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A Survey of Qualitative Spatial and Temporal Calculi — Algebraic and Computational Properties

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Qualitative Spatial and Temporal Reasoning (QSTR) is concerned with symbolic knowledge representation, typically over infinite domains. The motivations for employing QSTR techniques range from exploiting computational properties that allow efficient reasoning to capture human cognitive concepts in a computational framework. The notion of a qualitative calculus is one of the most prominent QSTR formalisms. This article presents the first overview of all qualitative calculi developed to date and their computational properties, together with generalized definitions of the fundamental concepts and methods, which now encompass all existing calculi. Moreover, we provide a classification of calculi according to their algebraic properties.

CCS Concepts: **•General and reference → Surveys and overviews; •Computing methodologies → Symbolic calculus algorithms; Temporal reasoning; Spatial and physical reasoning;**

Additional Key Words and Phrases: Qualitative Reasoning, Knowledge Representation, Relation Algebra

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1. INTRODUCTION

Knowledge about our world is densely interwoven with spatial and temporal facts. Nearly every knowledge-based system comprises means for representation of, and possibly reasoning about, spatial or temporal knowledge. Among the different options available to a system designer, ranging from domain-level data structures to highly abstract logics, qualitative approaches stand out for their ability to mediate between the domain level and the conceptual level. Qualitative representations explicate relational knowledge between (spatial or temporal) domain entities, allowing individual

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statements to be evaluated by truth values. The aim of qualitative representations is to focus on the aspects that are essential for a task at hand by abstracting away from other, unimportant aspects. As a result, a wide range of representations has been applied, using various kinds of knowledge representation languages. The most fundamental principles for representing knowledge qualitatively that are at the heart of virtually every representation language are captured by a construct called *qualitative (spatial or temporal) calculus*. In the past decades, a great variety of qualitative calculi have been developed, each tailored to specific aspects of spatial or temporal knowledge. They share common principles but differ in formal and computational properties.

This article presents an up-to-date comprehensive overview of *qualitative spatial and temporal reasoning (QSTR)*. We provide a general definition of QSTR (Section 2), give a uniform account of a calculus that is more integrative than existing ones (Section 3), identify and differentiate algebraic properties of calculi (Section 4), and discuss their role within other knowledge representation paradigms (Section 5) as well as alternative approaches (Section 6). Besides the survey character, the article provides a taxonomy of the most prominent reasoning problems, a survey of *all* existing calculi proposed so far (to the best of our knowledge), and the first comprehensive overview of their computational properties.

This article is accompanied by an electronic appendix that contains additional examples, observations, proofs and detailed experimental results, marked "$\lhd$" in the text.

Demarcation of Scope and Contribution

This article addresses researchers and engineers working with knowledge about space or time and wishing to employ reasoning on a symbolic level. We supply a thorough overview of the wealth of qualitative calculi available, many of which have emerged from concrete application scenarios, for example, proposed for geographical information systems (GIS) [Egenhofer 1991; Frank 1991] and now readily employed in current systems; for applications in general see also the overview given in [Ligozat 2011]. Our survey focuses on the calculi themselves (Tables I–II) and their computational and algebraic properties, i.e., characteristics relevant for reasoning and symbolic manipulation (Table IV, Figure 6). To this end, we also categorize reasoning tasks involving qualitative representations (Figure 2).

We exclusively consider qualitative formalisms for reasoning on the basis of finite sets of relations over an infinite spatial or temporal domain. As such, the mere use of symbolic labels is not surveyed. We also disregard approaches augmenting qualitative formalisms with an additional interpretation such as fuzzy sets or probability theory.

This article significantly advances from previous publications with a survey character in several regards. Ligozat [2011] describes in the course of the book "the main" qualitative calculi, describes their relations, complexity issues and selected techniques. Although an algebraic perspective is taken as well, we integrate this in a more general context. Additionally to mentioning general axioms in context of relation algebras we present a thorough investigation of calculi regarding these axioms. He also gives references to applications that employ QSTR techniques in a broad sense. Our survey supplements precise definitions of the underlying formal aspects, which will then be general enough to encompass *all* existing calculi that we are aware of. Chen et al. [2013] summarize the progress in QSTR by presenting selected key calculi for important spatial aspects. They give a brief introduction to basic properties of calculi, but neither detail formal properties nor picture the entire variety of formalisms achieved so far as provided by this article. Algebra-based methods for reasoning with qualitative constraint calculi have been covered by Renz and Nebel [2007]. Their description applies to calculi that satisfy rather strong properties, which we relax. We present revised definitions and an algebraic closure algorithm that generalizes to all existing calculi, and, to the best of our knowledge, we give the first comprehensive overview on computational properties. Cohn and Renz [2008] present an introduction to the field which extends the earlier article of Cohn and Hazarika [2001] by a more detailed discussion of logic theories for mereotopology and by presenting efficient reasoning algorithms.

2. WHAT IS QUALITATIVE SPATIAL AND TEMPORAL REASONING

We characterize QSTR by considering the reasoning problems it is concerned with. Generally speaking, reasoning is a process to generate new knowledge from existing one. Knowledge primarily refers to facts given explicitly, possibly implicating implicit ones. *Sound reasoning* is involved with explicating the implicit, allowing it to be processed further. Thus, sound reasoning is crucial for many applications. In QSTR it is a key characteristic and the applied reasoning methods are largely shaped by the specifics of qualitative knowledge about spatial or temporal domains as provided within the *qualitative domain representation*.

2.1. A General Definition of QSTR

Qualitative domain representations employ symbols to represent semantically meaningful properties of a *perceived domain*, abstracting away any details not regarded relevant to the context at hand. The perceived domain comprises the available raw information about objects. By *qualitative abstraction*, the perceived domain is mapped to the qualitative domain representation, called domain representation from now on. Various aims motivate research on qualitative abstractions, most importantly the desire to develop formal models of common sense relying on coarse concepts [Williams and de Kleer 1991; Bredeweg and Struss 2004] and to capture the catalog of concepts and inference patterns in human cognition [Kuipers 1978; Knauff et al. 2004], which in combination enables intuitive approaches to designing intelligent systems [Davis 1990] or human-centered computing [Frank 1992]. Within QSTR it is required that qualitative abstraction yields a *finite* set of elementary concepts. The following definition aims to encompass all contexts in which QSTR is studied in the literature.

Definition 2.1. Qualitative spatial and temporal representation and reasoning (QSTR) is the study of techniques for representing and manipulating spatial and temporal knowledge by means of relational languages that use a finite set of symbols. These symbols stand for classes of semantically meaningful properties of the represented domain (positions, directions, etc.).

Spatial and temporal domains are typically infinite and exhibit complex structures. Due to their richness and diversity, QSTR is confronted with unique theoretic and computational challenges. Consequently, there is a high variety of domain representations, each focusing on specific aspects relevant to specific tasks. To achieve qualitative abstraction, QSTR uses a relational language to formulate domain representations. It turns out that binary relations can capture most relevant facets of space and time – this class also received most attention by the research community. Expressive power is purely based on these pre-defined relations, no conjuncts or quantifiers are considered. Thus, the associated reasoning methods can be regarded as variants of constraint-based reasoning. Additionally, constraint-based reasoning techniques can be used to empower other methods, for example to assess the similarity of represented entities or logic inference.

Finally, to map a domain representation to the perceived domain a *realization* process is applied. This process instantiates entities in the perceived domain that are based on entities provided in the domain representation.

Figure 1 depicts the overall view on knowledge representation and aligns with the well-known view on intelligent agents considered in AI, which connects the environ-

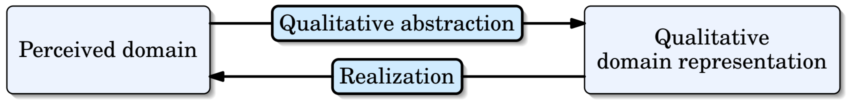


Fig. 1: Relation between perceived domain and domain representation

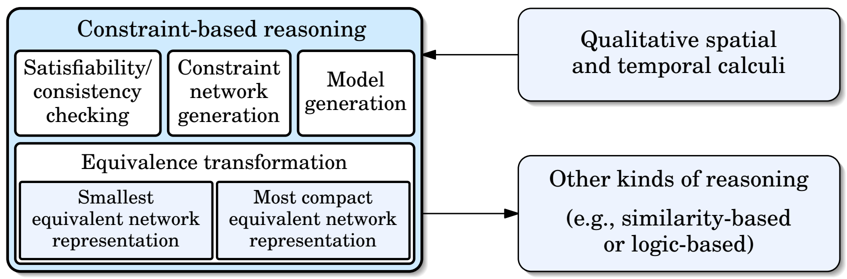


Fig. 2: Classification of fundamental reasoning tasks and representation formalisms

ment to the agent and its internal representation by means of perception (which is an abstraction process as well) and, vice versa, by actions (see, e.g., [Russell and Norvig 2009, Chapter 2]).

2.2. Taxonomy of Constraint-Based Reasoning Tasks

Figure 2 depicts an overview of constraint-based reasoning tasks in the context of QSTR. We now briefly describe these tasks and highlight some associated literature. The description is deliberately provided at an abstract level: each task may come in different flavors, depending on specific (application) contexts. Also, applicability of specific algorithms largely depends on the qualitative representation at hand. The following taxonomy is loosely based on the overview by Wallgrün et al. [2013].

In the following, we refer to the set of objects received from the perceived domain by applying qualitative abstraction as domain entities. These are for example geometric entities such as points, lines, or polygons. In general domain entities can be of any type regarding spatial or temporal aspects.

We further use the notion of a *qualitative constraint network (QCN)*, a special form of domain representation. Commonly, a QCN Q is depicted as a directed labeled graph, whose nodes represent abstract domain entities, i.e., with no specific values from the domain assigned, and whose edges are labeled with *constraints*, i.e., symbols representing relationships required to hold between these entities – see Figure 3 b. An assignment of concrete domain entities to the nodes in Q is called a *solution* of Q if the assigned entities satisfy all constraints in Q. Section 3.2 has precise definitions.

Constraint network generation. This task determines relational statements that describe given domain entities regarding specific aspects, using a predetermined qualitative language fulfilling certain properties, i.e., in our case provided by a qualitative spatial calculus. For example, Figure 3 b could be the QCN derived from the scene shown in Figure 3 a. Techniques for solving this task are described, e.g., in [Cohn et al. 1997; Worboys and Duckham 2004; Forbus et al. 2004; Dylla and Wallgrün 2007].

Consistency checking. This decision problem is considered the fundamental QSTR task [Renz and Nebel 2007]: given an input QCN Q, decide whether a solution exists. Applicable algorithms depend on the kind of constraints that occur in Q and are addressed in Sections 3.2 and 3.4.

Model generation. This task determines a solution for a QCN Q, i.e., a concrete assignment of a domain entity for each node in Q. This may be computationally harder than merely deciding the existence of a solution. For instance, Fig. 3 a could be the result of the model generation for the QCN shown in Fig. 3 c. Typically, a single QCN has infinitely many solutions, due to the abstract nature of qualitative representations. Implementations of model generation may thus choose to introduce further parameters for controlling the kind of solution determined. Techniques for solving this task are described, e.g., in [Schultz and Bhatt 2012; Kreutzmann and Wolter 2014; Schockaert and Li 2015].

Equivalence transformation. Taking a QCN Q as input, equivalence transformation methods determine a QCN Q' that has exactly the same solutions but meets additional criteria. Two variants are commonly considered.

Smallest equivalent network representation determines the strongest refinement of the input Q by modifying its constraints in order to remove redundant information. Figure 3 b depicts a refinement of Figure 3 c since in 3 c the relation between A and C is not constrained at all (i.e., being "$<, =, >$"), whereas 3 b involves the tighter constraint "$<$". Thus, the QCN Q in 3 c contains 5 base relations, whereas the QCN Q' in 3 b contains only 3. Methods for this task are addressed, e.g., by van Beek [1991], and Amaneddine and Condotta [2013].

Most compact equivalent network representation determines a QCN Q' with a minimal number of constraints: it removes *whole* constraints that are redundant. In that sense, Figure 3 c shows a more compact network than Figure 3 b. This task is addressed, e.g., by Wallgrün [2012], and Duckham et al. [2014].

With this taxonomy in mind, the next section studies properties of qualitative representations and their reasoning operations.

3. QUALITATIVE SPATIAL AND TEMPORAL CALCULI FOR DOMAIN REPRESENTATIONS

The notion of a qualitative (spatial or temporal) calculus is a formal construct which, in one form or another, underlies virtually every language for qualitative domain representations. In this section, we survey this fundamental construct, formulate minimal requirements to a qualitative calculus, discuss their relevance to spatial and temporal representation and reasoning, and list calculi described in the literature. As mentioned in Section 2.2, domain entities can be of any type representing spatial or temporal aspects. The notion of a qualitative calculus has been devised to deal with any entities; thus we omit an exhaustive list. Instead we refer to Table I listing entities covered by known calculi.

Existing calculi are entirely based on binary or ternary relations between entities, which comprise, for example, points, lines, intervals, or regions. *Binary* relations are used to represent the location or moving direction of two entities relative to one another *without* referring to a third entity as a reference object. Examples of relations are

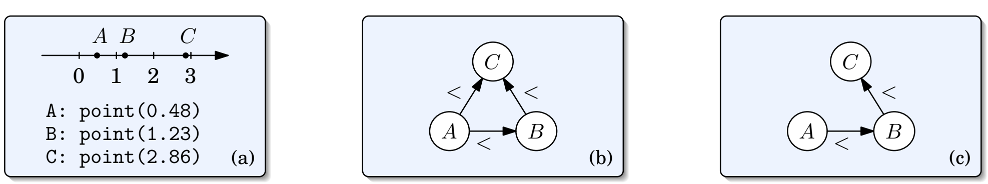


Fig. 3: One geometric (a) and two qualitative domain representations of a spatial scene, obtained via complete (b) or incomplete (c) abstraction. Furthermore, (b) can be obtained from (c) via constraint-based reasoning.

"overlaps with" (for intervals or regions) or "move towards each other" (for dynamic objects). Additionally, a binary calculus is equipped with a converse operation acting on single relation symbols and a binary composition operation acting on pairs of relation symbols, representing the natural converse and composition operations on the domain relations, respectively. Converse and composition play a crucial role for symbolic reasoning: from the knowledge that the pair (x, y) of entities is in relation r, a symbolic reasoner can conclude that (y, x) is in the converse of r; and if it is additionally known that the pair (y, z) is in s, then the reasoner can conclude that (x, z) is in the relation resulting in composing r and s. In addition, most calculi provide an identity relation which allows to represent the (explicit or derived) knowledge that, for example, x and y represent the same entity. $\triangleleft$ ***Ex. A.1,A.2,A.3***

Depending on the properties postulated for converse and composition, notions of a calculus of varying strengths exist [Nebel and Scivos 2002; Ligozat and Renz 2004]. The algebraic properties of binary calculi are well-understood, see Section 4.

The main motivation for using *ternary* relations is the requirement of directly capturing *relative frames of reference* which occur in natural language semantics [Levinson 2003]. In these frames of reference, the location of a target object is described from the perspective of an observer with respect to a reference object. For example, a hiker may describe a mountain peak to be to the left of a lake with respect to her own point of view. Another important motivation is the ability to express that an object is located between two others. Thus, ternary calculi typically contain projective relations for describing relative orientation and/or betweenness. The commitment to ternary (or n-ary) relations complicates matters significantly: instead of a single converse operation, there are now five (or $n! - 1$) nontrivial permutation operations, and there is no longer a unique choice for a natural composition operation. For capturing the algebraic structure of n-ary relations, Condotta et al. [2006] proposed an algebra but there are other arguably natural choices, and they lead to different algebraic properties, as shown in Section 4. These difficulties may be the main reason why algebraic properties of ternary calculi are not as deeply studied as for binary calculi. Fortunately, this will not prevent us from establishing our general notion of a qualitative spatial (or temporal) calculus with relation symbols of arbitrary arity. However, we will then restrict our algebraic study to binary calculi; a unifying algebraic framework for n-ary calculi has yet to be established.

