} = \mathbf{E}$, then $\mathcal{M}, q \models_f p$, i.e., $q \in v(p)$, whence Eloise wins the game. And if $\mathbf{P} = \mathbf{A}$, then we have $\mathcal{M}, q \not\models_f p$, i.e., $q \notin v(p)$, whence again Eloise wins. Hence the strategy we have described is indeed a winning strategy for Eloise in $\mathcal{G}(\mathcal{M}, q_{in}, \varphi, \omega)$, and thus $\mathcal{M}, q_{in} \models_f^g \varphi$. ☐

Concluding remarks

We argue that the systems of GTS for ATL introduced in this article are conceptually and technically natural from both logical and game-theoretic perspective. They offer novel complementary approaches to the semantics of ATL. In particular, our bounded GTS provides a framework where truth of ATL-formulae can be determined in finite time. In the future we will develop game-theoretic approaches to ATL^+, ATL^* and beyond. As already argued in the introduction, approaches via GTS have proved their worth in multiple fields of logic.

As mentioned in the introduction, *some* of our technical results could have alternatively been derived relatively directly using results for coalgebraic modal logic. This is because concurrent game models can be viewed as coalgebras for a *game functor* defined in [5], and the fixed-point extension of the coalitional coalgebraic modal logic for this functor links to ATL in a natural way. Game-theoretic semantics has been developed for coalgebraic fixed-point logics, e.g. in [13, 4, 6] and can be used to obtain *some* of our results concerning the unbounded game-theoretic semantics. However, that approach would be unhelpful for readers not familiar with coalgebras and coalgebraic modal logic, so the more direct and self-contained approach in this article has its benefits. Moreover, our work on the bounded and finitely bounded semantics is not directly related to existing work in coalgebraic modal logic. However, even there some natural shortcuts based on background theory could have been used. For example, using König's Lemma, it is not difficult to see that the finitely bounded and unbounded game-theoretic semantics are equivalent on image-finite models.

Acknowledgements

The work of Valentin Goranko was partly supported by a research grant 2015-04388 of the Swedish Research Council. The work of Antti Kuusisto was supported by the ERC grant 647289 "CODA."

References

[1] N. Alechina, N. Bulling, B. Logan, and H. N. Nguyen. On the boundary of (un)decidability: Decidable model-checking for a fragment of resource agent logic. In *Proc. of IJCAI 2015*, pages 1494–1501, 2015.

[2] N. Alechina, B. Logan, N. H. Nga, and A. Rakib. Logic for coalitions with bounded resources. *J. Log. Comput.*, 21(6):907–937, 2011.

[3] R. Alur, T. A. Henzinger, and O. Kupferman. Alternating-time temporal logic. *Journal of the ACM*, 49(5):672–713, 2002.

[4] C. Cîrstea, C. Kupke, and D. Pattinson. EXPTIME tableaux for the coalgebraic μ-calculus. In *CSL 2009*, pages 179–193, 2009.

[5] C. Cîrstea, A. Kurz, D. Pattinson, L. Schröder, and Y. Venema. Modal logics are coalgebraic. *Comput. J.*, 54(1):31–41, 2011.

[6] G. Fontaine, R. A. Leal, and Y. Venema. Automata for coalgebras: An approach using predicate liftings. In *ICALP 2010*, pages 381–392, 2010.

[7] J. Hintikka. *Logic, Language-games and Information: Kantian Themes in the Philosophy of Logic*. Clarendon Press, 1973.

[8] J. Hintikka and G. Sandu. Informational independence as a semantical phenomenon. In J. E. Fenstad, editor, *Logic, Methodology and Philosophy of Science VIII*, pages 571–589. North-Holland, Amsterdam, 1989.

[9] J. Hintikka and G. Sandu. Game-theoretical semantics. In J. van Benthem and A. ter Meulen, editors, *Handbook of Logic and Language*, pages 361–410. Elsevier, 1997.

[10] P. Lorenzen. Ein dialogisches konstruktivitätskriterium. In A. Mostowski, editor, *Proceedings of the Symposium on Foundations of Mathematics, Warsaw 1959*, pages 193–200. Panstwowe wydawnictwo naukowe, 1961.

[11] D. D. Monica, M. Napoli, and M. Parente. On a logic for coalitional games with priced-resource agents. *Electr. Notes Theor. Comput. Sci.*, 278:215–228, 2011.

[12] C. Smith. Graphs and composite games. *Journal of Combinatorial Theory*, 1:51–81, 1966.

[13] Y. Venema. Automata and fixed point logic: A coalgebraic perspective. *Inf. Comput.*, 204(4):637–678, 2006.