On the uniform one-dimensional fragment

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Abstract

The uniform one-dimensional fragment of first-order logic, U_1, is a recently introduced formalism that extends two-variable logic in a natural way to contexts with relations of all arities. We survey properties of U_1 and investigate its relationship to description logics designed to accommodate higher arity relations, with particular attention given to $\mathcal{DLR}_{reg}$. We also define a description logic version of a variant of U_1 and prove a range of new results concerning the expressivity of U_1 and related logics.

1 Introduction

Two-variable logic [10, 24] and the guarded fragment [1] are currently perhaps the most widely studied subsystems of first-order logic. Two-variable logic FO^2 was proved decidable in [19], and the satisfiability problem of FO^2 was shown to be NEXPTIME-complete in [6]. The extension of two-variable logic with counting quantifiers, FOC^2, was proved decidable in [8, 20] and subsequently shown to be NEXPTIME-complete in [21]. Research on extensions and variants of two-variable logic is currently very active. Recent research has mainly concerned decidability and complexity issues in restriction to particular classes of structures and also questions related to different built-in features and operators that increase the expressivity of the base language. Recent articles in the field include for example [3, 4, 11, 25] and several others.

The guarded fragment was shown 2EXPTIME-complete in [7] and in fact EXPTIME-complete over bounded arity vocabularies in the same article. The guarded fragment has since then generated a vast literature. The fragment has recently been significantly generalized in the article [2] which introduces the *guarded negation first-order logic* GNFO. Intuitively, GNFO only allows negations of formulae that are guarded in the sense of the guarded fragment. The guarded negation fragment has been shown complete for 2NEXPTIME in [2].

The recent article [9] introduced the *uniform one-dimensional fragment*, U_1, which is a natural generalization of FO^2 to contexts with relations of arbitrary arities. Intuitively, U_1 is a fragment of first-order logic obtained by restricting quantification to blocks of existential (universal) quantifiers that *leave at most one free variable* in the resulting formula. Additionally, a *uniformity condition* applies to the use of atomic formulae: if $n, k \geq 2$, then a Boolean combination of atoms $R(x_1, ..., x_k)$ and $S(y_1, ..., y_n)$ is allowed only if the sets $\{x_1, ..., x_k\}$ and $\{y_1, ..., y_n\}$ of variables are equal. Boolean combinations of formulae with at most one free variable can be formed freely, and the use of equality is unrestricted. Several variants of U_1 have also been investigated in [9] and the two subsequent papers [12, 13].

Perhaps the easiest way to gain intuitive insight on U_1 is to consider the *fully uniform fragment*, FU_1, which is a slight restriction of U_1 introduced in the current article. It turns out that FU_1 can be represented roughly as the *standard polyadic modal logic* where novel accessibility relations can be formed by the Boolean combination and permutation of atomic accessibility relations. Recall that polyadic modal logic is the extension of modal logic with formulae $\chi := \langle R\rangle(\varphi_1, ..., \varphi_k)$ interpreted such that $M, w \models \chi$ iff there exist points $u_1, ..., u_k$ such that $(w, u_1, ..., u_k) \in R$ and $M, u_i \models \varphi_i$ for each i. It also turns out, as we shall see, that over vocabularies with at most binary relations, FU_1 is in fact *equi-expressive* with FO^2. This result extends a similar observation from [18] concerning *Boolean modal logic* with the inverse operator and a built-in identity modality. It was proved in [18] that this logic is expressively complete for FO^2. The fact that FU_1 collapses to FO^2 over binary vocabularies can be taken to indicate that FU_1 is a natural and *in some sense* minimal generalization of FO^2 to higher arity contexts.

The uniform one-dimensional fragment U_1 was shown to have the finite model property and a NEXPTIME-complete decision problem in [12], thereby establishing that the transition from FO^2 to U_1 comes without a cost in complexity. It was also shown in [12] that U_1 is incomparable in expressivity with FOC^2; we will prove in the current article that U_1 is incomparable with GNFO, too. We note, however, that the article [9] already established a similar incomparability result concerning GNFO and the *equality-free* fragment of U_1. The article [12] also showed that the extension of U_1 with counting quantifiers is undecidable. The article [9], in turn, established that relaxing either of the two principal constraints of the syntax of U_1-formulae—leaving two free variables after quantification or violating the uniformity condition—leads to undecidability. Building on [9] and [12], the article [13] investigated variants of U_1 in the presence of built-in equivalence relations. It was shown, e.g., that while U_1 becomes 2NEXPTIME-complete when a built-in equivalence is added, a certain natural restriction of U_1 (which still contains FO^2) remains NEXPTIME-complete. In the current article we briefly discuss the above collection of results on U_1 and its variants and list a number of related open problems.

Unlike the guarded fragment and GNFO, two-variable logic does not cope well with relations of arities greater than two, and the same applies to

FOC^2. In database theory contexts, for example, this can be a major drawback. Therefore the scope of research on two-variable logics is significantly restricted. The uniform one-dimensional fragment U_1 extends two-variable logics in a way that leads to the possibility of investigating systems with relations of all arities.

Another possible advantage of U_1 is its one-dimensionality, i.e., the fact that its formulae are *essentially* of the type $\varphi(x)$, where x is a free variable. This links U_1 to description logics in a natural way, as formulae of U_1 can be regarded as *concepts* in the description logic sense. Below we make use of this issue and define a description logic $\mathcal{DL}_{FU_1}$, which we prove to be expressively equivalent to the fully uniform one-dimensional fragment FU_1. The logic $\mathcal{DL}_{FU_1}$ makes explicit the link between FU_1 and polyadic modal logic we mentioned above. It can be seen as *the canonical extension* of the description logic $\mathcal{ALBO}^{id}$ [22] to higher arity contexts. While $\mathcal{ALBO}^{id}$ is $\mathcal{ALC}$ extended with Boolean and inverse operators on roles, an identity role and singleton concepts, $\mathcal{DL}_{FU_1}$ is essentially the same system with roles of all arities. The relational inverse operator is generalized to an operator that slightly generalizes the relational permutation operator.

