On Decidability and Tractability of Querying in Temporal $\mathcal{EL}$

Víctor Gutiérrez-Basulto$^1$, Jean Christoph Jung$^1$, and Roman Kontchakov$^2$

$^1$ Department of Computer Science, Universität Bremen, Germany
$^2$ Dept. of Computer Science and Information Systems, Birkbeck, University of London, UK

Abstract. We study access to temporal data with $\mathcal{T\mathcal{EL}}$, a temporal extension of the tractable description logic $\mathcal{EL}$. Our aim is to establish a clear computational complexity landscape for the atomic query answering problem, in terms of both data and combined complexity. Atomic queries in full $\mathcal{T\mathcal{EL}}$ turn out to be undecidable even in data complexity. Motivated by the negative result, we identify well-behaved yet expressive fragments of $\mathcal{T\mathcal{EL}}$. Our main contributions are a semantic and sufficient syntactic conditions for decidability and three orthogonal tractable fragments, which are based on restricted use of rigid roles, temporal operators, and novel acyclicity conditions on the ontologies.

1 Introduction

Due to the increasing need to account for the temporal dimension of data available on the Web [30, 16], the DL community has recently investigated extensions of the ontology-based data access (OBDA) paradigm for temporal data. The initial efforts concentrated on temporal query languages with atemporal ontologies [22, 25, 5, 9, 10], but some applications, such as managing data from sensor networks, require temporal aspects in conceptual modelling; hence, there is a need for temporal ontology languages [1]. In this line, the research has focused on temporal extensions of $\mathcal{DL-Lite}$ that support rewritability of temporal queries into monadic second-order logic with order or two-sorted first-order logic with < and + [4, 1]. Standard relational databases have such built-in predicates and so, in principle, can evaluate FO($<, +$)-rewritings. However, no temporal extensions of other DLs have been investigated in the context of OBDA, partly due to intractability and often even undecidability of the standard reasoning tasks [2, 19, 20]. On the other hand, temporal data has also been studied in database theory [15]. In their seminal paper, Chomicki and Imielinski [13] identified $\mathcal{DATALOG}_{1S}$ as a decidable extension of $\mathcal{DATALOG}$ with one successor function. We make the first (to the best of our knowledge) attempt to link temporal OBDA with temporal deductive databases [12, 7].

In this paper, we study $\mathcal{T\mathcal{EL}}$, a temporal extension of $\mathcal{EL}$ [6]. The underlying DL component, $\mathcal{EL}$, underpins the OWL 2 EL profile of OWL 2 and the medical ontology SNOMED CT, which provides the vocabulary for electronic health records (EHRs). Indeed, applications managing EHRs must be able to provide information, e.g., on when and for how long some drug has been prescribed to a patient, so that drugs that interact adversely are not prescribed at the same time. Clinical trials [31, 29] also require a unified conceptual model for specifying temporal constraints of protocol entities such as
'a viable participant should have had a vaccination with live virus 5 days ago’ or ‘blood tests of a patient should be run every 3 days’. These statements can be expressed in TEL:

\[
\begin{align*}
\text{Patient} & \sqcap \circ_F^5 \exists \text{vaccinated.LiveVirus} \sqsubseteq \text{ViableParticipant}, \\
\text{Patient} & \sqcap \circ_P^3 \exists \text{RequiresBloodTest} \sqsubseteq \text{RequiresBloodTest}.
\end{align*}
\]

Our main objective is to establish the limits of decidability and tractability of the query answering problem over TEL ontologies, in terms of both data and combined complexity. In order to set the foundations, we focus on temporal atomic queries. On the one hand, an atomic query like \( \text{ViableParticipant}(x, t) \) together with the temporal concept inclusion (1) effectively encodes a tree-shaped temporal conjunctive query. On the other hand, using (1) to extend the vocabulary with the concept \( \text{ViableParticipant} \) is closer to the spirit of the OBDA paradigm than repeating the same conjunction in similar user queries. Moreover, the recurrent pattern \( \text{RequiresBloodTest}(x, t) \) is expressible as an atomic query with the temporal concept inclusion (2) but not expressible as a query without temporal concept inclusions such as (2). As we shall see, even for the atomic queries rather surprising (and challenging) results are obtained.

Our main contributions are complexity bounds, algorithms, and rewritability into DATALOG\(_{1S}\) for atomic query answering in fragments of TEL. Since query answering over full TEL turns out to be undecidable even in data complexity, we investigate its fragments to attain decidability and tractability. First, for TEL\(^C\), which allows only the ‘next’ \( \circ_F \) and ‘previous-time’ \( \circ_P \) operators, we identify ultimate periodicity as a natural semantic condition ensuring decidability, more precisely, \( \text{PSPACE} \) data complexity (the question of decidability of the full TEL\(^C\) is left open for future work). Then, we identify a number of fragments with better computational properties. (a) For the fragment of TEL\(^C\) without rigid (not changing over time) roles on the right-hand side of concept inclusions, we construct a polynomial rewriting into DATALOG\(_{1S}\), and so, establish \( \text{PSpace}\)-completeness for data complexity. This fragment contains all EL ontologies as well as both (1) and (2). (b) Over temporally acyclic TEL\(^C\)-ontologies (with rigid roles), query answering is \( \text{PTIME}\)-complete in both data and combined complexity. This tractable fragment contains (1) and all atemporal EL ontologies and may prove particularly useful in applications; it, however, does not contain (2). (c) Query answering over DL-acyclic TEL\(^C\)-ontologies is \( \text{NC}^1\)-complete for data complexity. This fragment contains many acyclic EL ontologies as well as both (1) and (2) (note that large parts of SNOMED CT are in fact acyclic). We remark that these novel acyclicity conditions (each constraining only one dimension) are inspired by the ‘traditional’ notion of acyclicity in (temporal) DLs [23, 21]. Finally, (d) we show that the language with only \( \circ_F \) and \( \circ_P \) (sometime in the past/future) on the left-hand side of concept inclusions enjoys \( \text{PTIME} \) query answering. All proofs can be found at http://tinyurl.com/TempEL16.