3.1. Requirements to Qualitative Spatial and Temporal Calculi

We start with minimal requirements used in the literature. We use the following standard notation. A *universe* is a non-empty set $\mathcal{U}$. With X^n we denote the set of all n-tuples with elements from X. An *n-ary domain relation* is a subset $r \subseteq \mathcal{U}^n$. We use the prefix notation $r(x_1, \ldots, x_n)$ to express $(x_1, \ldots, x_n) \in r$; in the binary case we will often use the infix notation $x\,r\,y$ instead of $r(x, y)$.

Abstract partition schemes. Ligozat and Renz [2004] note that most spatial and temporal calculi are based on a set of JEPD (jointly exhaustive and pairwise disjoint) domain relations. The following definition is predominant in the QSTR literature [Ligozat and Renz 2004; Cohn and Renz 2008].

Definition 3.1. Let $\mathcal{U}$ be a universe and $\mathcal{R}$ a set of non-empty domain relations of the same arity n. $\mathcal{R}$ is called a set of ***JEPD relations*** over $\mathcal{U}$ if the relations in $\mathcal{R}$ are jointly exhaustive, i.e., $\mathcal{U}^n = \bigcup_{r \in \mathcal{R}} r$, and pairwise disjoint.

An *n-ary abstract partition scheme* is a pair $(\mathcal{U}, \mathcal{R})$ where $\mathcal{R}$ is a set of JEPD relations over the universe $\mathcal{U}$. The relations in $\mathcal{R}$ are called *base relations*.

◁ *Ex. A.4*

In Definition 3.1, the universe $\mathcal{U}$ represents the set of all (spatial or temporal) entities. The main ingredients of a calculus will be relation symbols representing the base relations in the underlying partition scheme. A constraint linking an n-tuple t of entities via a relation symbol will thus represent complete information (modulo the qualitative abstraction underlying the partition scheme) about t. Incomplete information is modeled by t being in a *composite relation*, which is a set of relation symbols representing the union of the corresponding base relations. The set of all relation symbols represents the *universal relation* (the union of all base relations) and indicates that no information is available. ◁ *Ex. A.5*

The requirement that all base relations are JEPD ensures that every n-tuple of entities belongs to exactly one base relation. Thanks to PD (pairwise disjointness), there is a unique way to represent any composite relation using relation symbols and, due to JE (joint exhaustiveness), the empty relation can never occur in a consistent set of constraints, which is relevant for reasoning, see Section 3.2. ◁ *Ex. A.6*

Partition schemes, identity, and converse. Ligozat and Renz [2004] base their definition of a (binary) qualitative calculus on the notion of a *partition scheme*, which imposes additional requirements on an abstract partition scheme. In particular, it requires that the set of base relations contains the identity relation and is closed under the converse operation. The analogous definition by Condotta et al. [2006] captures relations of arbitrary arity. Before we define the notion of a partition scheme, we discuss the generalization of identity and converse to the n-ary case.

The binary identity relation is given as usual by

$$\mathsf{id}^2 = \{(u, u) \mid u \in \mathcal{U}\}. \tag{1}$$

◁ *Ex. A.7*

The most inclusive way to generalize (1) to the n-ary case is to fix a set M of numbers of all positions where tuples in id^n are required to agree. Thus, an *n-ary identity relation* is a domain relation id^n_M with $M \subseteq \{1, \ldots, n\}$ and $|M| \geqslant 2$, which is defined by

$$\mathsf{id}^n_M = \{(u_1, \ldots, u_n) \in \mathcal{U}^n \mid u_i = u_j \text{ for all } i, j \in M\}.$$

This definition subsumes the "diagonal elements" Δ_{ij} of Condotta et al. [2006] for the case $|M| = 2$. However, it is not enough to restrict attention to $|M| = 2$ because there are ternary calculi which contain all identities $\mathsf{id}^3_{1,2}$, $\mathsf{id}^3_{1,3}$, $\mathsf{id}^3_{2,3}$, and $\mathsf{id}^3_{1,2,3}$, an example being the LR calculus, which was described as "the finest of its class" [Scivos and Nebel 2005]. Since the relations in an n-ary abstract partition scheme are JEPD, all identities id^n_M are either base relations or subsumed by those. The stronger notion of a *partition scheme* should thus require that all identities be made explicit.

For binary relations, id^2 from (1) is the *unique* identity relation $\mathsf{id}^2_{\{1,2\}}$.

The standard definition for the converse operation $\breve{}$ on binary relations is

$$r^{\smile} = \{(v, u) \mid (u, v) \in r\}. \tag{2}$$

◁ *Ex. A.8*

In order to generalize the reversal of the pairs (u, v) in (2) to n-ary tuples, we consider arbitrary permutations of n-tuples. An *n-ary permutation* is a bijection $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$. We use the notation $\pi : (1, \ldots, n) \mapsto (i_1, \ldots, i_n)$ as an abbreviation for "$\pi(1) = i_1, \ldots, \pi(n) = i_n$". The identity permutation $\iota : (1, \ldots, n) \mapsto (1, \ldots, n)$ is called *trivial*; all other permutations are *nontrivial*.

A finite set P of n-ary permutations is called *generating* if each n-ary permutation is a composition of permutations from P. For example, the following two permutations form a (minimal) generating set:

$$\mathsf{sc} : (1,\dots,n) \mapsto (2,\dots,n,1) \qquad \text{(shortcut)}$$
$$\mathsf{hm} : (1,\dots,n) \mapsto (1,\dots,n-2,n,n-1) \qquad \text{(homing)}$$

The names have been introduced in Freksa and Zimmermann [1992] for ternary permutations, together with a name for a third distinguished permutation:

$$\mathsf{inv} : (1,\dots,n) \mapsto (2,1,3\dots,n) \qquad \text{(inversion)}$$

Condotta et al. [2006] call shortcut "rotation" ($r^{\frown}$) and homing "permutation" ($r^{\leftrightarrow}$). ◁ *Ex. A.9*

For $n = 2$, sc, hm and inv coincide; indeed, there is a unique minimal generating set, which consists of the single permutation $\breve{}$: $(1,2) \mapsto (2,1)$. For $n \geqslant 3$, there are several generating sets, e.g., $\{\mathsf{sc},\mathsf{hm}\}$ and $\{\mathsf{sc},\mathsf{inv}\}$.

Now an *n-ary permutation operation* is a map $\cdot^{\pi}$ that assigns to each n-ary domain relation r an n-ary domain relation denoted by r^{π}, where π is an n-ary permutation and the following holds: $r^{\pi} = \{(u_{\pi(1)},\dots,u_{\pi(n)}) \mid (u_1,\dots,u_n) \in r\}$

We are now ready to give our definition of a partition scheme, lifting Ligozat and Renz's binary version to the n-ary case, and generalizing Condotta et al.'s n-ary version to arbitrary generating sets.

Definition 3.2. An *n-ary partition scheme* $(\mathcal{U},\mathcal{R})$ is an n-ary abstract partition scheme with the following two additional properties.

(1) $\mathcal{R}$ contains all identity relations id^n_M, $M \subseteq \{1,\dots,n\}$, $|M| \geqslant 2$.
(2) There is a generating set P of permutations such that, for every $r \in \mathcal{R}$ and every $\pi \in P$, there is some $s \in \mathcal{R}$ with $r^{\pi} = s$.

◁ *Ob. B.1* ◁ *Ex. A.10, A.11*

It is important to note that violations of Definition 3.2 (e.g., depicted in Example A.11) are not necessarily bugs in the design of the respective calculi – in fact they are often a feature of the corresponding representation language, which is deliberately designed to be just as granular as necessary, and may thus omit some identity relations or converses/compositions of base relations. ◁ *Ex. A.12*

Thus violations of Definition 3.2 are unavoidable, and we adopt the more general notion of an abstract partition scheme.

Calculi. Intuitively, a qualitative (spatial or temporal) calculus is a symbolic representation of an abstract partition scheme and additionally represents the composition operation on the relations involved. As before, we need to discuss the generalization of binary composition to the n-ary case before we can define it precisely.

For binary domain relations, the standard definition of composition is:

$$r \circ s = \{(u,w) \mid \exists \underline{v} \in \mathcal{U} : (u,\underline{v}) \in r \text{ and } (\underline{v},w) \in s\} \tag{3}$$

◁ *Ex. A.13*

We are aware of three ways to generalize (3) to higher arities. The first is a binary operation on the ternary relations of the calculus double-cross (2-cross) [Freksa 1992b; Freksa and Zimmermann 1992] (see also Fig. 8 in the appendix):

$$r \circ^3_{\mathrm{FZ}} s = \{(u,v,w) \mid \exists \underline{x} : (u,v,\underline{x}) \in r \text{ and } (v,\underline{x},w) \in s\}$$

◁ *Ex. A.14*

A second alternative results in $n(n-1)$ binary operations ${}_i\circ_j^n$ [Isli and Cohn 2000; Scivos and Nebel 2005]: the composition of r and s consists of those n-tuples that belong to r (respectively, s) if the i-th (respectively, j-th) component is replaced by some uniform element v.

$$r \;{}_i\circ_j^n\; s = \{(u_1,\ldots,u_n) \mid \exists \underline{v} : (u_1,\ldots,u_{i-1},\underline{v},u_{i+1},\ldots,u_n) \in r \text{ and } (u_1,\ldots,u_{j-1},\underline{v},u_{j+1},\ldots,u_n) \in s \quad \}$$

In the ternary case, this yields, for example:

$$r \;{}_3\circ_2^3\; s = \{(u,v,w) \mid \exists \underline{x} : (u,v,\underline{x}) \in r \text{ and } (u,\underline{x},w) \in s\} \tag{4}$$

If we assume, for example, that the underlying partition scheme speaks about the relative position of points, we can consider (4) to say: if the position of x relative to u and v is determined by the relation r (as given by $(u,v,x) \in r$) and the position of w relative to u and x is determined by the relation s (as given by $(u,x,w) \in s$), then the position of w relative to u and v can be inferred to be determined by $r \;{}_3\circ_2^3\; s$.

The third is perhaps the most general, resulting in an n-ary operation [Condotta et al. 2006]: $\circ(r_1,\ldots,r_n)$ consists of those n-tuples which, for every $i = 1,\ldots,n$, belong to the relation r_i whenever their i-th component is replaced by some uniform v.

$$\circ(r_1,\ldots,r_n) = \{(u_1,\ldots,u_n) \mid \exists \underline{v} \in \mathcal{U} : (u_1,\ldots,u_{n-1},\underline{v}) \in r_1 \text{ and } (u_1,\ldots,u_{n-2},\underline{v},u_n) \in r_2 \text{ and } \ldots \text{ and } (\underline{v},u_2,\ldots,u_n) \in r_n\} \tag{5}$$

◁ ***Ex. A.15***

For binary domain relations, all these alternative approaches collapse to (3).

In the light of the diverse views on composition, we define a *composition operation on n-ary domain relations* to be an operation of arity $2 \leqslant m \leqslant n$ on n-ary domain relations, without imposing additional requirements. Those are not necessary for the following definitions, which are independent of the particular choice of composition.

We now define our minimal notion of a calculus, which provides a set of symbols for the relations in an *abstract* partition scheme (Rel), and for *some choice of* nontrivial permutation operations ($^{\smile 1},\ldots,{}^{\smile k}$) and *some* composition operation ($\diamond$).

Definition 3.3. An *n-ary qualitative calculus* is a tuple $(\mathsf{Rel},\mathsf{Int},{}^{\smile 1},\ldots,{}^{\smile k},\diamond)$ with $k \geqslant 1$ and the following properties.

— Rel is a finite, non-empty set of n-ary *relation symbols* (denoted $r,s,t,\ldots$). The subsets of Rel, including singletons, are called *composite relations* (denoted $R,S,T,\ldots$).
— $\mathsf{Int} = (\mathcal{U},\varphi,\cdot^{\pi_1},\ldots,\cdot^{\pi_k},\circ)$ is an *interpretation* with the following properties.
  — $\mathcal{U}$ is a universe.
  — $\varphi : \mathsf{Rel} \to 2^{\mathcal{U}^n}$ is an injective map assigning an n-ary relation over $\mathcal{U}$ to each relation symbol, such that $(\mathcal{U},\{\varphi(r) \mid r \in \mathsf{Rel}\})$ is an abstract partition scheme. The map φ is extended to composite relations $R \subseteq \mathsf{Rel}$ by setting $\varphi(R) = \bigcup_{r\in R}\varphi(r)$.
  — $\{\cdot^{\pi_1},\ldots,\cdot^{\pi_k}\}$ is a set of n-ary nontrivial permutation operations.
  — $\circ$ is a composition operation on n-ary domain relations that has arity $2 \leqslant m \leqslant n$.
— Every *permutation operation* $^{\smile i}$ is a map $^{\smile i} : \mathsf{Rel} \to 2^{\mathsf{Rel}}$ that satisfies

$$\varphi(r^{\smile i}) \supseteq \varphi(r)^{\pi_i} \tag{6}$$

  for every $r \in \mathsf{Rel}$. The operation $^{\smile i}$ is extended to composite relations $R \subseteq \mathsf{Rel}$ by setting $R^{\smile i} = \bigcup_{r\in R} r^{\smile i}$.
— The *composition operation* $\diamond$ is a map $\diamond : \mathsf{Rel}^m \to 2^{\mathsf{Rel}}$ that satisfies

$$\varphi(\diamond(r_1,\ldots,r_m)) \supseteq \circ(\varphi(r_1),\ldots,\varphi(r_m)) \tag{7}$$

for all $r_1, \ldots, r_m \in \mathsf{Rel}$. The operation $\diamond$ is extended to composite relations $R_1, \ldots, R_m \subseteq \mathsf{Rel}$ by setting $\diamond(R_1, \ldots, R_m) = \bigcup_{r_1 \in R_1} \cdots \bigcup_{r_m \in R_m} \diamond(r_1, \ldots, r_m)$.

In the special case of binary relations, the natural converse is the only non-trivial permutation operation, i.e., $k = 1$. $\lhd$ *Ob. B.2*
Due to the last sentence of Definition 3.3, the composition operation of a calculus is uniquely determined by the composition of each pair of relation *symbols*. This information is usually stored in an m-dimensional table, the *composition table*.
$\lhd$ *Ex. A.16, A.17, A.18*

Abstract versus weak and strong operations. We call permutation and composition operations with Properties (6) and (7) *abstract permutation* and *abstract composition*, following Ligozat's naming in the binary case [Ligozat 2005]. For reasons explained further below, our notion of a qualitative calculus imposes weaker requirements on the permutation operation than Ligozat and Renz's notions of a weak (binary) representation [Ligozat 2005; Ligozat and Renz 2004] or the notion of a (binary) constraint algebra [Nebel and Scivos 2002]. The following definition specifies those stronger variants, see, e.g., Ligozat and Renz [2004].