Higher arity relations arise naturally in contexts relevant to description logics. Consider for example the ternary role R such that $R(a, b, c)$ iff a has contracted a virus b in country c, or the quaternary role S such that $S(c, d, e, f)$ iff c and d have sold e to f. It is easy to see by a counting argument that a k-ary relation cannot be encoded by a finite number of relations of lower arity without changing the domain, and therefore—in addition to aesthetic considerations—a direct access to higher arity roles can be advantageous.

Higher arity roles have of course been investigated before in the desctiption logic literature, for example in [5, 17, 23]. Below we compare $\mathcal{DL}_{FU_1}$ and the system $\mathcal{DLR}_{reg}$ from [5], which includes, e.g., the union, composition and transitive reflexive closure operators for binary roles as well as operators that enable the creation of binary relations from higher arity roles. We show that $\mathcal{DL}_{FU_1}$ and $\mathcal{DLR}_{reg}$ are incomparable in expressivity. While this result itself is not at all surprizing, it is still worth proving since the related arguments directly demonstrate the relative expressivities of $\mathcal{DLR}_{reg}$ and $\mathcal{DL}_{FU_1}$. We end the article by identifying a fragment of $\mathcal{DLR}_{reg}$ which is in a certain sense maximal with the property that it embeds into $\mathcal{DL}_{FU_1}$. In the context of this investigation we discuss the curious fact that while U_1 can count, it cannot count well enough to express the number restriction operators of $\mathcal{DLR}_{reg}$. In the investigations below concerning expressivity issues, we make occasional use of the novel Ehrenfeucht-Fraïssé (EF) game for U_1 from [13]. The related concrete arguments shed light on the expressivity properties of U_1.

Finally, it is worth pointing out here that a rather nice and potentially fruitful feature of $\mathcal{DL}_{FU_1}$ is that it is based on the syntactically and semantically same approach as standard polyadic modal logic. Thereby $\mathcal{DL}_{\mathrm{FU}_1}$ extends the celebrated and fruitful link between modal and description logics to higher arity contexts in a way that *preserves the close relationship* between the two fields.

2 Preliminaries

We let VAR denote a countably infinite set of variable symbols. Let $X = \{x_1, ..., x_k\}$ be a finite set of variable symbols and let R be an n-ary relation symbol; R is not allowed to be the identity symbol here. An atomic formula $R(x_{i_1}, ..., x_{i_n})$ is called an X-*atom* if $\{x_{i_1}, ..., x_{i_n}\} = X$. For example, assuming x, y, z to be distinct variables, both $S(x, y)$ and $T(x, x, y, y, x)$ are $\{x, y\}$-atoms while $P(x)$ and $R(x, y, z)$ are not.

Let $\mathbb{Z}_+$ be the set of positive integers. We let V denote the infinite relational vocabulary $V := \bigcup_{k \in \mathbb{Z}_+} \tau_k$, where τ_k is a countably infinite set of k-ary relation symbols; the equality symbol is not in V. A *unary* V-atom is an atomic formula of the form $P(x)$ or $R(x, ..., x)$, where $P, R \in V$. Here $(x, ..., x)$ denotes the tuple that repeats x exactly n times, n being the arity of R.

The set of formulae of the *equality-free uniform one-dimensional fragment* $\mathrm{U}_1(wo =)$ of first-order logic is the smallest set $\mathcal{F}$ satisfying the following conditions (cf. [9]).

1. Every unary V-atom is in $\mathcal{F}$. Also $\bot, \top \in \mathcal{F}$.
2. If $\varphi \in \mathcal{F}$, then $\neg\varphi \in \mathcal{F}$.
3. If $\varphi, \psi \in \mathcal{F}$, then $(\varphi \wedge \psi) \in \mathcal{F}$.
4. Let $Y := \{x_0, ..., x_k\} \subseteq$ VAR and $X \subseteq Y$. Let φ be a Boolean combination of X-atoms over V and formulae in $\mathcal{F}$ whose free variables (if any) are in Y. Then $\exists x_1 ... \exists x_k\, \varphi \in \mathcal{F}$ and $\exists x_0 ... \exists x_k\, \varphi \in \mathcal{F}$.

For example $\exists y \exists z((\neg R(x, y, z) \vee T(z, y, x, x)) \wedge P(z))$ is a $\mathrm{U}_1(wo =)$-formula, while $\exists y \exists z(S(x, y) \wedge S(y, z) \wedge P(z))$ is not because $\{x, y\} \neq \{y, z\}$. This latter formula is said to *violate the uniformity condition* of U_1. The formula $\exists y R(x, y, z)$ is also illegitimate because it *violates one-dimensionality*, leaving two variables free instead of one. However, the sentence $\exists x \exists z \exists y R(x, y, z)$ is legitimate, and so is $\forall x \exists z \exists y(R(x, y, z) \wedge \exists u \neg U(y, u))$, while the sentence $\forall x \forall z \exists y R(x, y, z)$ is not.

The *fully uniform one-dimensional fragment* FU_1 is the logic whose formulae are obtained from formulae of $\mathrm{U}_1(wo =)$ by allowing the free substitution of any collection of *binary* relation symbols by the equality symbol $=$. The *uniform one-dimensional fragment* U_1 is obtained by adding to the above four clauses that define the set $\mathcal{F}$ of formulae of $\mathrm{U}_1(wo =)$ the additional clause $x = y \in \mathcal{F}$.

For example $\exists y \exists z (R(y,z,x) \wedge x \neq y \wedge \exists z S(y,z))$ is a formula of U_1 but not of FU_1. Clearly FU_1 is a fragment of U_1. The following proposition, where FO^2 denotes two-variable logic with equality, is easy to prove using disjunctive normal form representations of formulae.