2 Preliminaries

We begin by introducing TEL, a temporal extension of the classical DL EL. Let \( N_C, N_R, N_I \) be countably infinite sets of concept, role and individual names, respectively. We assume that \( N_R \) is partitioned into two infinite sets, \( N_R^{\text{rig}} \) and \( N_R^{\text{loc}} \), of rigid and local role...
names, respectively. \textit{TEL-concepts} are defined by the following grammar:

\[
C, D ::= A | C \cap D | \exists r.C | C \circ | \Diamond C,
\]

where \( A \in \mathbb{N}_C \), \( r \in \mathbb{N}_R \), and \( * \in \{ F, P \} \). A \textit{TEL-TBox} is a finite set of concept inclusions (\textit{CIs}) \( C \sqsubseteq D \) and concept definitions (\textit{CDs}) \( C \equiv D \), for \textit{TEL-concepts} \( C, D \).

Data is given in terms of \textit{temporal ABoxes} \( A \), which are finite sets of assertions of the form \( A(a, n) \) and \( r(a, b, n) \), where \( A \in \mathbb{N}_C \), \( r \in \mathbb{N}_R \), \( a, b \in \mathbb{N}_I \), and \( n \in \mathbb{Z} \). We denote by \text{ind}(\( A \)) the set of individual names occurring in \( A \), and by \text{tem}(\( A \)) the set \( \{ n \in \mathbb{Z} | \min A \leq n \leq \max A \} \), where \( \min A \) and \( \max A \) are, respectively, the minimal and maximal time points in \( A \). The size, \( |T| \) and \( |A| \), of \( T \) and \( A \) is the number of symbols required to write \( T \) and \( A \), resp., with time points \( n \in \mathbb{Z} \) encoded in unary.

An interpretation \( T \) is a structure \( (\Delta^3, (I_n)_{n \in \mathbb{Z}}) \), where each \( I_n \) is a classical DL interpretation with domain \( \Delta^3 \); we have \( A^{I_n} \subseteq \Delta^3 \) and \( r^{I_n} \subseteq \Delta^3 \times \Delta^3 \). Rigid roles \( r \in \mathbb{N}_R^{|}\) do not change their interpretation in time: \( r^{I_n} = r^{I_0} \) for all \( n \in \mathbb{Z} \). We usually write \( A^{I_n} \) and \( r^{I_n} \) instead of \( A^{I_0} \) and \( r^{I_0} \), respectively, and extend \( \cdot^{I_n} \) as follows:

\[
(C \cap D)^{I_n} = C^{I_n} \cap D^{I_n}, \quad (\exists r.C)^{I_n} = \{ d | \text{there is} \ e \in C^{I_n} \text{with} \ (d, e) \in r^{I_n} \},
\]

\[
(C \circ)^{I_n} = C^{I_n} \circ^{I_n} 1, \quad (\Diamond C)^{I_n} = \{ d | \text{there is} \ e \in C^{I_n} \circ^{I_n} k \text{ for some } k > 0 \},
\]

where \( \circ^{I_n} \) stands for + if \( * = F \) and for − if \( * = P \). Although we use strict \( \circ \), our results do not depend on the choice.

TBoxes are interpreted \textit{globally} an interpretation \( T \) is a model of \( C \sqsubseteq D \), written \( T \models C \sqsubseteq D \), if \( C^{I_n} \subseteq D^{I_n} \), for all \( n \in \mathbb{Z} \); and a model of \( C \equiv D \) if \( C^{I_n} = D^{I_n} \), for all \( n \in \mathbb{Z} \). We call \( T \) a model of a TBox \( T \), written \( T \models T \), if \( T \models a \) for all \( a \in T \). For ABoxes \( A \) we adopt the standard name assumption: \( a^{I_n} = a \) for all \( a \in \text{ind}(A) \), \( n \in \mathbb{Z} \); thus, \( \text{ind}(A) \subseteq \Delta^3 \). The relation \( \models \) is extended to ABoxes: \( T \models A(a, n) \) iff \( a \in A^{I_n} \) and \( T \models r(a, b, n) \) iff \( (a, b) \in r^{I_n} \); then, \( T \models A \) if \( T \models a \) for all \( a \in A \). An interpretation \( T \) is a model of a temporal knowledge base (KB) \( K = (T, A) \), written \( T \models K \), if \( T \models T \) and \( T \models A \). Finally, if \( T \models A(a, n) \), then \( T \models A(a, n) \) in every \( T \models K \).

A \textit{temporal atomic query (TAQ)} is of the form \( A(x, t) \), where \( A \in \mathbb{N}_C \), \( x \) an individual variable and \( t \) a temporal variable. A certain answer to \( A(x, t) \) over \( (T, A) \) is a pair \( (a, n) \in \text{ind}(A) \times \text{tem}(A) \) with \( (T, A) \models A(a, n) \). We study the problem of TAQ answering:

\textbf{Input:} TBox \( T \), ABox \( A \), TAQ \( A(x, t) \) and a pair \( (a, n) \).

\textbf{Question:} Is \( (a, n) \) a certain answer to \( A(x, t) \) over \( (T, A) \)?

Our results concern both the combined and data complexity of the problem: for data complexity, the TBox is fixed. As usual, for a complexity class \( C \) and a class \( X \) of TBoxes, we say that TAQ answering over \( X \) is \( C \)-hard in data complexity if there is some \( T \in X \) such that answering TAQs over \( T \) is \( C \)-hard. Conversely, TAQ answering over \( X \) is in \( C \) in data complexity if answering TAQs over \( T \) is in \( C \) for all \( T \in X \).