Definition 3.4. Let $(\mathsf{Rel}, \mathsf{Int}, {}^{\smile 1}, \ldots, {}^{\smile k}, \diamond)$ be a qualitative calculus based on the interpretation $\mathsf{Int} = (\mathcal{U}, \varphi, \cdot^{\pi_1}, \ldots, \cdot^{\pi_k}, \circ)$.
The permutation operation ${}^{\smile i}$ is a *weak permutation* if, for all $r \in \mathsf{Rel}$:

$$r^{\smile i} = \bigcap \{S \subseteq \mathsf{Rel} \mid \varphi(S) \supseteq \varphi(r)^{\pi_i}\} \tag{8}$$

The permutation operation ${}^{\smile i}$ is a *strong permutation* if, for all $r \in \mathsf{Rel}$:

$$\varphi(r^{\smile i}) = \varphi(r)^{\pi_i} \tag{9}$$

The composition operation $\diamond$ is a *weak composition* if, for all $r_1, \ldots, r_m \in \mathsf{Rel}$:

$$\diamond(r_1, \ldots, r_m) = \bigcap \{S \subseteq \mathsf{Rel} \mid \varphi(S) \supseteq \circ(\varphi(r_1), \ldots, \varphi(r_m))\} \tag{10}$$

The composition $\diamond$ is a *strong composition* if, for all $r_1, \ldots, r_m \in \mathsf{Rel}$:

$$\varphi(\diamond(r_1, \ldots, r_m)) = \circ(\varphi(r_1), \ldots, \varphi(r_m)) \tag{11}$$

In the literature, the equivalent variant $r^{\smile i} = \{s \in \mathsf{Rel} \mid \varphi(s) \cap \varphi(r)^{\pi_i} \neq \emptyset\}$ of Equation (8) is sometimes found; analogously for Equation (10). $\lhd$ *Ex. A.19, A.20*
In terms of composition tables, abstract composition requires that each cell corresponding to $\diamond(r_1, \ldots, r_m)$ contains *at least* those relation symbols t whose interpretation intersects with $\circ(\varphi(r_1), \ldots, \varphi(r_m))$. Weak composition additionally requires that each cell contains *exactly* those t. Strong composition, in contrast, imposes a requirement on the underlying *partition scheme:* whenever $\varphi(t)$ intersects with $\circ(\varphi(r_1), \ldots, \varphi(r_m))$, it has to be *contained* in $\circ(\varphi(r_1), \ldots, \varphi(r_m))$. Analogously for permutation.

The above "at least" is a crucial requirement: if some cell did not contain any relation symbol t as above, then the composition table would give rise to unsound inferences, (e.g., described in Example A.20). Abstractness as in Properties (6) and (7) thus captures *minimal requirements* to the operations in a qualitative calculus that ensure soundness of reasoning, as described in Section 3.2.

Along the same lines, adding unnecessary relations to a cell in the table leads to weaker inferences and thus amounts to a loss of knowledge. Weakness (Properties (8) and (10)) ensures that this loss is kept to the unavoidable minimum. This last observation is presumably the reason why existing calculi (see Section 3.4) typically have at least weak operations – we are not aware of any calculus with only abstract operations.

In Section 3.2, we will see that abstract composition is a minimal requirement for ensuring soundness of the most common reasoning algorithm, a-closure, and review the impact of the various strengths of the operations on reasoning algorithms.

The three notions form a hierarchy:

FACT 3.5. *Every strong permutation (composition) is weak, and every weak permutation (composition) is abstract.* $\lhd$ C.1

It suffices to postulate the properties weakness and strongness with respect to relation symbols only: they carry over to composite relations as shown in Fact 3.6.

FACT 3.6. *Given a qualitative calculus* $(\mathsf{Rel}, \mathsf{Int}, {}^{\smile_1}, \dots, {}^{\smile_k}, \diamond)$ *the following holds. For all composite relations* $R \subseteq \mathsf{Rel}$ *and* $i = 1, \dots, k$*:*

$$\varphi(R^{\smile_i}) \supseteq \varphi(R)^{\pi_i} \tag{12}$$

For all composite relations $R_1, \dots, R_m \subseteq \mathsf{Rel}$*:*

$$\varphi(\diamond(R_1, \dots, R_m)) \supseteq \circ(\varphi(R_1), \dots, \varphi(R_m)) \tag{13}$$

If ${}^{\smile_i}$ *is a weak permutation, then, for all* $R \subseteq \mathsf{Rel}$*:*

$$R^{\smile_i} = \bigcap\{S \subseteq \mathsf{Rel} \mid \varphi(S) \supseteq \varphi(R)^{\pi_i}\}$$

If ${}^{\smile_i}$ *is a strong permutation, then, for all* $R \subseteq \mathsf{Rel}$*:*

$$\varphi(R^{\smile_i}) = \varphi(R)^{\pi_i}$$

If $\diamond$ *is a weak composition, then, for all* $R_1, \dots, R_m \subseteq \mathsf{Rel}$*:*

$$\diamond\,(R_1, \dots, R_m) = \bigcap\{S \subseteq \mathsf{Rel} \mid \varphi(S) \supseteq \circ(\varphi(R_1), \dots, \varphi(R_m)\}$$

If $\diamond$ *is a strong composition, then, for all* $R_1, \dots, R_m \subseteq \mathsf{Rel}$*:*

$$\varphi(\diamond(R_1, \dots, R_m)) = \circ(\varphi(R_1), \dots, \varphi(R_m))$$ $\lhd$ C.2

Suppose that we want to achieve that the symbolic permutation operations provided by a calculus C capture all permutations at the domain level. Then C needs to be permutation-complete in the sense that at least weak permutation operations for all $n! - 1$ nontrivial permutations can be derived uniquely by composing the ones defined.

In the binary case, where the converse is the unique nontrivial (and generating) permutation, every calculus is permutation-complete. However, as noted above, the converse is not strong for the binary CDR [Skiadopoulos and Koubarakis 2005] and RCD [Navarrete et al. 2013] calculi (cf. Definition 3.2 ff.). There are also ternary calculi whose permutations are not strong: e.g., the shortcut, homing, and inversion operations in the single-cross and double-cross calculi [Freksa 1992b; Freksa and Zimmermann 1992] are only weak. Since these calculi provide no further permutation operations, they are not permutation-complete. However, it is easy to compute the two missing permutations and thus make both calculi permutation-complete.

Ligozat and Renz' [2004] basic notion of a binary qualitative calculus is based on a *weak representation* which requires an identity relation, abstract composition, and the converse being strong, thus excluding, for example, CDR and RCD. A *representation* is a weak representation with a strong composition and an injective map φ. Our basic notion of a qualitative calculus is more general than a weak representation by not requiring an identity relation, and by only requiring abstract permutations and composition, thus including CDR and RCD. On the other hand, it is slightly more restrictive by requiring the map φ to be injective. However, since base relations are JEPD, the only way for φ to violate injectivity is to give multiple names to the same relation, which is not really intuitive. It is even problematic because it leads to unintended behavior of the notion of weak composition (or permutation): if there are two relation symbols for every domain relation, then the intersections in Equations (8) and (10) will range over disjoint composite relations S and thus become empty.

Recently, Westphal et al. [2014] gave a new definition of a qualitative calculus that does not explicitly use a map – in our case the interpretation Int – that connects the symbols with their semantics. Instead, they employ the "notion of consistency" [Westphal et al. 2014, p. 211] for generating a weak algebra from the Boolean algebra of relation symbols. As with [Ligozat and Renz 2004] their definition of a qualitative calculus is confined to binary relations only.

3.2. Spatial and Temporal Reasoning

As in the area of classical constraint satisfaction problems (CSPs), we are given a set of variables and constraints: a constraint network or a *qualitative CSP*.[1] The task of constraint satisfaction is to decide whether there exists a valuation of all variables that satisfies the constraints. In calculi for spatial and temporal reasoning, all variables range over the entities of the specific domain of a qualitative calculus. The relation symbols defined by the calculus serve to express constraints between the entities. More formally, we have:

Definition 3.7 (*QCSP*). Let $C = (\mathsf{Rel}, \mathsf{Int}, {}^{\smile 1}, \ldots, {}^{\smile k}, \diamond)$ be an n-ary qualitative calculus with $\mathsf{Int} = (\mathcal{U}, \varphi, \cdot^{\pi_1}, \ldots, \cdot^{\pi_k}, \circ)$, and let X be a set of variables ranging over $\mathcal{U}$. An *n-ary qualitative constraint in C* is a formula $R(x_1, \ldots, x_n)$ with variables $x_1, \ldots, x_n \in X$ and a relation $R \subseteq \mathsf{Rel}$. We say that a valuation $\psi : X \to \mathcal{U}$ *satisfies* $R(x_1, \ldots, x_n)$ if $(\psi(x_1), \ldots, \psi(x_n)) \in \varphi(R)$ holds.

A *qualitative constraint satisfaction problem* (QCSP) is the task to decide whether there is a valuation ψ for a set of variables satisfying a set of constraints.

$\lhd$ *Ex. A.21*

For simplicity and without loss of generality, we assume that every set of constraints contains exactly one constraint per set of n variables. Thus, of binary constraints either r_{x_1,x_2} or r'_{x_2,x_1} is assumed to be given – the other can be derived using converse; multiple constraints regarding variables x_1, x_2 can be integrated via intersection. In the following, $r_{x_1,\ldots,x_n}$ stands for the unique constraint between the variables $x_1, \ldots, x_n$.

Several techniques originally developed for finite-domain CSPs can be adapted to spatial and temporal QCSPs. Since deciding CSP instances is already NP-complete for search problems with finite domains, heuristics are important. One particularly valuable technique is constraint propagation which aims at making implicit constraints explicit in order to identify variable assignments that would violate some constraint. By pruning away these variable assignments, a consistent valuation can be searched more efficiently. A common approach is to enforce k-consistency; the following definition is standard in the CSP literature [Dechter 2003].

Definition 3.8. A QCSP with variables X is *k-consistent* if, for all subsets $X' \subsetneq X$ of size $k-1$, we can extend any valuation of X' that satisfies the constraints to a valuation of $X' \cup \{z\}$ also satisfying the constraints, for any additional variable $z \in X \setminus X'$.

QCSPs are naturally 1-consistent as universes are nonempty and there are no unary constraints. An n-ary QCSP is n-consistent if $r^{\smile i}_{x_1,\ldots,x_k} = r_{\pi_i(x_1,\ldots,x_k)}$ for all i and

[1] In the CSP domain, "CSP" usually refers to a single instance, not the decision or computation problem. This the same as a qualitative constraint network (QCN) as introduced in Sec 2.2.

$r_{x_1,\dots,x_k} \neq \emptyset$: domain relations are typically *serial*, that is, for any r and $x_1, \dots, x_{k-1}$, there is some x_k with $r(x_1, \dots, x_k)$. In the case of binary relations, this means that 2-consistency is guaranteed in calculi with a strong converse by $r^{\smile}_{x,y} = r_{y,x}$ and $r_{x,y} \neq \emptyset$, and seriality of r means that, for every x, there is a y with $r(x, y)$.

Already examining $(n + 1)$-consistency may provide very useful information. The following is best explained for binary relations and then generalized to higher arities. A 3-consistent binary QCSP is called *path-consistent*, and Definition 3.8 can be rewritten using binary composition as

$$\forall x, y \in X \qquad r_{x,y} \subseteq \bigcap_{z \in X} r_{x,z} \circ r_{z,y}. \tag{14}$$

We can enforce 3-consistency by computing the fixpoint of the refinement operation

$$r_{x,y} \leftarrow r_{x,y} \cap (r_{x,z} \circ r_{z,y}), \tag{15}$$

applied to all variables $x, y, z \in X$. In finite CSPs with variables ranging over finite domains, composition is also finite and the procedure always terminates since the refinement operation is monotone and there can thus only be finitely many steps until reaching the fixpoint. Such procedures are called path-consistency algorithms and require $\mathcal{O}(|X|^3)$ time [Dechter 2003]. $\lhd$ *Ex. A.22*

Enforcing path-consistency with QCSPs may not be possible using a symbolic algorithm since Equation (15) may lead to relations not expressible in 2^{Rel}. This problem occurs when composition in a qualitative calculus is not strong. It is however straightforward to weaken Equation (15) using weak composition:

$$r_{x,y} \leftarrow r_{x,y} \cap (r_{x,z} \diamond r_{z,y}) \tag{16}$$

The resulting procedure is called enforcing *algebraic closure* or *a-closure* for short. The QCSP obtained as a fixpoint of the iteration is called *algebraically closed*. $\lhd$ *Ex. A.23*

If composition in a qualitative calculus is strong, a-closure and path-consistency coincide. Since there are finitely many relations in a qualitative calculus, a-closure shares all computational properties with the finite CSP case.

A natural generalization from binary to n-ary relations can be achieved by considering $(n + 1)$-consistency (recall that path-consistency is 3-consistency). In context of symbolic computation with qualitative calculi we thus need to lift Equations (14) and (15) to the particular composition operation available. For composition as defined by (5) one obtains

$$\forall x_1, \dots, x_n \in X \qquad r_{x_1,\dots,x_n} \subseteq \bigcap_{y \in X} \circ(r_{x_1,\dots,x_{n-1},y},\ r_{x_1,\dots,x_{n-2},y,x_n},\ \dots,\ r_{y,x_2,\dots,x_n}),$$

and the symbolic refinement operation (16) becomes

$$r_{x_1,\dots,x_n} \leftarrow r_{x_1,\dots,x_n} \cap \diamond(r_{x_1,\dots,x_{n-1},y},\ r_{x_1,\dots,x_{n-2},y,x_n},\ \dots,\ r_{y,x_2,\dots,x_n}). \tag{17}$$

The reason why, in Definition 3.3, we require composition to be at least abstract is that Inclusion (7) guarantees that reasoning via a-closure is sound: enforcing k-consistency or a-closure does not change the solutions of a CSP, as only impossible valuations are locally removed. If application of a-closure results in the empty relation, then the QCSP is known to be inconsistent. By contrast, an algebraically closed QCSP may not be consistent. However, for several qualitative calculi (or at least subalgebras thereof) a-closure and consistency coincide, see also Section 3.4. $\lhd$ *Ex. A.24*

Since domain relations are JEPD, deciding QCSPs with arbitrary composite relations can be reduced to deciding QCSPs with only *atomic relations* (i.e., relation symbols) by means of search (cf. [Renz and Nebel 2007]). The approach to reason in a full algebra is thus to *refine* a composite relation $R \cup S$ to either R or S in a backtracking search fashion, until a dedicated decision procedure becomes applicable. Computationally, reasoning with the complete algebra is typically NP-hard due to the exponential number of possible refinements to atomic relations. For investigating reasoning algorithms, one is thus interested in the complexity of reasoning with atomic relations. If they can be handled in polynomial time, maximal tractable sub-algebras that extend the set of atomic relations are of interest too. Efficient reasoning algorithms for atomic relations and the existence of large tractable sub-algebras suggest efficiency in handling practical problems. The search for maximal tractable sub-algebras can be significantly eased by applying the automated methods proposed by Renz [2007]. These exploit algebraic operations to derive tractable composite relations and, complementary, search for embeddings of NP-hard problems. Using a-closure plus refinement search has been regarded as the prevailing reasoning method. Certainly, a-closure provides an efficient cubic time method for constraint propagation, but Table IV clearly shows that the majority of calculi require further methods as decision procedures.

3.3. Tools to Facilitate Qualitative Reasoning

There are several tools that facilitate one or more of the reasoning tasks. The most prominent plain-QSTR tools are *GQR* [Westphal et al. 2009], a constraint-based reasoning system for checking consistency using a-closure and refinement search, and the *SparQ* reasoning toolbox [Wolter and Wallgrün 2012],[2] which addresses various tasks from constraint- and similarity-based reasoning. Besides general tools, there are implementations addressing specific aspects (e.g., reasoning with CDR [Liu et al. 2010]) or tailored to specific problems (e.g., *Phalanx* for sparse RCC-8 QCSPs [Sioutis and Condotta 2014]). In the contact area of qualitative and logical reasoning, the DL reasoners *Racer* [Haarslev et al. 2012] and *PelletSpatial* [Stocker and Sirin 2009] offer support for handling a selection of qualitative formalisms. For logical reasoning about qualitative domain representations, the tools *Hets* [Mossakowski et al. 2007], *SPASS* [Weidenbach et al. 2002], and *Isabelle* [Nipkow et al. 2002] have been applied, supporting the first-order Common Algebraic Specification Language CASL [Astesiano et al. 2002] as well as its higher-order variant HasCASL (see [Wölfl et al. 2007]).