Proposition 1. FU_1 *and* FO^2 *are equi-expressive over models with at most binary relations. That is, in restriction to models with relations of arity at most two, each formula of* FU_1 *with at most two free variables has an equivalent* FO^2*-formula, and each* FO^2*-formula has an equivalent* FU_1*-formula.*

However, U_1 is strictly more expressive than two-variable logic FO^2 even over the empty vocabulary, because U_1 can count better than FO^2: we observe that for example the sentence $\exists x \exists y \exists z (x \neq y \wedge x \neq z \wedge y \neq z)$ is a U_1-formula. It is well known and easy to show by a two-pebble-game argument (see [16] for pebble games) that this sentence is not expressible in FO^2.

It is easy to see that FO^2 and therefore FU_1 can define the property that $|P| = 1$ for a unary predicate P. Thus *nominals* can be simulated in those logics. The logic U_1 can define even the properties $|P| \leq k$, $|P| \geq k$ and $|P| = k$ for any finite k. However, the counting capacity of U_1 is restricted in an interesting way, as we will see later on; U_1 cannot make counting statements about the in-degrees and out-degrees of binary relations.

Finally, the U_1-sentence $\exists x \forall y \forall z (R(y,z) \rightarrow (x = y \vee x = z))$ provides a perhaps more interesting example of what is definable in U_1 but not in FO^2. This sentence states that there is an element that belongs to every edge of R. It is easy to see by a two-pebble-game argument that this property is not expressible in FO^2: the *Duplicator* wins the two-pebble-game played on K_2 and K_3, where K_n is the n-clique. Recall that the n-clique is the structure with n elements where R is the total binary relation with the reflexive loops removed.

3 Complexity of U_1 and its variants

The complexity of U_1 was identified in [12] by showing that the logic has the exponential model property.

Theorem 1 ([12])**.** *Every satisfiable* U_1*-formula* φ *has a model whose size is bounded exponentially in* $|\varphi|$.

Theorem 2 ([12])**.** *The satisfiability problem (=finite satisfiability problem) for* U_1 *is* NEXPTIME*-complete.*

The argument in [12] leading to the above results bears *at least some degree* of resemblance to the NEXPTIME upper bound proof of FO^2 by Grädel, Kolaitis and Vardi in [6]. It turns out that U_1-formulae can be transferred into equisatisfiable formulae in a generalized version of the *Scott normal form* specially designed for U_1, and the exponential model property can then be established by appropriately modifying and extending the arguments applied in [6].

The complexity results of the article [12] were extended in [13]. If L denotes a fragment of first-order logic and $R_1, ..., R_k$ are binary relation symbols, then we let $L(R_1, ..., R_k)$ denote the language obtained by allowing for the free substitution of identity symbols in L-formulae by the special symbols R_i. The article [13] investigated U_1 and its variants over models with a built-in equivalence relation $\sim$. It was shown that the satisfiability (SAT) and finite satisfiability (FINSAT) problems for $\mathrm{U}_1(\sim)$ are 2NEXPTIME-complete. The article [13] also identified a natural restriction SU_1 of U_1 that still extends FO^2 and showed that the SAT and FINSAT problems for $\mathrm{SU}_1(\sim)$ are only NEXPTIME-complete; see [13] for the formal definition of SU_1. Furthermore, the article [13] established that the SAT and FINSAT-problems of $\mathrm{SU}_1(\sim_1, \sim_2)$, i.e., SU_1 with two built-in equivalences, is undecidable. This contrasts with the case for FO^2 which remains decidable with two equivalences (SAT [14] and FINSAT [15]).

Several immediately interesting open problems remain, for example the decidability issue for $\mathrm{U}_1(\leq)$, where $\leq$ denotes a built-in linear order. Also, while $\mathrm{U}_1(tr)$ (i.e., U_1 with a built-in transitive relation tr) was shown undecidable in [13], it was left open whether $\mathrm{U}_1(tr(uniform))$ is decidable; here $\mathrm{U}_1(tr(uniform))$ denotes the language obtained from U_1 by allowing the free substitution of any instances of a binary relation (rather than the equality symbol) by the built-in transitive relation tr.

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4 Expressivity issues

In this section we provide an overview on the expressivity of U_1 and its variants. The following theorem from [12] relates the expressivities of U_1 and FOC^2.

Theorem 3 ([12]). U_1 *and* FOC^2 *are incomparable in expressivity.*

Proof. It is easy to show that the U_1-sentence $\exists x \exists y \exists z R(x, y, z)$ cannot be expressed in FOC^2, and therefore $\mathrm{U}_1 \not\leq \mathrm{FOC}^2$. To prove that $\mathrm{FOC}^2 \not\leq \mathrm{U}_1$, let S be a binary relation symbol. We will show that U_1 cannot express the FOC^2-definable condition that the in-degree (with respect to the relation S) at every node is at most one. Assume $\varphi(S)$ is a U_1-formula that defines the property. Consider the formula $\varphi(S) \wedge \forall x \exists y S(x, y) \wedge \exists x \forall y \neg S(y, x)$. It is clear that this formula has only infinite models, and thereby the assumption that U_1 can express $\varphi(S)$ is false by the finite model property of U_1 (Theorem 1). $\square$

We next consider U_1 over vocabularies with at most binary relations.

Theorem 4 ([12])**.** *Consider models over a relational vocabulary τ with the arity bound two. Suppose that τ indeed contains at least one binary relation symbol. Then* $FO^2 < U_1 < FOC^2$.

Proof. We already discussed the strict inclusion $FO^2 < U_1$ above in the preliminaries section. A lengthy proof of the inclusion $U_1 \leq FOC^2$ is given in [12]. The strictness of this inclusion follows from the proof of Theorem 3 where we showed that U_1 cannot express that the in-degree of a binary relation is at most one. □

We then compare the expressivities of U_1 and the guarded negation fragment GNFO [2]. The first non-inclusion ($U_1 \not\leq$ GNFO) of the following theorem has been proved in [9], where only the equality-free fragment of U_1 was investigated. The second non-inclusion (GNFO $\not\leq U_1$) is new.