As classes \( X \), we will in particular look at full \( \text{TEL} \) and its fragments \( \text{TEL}^0 \) and \( \text{TEL}^\circ \), in which, respectively, only the temporal operators \( \Diamond \) and \( \circ \) are allowed. Note that \( \circ \) on the left-hand side (and \( \Box \), with the usual semantics on the right-hand side) of CIs can be expressed in \( \text{TEL}^\circ \), e.g., instead of \( \Diamond p, A \sqsubseteq C \) or, equivalently, \( A \sqsubseteq \Box p, C \), take \( A \sqsubseteq A' \) and \( \Diamond p, A' \sqsubseteq A' \cap C \), for a fresh concept name \( A' \). Observe that rigid concepts, which do not change their interpretation in time, can be expressed in these two fragments using \( \Diamond p, \Diamond p, C \ sqsubseteq C \) and \( \Diamond p, C \ sqsubseteq C \), respectively.
3 Query Answering in $\mathcal{TEL}$: Undecidability

We first pinpoint different sources of complexity for the query answering problem in $\mathcal{TEL}$ in order to identify computationally well-behaved fragments later.

We begin by showing that TAQ answering over $\mathcal{TEL}^3$ is undecidable. The known undecidability of subsumption in $\mathcal{TEL}^3$ [2] translates only into the combined complexity of TAQ answering. We strengthen the result to obtain undecidability in data complexity by reducing the halting problem for the universal Turing machine. We exploit the crucial observation that disjunction, although not in the syntax, can be simulated with $\Diamond$.

**Theorem 1.** TAQ answering over $\mathcal{TEL}^3$ is undecidable in data complexity.

The proof can also be adapted to the non-strict semantics of $\Diamond$, using the chessboard technique [17]. Next, we show that over $\mathcal{TEL}^\circ$ — although it is not capable of expressing disjunction — TAQ answering is hard.

**Theorem 2.** TAQ answering over $\mathcal{TEL}^\circ$ is non-elementary in combined complexity and PSPACE-hard in data complexity.

The proof of PSPACE-hardness is close in spirit to that for DATALOG$_{LS}$ [13]; we only remark that the lower bound holds even for the sublanguage of $\mathcal{TEL}^\circ$ without $\exists r.C$ on the right-hand side of CIs. For the non-elementary lower bound, we take inspiration in the construction for the product modal logic LTL $\times$ K [17, Theorem 6.34]. Our proof requires a careful implementation of the yardstick technique [33] with only Horn formulas.

Decidability of TAQ answering over full $\mathcal{TEL}^\circ$ is left open as interesting and challenging future work; more insights on the difficulty of the problem are given in Section 4. Nevertheless, we show that extending $\mathcal{TEL}^\circ$ with certain DL constructs that are harmless for data complexity of atemporal query answering [26] immediately leads to undecidability. Let $\mathcal{TELI}^\circ$ and $\mathcal{TELF}^\circ$ be the extensions of $\mathcal{TEL}^\circ$ with inverse roles $r^-$ and functionality axioms $\mathit{func}(r)$, respectively.

**Theorem 3.** TAQ answering over $\mathcal{TELI}^\circ$ and $\mathcal{TELF}^\circ$ is undecidable in data complexity.

4 Foundations of Query Answering in $\mathcal{TEL}^\circ$

In the rest of the paper, we study decidability and complexity of TAQ answering in various fragments of $\mathcal{TEL}^\circ$ and $\mathcal{TEL}^\circ$. To this end, we first lay the groundwork for the development of algorithms for query answering in those fragments by introducing canonical quasimodels, which are succinct abstract representations of the universal models of the KBs, see also [4, 1]. They can also be viewed as a generalization of the canonical structures used for query answering in atemporal EL [27].

We assume that $\mathcal{TEL}^\circ$ -TBoxes are in normal form: they consist of CIs of the form
\[
A \sqcap A' \sqsubseteq B, \quad A \sqsubseteq \exists r.B, \quad X \sqsubseteq A,
\]
where $A, A', B \in \mathbb{N}_C$ and $X$ is a basic concept of the form $A, \circ A$, or $\exists r.A$, for $A \in \mathbb{N}_C$. Observe that, without loss of generality, $\circ$, is restricted to the left-hand side of CIs: e.g.,
\[
(A \sqcap (A') \sqsubseteq B), \quad A \sqsubseteq \exists r.B, \quad X \sqsubseteq A,
\]
where $A, A', B \in \mathbb{N}_C$ and $X$ is a basic concept of the form $A, \circ A$, or $\exists r.A$, for $A \in \mathbb{N}_C$. Observe that, without loss of generality, $\circ$, is restricted to the left-hand side of CIs: e.g.,

1 with the usual semantics: $(r^-)^{2,n} = \{(e, d) \mid (d, e) \in r^{3,n}\}$ and $I \models \mathit{func}(r)$ iff $e_1 = e_2$, for all $(d, e_1), (d, e_2) \in r^{3,n}$ and $n \in \mathbb{Z}$. 

---

---
A ⊆ O_r B is equivalent to O_r A ⊆ B. It is routine to show that every $\mathcal{TEC}^0$-TBox can be transformed into the normal form by introducing fresh concept names; see, e.g., [6].

Fix a KB $(\mathcal{T}, \mathcal{A})$ with a $\mathcal{TEC}^0$-TBox $\mathcal{T}$ in normal form. Let $CN$ be the set of concept names in $(\mathcal{T}, \mathcal{A})$. A map $\pi : \mathbb{Z} \to 2^{CN}$ is a trace for $\mathcal{T}$ if it satisfies the following:

(t1) if $A \cap A' \subseteq B \in \mathcal{T}$ and $A, A' \in \pi(n)$, then $B \in \pi(n)$;
(t2) if $\bigcirc_r A \subseteq B \in \mathcal{T}$ and $A \in \pi(n)$, then $B \in \pi(n \circ_r 1)$.