3.4. Existing Qualitative Spatial and Temporal Calculi

In the following, we present an overview of existing calculi obtained from a systematic literature survey, covering publications in the relevant conferences and journals in the past 25 years, and following their citation graphs. To be included in our overview, a qualitative calculus has to be based on a spatial and/or temporal domain, fall under our general definition of a qualitative calculus (Def. 3.3: provide symbolic relations, the required symbolic operations, and semantics based on an abstract partition scheme), and be described in the literature either with explicit composition/converse tables, or with instructions for computing them. These selection critera exclude sets of qualitative relations that have been axiomatized in the context of logical theories, see Section 5.2, or qualitative calculi designed for other domains, such as ontology alignment [Inants and Euzenat 2015]).

Tables I–II list, to the best of our knowledge, all calculi satisfying these criteria. Table I lists the names of families of calculi and their domains. Table II lists all variants of these families with original references, arity and number of their base relations (which is an indicator for the level of granularity offered and for the average branching factor to expect in standard reasoning procedures). Additionally we indicate which calculi are implemented in *SparQ* and can be obtained from there.

[2] available at https://github.com/dwolter/sparq

Abbrev.	Name	Domain	Aspect
1-,2-cross	Single/Double Cross Calculus	points in 2-d	relative location
9-int	Nine-Intersection Model	simple n-d regions	topology
$9^{(+)}$-int	9- and 9^{+}-Intersection Calculi	9-int & bodies, lines, points in 2-d/3-d	
ABA^{8}_{23}	Alg. of Bipartite Arrangements	1-d intervals in 2-d	rel. loc./orientation
BA	Block algebra (aka Rectangle Algebra or Rectangle Calculus)	n-d blocks	order
CBM	Calculus Based Method	2-d regions, lines, and points	topology
CDA	Closed Disk Algebra	2-d closed disks	topology
CDC	Cardinal Direction Calculus	points in 2-d	cardinal directions
CDR	Cardinal Direction Relations	2-d regions	cardinal directions
CI	Algebra of Cyclic Intervals	intvls. on closed curves	cyclic order
CYC	Cyclic Ordering (CYC_b aka Geometric Orientation)	oriented lines in 2-d	relative orientation
DepCalc	Dependency Calculus	partially ordered points	partial order
DIA	Directed Intervals Algebra	directed 1-d intvls. in 1-d	order/orientation
DRA	Dipole Calculus	oriented line segms. in $\mathbb{R}^2$	rel. loc./orientation
DRA-conn	Dipole connectivity	connectivity of the above	connectivity
EIA	Extended Interval Algebra	1-d intervals in 1-d	order
EOPRA	Elevated Oriented Point Rel. Alg.	OPRA & local distance	
EPRA	Elevated Point Relation Algebra	CDC & local distance	
GenInt	Generalized Intervals	unions of 1-d intvls.	order
IA	(Allen's) Interval Algebra	1-d intervals in 1-d	order
INDU	Intvl. and Duration Network	IA & relative duration	
LOS	Lines of Sight	2-d regions in 3-d	obscuration
LR	LR Calculus (aka Flip-Flop)	points in 2-d	relative location
MC-4	MC-4	regions in 2-d	congruence
OCC	Occlusion Calculus	2-d regions in 3-d	obscuration
OM-3D	3-D Orientation Model	points in 3-d	relative location
OPRA	Oriented Point Rel. Algebra	oriented points in 2-d	rel. loc./orientation
PC	Point Calculus (aka Point Algebra)	points in n-d	total order
QRPC	Qualitative Rectilinear Projection Calculus	oriented points in 2-d	relative motion
QTC	Qualitative Trajectory Calculus	moving points in 1-d/2-d	relative motion
RCC	Region Connection Calculus	general regions	topology
RCD	Rectang. Card. Dir. Calculus	bounding boxes in 2-d	cardinal directions
RfDL-3-12	Region-in-the-frame-of-Directed-Line	regions & paths in 2-d	relative motion
ROC	Region Occlusion Calculus	2-d regions in 3-d	obscuration
SIC	Semi-Interval Calculus	1-d intervals in 1-d	order
STAR	Star Calculi	points in 2-d	direction
SV	StarVars	oriented points in 2-d	relative direction
TPCC	Ternary Point Config. Calc.	points in 2-d	relative location
TPR	Ternary Projective Relations	points or regions in 2-d	relative location
VR	Visibility Relations	convex regions	obscuration

Table I: Existing families of spatial and temporal calculi

Variant	Specifics	Reference(s)	Params		St
1-, 2-cross		[Freksa and Zimmermann 1992]	t	$8, 15$	● Ⓢ
9-int		[Egenhofer 1991]	b	8	● Ⓢ
$9^{(+)}$-int	10 variants[a]	[Kurata 2010]	b	$\leqslant 233$	○[c]
ABA^{8}_{23}	[b]	[Gottfried 2004]	b	125	○[c]
BA_n	n dimensions	[Balbiani et al. 1998; 1999]	b	13^n	● Ⓢ[1,2]
CBM		[Clementini et al. 1993]	b	7	○
CDA		[Egenhofer and Sharma 1993]	b	8	◐
CDC		[Frank 1991; Ligozat 1998]	b	9	● Ⓢ
CDR	original version	[Skiadopoulos and Koubarakis 2004]	b	511	◑[c]
cCDR	connected variant	[Skiadopoulos and Koubarakis 2005]	b	289	● Ⓢ
CI		[Balbiani and Osmani 2000]	b	16	●
CYC_b	binary	[Isli and Cohn 2000]	b	4	◐ Ⓢ
CYC_t	ternary	ibid.	t	24	● Ⓢ
DepCalc		[Ragni and Scivos 2005]	b	5	● Ⓢ
DIA		[Renz 2001]	b	26	○[c]
DRA_c	coarse-grained[b]	[Moratz et al. 2000]	b	24	◐ Ⓢ
DRA_f	fine-grained	ibid.	b	72	● Ⓢ
DRA_{fp}	f+parallelism	[Moratz et al. 2011]	b	80	● Ⓢ
DRA-conn		[Wallgrün et al. 2010]	b	7	● Ⓢ
EIA		[Zhang and Renz 2014]	b	27	◑[c]
EOPRA_n	granularity n	[Moratz and Wallgrün 2012]	b	$\mathcal{O}(n^3)$	○[c]
EPRA_n	granularity n	[Moratz and Wallgrün 2012]	b	$\mathcal{O}(n^3)$	○[c]
IA×EIA	coarser variant	[Zhang and Renz 2014]	b	351	○[c]
EIA×EIA	finer variant	ibid.	b	729	○[c]
GenInt		[Condotta 2000]	b	13	◑[c]
IA		[Allen 1983]	b	13	● Ⓢ
INDU		[Pujari et al. 1999]	b	25	● Ⓢ
LOS-14	convex regions	[Galton 1994]	b	14	○[c]
LR		[Scivos and Nebel 2005; Ligozat 1993]	t	9	● Ⓢ
MC-4		[Cristani 1999]	b	4	● Ⓢ
OCC	convex regions	[Köhler 2002]	b	8	◐
OM-3D		[Pacheco et al. 2001]	t	75	◑[c]
OPRA_n	granularity n	[Moratz 2006; Mossakowski & M. 2012]	b	$\mathcal{O}(n^2)$	● Ⓢ
OPRA^*_n	plus alignment	[Dylla and Lee 2010]	b	$\mathcal{O}(n^2)$	●
PC_n	n dimensions	[Vilain and Kautz 1986] [Balbiani and Condotta 2002]	b	3^n	● Ⓢ[1]
QRPC		[Glez-Cabrera et al. 2013]	b	48	○
QTC-B1x, $x = 1, 2$	1-d variants	[Van de Weghe et al. 2005]	b	$9, 27$	◐ Ⓢ
QTC-B2x, -C2x	2-d variants	ibid.	b	9–305	◐ Ⓢ
QTC-N	network variant	[Delafontaine et al. 2011]	b	17	○[c]
RCC-5	without tangentiality	[Randell et al. 1992]	b	5	● Ⓢ
RCC-8	with tangentiality	ibid.	b	8	● Ⓢ
RCC-15, -23	concave regions	[Cohn et al. 1997]	b	$15, 23$	○
RCC-62	”	[OuYang et al. 2007]	b	62	○
RCC*-7, -9	+ lower-dim. features	[Clementini and Cohn 2014]	b	$7, 9$	◐
(V)RCC-3D(+)	with occlusion	[Sabharwal and Leopold 2014]	b	13–37	○[c]
RCD		[Navarrete et al. 2013]	b	36	● Ⓢ
RfDL-3-12		[Kurata and Shi 2008]	b	1772	○
ROC-20		[Randell et al. 2001]	b	20	○
SIC		[Freksa 1992a]	b	13	○[c]
STAR_n	granularity n	[Renz and Mitra 2004]	b	$\mathcal{O}(n)$	◑[c]
STAR^r_n	revised variants	ibid.	b	$\mathcal{O}(n)$	● Ⓢ[4]
SV_n	granularity n	[Lee et al. 2013]	b	$\mathcal{O}(n)$	◑
TPCC		[Moratz and Ragni 2008]	t	25	● Ⓢ
TPR-p	for points	[Clementini et al. 2006; 2010]	t	7	◐
TPR-r	for regions	ibid.	t	34	○[c]
VR		[Tarquini et al. 2007]	t	7	◐

Table II: Overview of existing spatial and temporal calculi, legend in Table III

Params	Arity – (b)inary, (t)ernary – and number of relation symbols
St	Status of availability: ○ base relations, ◐ composition table, ◑ complexity results ● table *and* complexity, Ⓢ *SparQ* implementation, https://github.com/dwolter/sparq
a	2 variants over 5 domains each
b	Not based on abstract partition scheme (violates JEPD over $\mathcal{U} \times \mathcal{U}$)
c	Original work describes how to compute the composition table
1	For $n = 1$
2	For $n = 2$
4	For $n = 4$, regular version only

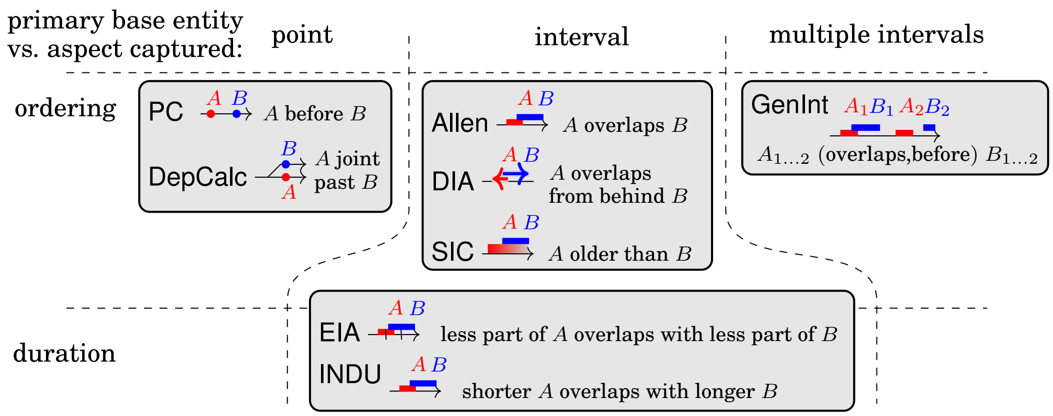
Table III: Legend for Table II

Representational aspects of calculi are shown in Figure 4, grouping calculi by the type of their basic entities and the key aspects captured. For all temporal and selected spatial calculi we iconographically show one exemplary base relation to illustrate the kind of statements it permits. For a complete understanding of the respective calculus, the interested reader is referred to the original research papers cited in Table II. We sometimes use a more descriptive relation name than the original work.

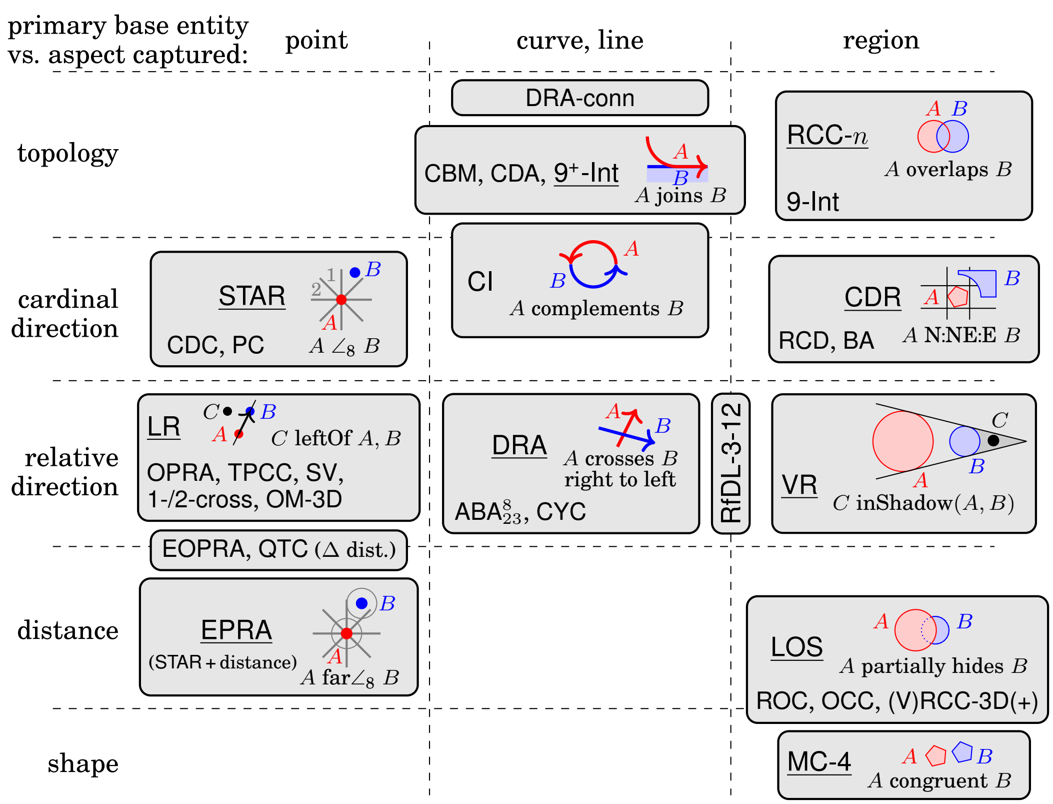
Figure 5 shows the known relations between the expressivity of existing calculi. There are several ways to measure these, via the existence of faithful translations not only between base relations over the same domain, but also between representations of related domains or between representations concerned with a different domain. For example, the dependency calculus DepCalc representing dependency between points is isomorphic to RCC-5 representing topology of regions. Both calculi feature the same algebraic structure representing partial-order relationships in the domain.

Since expressivity of qualitative representations solely relies on how relations are defined, there are distinct calculi which exhibit the same expressivity when Boolean combinations of constraints are considered [Wolter and Lee 2016]. These connections are particularly interesting, not only from the perspective of selecting an appropriate representation, but also in view of computational properties. For example, deciding consistency of atomic constraint networks over the point calculus PC is polynomial. Using Boolean combinations of PC relations one can simulate Allen interval relations. Nebel and Bürckert [1995] have exploited this relationship to lift a tractable subset to Allen. In Figure 5 we give an overview of these expressivity relations. An arrow $A \to B$ indicates that sets of constraints over relations from calculus A can be expressed by Boolean formulas of constraints over relations from calculus B. For clarity we only show direct relations, not their transitive closure. Calculi in a joint box are of equivalent expressivity. For those expressivity relations that do not follow directly from the original papers defining the respective calculi, proof sketches are provided by Wolter and Lee [2016] and in Appendix D.

Computational aspects of calculi are shown in Table IV, as far as they have already been identified. Some fairly straightforward supplements have been made while compiling this table; their proofs are in Appendix E. According to the discussion in the previous section, we give the computational complexity for deciding consistency with atomic QCSPs and the best known complete decision procedure, which is different from a-closure in those cases where a-closure is incomplete. We only indicate the type of algorithm applicable (e.g., linear programming), but not its most efficient realization. We furthermore list tractable subalgebras that cover at least all atomic relations – these subalgebras allow for reasoning in the full algebra via combining the named decision procedure with a search for a refinement. The complexity is given as “P” (in polynomial time), “NPc” (NP-complete), and “NPh” (NP-hard, NP-membership unknown).