Theorem 5. U_1 *and* GNFO *are incomparable in expressivity.*

Proof. Define the two structures $\big(\{a\}, \{(a,a)\}\big)$ and $\big(\{a,b\}, \{(a,a),(b,b)\}\big)$. It is straightforward to establish by using the bisimulation for GNFO, provided in [2], that these two structures are bisimilar in the sense of GNFO. Thus the U_1-sentence $\exists x \exists y \neg R(x,y)$ is not expressible in GNFO. Hence $U_1 \not\leq$ GNFO.

Consider then the GNFO-sentence $\varphi := \exists x \exists y \exists z (Rxy \wedge Ryz \wedge Rzx)$. Let $\mathfrak{A}$ denote the model consisting of four disjoint copies of the directed cycle with three elements. Let $\mathfrak{B}$ be the model with three disjoint copies of the directed cycle with four elements. It follows rather directly from the Ehrenfeucht-Fraïssé game for U_1 (which is defined in [13]) that $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same U_1-sentences. For the game-based argument to work, it is essential that the two models $\mathfrak{A}$ and $\mathfrak{B}$ have the same cardinality, because bijections between subsets of the domains of $\mathfrak{A}$ and $\mathfrak{B}$ are used in the game. (See [13] for a detailed discussion of the game.) With $\mathfrak{A}$ and $\mathfrak{B}$ defined in this way, the rest of the game-based argument is straightforward. We can therefore now conclude that U_1 cannot express the GNFO-sentence φ we fixed above, and hence GNFO $\not\leq U_1$. □

Before we close the current section, we observe that all the above results concerning expressivity hold even if attention is limited to finite models only. The same proofs apply without modification, as the reader can check. This is especially interesting in the case of Theorem 3, whose proof makes use of the finite model property of U_1.

5 Undecidability of U_1 with counting quantifiers

Since FOC^2 and U_1 are both NEXPTIME-complete, it is natural to ask whether the extension of U_1 by counting quantifiers (UC_1) remains decidable. Formally, UC_1 is obtained from U_1 by allowing the free substitution of quantifiers $\exists$ by quantifiers $\exists^{\geq k}, \exists^{\leq k}, \exists^{=k}$.

While the transition from FO^2 to FOC^2 preserves NEXPTIME-completeness, the analogous step from U_1 to UC_1 crosses the undecidability barrier.

Theorem 6 ([12]). *The satisfiability and finite satisfiability problems of* UC_1 *are* Π^0_1*-complete and* Σ^0_1*-complete, respectively.*

Thereby UC_1 has the same complexity as full first-order logic. It is an interesting open problem to identify natural logics that extend FOC^2 into higher arity contexts in a way that preserves decidability. Possible research directions here could involve for example investigating restrictions of UC_1 based on somewhat more limited ways of using the quantifiers $\exists^{\geq k}, \exists^{\leq k}, \exists^{=k}$.

6 U_1 and description logics

In this section we define a novel logic $\mathcal{DL}_{\mathrm{FU}_1}$ which is a description logic version of FU_1 and compare it to $\mathcal{DLR}_{reg}$ [5], which is a well-known description logic that accommodates higher arity relations.

We first generalize the relational inverse operation to contexts with higher arity relations. When n is a positive integer, we let $[n]$ denote the set $\{1, ..., n\}$. We let SRJ denote the set of all surjections $\sigma : [k] \to [m]$, such that $2 \leq m \leq k$. When $m = k$, then σ is a permutation; permutations are natural generalizations of the relational inverse operator into higher arity contexts, and surjections generalize permutations an inch further. When we use SRJ in constructing the syntax of $\mathcal{DL}_{\mathrm{FU}_1}$ below, we assume each function $\sigma \in$ SRJ to be a suitable string listing the ordered pairs (n, k) such that $\sigma(n) = k$ in binary.

The set $\mathcal{R}$ of roles of $\mathcal{DL}_{\mathrm{FU}_1}$ is defined by the grammar

$$\mathcal{R} \ ::= \ R \,|\, \varepsilon \,|\, \neg\mathcal{R} \,|\, (\mathcal{R}_1 \cap \mathcal{R}_2) \,|\, \sigma\mathcal{R}$$

where R denotes an atomic role, ε the binary identity role and $\sigma \in$ SRJ. Here R can have any arity greater or equal to two, and the arity of ε is two. The intersection of relations of different arity will produce the empty relation, so we may as well allow such terms. (We fix the arity of the empty relation in such cases to be two.) The set of concepts of $\mathcal{DL}_{\mathrm{FU}_1}$ is given by the grammar

$$C \ ::= \ A \,|\, \neg C \,|\, (C_1 \sqcap C_2) \,|\, \exists\mathcal{R}.(C_1, ..., C_n)$$

where A is an atomic concept and the arity of the relation term $\mathcal{R}$ is $n + 1$. An *interpretation* $\mathcal{I}$ is a pair $(\Delta, \cdot^{\mathcal{I}})$, where Δ is a nonempty set and $\cdot^{\mathcal{I}}$ a function such that $R^{\mathcal{I}} \subseteq \Delta^k$ and $A^{\mathcal{I}} \subseteq \Delta$ for atomic roles R and atomic concepts A; here k is the arity of R. The operators of $\mathcal{DL}_{\mathrm{FU}_1}$ are defined as follows.