Traces are the building blocks of quasimodels and are used to represent the temporal evolution of individual domain elements. For example, for $\mathcal{T} = \{ \bigcirc_r C \subseteq B, \bigcirc_r B \subseteq C \}$, the map $\pi$ such that $\pi(i) = \{ B \}$ for odd $i$ and $\pi(i) = \{ C \}$ for even $i$ is a trace for $\mathcal{T}$.

In order to describe interactions of domain elements, we require more notation. Let $\pi$ be a trace for $\mathcal{T}$. For a rigid role $r \in N^r_{\text{rig}}$, the $r$-projection of $\pi$ is a map $\text{proj}_r(\pi) : \mathbb{Z} \to 2^{\mathcal{CN}}$ that sends each $i \in \mathbb{Z}$ to $\{ A \mid \exists r.B \subseteq A \in \mathcal{T}, B \in \pi(i) \}$; for a local role $r \in N^r_{\text{loc}}$, $\text{proj}_r(\pi)$ is defined in the same way on 0 but is $\emptyset$ for all other $i \in \mathbb{Z}$.

Given a map $\varrho : \mathbb{Z} \to 2^{\mathcal{CN}}$ and $n \in \mathbb{Z}$, we say that $\pi$ contains the $n$-shift of $\varrho$ and write $\varrho \subseteq^n \pi$ if $\varrho(i - n) \subseteq \pi(i)$, for all $i \in \mathbb{Z}$. For example, let $\mathcal{T} = \{ \exists r.B \subseteq B' \}$ with rigid role $r$. In the picture below, trace $\pi_a$ contains the 1-shift of the $r$-projection of $\pi_B$:

![Diagram](https://example.com/diagram.png)

Note that, if $r$ were local then $\pi_a$ would have to contain $B'$ only at 1 (but not at 3, etc.).

We are now fully equipped to define quasimodels. Henceforth, let $D = \text{ind}(\mathcal{A}) \cup \mathcal{CN}$.

A quasimodel $\mathcal{Q}$ for $(\mathcal{T}, \mathcal{A})$ is a set of traces $\pi_d$ for $\mathcal{T}$ ($d \in D$) such that

(q1) $A \in \pi_a(n)$, for all $A(a, n) \in \mathcal{A}$;
(q2) $B \in \pi_B(0)$, for all $B \in \mathcal{CN}$;
(q3) $\text{proj}_r(\pi_b) \subseteq^n \pi_a$, for all $r(a, b, n) \in \mathcal{A}$;
(q4) if $A \in \pi_d(n)$ then $\text{proj}_r(\pi_B) \subseteq^n \pi_d$, for all $d \in D$, $n \in \mathbb{Z}$ and $A \subseteq \exists r.B$ in $\mathcal{T}$.

Intuitively, quasimodels represent models of $(\mathcal{T}, \mathcal{A})$: each $\pi_a$ stands for the ABox individual $a$; each $\pi_B$, on the other hand, represents all individuals that witness $B$ for CIs $A \subseteq \exists r.B$ in $\mathcal{T}$. The latter is, in fact, the crucial abstraction underlying quasimodels. Note that traces $\pi_B$ are normalized: $B$ occurs at time point 0, which has to be compensated by the shift operation in (q4). For example, in the picture above, if $A \subseteq \exists r.B \in \mathcal{T}$ then, in any model, $a$ has an $r$-successor that belongs to $B$ at moment 1. Such a successor can be obtained as a ‘copy’ of trace $\pi_B$ shifted by 1 so that its origin, 0, matches moment 1 for $a$. Then, by (q4), $a$ belongs to $B'$ at all odd moments.

For the purposes of query answering we need to identify canonical (minimal) quasimodels. We define the canonical quasimodel as the limit of the following saturation (chase-like) procedure. Start with initially empty maps $\pi_d$, for $d \in D$, and apply (t1)–(t2), (q1)–(q4) as rules: (q3), for example, says 'if $r(a, b, n) \in \mathcal{A}$ and $A \in \text{proj}_r(\pi_b)(i)$, then add $A$ to $\pi_a(i)$.' Then we have the following characterization:

**Theorem 4.** Let $\mathcal{T}$ be a $\mathcal{TEC}^0$-TBox and $\Omega = \{ \pi_d \mid d \in D \}$ the canonical quasimodel of $(\mathcal{T}, \mathcal{A})$. Then, $(\mathcal{T}, \mathcal{A}) \models A(a, i)$ iff $A \in \pi_a(i)$, for any $A \in \mathcal{CN}$, $a \in \text{ind}(\mathcal{A})$, $i \in \mathbb{Z}$. 
The condition of ultimate periodicity is rather natural. On the practical side, it is motivated by applications with recurrent patterns such as health care support [31], see concept inclusions (1) and (2) in Section 1. From the theoretical point of view, any satisfiable LTL formula has an ultimately periodic model [28].