Temporal calculi



Spatial calculi

Fig. 4: Classification of qualitative calculi by representable statements with selected example relations

Abbrev.	Complexity[1] (atomic QCSP)		Decision procedure[2] (atomic QCSP)	Largest known tractable subalgebra[3]	and its coverage[4]
1,2-cross	NPh	[WL10]	PS	–	–
9-int	NPc	[SSD03]	recognizing string graphs [SSD03]	–	–
BA$_n$	$\mathcal{O}(n^3)$	[BCC02]	AC	Strongly preconvex relations [BCF99]	
CDC	$\mathcal{O}(n^3)$	[Lig98]	AC	pre-convex relations	$\geq$ 25%
CDR	$\mathcal{O}(n^3)$	[LZLY10]	dedicated [LZLY10]		
cCDR	NPc	[LL11]	dedicated [LZLY10]	–	–
CI	$\mathcal{O}(n^3)$	[BO00]	AC	nice relations	0.75‰
CYC$_t$	$\mathcal{O}(n^4)$	[IC00]	strong 4-consistency	$\mathcal{CT}_t$	0.01‰
DepCalc	$\mathcal{O}(n^3)$	[RS05]	AC	τ_{28} [RS05]	87.5% [RS05]
DIA	$\mathcal{O}(n^3)$	[Ren01]	AC	$\mathcal{H}^{\pm}$ (M) (ORD-Horn)	
DRA$_{c/f/fp}$	NPh	[WL10]	PS	–	–
DRA-conn	$\mathcal{O}(n^3)$	$\lhd$ *E.1*	AC	DRA-conn	100%
EIA	P	$\lhd$ *E.2*	translation to INDU		
GenInt	P	[Con00]	AC	strongly pre-convex general relations	$\ll$ 1‰ for 3-intvls $\lhd$ *E.3*
IA	$\mathcal{O}(n^3)$	[VKvB89]	AC	ORD-Horn [NB95, KJJ03]	10.6%
INDU	P	[BCL06]	translation to Horn-ORD SAT	strongly pre-convex relations	13.6%
LR	NPh	[WL10]	PS	–	–
MC-4	P		dedicated [Cri99]	M-99	75.0%
OM-3D	NPh	$\lhd$ *E.4*	PS	–	–
OPRA$_1^{(*)}$	NPh	[WL10]	PS	–	–
PC$_m$	$\mathcal{O}(n^2)$	[vB92]	dedicated	PC$_m$	100% [VK86]
RCC-5[a]	$\mathcal{O}(n^3)$	[Ren02]	AC [JD97]	R_5^{28} [JD97]	87.5% [JD97]
RCC-8[a]	$\mathcal{O}(n^3)$	[Ren02]	AC [Ren02]	$\widehat{\mathcal{H}}_8$ [Ren99]	62.6% [Ren99]
RCD	$\mathcal{O}(n^3)$	[NMSC13]	translat. to IA; AC	convex relations	$\lll$ 0.01‰
STAR$_m$	P	[LRW13]	LP	convex relations $\lhd$ *E.5*	$m = 4 : <1\%$
STAR$_m^r$ [b]	$\mathcal{O}(n^3)$	[RM04]	AC	convex relations	$m = 3 : 28\%$
STAR$_m^r$ [c]	$\mathcal{O}(n^4)$	[RM04]	4-consistency	convex relations	$m = 4 : 12.5\%$ $m = 8 : <1\%$
SV$_m$	NPc	[LRW13]	LP with search	–	–
TPCC	NPh	[WL10]	PS	–	–

[1] Complexity of deciding consistency (atomic relations plus universal relation)
[2] Best known algorithm
[3] Name of largest known tractable subalgebra that includes all base relations (LKTS)
[4] Percentage of LKTS compared to the complete algebra

[a] For unconstrained regions; connectedness constraints can increase complexity up to PSpace [KPWZ10]
[b] for $m < 3$
[c] for $m \geqslant 3$

Table IV: Overview of the known complexity landscape of deciding consistency for existing spatial and temporal calculi. Legend: see Table V

AC	Algebraic closure
ACS	Algebraic closure plus search
PS	(Multivariate) polynomial systems solving [Basu et al. 2006]
LP	Reducible to linear programming and thus polynomial
NPc; NPh	NP-complete; NP-hard (NP-membership unknown)
P; PSpace	In polynomial time; in polynomial space

[BCC02]	[Balbiani et al. 2002]	[LZLY10]	[Liu et al. 2010]
[BCF99]	[Balbiani et al. 1999]	[NB95]	[Nebel and Bürckert 1995]
[BCL06]	[Balbiani et al. 2006]	[NMSC13]	[Navarrete et al. 2013]
[BO00]	[Balbiani and Osmani 2000]	[Ren99]	[Renz 1999]
[Con00]	[Condotta 2000]	[Ren01]	[Renz 2001]
[Cri99]	[Cristani 1999]	[Ren02]	[Renz 2002]
[GPP95]	[Grigni et al. 1995]	[RM04]	[Renz and Mitra 2004]
[IC00]	[Isli and Cohn 2000]	[RS05]	[Ragni and Scivos 2005]
[JD97]	[Jonsson and Drakengren 1997]	[SSD03]	[Schaefer et al. 2003]
[KJJ03]	[Krokhin et al. 2003]	[vB92]	[van Beek 1992]
[KPWZ10]	[Kontchakov et al. 2010]	[VK86]	[Vilain and Kautz 1986]
[Lig98]	[Ligozat 1998]	[VKvB89]	[Vilain et al. 1990]
[LL11]	[Liu and Li 2011]	[WL10]	[Wolter and Lee 2010]
[LRW13]	[Lee et al. 2013]		

Table V: Legend for Table IV

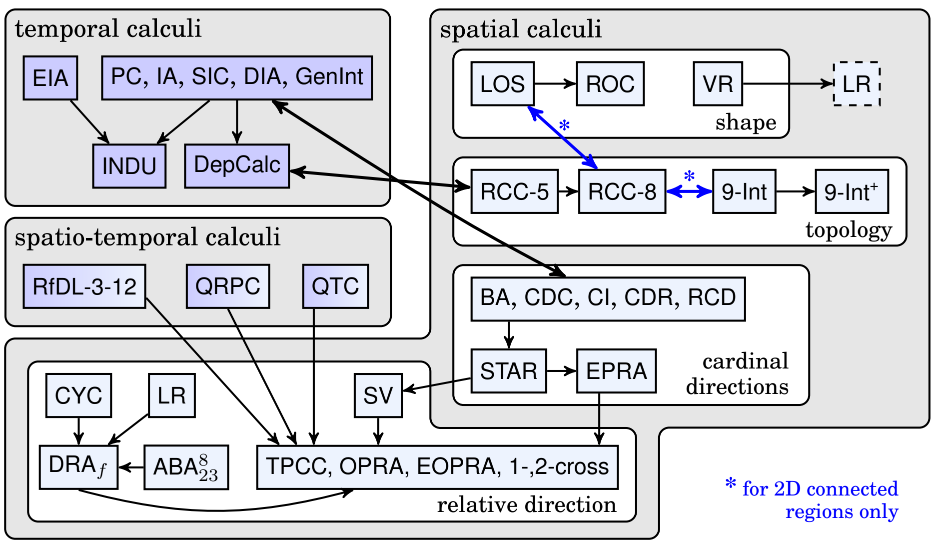


Fig. 5: Expressivity relations between calculi

4. ALGEBRAIC PROPERTIES OF SPATIAL AND TEMPORAL CALCULI

Algebraic properties have been recognized as a formal tool for measuring the information preservation properties of a calculus and for providing the theoretical underpinnings for vital optimizations to reasoning procedures [Isli and Cohn 2000; Ligozat and Renz 2004; Düntsch 2005; Dylla et al. 2013].

To start with information preservation, it is important to distinguish two sources for a loss of information: one is qualitative abstraction, which maps the perceived, continuous domain to a symbolic, discrete representation using n-ary domain relations and operations on them (such as composition and permutation operations). The loss of information associated with this mapping is mostly intended. To understand the other, we recall that a qualitative calculus consists of *symbolic* relations and operations, representing the domain relations and operations. While the domain operations are known to satisfy strong algebraic properties, those do not necessarily carry over to the symbolic operations – for example, if the operation $\cdot^{\mathsf{hm}}$ representing homing (Section 3.1) is only abstract or weak, then there will be symbolic relations r with $(r^{\mathsf{hm}})^{\mathsf{hm}} \neq r$ although, at the domain level, $(R^{\mathsf{hm}})^{\mathsf{hm}} = R$ holds for any n-ary relation R, including the interpretation $\varphi(r)$ of r. This loss of information indicates an unintended structural misalignment between the domain level and the symbolic level. Having its roots in the abstraction step, where the set of domain relations and operations is determined, the information loss becomes noticeable only with the symbolic representation.

If we want to measure how well the symbolic operations in a calculus preserve information, we can compare their algebraic properties with those of their domain-level counterparts. If they share all algebraic properties, this indicates that they maximally preserve information. In addition, algebraic properties seem to supply a finer-grained measure than the mere distinction between abstract, weak, and strong operations: there are 14 axioms for binary relation algebras and variants, each containing two inclusions or implications that may or may not hold independently.

Several algebraic properties can be exploited to justify and implement optimizations in constraint reasoners. For example, associativity of the composition operation $\diamond$ for binary symbolic relations ensures that, if the reasoner encounters a path $ArBsCtD$ of length 3, then the relationship between A and D can be computed “from left to right”. Without associativity, it may be necessary to compute $(r \diamond s) \diamond t$ as well as $r \diamond (s \diamond t)$.

In order to study the algebraic properties of spatial and temporal calculi, the classical notion of a *relation algebra (RA)* [Maddux 2006] plays a central role [Isli and Cohn 2000; Ligozat and Renz 2004; Düntsch 2005; Mossakowski 2007]. The axioms in the definition of an RA reflect the algebraic properties of the relevant operations on *binary* domain relations – the operations are union, intersection, complement, converse, and binary compositions; the properties include commutativity, several variants of associativity and distributivity. These properties have been postulated for binary calculi [Ligozat and Renz 2004; Düntsch 2005], but it has been shown that not all existing calculi satisfy these strong properties [Mossakowski 2007]. It is the main aim of this subsection to study the algebraic properties of existing binary calculi and derive from the results a taxonomy of calculus algebras.

Unfortunately, it is far from straightforward to extend this study to arity 3 or higher: while algebraic properties of ternary and n-ary calculi have been studied [Isli and Cohn 2000; Scivos and Nebel 2005; Condotta et al. 2006], we are aware of only one axiomatization for a ternary RA [Isli and Cohn 2000], based on one particular choice of permutation (homing and shortcut) and composition (the *binary* variant (4)). However, existing calculi are based on different choices of these operations, and each choice comes with different algebraic properties at the domain level, for example:

— Not all permutations are involutive: e.g., in the ternary case, we do *not* have $(R^{\mathsf{sc}})^{\mathsf{sc}} = R$ for all domain relations R, but rather $((R^{\mathsf{sc}})^{\mathsf{sc}})^{\mathsf{sc}} = R$.
— Each variant of the composition operation has its own neutral element, that is, a relation E such that $R \circ E = E \circ R = R$ for all relations R: e.g., in the ternary case, ${}_3\circ^3_2$ (Section 3.1) has $\mathsf{id}^3_{\{2,3\}}$ as the neutral element while $\circ^3_{\mathrm{FZ}}$ has $\mathsf{id}^3_{\{1,2\}}$.
— Some variants of the composition operation have stronger properties than others: e.g., ${}_3\circ^3_2$ is associative while $\circ^3_{\mathrm{FZ}}$ is not.

Establishing a unifying algebraic framework for n-ary qualitative calculi and determining the algebraic properties of existing calculi would require a whole new research program. In the remainder of this section, we will therefore restrict our attention to the binary case.

4.1. The Notion of a Relation Algebra

The notion of an (abstract) RA is defined in [Maddux 2006] and makes use of the axioms listed in Table VI.

Definition 4.1. Let Rel be a set of relation symbols containing id and 1 (the symbols for the identity and universal relation), and let $\cup$, $\diamond$ be binary and $^{-}$, $^{\smile}$ unary operations on Rel. The tuple $(\mathsf{Rel}, \cup, ^{-}, 1, \diamond, ^{\smile}, \mathsf{id})$ is a

— *non-associative relation algebra (NA)* if it satisfies Axioms R_1–R_3, R_5–R_{10};
— *semi-associative relation algebra (SA)* if it is an NA and satisfies Axiom S,
— *weakly associative relation algebra (WA)* if it is an NA and satisfies W,
— *relation algebra (RA)* if it satisfies R_1–R_{10},

for all $r, s, t \in \mathsf{Rel}$.

Clearly, every RA is a WA; every WA is an SA; every SA is an NA.

In the literature, a different axiomatization is sometimes used, for example in [Ligozat and Renz 2004]. The most prominent difference is that R_{10} is replaced by PL, “a more intuitive and useful form, known as the Peircean law or De Morgan’s Theorem K” [Hirsch and Hodkinson 2002]. It is shown in [Hirsch and Hodkinson 2002, Section 3.3.2] that, given R_1–R_3, R_5, R_7–R_9, the axioms R_{10} and PL are equivalent. The implication $\mathsf{PL} \Rightarrow \mathsf{R}_{10}$ does not need R_5 and R_8.

All axioms except PL can be weakened to only one of two inclusions, which we denote by a superscript $^{\supseteq}$ or $^{\subseteq}$. For example, $\mathsf{R}_7^{\supseteq}$ denotes $(r^{\smile})^{\smile} \supseteq r$. Likewise, we use $\mathsf{PL}^{\Rightarrow}$ and $\mathsf{PL}^{\Leftarrow}$. Furthermore, Table VI contains the redundant axiom R_{6l} because it may be satisfied when some of the other axioms are violated. It is straightforward to establish that R_6 and R_{6l} are equivalent given R_7 and R_9. $\lhd$ F.1

Thanks to Def. 3.3, certain axioms are satisfied by every calculus:

FACT 4.2. *Every qualitative calculus (Def. 3.3) satisfies* R_1–R_3, R_5, $\mathsf{R}_7^{\supseteq}$, R_8, $\mathsf{W}^{\supseteq}$, $\mathsf{S}^{\supseteq}$ *for all (atomic and composite) relations. This axiom set is maximal: each of the remaining axioms in Table VI is not satisfied by some qualitative calculus.* $\lhd$ *F.2*

R_1	$r \cup s$	$=$	$s \cup r$	$\cup$-commutativity
R_2	$r \cup (s \cup t)$	$=$	$(r \cup s) \cup t$	$\cup$-associativity
R_3	$\overline{\overline{r} \cup \overline{s}} \cup \overline{\overline{r} \cup s}$	$=$	r	Huntington’s axiom
R_4	$r \diamond (s \diamond t)$	$=$	$(r \diamond s) \diamond t$	$\diamond$-associativity
R_5	$(r \cup s) \diamond t$	$=$	$(r \diamond t) \cup (s \diamond t)$	$\diamond$-distributivity
R_6	$r \diamond \mathsf{id}$	$=$	r	identity law
R_7	$(r^{\smile})^{\smile}$	$=$	r	$^{\smile}$-involution
R_8	$(r \cup s)^{\smile}$	$=$	$r^{\smile} \cup s^{\smile}$	$^{\smile}$-distributivity
R_9	$(r \diamond s)^{\smile}$	$=$	$s^{\smile} \diamond r^{\smile}$	$^{\smile}$-involutive distributivity
R_{10}	$r^{\smile} \diamond \overline{r \diamond s} \cup \overline{s}$	$=$	$\overline{s}$	Tarski/de Morgan axiom
W	$((r \cap \mathsf{id}) \diamond 1) \diamond 1$	$=$	$(r \cap \mathsf{id}) \diamond 1$	weak $\diamond$-associativity
S	$(r \diamond 1) \diamond 1$	$=$	$r \diamond 1$	$\diamond$ semi-associativity
R_{6l}	$\mathsf{id} \diamond r$	$=$	r	left-identity law
PL	$(r \diamond s) \cap t^{\smile} = \emptyset$	$\Leftrightarrow$	$(s \diamond t) \cap r^{\smile} = \emptyset$	Peircean law

Table VI: Axioms for relation algebras and weaker variants [Maddux 2006].