1. $\varepsilon^{\mathcal{I}} := \{ (u,u) \mid u \in \Delta \}$, $(\neg\mathcal{R})^{\mathcal{I}} := \Delta^{n+1} \setminus \mathcal{R}^{\mathcal{I}}$ and $(\mathcal{R}_1 \cap \mathcal{R}_2)^{\mathcal{I}} := \mathcal{R}_1^{\mathcal{I}} \cap \mathcal{R}_2^{\mathcal{I}}$.
2. $(\sigma\mathcal{R})^{\mathcal{I}} := \{(u_1, ..., u_m) \mid (u_{\sigma(1)}, ..., u_{\sigma(n+1)}) \in \mathcal{R}^{\mathcal{I}}\}$. Here σ maps $[n+1]$ onto $[m]$. The arity of $(\sigma\mathcal{R})^{\mathcal{I}}$ is of course m.
3. $(\neg C)^{\mathcal{I}} := \Delta \setminus C^{\mathcal{I}}$ and $(C \sqcap D)^{\mathcal{I}} := C^{\mathcal{I}} \cap D^{\mathcal{I}}$.
4. $(\,\exists\mathcal{R}.(C_1, ..., C_n)\,)^{\mathcal{I}} :=$
   $\{\, u \in \Delta \mid \text{there is a tuple } (u, v_1, ..., v_n) \in \mathcal{R}^{\mathcal{I}} \text{ s.t. } v_i \in C_i^{\mathcal{I}} \text{ for each } i \,\}$

In the pathological case where $\sigma : [n] \to [m]$ acts on a relation $\mathcal{R}$ whose arity is not equal to n, the empty binary relation is produced. We need the surjection operators (rather than simply permutations) in order to express in $\mathcal{DL}_{\mathrm{FU}_1}$ conditions such as the one given by the FU_1-formula $\exists y(R(x,y) \wedge S(x,y,x) \wedge P(y))$. In the following theorem, equivalence means equivalence in the standard sense in which formulae of modal and predicate logic are compared.

Theorem 7. *$\mathcal{DL}_{\mathrm{FU}_1}$ and FU_1 are equi-expressive: each FU_1-formula $\varphi(x)$ has an equivalent $\mathcal{DL}_{\mathrm{FU}_1}$-concept, and vice versa.*

Proof. We only provide a rough sketch the proof. The most involved issue here is the translation of FU_1-formulae of the type $\exists x_1 ... \exists x_k \varphi$ into $\mathcal{DL}_{\mathrm{FU}_1}$, where φ is a Boolean combination of higher arity atoms and at most unary FU_1-formulae. Here we put φ into disjunctive normal form and distribute the quantifier prefix over the disjunctions in order to obtain a disjunction of formulae of the type

$$\exists x_1 ... \exists x_k \big(\mathcal{T}(y_1, ..., y_n) \wedge \chi_1(u_1) \wedge ... \wedge \chi_m(u_m)\big),$$

where $\{y_1, ..., y_n\} \subseteq \{x_0, x_1, ..., x_k\}$, $\{u_1, ..., u_m\} \subseteq \{x_0, x_1, ..., x_k\}$, and where the term $\mathcal{T}(y_1, ..., y_n)$ is a *conjunction* of higher arity literals (atoms and negated atoms) such that each literal has exactly the same set $\{y_1, ..., y_n\}$ of variables. Such formulae can easily be translated into $\mathcal{DL}_{\mathrm{FU}_1}$, assuming inductively that we already know how to translate the unary FU_1-formulae $\chi_i(u_i)$. $\square$

We then define the description logic $\mathcal{DLR}_{reg}$ from [5] and compare it to $\mathcal{DL}_{\mathrm{FU}_1}$. $\mathcal{DLR}_{reg}$ is defined by the grammar

$$\begin{aligned}
\mathcal{R} &::= \top_n \mid R \mid (\$i/n : C) \mid \neg\mathcal{R} \mid (\mathcal{R}_1 \cap \mathcal{R}_2) \\
\mathcal{E} &::= \varepsilon \mid \mathcal{R}_{|\$i,\$j} \mid (\mathcal{E}_1 \circ \mathcal{E}_2) \mid (\mathcal{E}_1 \cup \mathcal{E}_2) \mid \mathcal{E}^* \\
C &::= \top_1 \mid A \mid \neg C \mid (C_1 \sqcap C_2) \mid \exists\mathcal{E}.C \mid \exists[\$i]\mathcal{R} \mid (\leq k\,[\$i]\mathcal{R})
\end{aligned}$$

where R is an atomic role and A an atomic concept from a finite set $\mathcal{V}$ of atomic role and concept symbols. The indices i and j denote integers between 1 and n_{max} (where n_{max} is the maximum arity of the symbols in $\mathcal{V}$), n denotes an integer between 2 and n_{max} and k denotes a non-negative integer. All these numbers are encoded in binary.

An interpretation $\mathcal{I} = (\Delta, \cdot^{\mathcal{I}})$ for $\mathcal{DLR}_{reg}$ over $\mathcal{V}$ is any structure such that the following conditions are met (cf. [5]).

1. For each atomic concept $A \in \mathcal{V}$ and atomic role $R \in \mathcal{V}$, we have $A \subseteq \Delta$ and $R \subseteq \Delta^n$, where n is the arity of R.
2. For each $n > 1$, $(\top_n)^{\mathcal{I}}$ is a subset of Δ^n that covers the relations of arity n.
3. $(\$i/n : C)^{\mathcal{I}}$ is the set of tuples $(u_1, ..., u_n) \in (\top_n)^{\mathcal{I}}$ such that $u_i \in C^{\mathcal{I}}$.
4. $(\neg\mathcal{R})^{\mathcal{I}} = (\top_n)^{\mathcal{I}} \setminus \mathcal{R}^{\mathcal{I}}$ when $\mathcal{R}$ is an n-ary term and $(\mathcal{R}_1 \cap \mathcal{R}_2)^{\mathcal{I}} = \mathcal{R}_1^{\mathcal{I}} \cap \mathcal{R}_2^{\mathcal{I}}$.
5. $\varepsilon^{\mathcal{I}} = \{\, (u, u) \,|\, u \in \Delta \,\}$ and $(\mathcal{R}_{|\$i,\$j})^{\mathcal{I}}$ is the relation

$$\{\, (u, v) \,|\, u = w_i \text{ and } v = w_j \text{ for some tuple } (w_1, ..., w_n) \in \mathcal{R}^{\mathcal{I}} \,\}.$$