The starting point of the rest of the paper is a semantic condition on the canonical quasimodel, ultimate periodicity, which ensures decidability, at least in data complexity. Let $T$ be a $\mathcal{TEL}^\circ$-TBox and $\Omega$ the canonical quasimodel for $(T, \emptyset)$. We say that $T$ is ultimately periodic, if there is $p \in \mathbb{N}$ such that all $\pi_B$, $B \in CN$, in $\Omega$ are ultimately $p$-periodic, that is, for each $B \in CN$, there are positive integers $m_p, p, m_p, p \leq p$ satisfying the following conditions:

$$\pi_B(n - p) = \pi_B(n), \text{ for all } n \leq -m_p, \quad \pi_B(n + p) = \pi_B(n), \text{ for all } n \geq m_p.$$  

Intuitively, an ultimately $p$-periodic trace has repeating sections on the left and on the right:

$$-m_p - 2p, \quad -m_p - p, \quad -m_p, \quad 0, \quad m_p, \quad m_p + p, \quad m_p + 2p.$$  

The condition of ultimate periodicity is rather natural. On the practical side, it is motivated by applications with recurrent patterns such as health care support [31], see concept inclusions (1) and (2) in Section 1. From the theoretical point of view, any satisfiable LTL formula has an ultimately periodic model [28].

**Theorem 5.** $TAQ$ answering over ultimately periodic $\mathcal{TEL}^\circ$-TBoxes is PSPACE-complete in data complexity.

PSPACE-hardness follows from (the proof of) Theorem 2. We prove the matching upper bound by rewriting an ultimately periodic $\mathcal{TEL}^\circ$-TBox $T$ into $\text{DATALOG}_{IS} [13]$. First, we take temporal rules reflecting rigid roles and standard $\mathcal{EL}$ concept inclusions:

$$r(x, y, t \pm 1) \leftarrow r(x, y, t), \quad \text{for } r \in \text{N}_{\text{rig}}^{\text{rig}} \text{ in } T,$$

$$B(x, t) \leftarrow A(x, t), A'(x, t), \quad \text{for } A \sqcap A' \subseteq B \text{ in } T,$$

$$B(x, t) \leftarrow r(x, y, t), A(y, t), \quad \text{for } \exists r.A \sqsubseteq B \text{ in } T.$$  

Second, we observe that, for any trace $\pi_a$ in the canonical quasimodel $\Omega$ of any $(T, A)$, if $A \in \pi_a(n)$ and $A \sqsubseteq \exists r.B \in T$ then, by (q4), $\pi_a$ contains the $n$-shift not only of $\text{proj}_i(\pi_B)$ but also of $\pi_A$. Since $T$ is ultimately periodic, for each trace $\pi_B$, we fix integers $m_p, p, m_p, p$ and take the following rules with a fresh predicate $F_B$:

$$A(x, t + i) \leftarrow B(x, t), \quad \text{for } 0 \leq i < m_p \text{ and } A \in \pi_B(i),$$

$$A(x, t + i) \leftarrow F_B(x, t), \quad \text{for } 0 \leq i < p \text{ and } A \in \pi_B(m_p + i),$$

$$F_B(x, t + m_p) \leftarrow B(x, t) \quad \text{and} \quad F_B(x, t + p) \leftarrow F_B(x, t),$$

and symmetric rules with $m_p, p$, and fresh $F_B$. Intuitively, the rules in the first line replicate the (irregular) part of $\pi_B$ from 0 to $m_p$. The last two rules add recurring markers.
We take a proposition \( \pi_B \) starting from each marker \( F_B \).

The required DATALOG \(_{1S}\)-program \( \Pi_T \) contains all the rules above (note that CIs of the form \( \bigcirc, A \sqsubseteq B \) are also covered by the rules for traces \( \pi_B \)). Using the canonical quasimodel and Theorem 4, it is readily seen that \( \Pi_T \) is equivalent to \( \varphi \): for every temporal ABox \( A \), the answers to \( \Pi_T \) over \( A \) coincide with the certain answers to \( (T, A) \). Theorem 5 follows from the PSPACE data complexity in DATALOG \(_{1S} \) [13] and independence of \( \Pi_T \) from \( A \).

Observe that Theorem 5 does not imply decidability of full \( \mathcal{TEL} \), since it is open whether every \( \mathcal{TEL} \)-TBox is ultimately periodic. We thus turn our attention to *sufficient syntactic conditions* for ultimate periodicity and obtain tight complexity bounds for both data and combined complexity for the resulting fragments. We consider two types of conditions: restricted use of rigid roles and acyclicity of concept inclusions.

## 5 Restricted Use of Rigid Roles

We consider \( \mathcal{TEL}_{\text{loc}} \), the restriction of \( \mathcal{TEL} \) in which only local roles are allowed. Due to the reduced interaction between temporal and DL component, we obtain data tractability.

**Theorem 6.** TAQ answering over \( \mathcal{TEL}_{\text{loc}} \) is PSPACE-complete in combined and PTIME-complete in data complexity.

Lower bounds follow from (the proof of) PSPACE- and PTIME-hardness of entailment in Horn-LTL [11] and atomic query answering in EL, respectively. For the upper bounds, let \( (T, A) \) be a KB with a \( \mathcal{TEL}_{\text{loc}} \)-TBox and \( \Omega = \{ \pi_d \mid d \in D \} \) its canonical quasimodel. We take a proposition \( P_{A, d} \) for each \( A \in \text{CN} \) and \( d \in D \) and construct a Horn-LTL formula \( \varphi_{T, A} \) whose minimal model is isomorphic to \( \Omega \): variable \( P_{A, d} \) is true in the model at moment \( n \) just in case \( A \in \pi_d(n) \). We take the conjunction of the following formulas, for \( d \in D \):

\[
\begin{align*}
\Box(P_{A, d} \land P_{A', d} \rightarrow P_{B, d}), & \quad \text{for } A \sqsubseteq A' \sqsubseteq B \text{ in } T, \\
\Box(\bigcirc, P_{A, d} \rightarrow P_{B, d}), & \quad \text{for } \bigcirc, A \sqsubseteq B \text{ in } T, \\
\bigcirc^n P_{A, a}, & \quad \text{for } A(a, n) \in A, \\
P_{B, B}, & \quad \text{for } B \in \text{CN}, \\
\bigcirc^n P_{B, b} \rightarrow \bigcirc^n P_{A, a}, & \quad \text{for } r(a, b, n) \in A \text{ and } \exists r. B \sqsubseteq A \text{ in } T, \\
P_{B', B} \rightarrow \Box(P_{A, d} \rightarrow P_{A', d}), & \quad \text{for } A \sqsubseteq \exists r. B \text{ and } \exists r. B' \sqsubseteq A' \text{ in } T,
\end{align*}
\]