4.2. Discussion of the Axioms

We will now discuss the relevance of the above axioms for spatial and temporal representation and reasoning. Due to Fact 4.2, we only need to consider axioms R_4, R_6, R_7, R_9, R_{10} (or PL) and their weakenings R_{6l}, S, W.

R_4 (and S, W). Axiom R_4 is helpful for modeling since it allows parentheses in chains of compositions to be omitted. For example, consider the following statement in natural language about the relative length and location of two intervals A and D. *Interval A is before some equally long interval that is contained in some longer interval that meets the shorter interval D.* This statement is just a conjunction of relations between A, the unnamed intermediary intervals B, C, and D. Although it intuitively does not matter whether we give priority to the composition of the relations between A, B and B, C or to the composition of the relations between B, C and C, D, there are calculi such as INDU which do not satisfy Axiom R_4 – then the example statement needs to be interpreted as a Boolean formula consisting of a conjunction over both alternatives.

We note that violation of R_4 is independent of composition not being strong, as shown in Section 4.4. Presence of strong composition however implies R_4 since composition of binary domain relations over $\mathcal{U}$ is associative:

FACT 4.3. *Every qualitative calculus where composition is strong satisfies* R_4.

Furthermore, already a weakening $\mathsf{R}_4^{\supseteq}$ or $\mathsf{R}_4^{\subseteq}$ is useful for optimizing reasoning algorithms, allowing the "finer" composition – say, $r \diamond (s \diamond t)$ in case of $\mathsf{R}_4^{\subseteq}$ – to be computed when a chain of compositions needs to be evaluated.

R_6 and R_{6l}. Presence of an id relation allows the standard reduction from the correspondence problem to satisfiability: to test whether a constraint system admits the equality of two variables x, y, one can add an id-constraint between x, y and test the extended system for satisfiability.

R_7 and R_9. These axioms allow for certain optimizations in symbolic reasoning, in particular algebraic closure. If a relation r satisfies R_7, then reasoning systems do not need to store both constraints $A\,r\,B$ and $B\,r'\,A$, since r' can be reconstructed as $r^{\smile}$ if needed. Similarly, R_9 grants that, when enforcing algebraic closure by using Equation (16) to refine constraints between variable A and B, it is sufficient to compute composition once and, after applying the converse, reuse it to refine the constraint between B and A too. Current reasoning algorithms and their implementations use the described optimizations; they produce incorrect results for calculi violating R_7 or R_9.

R_{10} and PL. These axioms reflect that the relation symbols of a calculus indeed represent binary domain relations, i.e., pairs of elements of a universe. This can be explained from two different points of view.

(1) If binary domain relations are considered as sets, R_{10} is equivalent to $r^{\smile} \diamond \overline{r \diamond s} \subseteq \bar{s}$. If we further assume the usual set-theoretic interpretation of the composition of two domain relations, the above inclusion reads as: *For any X, Y, if $Z\,r\,X$ for some Z and, $Z\,r\,U$ implies not $U\,s\,Y$ for any U, then not $X\,s\,Y$.* This is certainly true because X is one such U.
(2) Under the same assumptions, each side of PL says (in a different order) that there can be no triangle $X\,r\,Y, Y\,s\,Z, Z\,t\,X$. The equality then means that the "reading direction" does not matter, see also [Düntsch 2005]. This allows for reducing nondeterminism in the a-closure procedure, as well as for efficient refinement and enumeration of consistent scenarios.

4.3. Prerequisites for Being a Relation Algebra

The following correspondence between properties of a calculus and notions of a relation algebra is due to Ligozat and Renz [2004]: every calculus C based on a partition scheme is an NA. If, in addition, the interpretations of the relation symbols are *serial* base relations, then C is an SA. Furthermore, R_7 is equivalent to the requirement that the converse operation is strong. This is captured by the following lemma.

LEMMA 4.4. *Let* $C = (\mathsf{Rel}, \mathsf{Int}, \breve{\ }, \diamond)$ *be a qualitative calculus. Then the following properties are equivalent.*

(*1*) *C has a strong converse.*
(*2*) *Axiom R_7 is satisfied for all relation symbols $r \in \mathsf{Rel}$.*
(*3*) *Axiom R_7 is satisfied for all composite relations $R \subseteq \mathsf{Rel}$.*

PROOF. Items (2) and (3) are equivalent due to distributivity of $\breve{\ }$ over $\cup$, which is introduced with the cases for composite relations in Definition 3.3.

For "(1) $\Rightarrow$ (2)", the following chain of equalities, for any $r \in \mathsf{Rel}$, is due to C having a strong converse: $\varphi(r^{\breve{}\breve{}}) = \varphi(r^{\breve{}})^{\breve{}} = \varphi(r)^{\breve{}\breve{}} = \varphi(r)$. Since Rel is based on JEPD relations and φ is injective, this implies that $r^{\breve{}\breve{}} = r$.

For "(2) $\Rightarrow$ (1)", we show the contrapositive. Assume that C does not have a strong converse. Then $\varphi(r^{\breve{}}) \supsetneq \varphi(r)^{\breve{}}$, for some $r \in \mathsf{Rel}$; hence $\varphi(r^{\breve{}})^{\breve{}} \supsetneq \varphi(r)^{\breve{}\breve{}}$. We can now modify the above chain of equalities replacing the first two equalities with inequalities, the first of which is due to Requirement (6) in the definition of the converse (Def. 3.3): $\varphi(r^{\breve{}\breve{}}) \supseteq \varphi(r^{\breve{}})^{\breve{}} \supsetneq \varphi(r)^{\breve{}\breve{}} = \varphi(r)$. Since $\varphi(r^{\breve{}\breve{}}) \neq \varphi(r)$, we have that $r^{\breve{}\breve{}} \neq r$. □

4.4. Algebraic Properties of Existing Spatial and Temporal Calculi

We study the algebraic properties of individual calculi, aiming to find those which are abstract relation algebras, and identifying relevant weaker algebraic properties. We have analyzed the calculi listed in Table II, restricting our selection to the 31 calculi[3] with (a) binary relations – because the notion of a relation algebra is best understood for binary relations – and (b) available SparQ implementations (marked ⓢ).

We have written a CASL specification of the axioms listed in Table VI along with weakenings thereof. These have been used with *Hets* to determine congruence of calculus and axioms. Additionally, *SparQ* and its built-in analysis tools have been employed to double-check results. Due to Fact 4.2, it suffices to test Axioms R_4, R_6, R_7, R_9, R_{10} (or PL) and, if necessary, the weakenings S, W, and R_{6l}.

Figure 6 shows the results of our tests; for further details see Appendix G. Figure 6 arranges the analyzed calculi as a hierarchy, with the strongest notion (relation algebra) at the top and the weakest (weakly associative Boolean algebra) at the bottom. Arrows represent the *is-a* relation; i.e., every relation algebra (RA) is an "RA minus id law" as well as a semi-associative RA and a weakly associative Boolean algebra.

With the exceptions of RCD, cCDR and all QTC variants, all tested calculi are at least semi-associative relation algebras; most of them are even relation algebras. Hence, only these calculi enjoy all advantages for representation and reasoning optimizations discussed in Section 4.2. For other groups of calculi, special care in implementations of reasoning procedures need to be taken. In Section 4.5 we present a revised algorithm to compute algebraic closure that respects all eventualities.

The three groups of calculi that are SAs but not RAs are the Dipole Calculus variant DRA_f (refined DRA_{fp} and coarsened DRA-conn are even RAs!), as well as INDU and OPRA_m for at least $m = 1, \ldots, 8$. These calculi do not even satisfy one of the inclusions $\mathsf{R}_4^{\supseteq}$ and $\mathsf{R}_4^{\subseteq}$, which implies that the reasoning optimizations described in Section 4.2 for

[3]For the parametrized calculi DRA, OPRA, QTC, we count every variant separately.

Axiom R_4 cannot be applied. As a side note, our observations suggest that the meaning of the letter combination "RA" in the abbreviations "DRA" and "OPRA" should stand for "Reasoning Algebra", not for "Relation Algebra".

In principle, one cannot completely rule out that the violations of associativity are due to errors in the published operation tables or in the experimental setup. This applies to non-violations too, but systematic non-violations are less likely to be caused by errors than sporadic violations. In the case of DRA_f, INDU and OPRA_m, $m = 1, \ldots, 8$, the relatively high percentage of violations seems to rule out implementation errors. However, to be certain that these calculi indeed violate R_4, one has to find counterexamples and verify them using the original definition of the calculus. For DRA_f and INDU, this was done by Moratz et al. [2011] and Balbiani et al. [2006]. Interestingly, the violation of associativity was attributed to the converse or composition not being strong. We remark, however, that composition cannot be the culprit as, for example, DRA_{fp} has an associative, but only weak, composition operation. While DRA_{fp} is associative due to strong composition [Moratz et al. 2011], *none* of the OPRA_m calculi are associative [Mossakowski and Moratz 2015].

The B-variants of QTC violate only the identity laws R_6, R_{6l}. As observed in [Mossakowski 2007], it is possible to add a new id relation symbol, modify the interpretation of the remaining relation symbols such that they become JEPD, and adapt the converse and composition tables accordingly, thus obtaining relation algebras.

The C-variants of QTC additionally violate R_4, R_9, R_{10}, and PL. Consequently, most of the reasoning optimizations described in Section 4.2 cannot be applied to the C-variants of QTC. The remarkably few violations of R_9, R_{10}, and PL might be due to errors in the composition table, but the non-trivial verification is part of future work.

cCDR and RCD are the only calculi with a weak converse in our tests. cCDR satisfies only W in addition to the axioms that are always satisfied by a Boolean algebra with distributivity. Hence, cCDR enjoys none of the advantages for representation and reasoning discussed before. Similarly to the C-variants of QTC, the relatively small number of violations of PL may be due to errors in the tables published. RCD additionally satisfies R_4. Since both calculi satisfy neither R_7 nor R_9, current reasoning algorithms and their implementations yield incorrect results for them, as seen in Section 4.2.

4.5. Universal Procedure for Algebraic Closure

We noted in Section 4.2 that existing descriptions and implementations of a-closure (e.g., in *GQR* and *SparQ*) use optimizations based on certain relation algebra axioms. Our analysis in Section 4.4 reveals that there are calculi which violate some of these axioms, e.g., R_9; hence those optimizations lead to incorrect results. In Algorithm 1 we present a universal algorithm that computes a-closure correctly for all calculi and uses optimizations only when they are justified. Its input is a graph $(\mathcal{V}, C)$ representing a constraint network, and $C_{i,j}$ denotes the relation between the i-th and j-th node ($r_{x,y}$ in Eq. (15)). Its main control structure is that of the popular path-consistency algorithm PC-2 [Mackworth 1977]. Algorithm 1 enforces 2- and 3-consistency and relies on its input being 1-consistent by implicitly assuming all $C_{i,i}$ to cover identity.

Algorithm 1's main function is `A-CLOSURE`, which employs a queue to store constraint relations that may give rise to an application of the refinement operation according to Eq. (15). The function `REVISE` implements Eq. (15). If R_9 is violated (the converse is not distributive over composition) the refinement from $C_{j,i}{}^{\smile}$ needs to be computed in addition to $C_{i,j}$. In addition, both `A-CLOSURE` and `REVISE` exploit conformance of a calculus with R_7 (strong converse) to halve the space for storing the constraints. Flag s indicates whether full storage is required. If R_7 is satisfied (s is false), then $C_{i,j}$ can be obtained by computing $C_{j,i}{}^{\smile}$; this is done in the auxiliary function `LOOKUP`.

ALGORITHM 1: Universal algebraic closure algorithm A-CLOSURE

```
Function LOOKUP $(C, i, j, s)$ :        ----- RETRIEVE RELATION FROM CONSTRAINT MATRIX -----
    if $s \vee (i < j)$ then
        return $C_{i,j}$                                          complete matrix stored?
    else
        return $(C_{j,i})^{\smile}$

Function REVISE $(C, i, j, k, s)$ :     ----- REVISE RELATION $r_{i,j}$ ACCORDING TO EQ. (15) -----
    $u \leftarrow$ false                  update flag to signal whether relation was updated
    $r \leftarrow C_{i,j} \cap$ LOOKUP$(C, i, k, s) \diamond$ LOOKUP$(C, k, j, s)$
    if $\mathsf{R}_9$ does not hold $\vee\ s$ then
        $r' \leftarrow$ LOOKUP$(C, j, i, s) \cap ($LOOKUP$(C, j, k, s) \diamond$ LOOKUP$(C, k, i, s))$
        $r \leftarrow r \cap r'^{\smile}$
        $r' \leftarrow r' \cap r^{\smile}$
        if $r' \neq C_{j,i}$ then
            assert $r' \neq \emptyset$                          stop if inconsistency is detected
            $u \leftarrow$ true
            $C_{j,i} \leftarrow r'$
    if $r \neq C_{i,j}$ then
        assert $r \neq \emptyset$                               stop if inconsistency is detected
        $u \leftarrow$ true
        $C_{i,j} \leftarrow r$
    return $(C, u)$

Function A-CLOSURE $(\mathcal{V}, C = \{C_{i,j} | i, j \in \mathcal{V}\})$ :      ----- MAIN ALGORITHM -----
    for $i, j \in \mathcal{V}$ do                                   Enforce strong 2-consistency
        $C_{i,j} \leftarrow C_{i,j} \cap C_{j,i}^{\smile}$
    if $R_7$ does not hold then                     full $|\mathcal{V}| \times |\mathcal{V}|$ matrix must be stored
        $s \leftarrow$ True
        $Q \leftarrow$ queue with elements $\{(i, j) | i, j \in \mathcal{V}\}$
    else                                                only triangular matrix is stored
        $s \leftarrow$ False
        $Q \leftarrow$ queue with elements $\{(i, j) | i, j \in \mathcal{V}, i < j\}$
    while $Q$ not empty do
        dequeue $(i, j)$ from $Q$
        for $k \in \mathcal{V}, k \neq i, k \neq j$ do
            $(C, u) \leftarrow$ REVISE$(C, i, k, j, s)$
            if $u$ then
                if $s$ then
                    enqueue $(i, k)$ in $Q$ unless already in queue
                else                           $R_7 \Rightarrow$ only one of $(i, k)$ and $(k, i)$ is required
                    enqueue $(\min\{i, k\}, \max\{i, k\})$ in $Q$ unless already in queue
            $(C, u) \leftarrow$ REVISE$(C, k, j, i, s)$
            if $u$ then
                if $s$ then
                    enqueue $(k, j)$ in $Q$ unless already in queue
                else                           $R_7 \Rightarrow$ only one of $(i, k)$ and $(k, i)$ is required
                    enqueue $(\min\{k, j\}, \max\{k, j\})$ in $Q$ unless already in queue
    return $C$
```

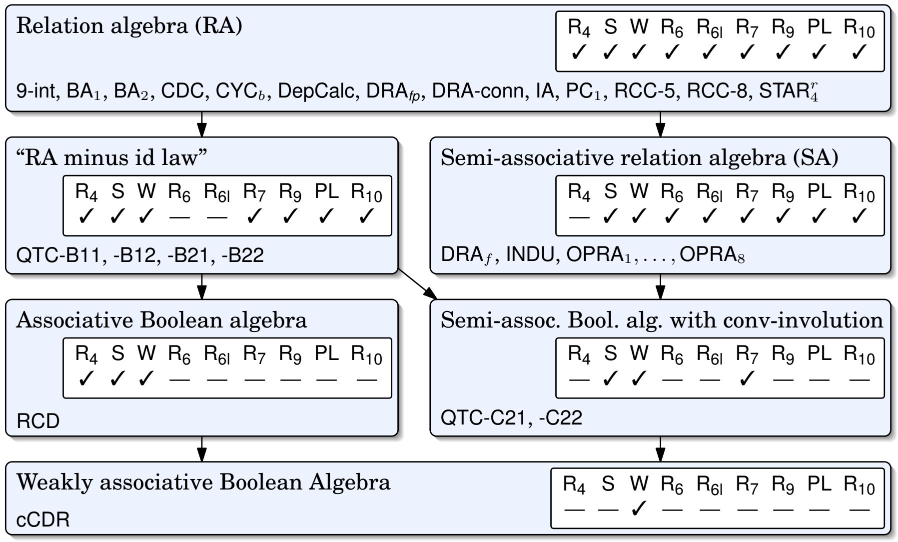


Fig. 6: Overview of algebra notions and calculi tested

5. COMBINATION AND INTEGRATION

Although qualitative calculi and constraint-based reasoning are predominant features of qualitative knowledge representation languages, they are rarely used by themselves in applications. For example, many applications involve several aspects of spatial and temporal knowledge simultaneously, e.g., topology and orientation of spatial objects. Others require additional forms of symbolic reasoning, such as logical reasoning. These requirements can best be solved by combining calculi or integrating them with other symbolic formalisms. In this section we review the interaction of qualitative calculi with other components of knowledge representation languages.