6. The operators $\circ$, $\cup$ and $\cdot^*$ in the terms $(\mathcal{E}_1 \circ \mathcal{E}_2)$, $(\mathcal{E}_1 \cup \mathcal{E}_2)$ and $\mathcal{E}^*$ are interpreted in the usual way, i.e., $\circ$ is the relational composition operator, $\cup$ the union and $\cdot^*$ the transivitive reflexive closure operator.
7. $(\top_1)^{\mathcal{I}} = \Delta$, $(\neg C)^{\mathcal{I}} = (\top_1)^{\mathcal{I}} \setminus C^{\mathcal{I}}$ and $(C \sqcap D)^{\mathcal{I}} = C^{\mathcal{I}} \cap D^{\mathcal{I}}$.
8. $(\exists\mathcal{E}.C)^{\mathcal{I}} = \{\, u \,|\, \text{ exists } (u, v) \in \mathcal{E}^{\mathcal{I}} \text{ such that } v \in C^{\mathcal{I}} \,\}$
9. $(\exists[\$i]\mathcal{R})^{\mathcal{I}} = \{\, u \,|\, \text{ exists } (v_1, ..., v_n) \in \mathcal{R}^{\mathcal{I}} \text{ such that } u = v_i\}$
10. $(\leq k\, [\$i]\mathcal{R})^{\mathcal{I}} = \{\, u \,|\, \; |\{\, u \,|\, \text{exists } (v_1, ..., v_n) \in \mathcal{R}^{\mathcal{I}} \text{ s.t. } u = v_i\}| \leq k \,\}$.

$\mathcal{DLR}_{reg}$ interpretations are associated with the atomic built-in relations $\top_n$. When comparing the expressivity of $\mathcal{DLR}_{reg}$ with $\mathcal{DL}_{\mathrm{FU}_1}$ below, we consider interpretations $\mathcal{I}$ where the relations $\top_n$ are appropriate atomic built-in roles and thus directly available also in $\mathcal{DL}_{\mathrm{FU}_1}$.

Proposition 2. *$\mathcal{DLR}_{reg}$ and $\mathcal{DL}_{\mathrm{FU}_1}$ are incomparable in expressvity.*

Proof. It is easy to see that $\mathcal{DLR}_{reg}$ is closed under disjoint copies such that if $C^{\mathcal{I}} = U$ for some $\mathcal{DLR}_{reg}$-concept C, then $C^{\mathcal{I}_1+\mathcal{I}_2} = U_1 \cup U_2$, where $\mathcal{I}_1+\mathcal{I}_2$ consists of two disjoint copies of $\mathcal{I}$ and obviously U_1 and U_2 are the related copies of U. Because of the free use of role negation in $\mathcal{DL}_{\mathrm{FU}_1}$, the same does not hold in that logic. For example the $\mathcal{DL}_{\mathrm{FU}_1}$-concept $\neg\exists(\neg R).A$, where R is a binary role, is satisfied in an interpretation consisting of a single element u that satisfies A and connects to itself via R. This interpretation together with a disjoint copy of itself does not satisfy $\neg\exists(\neg R).A$. Thus $\mathcal{DL}_{\mathrm{FU}_1}$ is not contained in $\mathcal{DLR}_{reg}$.

For the converse, it suffices to observe that $\mathcal{DL}_{\mathrm{FU}_1}$ cannot define the concept $\exists(R^*).A$. It is well known that this property is not first-order expressible, and thus it is not definable in $\mathcal{DL}_{\mathrm{FU}_1}$. $\square$

We finish up the current section by identifying a maximal fragment of $\mathcal{DLR}_{reg}$ that embeds into $\mathcal{DL}_{\mathrm{FU}_1}$. What exactly we mean by maximality in this context will become clear below.

Let $\mathcal{DLR}^0_{reg}$ denote the fragment of $\mathcal{DLR}_{reg}$ without Kleene star and counting, i.e., $\mathcal{DLR}^0_{reg}$ is obtained by the grammar that drops the terms $\mathcal{E}^*$ and $(\leq k\,[\$i]\mathcal{R})$ from the grammar of $\mathcal{DLR}_{reg}$. For each *positive* integer k, we let $\mathcal{DLR}^0_{reg}[\leq k]$ denote the system we obtain if we add the terms $(\leq k\,[\$i]\mathcal{R})$ (with each arity for $\mathcal{R}$ and each related i included) to the grammar of $\mathcal{DLR}^0_{reg}$. (Note that $(\leq 0\,[\$i]\mathcal{R})$ is equivalent to $\neg\exists[\$i]\mathcal{R}$.) Similarly, we let $\mathcal{DLR}^0_{reg}[*]$ be the logic we obtain by adding the term $\mathcal{E}^*$ to the grammar of $\mathcal{DLR}^0_{reg}$.

We will show that while $\mathcal{DLR}^0_{reg}$ embeds into $\mathcal{DL}_{\mathrm{FU}_1}$ (Theorem 8), neither $\mathcal{DLR}^0_{reg}[*]$ nor any of the logics $\mathcal{DLR}^0_{reg}[\leq k]$ does (Theorem 9). We already observed above that the operator $\cdot^*$ of $\mathcal{DLR}_{reg}$ is inexpressible in $\mathcal{DL}_{\mathrm{FU}_1}$. The fact that the number restriction operators $(\leq k\,[\$i]\mathcal{R})$ are definable neither in $\mathcal{DL}_{\mathrm{FU}_1}$ nor in U_1, as we shall prove, is somewhat more surprising since U_1 can do *some* counting. However, as we already discussed earlier, the counting ability of U_1 is limited.

Finally, it is not entirely trivial that we can indeed keep the composition operator in $\mathcal{DLR}^0_{reg}$ and still embed this logic into $\mathcal{DL}_{\mathrm{FU}_1}$. This is because the use of the composition operator often requires the three-variable fragment of first-order logic, and $\mathcal{DL}_{\mathrm{FU}_1}$ collapses to FO^2 on binary vocabularies.