where \( \bigcirc^n \) is \( \bigcirc^n \bigcirc \) if \( n \geq 0 \) and \( \bigcirc^{-n} \) if \( n < 0 \) and \( \Box \) is the ‘globally’ operator. It is readily verified that \( \varphi_{T, A} \) is as required. Crucially, \( \mathbf{(q4)} \) for local roles boils down to the last formula above. Since entailment in LTL is in PSPACE [32] and \( \varphi_{T, A} \) is polynomial in the size of \( (T, A) \), we obtain membership in PSPACE for combined complexity.

For PTIME data complexity, observe that traces \( \pi_B \), for \( B \in \text{CN} \), are ultimately \( 2^{|T|} \)-periodic because they are traces of the canonical quasimodel for \( (T, \emptyset) \); so, they can be stored in constant space. Next, traces \( \pi_a, a \in \text{ind}(A) \), are ultimately \( 2^{|T|+|A|} \)-periodic, but a closer inspection reveals that the middle irregular section, \( m_\ell + m_r \), is bounded.
We establish the $L_S$ where we consider below are well-suited for temporal extensions of such ontologies. From the ABox and so, $\phi$ range over notions of acyclicity that restrict only one dimension, either DL or temporal.

Theorem 8. TAQ answering over $TEL_{\text{rig}}^0$ is $\text{PSpace}$-complete in data complexity and in Exptime in combined complexity.

6 Acyclicity Conditions

It is known that acyclicity conditions often lead to better complexity. For example, acyclic TBoxes are a way of obtaining CTL-based temporal extensions of $EL$ that have rigid roles and enjoy PTIME subsumption [21]. In $\text{DATALOG}_{1S}$, a restriction on recursion has also been used to attain tractability [12]. From the application point of view, large parts of SNOMED CT and GO [18] are indeed acyclic. So, we believe that the fragments we consider below are well-suited for temporal extensions of such ontologies.

Acyclic TBoxes are finite sets of CDs $A \equiv C$, $A \in \mathbb{N}_C$, such that no two CDs have the same left-hand side, and there are no CDs $A_1 \equiv C_1, \ldots, A_k \equiv C_k$ in $T$ such that $A_{i+1}$ occurs in $C_i$, for all $1 \leq i \leq k$ (where $A_{k+1} := A_1$). We say $A$ is defined in $T$ if $A \equiv C \in T$ and primitive otherwise. We obtain the following basic tractability result.

Theorem 7. TAQ answering over $TEL_{\text{rig}}^0$ is $\text{LogTime}$-uniform $\mathcal{AC}^0$ in data complexity and in $\text{PTime}$ in combined complexity.

We establish the $\text{LogTime}$-uniform $\mathcal{AC}^0$ upper bound by rewriting into FO(+) in Section 4. For a given TAQ $A(x,t)$ and TBox $T$, we construct a two-sorted first-order formula $\varphi_{T,A}(x,t)$ with functions $\pm$ on temporal terms such that $(T,A) \models A(a,i)$ iff $A$ (viewed as an interpretation) is a model of $\varphi_{T,A}(a,i)$, for all ABoxes $A$, $a \in \text{ind}(A)$, $i \in \mathbb{Z}$. We construct $\varphi_{T,A}(x,t)$ by adapting the strategy developed for atemporal $EC$ [8]:

$$
\begin{align*}
\varphi_{T,A}(x,t) &= S_A(x,t), & \text{if } A \text{ is primitive}, \\
\varphi_{T,A}(x,t) &= S_A(x,t) \lor \varphi_{T,C}(x,t), & \text{if } A \equiv C \in T, \\
\varphi_{T,B_1 \lor B_2}(x,t) &= \varphi_{T,B_1}(x,t) \land \varphi_{T,B_2}(x,t), \\
\varphi_{T,\exists B}(x,t) &= \exists y(\forall r, \varphi_{T,B}(x, y, t)), \\
\varphi_{T,\cup B}(x,t) &= \varphi_{T,B}(x, t \oplus 1),
\end{align*}
$$

where $S_A(x,t)$ is a disjunction of all $B(x,t)$, for a concept name $B$, with $T \models B \subseteq A$, and $R_r(x,y,t)$ is $r(x,y,t)$ for $r \in \mathbb{N}_R$ and $\exists t^r r(x,y,t')$ for $r \in \mathbb{N}_R$. Note that $\varphi_{T,A}$ is an FO$^2$-rewriting in the terminology of Artale et al. [4, 1] because the temporal variables range over $\mathbb{Z}$. However, the infinite interpretation of $A$ is empty after at most $|T|$ steps from the ABox and so, $\varphi_{T,A}$ can be converted into an FO-rewriting whose temporal variables range over $\text{tem}(A)$ only; see [1].

To address the restricted expressiveness of acyclic TBoxes, we next introduce novel notions of acyclicity that restrict only one dimension, either DL or temporal.
DL Acyclicity

First, we introduce DL-acyclic $TEL^O$-TBoxes, which are well-suited as temporal extensions of, say, biomedical ontologies that may require recurrent patterns but have an acyclic DL component. A $TEL^O$-TBox $T$ with concept names CN is called DL-acyclic if there is a mapping $\ell_{\alpha}: \text{CN} \to \mathbb{N}$ such that:

(i) $A \sqsubseteq \exists r.B$ or $\exists r.B \sqsubseteq A \in T$ implies $\ell_{\alpha}(A) > \ell_{\alpha}(B)$;
(ii) $\circ A \sqsubseteq B$ implies $\ell_{\alpha}(A) = \ell_{\alpha}(B)$;
(iii) $A \sqcap A' \subseteq B \in T$ implies $\ell_{\alpha}(A) = \ell_{\alpha}(A') = \ell_{\alpha}(B)$.