5.1. Qualitative Calculi in Constraint-Based Knowledge Representation Languages

The simplest case of a qualitative knowledge representation language is a single qualitative calculus. Sometimes further elements of constraint languages are used in addition, for example, constants and difference operators as in the case of PIDN [Pujari and Sattar 1999], or a restricted form of disjunction [Li et al. 2013].

To model several aspects of spatial and temporal knowledge and their interdependencies, combinations of calculi are studied. Wölfl and Westphal [2009] identify two general approaches to such combinations and reasoning therein: *loose integration* is based on the simple cross product of the base relations plus *interdependency constraints* [Gerevini and Renz 2002; Westphal and Wölfl 2008]; *tight integration* designs a new calculus internalizing the interdependencies [Wölfl and Westphal 2009]. For example, INDU combines IA and PC_1 tightly, reducing the 13×3 pairs of relations to the 25 semantically possible. A combination of RCC-8 with IA was introduced in [Gerevini and Nebel 2002]; several combinations of RCC-8 with direction calculi were analyzed [Liu et al. 2009; Cohn et al. 2014]. In general, combinations do not inherit algebraic and reasoning properties from their constituent calculi (cf. Fig. 5 and 6 for INDU).

Hernández [1994] describes the use of topological and orientation relations, which does not result in a dedicated calculus, but reveals the effects of constraining one aspect on reasoning in the other.

Alternative ways to solve the combination problem include formalizing the domain and qualitative relations in an abstract logic – which typically are computationally more expensive – or applying the efficient paradigm of linear programming to qualitative calculi over real-valued domains [Kreutzmann and Wolter 2014].

5.2. Qualitative Relations and Classical Logics: Spatial Logics

There are several developments to enrich qualitative representation with concepts found in classical logics or to combine the two strands. Domain representations purely based on qualitative relations can be viewed as quantifier-free formulae with variables ranging over a certain spatial or temporal domain. QCSP instances can be posed as satisfiability problems of conjunctive constraint formulae with existentially quantified variables. Adopting this logic view for QCSPs leads to the field of spatial logics [Aiello et al. 2007], which is involved with combinations of qualitative calculi and logics. Already in the 1930s topological statements as those expressible in RCC were found to constitute a fragment of the modal logic S4 plus the universal modality ($S4_u$), comprehensively described by Bennett [1997]. The cartesian product of $S4_u$ with linear temporal logic captures topological relationships changing over time [Bennett et al. 2002]. Qualitative relations and their interrelations can also be described by axiomatic systems; this approach was argued to comprise the composition-table approach and support the construction of composition tables [Eschenbach 2001]. Axiomatic systems are given, e.g., in [Eschenbach and Kulik 1997; Gotts 1996; Hahmann and Grüninger 2011]. The field of spatial logics can thus be viewed as a continuum between purely qualitative knowledge representation languages and logics. Current work studies the computational complexity of increasing expressivity of qualitative relations, e.g., by introducing Boolean expressions of spatial variables $\mathsf{PO}(A \cap B, C)$ [Wolter and Zakharyaschev 2000], introducing a temporal modality [Kontchakov et al. 2007], or even combining spatial and temporal logics [Gabelaia et al. 2005].

5.3. Qualitative Calculi and Description Logics

Description logics (DLs) are a successful family of knowledge representation languages tailored to capturing conceptual knowledge in ontologies and reasoning over it [Baader et al. 2007]. The most prominent DL-based ontology language is the W3C standard OWL.[4] Several approaches to combining DLs and qualitative calculi have evolved, aiming at describing spatial and temporal qualities of application domains. A principal approach developed by Lutz and Milićič [2007] allows adding qualitative calculi that satisfy certain admissibility conditions to $\mathcal{ALC}$, the basic DL, incorporating spatial/temporal reasoning into a standard DL reasoning procedure. According to the authors, a practical implementation would be challenging. Stocker and Sirin [2009] describe *PelletSpatial*, an extension of the DL reasoner *Pellet* [Sirin et al. 2007] for query answering over non-spatial (DL) and spatial (RCC-8) knowledge. Batsakis and Petrakis [2011] describe SOWL, an OWL ontology capturing static, spatial, and temporal information, using a DL axiomatization of spatial relations from the calculi CDC and RCC-8. Temporal and spatial reasoning are separated (a-closure and Pellet, resp.). Ben Hmida et al. [2012] sketch an implementation of logic programming that combines 9-int with OWL ontologies and constructive solid geometry.

5.4. Qualitative Calculi and Situation Calculus

The situation calculus is a popular framework for reasoning about action and change; runtime systems such as *DTGolog* [Ferrein et al. 2004] and *ReadyLog* [Ferrein and

[4]http://www.w3.org/TR/owl2-overview

Lakemeyer 2008] are used in robotic applications. Qualitative relations are relevant to world modeling and underlie high-level behavior specifications [Schiffer et al. 2012].

Bhatt et al. [2006] aim at general integration of QSTR into reasoning about action and change, i.e., a general domain-independent theory, in order to reason about dynamic and causal aspects of spatial change. With a naive characterization of objects based on their physical properties they particularly investigate key aspects of a topological theory of space on the basis of RCC-8 [Bhatt and Loke 2008].

6. ALTERNATIVE APPROACHES

This section presents an overview of reasoning techniques that have also been used to address QSTR reasoning problems, but are not based on QSTR techniques. Since spatial reasoning connects to fields in mathematics related to geometry or topology, there are manifold possible connections to make. In the following we only hint at fields that have already proven to provide impulses to QSTR research.

6.1. Algebraic Topology

Fundamental concepts of algebraic topology resemble expressivity of topological QSTR calculi such as RCC-8. For example, Euler's well-known polyhedron formula "vertices - edges + faces = 2" is a representative of Euler characteristics that characterize topological invariants of a space or body. The PLCA framework [Takahashi 2012] exploits the Euler characteristics to reason about topological space by invariants.

6.2. Combinatorial Geometry

A set of Jordan curves (i.e., sets that are homeomorphic to the interval $[0, 1]$ in the plane) induce an *intersection graph*. The *string graph problem* poses the question, whether a given graph can be an intersection graph of a set of curves in the plane. While the problem itself already is of a spatial nature, Schaefer and Štefankovič [2004] reduced reasoning about topological relations in RCC-8 about planar regions to the string graph problem and later proved the string graph problem to be NP-complete [Schaefer et al. 2003], directly contributing to QSTR research.

An alternative approach to reasoning with directional relations can be found in *oriented matroid* theory, which comprise several equivalent combinatorial structures such as directed graphs, point and vector configurations, pseudoline arrangements, arrangements of hyperplanes [Björner et al. 1999]. Already Knuth [1992] points out the importance of oriented matroids for qualitative spatial reasoning. In the context of LR constraint networks, a connection to the oriented matroid axiomatization of so-called chirotopes lead to complexity results in QSTR [Wolter and Lee 2010; Lee 2014].

6.3. Graph Theoretical Approaches

Worboys [2013] describes topological configurations through their representation as labeled trees, called *map trees*. Graph edit operations on map trees can be defined to correspond to spatial change of the topological configuration, providing an efficient approach to reason about spatial change.

A different way to represent qualitative spatial change consists in describing the change on two levels of detail. Stell [2013] represents a scene of regions via a bipartite graph (U, V, E) where the elements of U (V) represent regions that can be seen as connected at a coarse level of detail (when accounting for finer details). This way it is possible to describe the splitting, connecting and change of distance of regions, as well as the creation, deletion and change of size of a (part of a) region.

6.4. Logic Frameworks

Viewing vectors in a vector space as abstract arrows, Aiello and Ottens [2007] introduce a hybrid modal logic (arrow logic) for capturing mereotopological relations between sets of vectors. Inversion and composition of arrows are modeled by morphological operators such as *dilation*, *erosion* and *difference*. A resolution calculus allows for automated reasoning about topological relations and relative size.

6.5. Model-theoretic and Constraint Reasoning Methods

Qualitative constraint satisfaction problems can be reformulated as general constraint satisfaction problems. Then, the consistency problem can be tackled using model-theoretic methods [Bodirsky and Wölfl 2011; Westphal 2015] or using SAT solving or datalog programs [Westphal 2015], leading to greater flexibility.

6.6. Quantitative Methods

Linear programming (LP) techniques have been used to decide constraint problems posed as linear inequalities, allowing polyhedral regions, lines, and points to be represented. LP can mix free-ranging variables with concrete values (e.g., points at known positions) and, beyond consistency checking, determine a model in polynomial time. By posing QCSP instances as LPs, constraints originating in distinct calculi can easily be mixed. While some QSTR problems can almost directly be posed as LPs [Jonsson and Bäckström 1998; Ligozat 2011; Lee et al. 2013], disjunctive LP formulae allow several QSTR calculi to be handled simultaneously [Kreutzmann and Wolter 2014]. In a similar fashion, Schockaert et al. [2011] combine qualitative and quantitative reasoning of relations about different spatial aspects by using genetic optimization. Techniques for deciding satisfiability of equations yield advancements on the inherent problem of consistency checking for directional constraints such as those present in the LR calculus, as (disjunctions of) linear equations can capture relevant geometric invariances [Lücke and Mossakowski 2010; van Delden and Mossakowski 2013].

7. CONCLUSION AND FUTURE RESEARCH DIRECTIONS

Qualitative spatial and temporal reasoning explores potentially interesting domain conceptualizations and their computational effects. As a consequence, QSTR is connected to various research areas in and around artificial intelligence, such as knowledge representation, linguistics and spatial cognition. Thus QSTR plays the role of a hub for connecting symbolic techniques to real-world applications. The notion of a *qualitative calculus* attests to this role by representing knowledge about spatial and temporal domains as an abstract algebra that provides the semantics to knowledge representation languages. Reasoning with qualitative representations occurs in several forms, with deductive forms of inference, such as deciding consistency, being in a central position. This is captured in the *qualitative constraint satisfaction problem*, which is decidable for all qualitative calculi (in the strict sense of Definition 3.3), ranging from low-order polynomial time complexity to within PSPACE (cf. Table IV). With this survey we present the first comprehensive overview of the known computational properties of all qualitative calculi proposed so far.

7.1. Beneficiaries of This Survey

This survey addresses a broad range of researchers and engineers from different research communities and application areas. We expect three groups of beneficiaries.

The first group comprises researchers and engineers who apply QSTR and build systems for their applications. Our survey provides them with a comprehensive and concise overview of the formalisms available, allowing objective design choices.

The second group consists of researchers contributing to QSTR to whom we provide revised definitions that are general enough to address all formalisms proposed so far. The overview of domain conceptualizations studied so far fosters identification of interesting new conceptualizations to be studied. Moreover, the summary of algebraic and computational properties of existing formalisms reveals open research questions: for calculi not listed in Table V reasoning properties have still to be analyzed.

Last, but not least, the third group benefiting from this presentation consists of developers of reasoning tools. In order to accrete the position of QSTR as hub, sophisticated tools are necessary that disseminate formalisms and algorithms, linking basic research to application development. On the one hand, we provide pointers to all formalisms proposed and the decision methods necessary to perform reasoning. This also reveals commonalities between formalisms, hopefully gearing tools towards becoming universal in the sense that they allow many variants of representations to be handled. On the other hand – and related to the discrepancy between the amount of formalisms proposed and those fully analyzed discussed before – the most efficient algorithms to decide QCSP instances have often not yet been identified and solid algorithm engineering can likely yield a great leap ahead for QSTR.

7.2. Open Problem Areas in QSTR

Combining qualitative abstractions. Despite the work reported in Section 5.1, *generally applicable* methods for combining existing abstractions for different spatial and temporal aspects are missing – a potential threat to the applicability of qualitative methods. It is clearly not feasible to identify all potentially useful combinations individually: there are infinitely many abstractions that give rise to a qualitative calculus.

Integration with other symbolic methods. In addition to the above observation that an application may need to handle more than one calculus at the same time, expressivity provided by domain-independent knowledge representation techniques may be important too. There are first contributions (e.g., combining description logic with QSTR), but these are limited to specific combinations using specific methods. A promising approach is the integration of a variety of QSTR formalisms into a first-order framework [Bhatt et al. 2011]—the challenge being the development of efficient reasoning methods. We expect that this will result in a combination of first-order methods, constraint-solving methods, relation-algebraic methods and specialised methods for the existential theory over the reals, see [van Delden and Mossakowski 2013] for some first steps.

Integration with quantitative approaches. Qualitative approaches link metric data and symbolic reasoning, but consistent interpretation of sensor data considering its inevitable uncertainty is a recurring and challenging task. An algorithmic understanding of this problem has to the best of our knowledge not been developed yet. Conversely, it can also be helpful to link qualitative inference with quantitative or other kinds of constraints. As Liu and Li [2012] recently discovered, constraint-based qualitative reasoning with information partially grounded in data can differ significantly from classic qualitative reasoning and thus calls for further exploration.

Algebras for higher-arity qualitative calculi. Abstract algebras provide the foundations for symbolic knowledge manipulation and enable optimizations to reasoning procedures. Our study gives an extensive account of algebraic properties of existing binary calculi, but we have also seen that it is highly non-trivial to extend this study to ternary calculi. The main problem is a missing notion of relation algebra already for ternary relations that is general enough to encompass the variety of existing calculi.

Practical reasoning algorithms. Few of the various methods required in qualitative reasoning (see Table V) have been studied rigorously in a practical context. In the light of continuously growing data bases, identifying best-practice algorithms, evaluating the scaling behavior, and potentially developing heuristic approximations will be crucial to foster the relevance of QSTR methods.

By completing the picture of computational complexity and identifying practical solutions to reasoning with all individual calculi, either individually or in combination with one another or even other KR techniques, it will be possible to realize truly universal QSTR tools. These tools will foster the position of QSTR as a hub, not only conceptually, but implemented in almost all knowledge-based systems.

ELECTRONIC APPENDIX

The electronic appendix for this article can be accessed in the ACM Digital Library. It contains additional examples, observations, proofs, and details for Sections 3 and 4.