Theorem 8. *$\mathcal{DLR}^0_{reg}$ embeds into $\mathcal{DL}_{\mathrm{FU}_1}$.*

Proof. We begin by showing that we can eliminate the composition operator $\circ$ from $\mathcal{DLR}^0_{reg}$ altogether. Consider a concept D of $\mathcal{DLR}^0_{reg}$. We first observe that we can use the the standard identity $R\circ(S\cup T) = (R\circ S)\cup(R\circ T)$ of relation algebra to obtain from D an expression where the composition operators are on the "atomic" level, with the relational terms ε and $\mathcal{R}_{|\$i,\$j}$ of the grammar of $\mathcal{DLR}_{reg}$ regarded as atoms. We then use the equivalence $\exists\big(\mathcal{E}_1 \cup ... \cup \mathcal{E}_m\big).C \equiv (\exists\mathcal{E}_1.C) \sqcup ... \sqcup (\exists\mathcal{E}_m.C)$ to obtain a disjunction of formulae $\exists\mathcal{E}_i.C$ where $\mathcal{E}_i$ is a composition of "atomic" terms $\mathcal{S}$. To eliminate the composition operators from the terms $\mathcal{E}_i = \mathcal{S}_1 \circ ... \circ \mathcal{S}_n$, we use the equivalence $\exists\big(\,\mathcal{S}_1 \circ ... \circ \mathcal{S}_n\,\big).C \equiv \exists\mathcal{S}_1.\exists\mathcal{S}_2.\exists\mathcal{S}_3 \;\; ... \;\; \exists\mathcal{S}_n.C$. Thus we can eliminate instances of $\circ$ from $\mathcal{DLR}^0_{reg}$.

Next we note that all the remaining union operators are also eliminable, using the equivalence $\exists\big(\mathcal{E}_1 \cup ... \cup \mathcal{E}_m\big).C \equiv (\exists\mathcal{E}_1.C) \sqcup ... \sqcup (\exists\mathcal{E}_m.C)$

We then show how to translate the obtained formula (which is free of union and composition operators) into $\mathcal{DL}_{\mathrm{FU}_1}$. For presentational reasons, we will translate the formula into the first-order fragment FU_1. The syntax of $\mathcal{DLR}^0_{reg}$ without composition and union is given by the grammar

$$\begin{aligned}
\mathcal{R} &::= \top_n \mid R \mid (\$i/n : C) \mid \neg\mathcal{R} \mid (\mathcal{R}_1 \cap \mathcal{R}_2) \\
\mathcal{E} &::= \varepsilon \mid \mathcal{R}_{|\$i,\$j} \\
C &::= \top_1 \mid A \mid \neg C \mid (C_1 \sqcap C_2) \mid \exists\mathcal{E}.C \mid \exists[\$i]\mathcal{R}
\end{aligned}$$

where $\mathcal{R}_{|\$i,\$j}$ with $i = j$ is not allowed; these are easy to eliminate. Our translation will be defined with three translation operators s, t, T that correspond to, respectively, the terms for $\mathcal{R}, \mathcal{E}, C$ above. Each of these operators is parameterized by an appropriate tuple of variables. We first define T as follows.

1. $T[x](\top_1) := \top$ and $T[x](A) := A(x)$.
2. $T[x](\neg C) := \neg T[x](C)$ and $T[x](C_1 \sqcap C_2) := T[x](C_1) \wedge T[x](C_2)$.
3. $T[x](\exists\mathcal{E}.C) := \exists y\big(t[x,y](\,\mathcal{E}\,) \wedge T[y]C\,\big)$, where t is the translation for terms $\mathcal{E}$ to be defined below.
4. $T[x](\exists[\$i]\mathcal{R}) := \exists x_1 ... \exists x_{i-1} \exists x_{i+1} ... \exists x_n\big(\,s[x_1,...,x_{i-1},x,x_{i+1},...,x_n](\mathcal{R})\,\big)$, where s is a translation for $\mathcal{R}$ and n is the arity of $\mathcal{R}$.

We then define the operator t.

1. $t[x,y](\varepsilon) := x = y$.
2. $t[x,y](\mathcal{R}_{|\$i,\$j}) := \exists\overline{z}\big(s[\overline{u}](\mathcal{R})\big)$, where $\exists\overline{z}$ quantifies existentially each of the variables $x_1,...,x_n$ except for x_i and x_j, and where $\overline{u}$ is obtained from the tuple $(x_1,...,x_n)$ by replacing x_i by x and x_j by y. Here n is the arity of the relation $\mathcal{R}$ and s is the translation for $\mathcal{R}$.

We finally define the operator s as follows.

1. $s[x_1,...,x_n](\top_n) := \top_n(x_1,...,x_n)$ and $s[x_1,...,x_n](R) := R(x_1,...,x_n)$ for atomic roles R and the built-in relation $\top_n$.
2. $s[x_1,...,x_n](\,(\$i/n : C)\,) := T[x_i](C) \wedge \top_n(x_1,...,x_n)$, where T is the translation for C.
3. $s[x_1,...,x_n](\neg\mathcal{R}) := \top_n(x_1,...,x_n) \wedge \neg\, s[x_1,...,x_n](\mathcal{R})$.
4. $s[x_1,...,x_n](\,\mathcal{R}_1 \cap \mathcal{R}_2\,) := s[x_1,...,x_n](\mathcal{R}_1) \wedge s[x_1,...,x_n](\mathcal{R}_2)$.

The translated formula is now easily modified to a formula of FU_1. This involves shifting the quantifiers introduced in clause 2 of the translation $t[x,y]$. $\square$

We then show that none of the operators of $\mathcal{DLR}_{reg}$ missing from $\mathcal{DLR}^0_{reg}$ could be added to $\mathcal{DLR}^0_{reg}$ without losing the embedding into $\mathcal{DL}_{\mathrm{FU}_1}$. By an operator we here mean $\cdot^*$ and each term $(\leq k\,[\$i]\mathcal{R})$ with $k \in \mathbb{Z}_+$. Note that for a fixed k, the term $(\leq k\,[\$i]\mathcal{R})$ strictly speaking denotes a collection of operators, because we could vary i and the arity of $\mathcal{R}$. Thus a more fine-grained analysis than the one below could be given. We ignore this issue for the sake of simplicity.