We say that a DL-acyclic TBox is of depth $k$ if $k$ is the smallest integer $m$ such that there is such a mapping $\ell_{\alpha}$ satisfying $\ell_{\alpha}(B) \leq m$ for all $B \in \text{CN}$.

Theorem 9. TAQ answering over DL-acyclic $TEL^O$-TBoxes of depth $k$, $k \geq 1$, is $k$-EXPSPACE-complete in combined complexity and NC$^1$-complete in data complexity.

A closer inspection of the non-elementary lower bound proof in Theorem 2 reveals that the TBox used is DL-acyclic and TAQ answering over TBoxes of depth $k$ is $k$-EXPSPACE-hard. NC$^1$-hardness in data complexity follows by reduction of the word problem of NFAs to TAQ answering (even without the DL dimension); see [1].

For the matching upper bounds, fix $(T, A)$ with $T$ of depth $k$. We devise a completion procedure, which is based on special LTL-formulas and implies ultimate periodicity of all traces in the canonical quasimodel of $(T, A)$; cf. Section 5. Given any $A$, let $A_i$ consist of all $A(a, i)$ and $r(a, b, i)$ in $A$ as well as all assertions $r(a, b, i)$ such that $r \in \mathbb{N}_{\mathbb{R}}$ and $r(a, b, j) \in A$, for some $j \in \mathbb{Z}$. The algorithm separates consequences coming from the role structure in the ABox and local temporal consequences of $T$. In particular, it exhaustively adds assertions $A(a, i)$ to $A$ if either

$$(T, A_i) \models A(a, i) \quad \text{or} \quad B(a, i \text{ op } 1) \in A \text{ and } \circ B \sqsubseteq A \in T. \quad (3)$$

It turns out that $A_i$ in (3) can be replaced by its suitably defined quotient $B_i$. Intuitively, the logic can only distinguish distinct trees of depth $k$, whose number depends on $|T|$ only; so, the size of $B_i$ is independent of $|A|$. By induction on depth $k$, we define LTL-formulas $\varphi_{a,i}$ of $k$-fold-exponential size characterizing all $A \in \text{CN}$ such that $(T, B_i) \models A(a, i)$; we start from formulas as in Theorem 6; the induction step takes account of the structure of $B_i$ and incurs an exponential blowup.

For the combined complexity upper bound, observe that each of the polynomially many $\varphi_{a,i}$ can be analyzed in $k$-EXPSPACE. For the data complexity upper bound, note that checking $(T, B_i) \models A(a, i)$ can be done in constant time. The second option in (3), however, cannot be implemented directly as the number of steps depends on $|A|$. Instead, by using Büchi automata, we show that the question of whether all traces extending $A$ have $A$ at position $i$ is a regular property and so, is in NC$^1$.

Temporal Acyclicity

We next relax acyclicity by admitting recursion in the DL dimension (but not in temporal); thus, temporally acyclic $TEL^O$-TBoxes include general $EL^O$-TBoxes. A $TEL^O$-TBox $T$ with concept names CN is temporally acyclic if there is $\ell_{\alpha}: \text{CN} \to \mathbb{N}$ such that
We establish a special shape of the traces in the canonical model of any (\(T, A\)) implies \(\ell_\circ(B) = \ell_\circ(A) + 1\); 
(ii) \(\exists r. B \sqsubseteq A\) or \(A \sqsubseteq \exists r. B \in \mathcal{T}\) implies \(\ell_\circ(A) = \ell_\circ(B)\); 
(iii) \(A \sqcap A' \sqsubseteq B \in \mathcal{T}\) implies \(\ell_\circ(A) = \ell_\circ(A') = \ell_\circ(B)\).

Temporally acyclic TBoxes cannot, unlike DL acyclic ones, express rigid concepts. Still, we can partition concept names into local \(N_C\) and rigid \(N_N^R\) ones and obtain:

**Theorem 10.** TAQ answering over temporally acyclic \(\mathcal{TEL}^\circ\) (with rigid concepts) is PTIME-complete in data and combined complexity.

The lower bounds are from \(\mathcal{EL}\). For the upper bounds, we show a small quasimodel property: traces of the canonical quasimodel of any (\(T, A\)) with such a TBox \(T\) satisfy

\[
\begin{align*}
\pi_a(j) &= \pi_a(j'), \quad \text{if } j, j' > \max A + |T| \text{ or } j, j' < \min A - |T|, \\
\pi_B(j) &= \pi_B(j'), \quad \text{if } j, j' > |T| \text{ or } j, j' < -|T|.
\end{align*}
\]

Intuitively, the canonical quasimodel has a restricted temporal extension that stretches only \(|T|\) time points beyond \(A\). By the small quasimodel property, the procedure for constructing the canonical quasimodel can be implemented in polynomial time: traces \(\pi_a\) require only polynomial space, and rules (q1)–(q4) extend the traces.

**Inflationary \(\mathcal{TEL}^\circ\)**

Finally, we follow an approach suggested by Artale et al. [4] (in the context of temporal DL-Lite) and restrict \(\mathcal{TEL}^\circ\) by allowing \(\circ\) only on the left-hand side of CIs. We denote this fragment by \(\mathcal{TEL}_{\text{infl}}^\circ\), for inflationary \(\mathcal{TEL}\) (related to inflationary DATALOG\(_{1S}\) [12]). Note that \(\mathcal{TEL}_{\text{infl}}^\circ\) extends general \(\mathcal{EL}\)-TBoxes. Yet, the complexity remains the same:

**Theorem 11.** TAQ answering over \(\mathcal{TEL}_{\text{infl}}^\circ\) is PTIME-complete in both data and combined complexity.

We need to show only the upper bounds. Let \(T\) be a \(\mathcal{TEL}_{\text{infl}}^\circ\)-TBox with concept names in CN. Observe that \(\mathcal{TEL}_{\text{infl}}^\circ\) can still be viewed as a fragment of \(\mathcal{TEL}^\circ\); see Section 2. In fact, one can show an analogue of Theorem 4 with the following replacement of (t2):

(t2') if \(\circ\), \(A \subseteq B \in \mathcal{T}\) and \(A \in \pi_d(n), \) then \(B \in \pi_d(n \circ_{\text{op}} k)\) for all \(k > 0\).

We establish a special shape of the traces in the canonical model of any (\(T, A\)). Let \(\varrho: Z \rightarrow 2^{CN}\) be a map and let \(l, u \in Z\) with \(l \leq u\). We say that \(\varrho\) is an \([l, u]\)-bow tie if

- for all \(i > u\), we have \(\varrho(i) \subseteq \varrho(i + 1)\), and if \(\varrho(i + 1) = \varrho(i)\) then all \(\varrho(i')\), for \(i' \geq i\), coincide;
- symmetrically, for all \(i < l\), we have \(\varrho(i) \subseteq \varrho(i - 1)\), and if \(\varrho(i - 1) = \varrho(i)\) then all \(\varrho(i')\), for \(i' \leq i\), coincide.

These properties mean that \(\varrho\) grows monotonically to the right of \(u\) and to the left of \(l\); in other words, \(\varrho\) has inflationary behaviour. We prove that the traces \(\pi_d\) in the canonical quasimodel \(\Omega\) of (\(T, A\)), for any \(A\), enjoy the following properties:

- \(\pi_a\) is a \([\min A, \max A]\)-bow tie, for each \(a \in \text{ind}(A)\);
- \(\pi_B\) is a \([0, 0]\)-bow tie, for each \(B \in CN\).

Thus, the traces in \(\Omega\) can be represented in polynomial space because only the middle section and at most \(|CN|\) steps at both ends need to be stored. Since traces are extended with every rule application, the procedure terminates after polynomially many steps.
7 Discussion and Future Work

We summarize the fragments of $\mathcal{TEL}$, their relationships and the obtained complexity results in the following diagram:

```
<table>
<thead>
<tr>
<th>$\geq \text{PSPACE}$</th>
<th>$\geq \text{non-elem}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{PSpace}$</td>
<td></td>
</tr>
<tr>
<td>DL-acyclic $\mathcal{TEL}^{\circ}$</td>
<td>non-elem</td>
</tr>
<tr>
<td>$\text{NC}^1$</td>
<td></td>
</tr>
<tr>
<td>$\text{AC}^0$</td>
<td></td>
</tr>
<tr>
<td>$\geq \text{PTIME}$</td>
<td></td>
</tr>
<tr>
<td>$\text{PTIME}$</td>
<td></td>
</tr>
<tr>
<td>$\mathcal{TEC}^{\circ}$</td>
<td>$\geq \text{non-elem}$</td>
</tr>
<tr>
<td>ultim. period. $\mathcal{TEL}^{\circ}$</td>
<td></td>
</tr>
<tr>
<td>$\mathcal{TEL}^{\circ}_{\text{rg}}$</td>
<td>EXPTime</td>
</tr>
<tr>
<td>$\text{PTIME}$</td>
<td></td>
</tr>
<tr>
<td>$\text{PTIME}$</td>
<td></td>
</tr>
<tr>
<td>$\mathcal{TEC}^{\circ}_{\text{infl}}$</td>
<td>$\text{PTIME}$</td>
</tr>
<tr>
<td>$\mathcal{TEC}^{\circ}$</td>
<td>$\text{undecidable}$</td>
</tr>
</tbody>
</table>
```

where the solid lines are inclusions between DLs, the dashed line is a reduction that preserves answers to all queries (model conservative extension). The data complexity is indicated by shading and the combined complexity is specified below the language.

Our data-tractability results show theoretical adequacy of the identified fragments of $\mathcal{TEL}$ for data-intensive applications. Our two novel forms of acyclicity, DL- and temporal, are somewhat close in spirit to multi-separability [12]; however, the latter puts a weaker restriction on recursion but a stricter one on the interaction between the temporal and data component. DL-acyclic $\mathcal{TEL}^{\circ}$ is the first (to the best of our knowledge) DL shown to have $\text{NC}^1$-complete query answering (the large gap between data and combined complexity is also remarkable). On the practical side, there is evidence that such data-tractable fragments should be sufficient for many biomedical applications. Following the principles of OBDA, our framework provides a means of defining temporal concepts in the ontology for these applications: temporal concepts capture both (restricted) tree-shaped temporal conjunctive queries (CQs) and recurring temporal patterns.

As our immediate future work, we will address decidability of (full) $\mathcal{TEL}^{\circ}$ and then consider CQs with the $+$ operation on temporal terms. We expect that our positive results can be lifted to CQs using the combined approach [27], which utilizes a structure similar to our canonical quasimodel to compute CQ certain answers in atemporal $\mathcal{EL}$. We will also study succinct and expressive representations of temporal data. For example, the only known algorithm for DATALOG$_{1S}$ with binary encoding of timestamps in the data runs in EXPTime in the size of the data [13]. We, however, conjecture that careful materialization should be sufficient to deal with the issue. We will also consider interval encoding of temporal ABoxes, e.g., $A(a, [n_1, n_2])$, and settings capturing infinite temporal periodic data as introduced in [24, 14].
References