Theorem 9. *$\mathcal{DLR}^0_{reg}[*]$ and $\mathcal{DLR}^0_{reg}[\leq k]$ for each $k \in \mathbb{Z}_+$ are all incomparable with $\mathcal{DL}_{\mathrm{FU}_1}$*

Proof. We already observed in the proof of Proposition 2 that $\mathcal{DL}_{\mathrm{FU}_1}$ cannot define the concept $\exists(R^*).A$ and that $\mathcal{DLR}_{reg}$ cannot define $\neg\exists(\neg R).A$, where $\neg$ is the full negation of $\mathcal{DL}_{\mathrm{FU}_1}$. Thus it now suffices to show that for each $k \in \mathbb{Z}_+$, the concept $(\leq k\,[\$2]R)$ is not expressible in $\mathcal{DL}_{\mathrm{FU}_1}$. Here R is a binary relation.

In the proof of Theorem 3, we already dealt with the special case where $k = 1$: if $\varphi(x)$ was an FU_1-formula defining the concept $(\leq 1\,[\$2]R)$, then the FU_1-sentence $\forall x\varphi(x)$ would define that the in-degree of R is at most one. Thus we can now fix a $k \geq 2$ and define two interpretations, one consisting of $k+1$ copies of the clique of size k and the other one of k copies of the clique of size $k+1$. (Recall that a clique is a structure where the binary relation R is the total relation with the reflexive loops removed).

We have prepared the setting in such a way that it is now easy to show, using once again the EF-game for U_1 (defined in [13]), that the two structures satisfy exactly the same U_1-sentences. However, the concept $(\leq k-1\,[\$2]R)$ is satisfied by every element in the first structure and by none of the elements of the second one. Thus no U_1-formula is equivalent to $(\leq k-1\,[\$2]R)$, because if $\varphi(x)$ was equivalent to $(\leq k-1\,[\$2]R)$, the U_1-sentence $\exists x\varphi(x)$ would be satisfied by the first structure but not the second one. □

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References

[1] Hajnal Andréka, Johan van Benthem, and Istvan Németi. Modal languages and bounded fragments of predicate logic. *Journal of Philosophical Logic*, 27(3):217–274, 1998.

[2] Vince Bárány, Balder ten Cate, and Luc Segoufin. Guarded negation. In *Proc. of ICALP (2)*, pages 356–367, 2011.

[3] Saguy Benaim, Michael Benedikt, Witold Charatonik, Emanuel Kieroński, Rastislav Lenhardt, Filip Mazowiecki, and James Worrell. Complexity of two-variable logic on finite trees. In *Proc. of ICALP (2)*, pages 74–88, 2013.

[4] M. Bojańczyk, A. Muscholl, T. Schwentick, and L. Segoufin. Two-variable logic on data trees and XML reasoning. *Journal of the ACM*, 56(3), 2009.

[5] Diego Calvanese, Giuseppe De Giacomo, and Maurizio Lenzerini. On the decidability of query containment under constraints. In *Proc. of PODS*, pages 149–158, 1998.

[6] E. Grädel, P. Kolaitis, and M. Vardi. On the decision problem for two-variable first-order logic. *Bulletin of Symbolic Logic*, 3(1):53–69, 1997.

[7] Erich Grädel. On the restraining power of guards. *Journal of Symbolic Logic*, 64(4):1719–1742, 1999.

[8] Erich Grädel, Martin Otto, and Eric Rosen. Two-variable logic with counting is decidable. In *Proc. of LICS*, pages 306–317, 1997.

[9] Lauri Hella and Antti Kuusisto. One-dimensional fragment of first-order logic. In *Proc. of AiML*, pages 274–293, 2014.

[10] Leon Henkin. Logical systems containing only a finite number of symbols. *Presses De l'Université De Montréal*, 1967.

[11] E. Kieroński, J. Michaliszyn, I. Pratt-Hartmann, and L. Tendera. Two-variable first-order logic with equivalence closure. *SIAM Journal on Computing*, 43(3), 2014.

[12] Emanuel Kieroński and Antti Kuusisto. Complexity and expressivity of uniform one-dimensional fragment with equality. In *Proc. of MFCS (1)*, pages 365–376, 2014.

[13] Emanuel Kieroński and Antti Kuusisto. Uniform one-dimensional fragments with one equivalence relation. In *Proc. of CSL*, pages 597–615, 2015.

[14] Emanuel Kieroński and Martin Otto. Small substructures and decidability issues for first-order logic with two variables. *Journal of Symbolic Logic*, 77(3):729–765, 2012.

[15] Emanuel Kieroński and Lidia Tendera. On finite satisfiability of two-variable first-order logic with equivalence relations. In *Proc. of LICS*, pages 123–132, 2009.

[16] Leonid Libkin. *Elements of Finite Model Theory.* Springer, 2004.

[17] Carsten Lutz, Ulrike Sattler, and Stephan Tobies. A suggestion for an n-ary description logic. In *Proc. of DL'99*, 1999.

[18] Carsten Lutz, Ulrike Sattler, and Frank Wolter. Modal logic and the two-variable fragment. In *Proc. of CSL*, pages 247–261, 2001.

[19] M. Mortimer. On languages with two variables. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 21(1):135–140, 1975.

[20] L. Pacholski, W. Szwast, and L. Tendera. Complexity of two-variable logic with counting. In *Proc. of LICS*, pages 318–327, 1997.

[21] Ian Pratt-Hartmann. Complexity of the two-variable fragment with counting quantifiers. *Journal of Logic, Language and Information*, 14(3):369–395, 2005.

[22] Renate A. Schmidt and Dmitry Tishkovsky. Using tableau to decide description logics with full role negation and identity. *ACM Transactions on Computational Logic*, 15(1), 2014.

[23] James G. Schmolze. Terminological knowledge representation systems supporting n-ary terms. In *Proc. of KR'89*, pages 432–443, 1989.

[24] D. Scott. A decision method for validity of sentences in two variables. *Journal of Symbolic Logic*, 27, 1962.

[25] Wieslaw Szwast and Lidia Tendera. FO^2 with one transitive relation is decidable. In *Proc. of STACS*, pages 317–328, 2013.