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Online Appendix to: A Survey of Qualitative Spatial and Temporal Calculi — Algebraic and Computational Properties

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A. EXAMPLES FOR SECTION 3

Example A.1. The one-dimensional point calculus PC_1 [Vilain and Kautz 1986] symbolically represents the relations $<, =, >$ between points on a line (which may model points in time), see Figure 7 a. These three relations are called *base relations* in Def. 3.1; PC_1 additionally represents all their unions and intersections: the empty relation and $\leqslant, \geqslant, \neq, \lesseqqgtr$. The calculus provides the relation symbols `<`, `=`, and `>`; sets of symbols represent unions of base relations, e.g., $\{\texttt{<}, \texttt{=}\}$ represents $\leqslant$. The symbol `=` represents the identity $=$.

PC_1 further provides converse and composition. For example, the converse of `<` is `>`: whenever $x < y$, it follows that $y > x$; the composition of `<` with itself is again `<`: whenever $x < y$ and $y < z$, we have $x < z$. PC_1 represents the converse as a list of size 3 (the converses of all relation symbols) and the composition as a table of size 3×3 (one composition result for each pair of relation symbols). •

Example A.2. The calculus RCC-5 [Randell et al. 1992] symbolically represents five binary topological relations between regions in space (which may model objects): "is discrete from", "partially overlaps with", "equals", "is proper part of", and "has proper part", plus their unions and intersections, see Figure 7 b. For this purpose, RCC-5 provides the relation symbols `DC`, `PO`, `EQ`, `PP`, and `PPi`. The latter two are each other's converses; the first three are their own converses. The composition of `DC` and `PO` is $\{\texttt{DC}, \texttt{PO}, \texttt{PP}\}$ because, whenever region x is disconnected from y and y partially overlaps with z, there are three possible configurations between x and z: those represented by $\texttt{DC}, \texttt{PO}, \texttt{PP}$. •

Example A.3. The calculus CYC_b [Isli and Cohn 2000] symbolically represents four binary topological relations orientated lines in the plane (which may model observers and their lines of vision): "equals", "is opposite to", and "is to the left/right of", plus their unions and intersections, see Figure 7 c. For this purpose, CYC_b provides the relation symbols `e`, `o`, `l`, and `r`. The latter two are each other's converses; `e` and `o` are their own converses. The composition of `l` and `r` is $\{\texttt{e}, \texttt{l}, \texttt{r}\}$: whenever orientation x is to the left of y and y is to the left of z, then x can be equal to, to the left of, or to the right of z. •

Example A.4. The calculus PC_1 is based on the binary abstract partition scheme $\mathcal{S}(\mathsf{PC}_1) := (\mathbb{R}, \{<, =, >\})$ where $\mathbb{R}$ is the set of reals and $\{<, =, >\}$ is clearly JEPD. For

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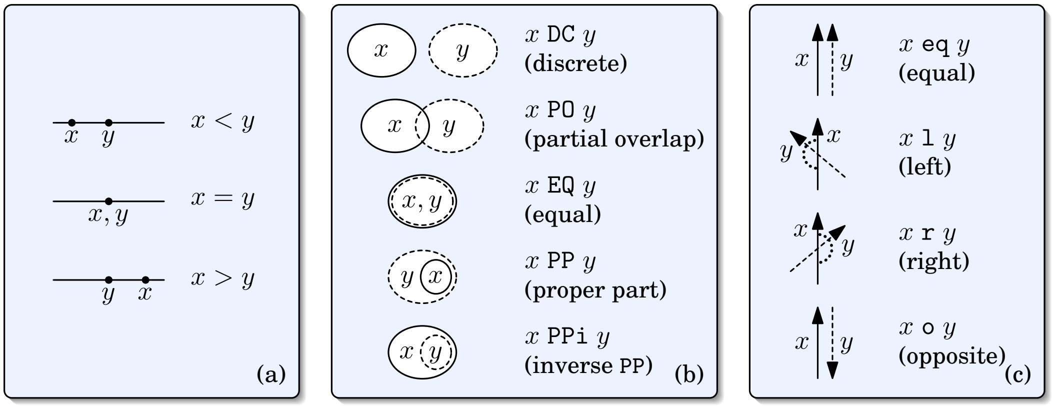

Fig. 7: Illustration of the base relations for the calculi (a) PC_1, (b) RCC-5, and (c) CYC_b

RCC-5, the universe is often chosen to be the set of all regular closed subsets of the 2- or 3-dimensional space $\mathbb{R}^2$ or $\mathbb{R}^3$. The five base relations from Figure 7 b are JEPD. For CYC_b, the universe is the set of all oriented line segments in the plane $\mathbb{R}^2$, given by angles between 0° and 360°. The four base relations from Figure 7 c are JEPD. •

Example A.5. In PC_1, "x < y" represents the relationship $a < b$, which holds complete information because $<$ is atomic in $\mathcal{S}(\mathsf{PC}_1)$. The statement "$x$ {<, =} y" represents the coarser relationship $a \leqslant b$ holding the incomplete information "$a < b$ or $a = b$". Clearly "x {<, =, >} y" holds no information: "$a < b$ or $a = b$ or $a > b$" is always true. •

Example A.6. Consider the modification PC_1' based on the non-PD set $\{\leqslant, \geqslant\}$. Then the relationship $a = b$ can be expressed in two ways using relation symbols <= and >= representing $\leqslant$ and $\geqslant$: "x <= y" and "x >= y".

Conversely, consider the variant PC_1'' based on the non-JE set $\{<, >\}$. Then the constraint $a = b$ cannot be expressed. Therefore, in any given set of constraints where it is known that x, y stand for identical entities, we would find the empty relation between x, y. The standard reasoning procedure described in Section 3.2 would declare such sets of constraints to be inconsistent, although they are not – we have simply not been able to express $x = y$. •

Example A.7. Clearly, $=$ in $\mathcal{S}(\mathsf{PC}_1)$ and "equals" in $\mathcal{S}(\text{RCC-5})$ and $\mathcal{S}(\mathsf{CYC}_b)$ are the identity relation over the respective domain. •

Example A.8. In $\mathcal{S}(\mathsf{PC}_1)$ we have that $<^{\smile}$ is $>$; $=^{\smile}$ is $=$; $>^{\smile}$ is $<$. The converses of the base relations in $\mathcal{S}(\text{RCC-5})$ and $\mathcal{S}(\mathsf{CYC}_b)$ were named in Examples A.2 and A.3. •

Example A.9. In Figure 8 we depict the permutations sc (rotation), hm (permutation), and inv for one relation from the ternary Double Cross Calculus (2-cross) [Freksa and Zimmermann 1992]. The 2-cross relations specify the location of a point P_3 relative to an oriented line segment given by two points P_1, P_2. Figure 8 a shows the relation `right-front`. The relations resulting from applying the permutations are depicted in Figure 8 b; e.g., sc(`right-front`) = `right-back` because the latter is P_1's position relative to the line segment $\overrightarrow{P_2P_3}$. Figure 8 c will be relevant later.

Example A.10. It follows that $\mathcal{S}(\mathsf{PC}_1)$, $\mathcal{S}(\text{RCC-5})$, and $\mathcal{S}(\mathsf{CYC}_b)$ are even partition schemes. In contrast, the abstract partition scheme $(\mathbb{R}, \{\leqslant, >\})$ is not a partition

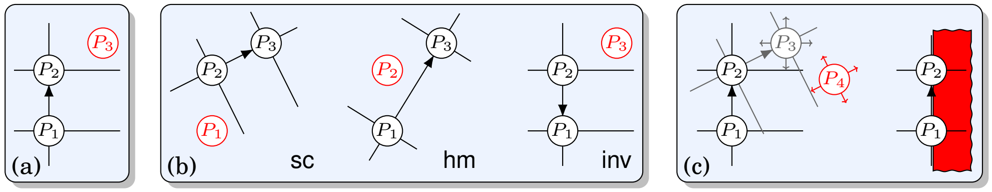

Fig. 8: (a) The 2-cross relation `right-front`; (b) permutations of `right-front`; (c) the composition `right-front` $\circ$ `right-front`

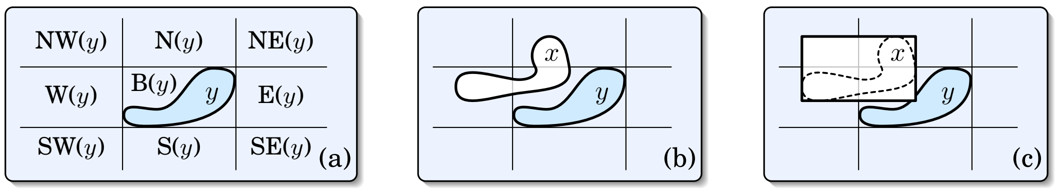

Fig. 9: Calculi CDR and RCD: (a) reference tiles; (b) the CDR base relation x N:W:B y; (c) the RCD base relation x NW:N:W:B y

scheme: it violates both conditions of Observation B.1 (and thus of Definition 3.2). •

Example A.11. As an example of an intuitive and useful abstract partition scheme that is *not* a partition scheme, consider the calculus *Cardinal Direction Relations* (CDR) [Skiadopoulos and Koubarakis 2005]. CDR describes the placement of regions in a 2D space (e.g., countries on the globe) relative to each other, and with respect to a fixed coordinate system. The axes of the bounding box of the *reference region* y divide the space into nine *tiles*, see Fig. 9 a. The binary relations in $\mathcal{S}$(CDR) now determine which tiles relative to y are occupied by a *primary region* x: e.g., in Fig. 9 b, tiles N, W, and B of y are occupied by x; hence we have x N:W:B y. Simple combinatorics yields $2^9 - 1 = 511$ base relations.

Now $\mathcal{S}$(CDR) is not a partition scheme because it violates Condition 2 of Observation B.1 (and thus of Definition 3.2): e.g., the converse of the base relation S (south) is not a base relation. To justify this claim, assume the contrary. Take two specific regions x, y with x S y, namely two unit squares, where y is exactly above x. Then we also have y N x; therefore the converse of S is N. Now stretch the width of x by any factor > 1. Then we still have y N x, but no longer x S y. Hence the converse of S cannot comprise all of N, which contradicts the assumption that the converse of S is a base relation.

The related calculus RCD [Navarrete et al. 2013] abstracts away from the concrete shape of regions and replaces them with their bounding boxes, see Fig. 9 c. $\mathcal{S}$(RCD) is not a partition scheme, with the same argument from above. •

Example A.12. To turn, say, CDR into a partition scheme, one would have to break down the 511 base relations into smaller ones, resulting in even more, less cognitively plausible ones.

Example A.13. In $\mathcal{S}(\mathsf{PC}_1)$ we have, e.g., that $< \circ <$ equals $<$ because $a < b$ and $b < c$ imply $a < c$. Furthermore, $< \circ >$ yields the universal relation, i.e., the union of $<$, $=$, and $>$, because "$a < b$ and $b > c$" is consistent with each of $a < c$, $a = c$, and $a > c$. •

Example A.14. It says: if the location of x relative to u and v is determined by r and the location of w relative to v and x is determined by s, then the location of

w relative to u and v is determined by $r \circ^3_{\mathrm{FZ}} s$. Fig. 8 c shows the composition of the 2-cross relation `right-front` with itself; i.e., `right-front` $\circ^3_{\mathrm{FZ}}$ `right-front`. The red area indicates the possible locations of the point P_4; hence the resulting relation is {`right-front`, `right-middle`, `right-back`}. A generalization to other calculi and arities $n > 3$ is obvious.

Example A.15. As an example, consider again $n = 3$ and 2-cross. Equation (5) says that the composition result of the relations `right-front`, `right-front`, and `left-back` is the set of all triples (u_1, u_2, u_3) such that there is an element v with $(u_1, u_2, v) \in$ `right-front`, $(u_1, v, u_3) \in$ `right-front`, and $(v, u_2, u_3) \in$ `right-back`. That set is exactly the relation `right-front`, which can be seen drawing pictures similar to Fig. 8.

Example A.16. We can now observe that PC_1 is indeed a binary calculus with the following components.

— The set of relation symbols is $\mathsf{Rel} = \{\texttt{<}, \texttt{=}, \texttt{>}\}$, denoting the relations depicted in Figure 7 a. The $2^3 = 8$ composite relations include, for example, $R_1 = \{\texttt{<}, \texttt{=}\}$ and $R_2 = \{\texttt{<}, \texttt{=}, \texttt{>}\}$.
— There are several possible interpretations, depending largely on the chosen universe. One of the most natural choices leads to the interpretation $\mathsf{Int} = \{\mathcal{U}, \varphi, \pi, \circ\}$ with the following components.
  — The universe $\mathcal{U}$ is the set of reals.
  — The map φ maps <, =, and > to $<$, $=$, and $>$, respectively; see Figure 7 a. Its extension to composite relations maps, for example, R_1 from above to $\geqslant$ and R_2 to the universal relation.
  — The operations π and $\circ$ are the standard binary converse and composition operations from (2) and (3).
— The converse operation $\breve{}$ is given by Table VII a. For its extension to composite relations, we have, e.g., $R_1\breve{} = \{\texttt{>}, \texttt{=}\}$ and $R_2\breve{} = R_2$.

(a)

r	$r\breve{}$
<	>
=	=
>	<

(b)

r\s	<	=	>
<	{<}	{<}	{<, =, >}
=	{<}	{=}	{>}
>	{<, =, >}	{>}	{>}

Table VII: Converse and composition tables for the point calculus PC_1.

— The composition operation $\diamond$ is given by a 3×3 table where each cell represents $r \diamond s$, see Table VII b. For its extension to composite relations, we have, for example:

$$\begin{aligned} R_1 \diamond R_2 &= \{\texttt{<}, \texttt{=}\} \diamond \{\texttt{<}, \texttt{=}, \texttt{>}\} \\ &= \{\texttt{<}\} \diamond \{\texttt{<}\} \;\cup\; \{\texttt{<}\} \diamond \{\texttt{=}\} \;\cup\; \ldots \;\cup\; \{\texttt{=}\} \diamond \{\texttt{>}\} \\ &= \{\texttt{<}\} \cup \{\texttt{<}\} \cup \cdots \cup \{\texttt{>}\} \\ &= R_2 \end{aligned}$$

•

Example A.17. RCC-5 too is a binary calculus, with the following components.

— The set of relation symbols is $\mathsf{Rel} = \{\texttt{EQ}, \texttt{DC}, \texttt{PO}, \texttt{PP}, \texttt{PPi}\}$, denoting the relations depicted in Figure 7 b. The $2^5 = 32$ composite relations include, for example, $R_1 = \{\texttt{DC}, \texttt{PO}, \texttt{PP}, \texttt{PPi}\}$ ("both regions are distinct") and $R_2 = \{\texttt{PP}, \texttt{PPi}\}$ ("one region is a proper part of the other").

— Similarly to PC_1, there are several possible interpretations, a natural choice being $\mathsf{Int} = \{\mathcal{U}, \varphi, \pi, \circ\}$ with the following components.
 — The universe $\mathcal{U}$ is the set of all regular closed subsets of $\mathbb{R}^2$.
 — The map φ maps, for example, `DC` to all pairs of regions that are disconnected or externally connected. Figure 7 b illustrates $\varphi(r)$ for all relation symbols $r =$ `EQ`, `DC`, `PO`, `PP`, `PPi`.
 — The operations π and $\circ$ are the standard binary converse and composition operations from (2) and (3).
— The converse operation $\smile$ is given by Table VIII a. we have, e.g., $R_2{}^{\smile} = R_2$.

(a)

r	$r^{\smile}$
EQ	EQ
DC	DC
PO	PO
PP	PP
PPi	PPi

(b)

r\s	EQ	DC	PO	PP	PPi
EQ	{EQ}	{DC}	{PO}	{PO}	{PPi}
DC	{DC}	$\mathcal{U}$	{DC, PO, PP}	{DC, PO, PP}	{DC}
PO	{PO}	{DC, PO, PPi}	$\mathcal{U}$	{PO, PP}	{DC, PO, PPi}
PP	{PP}	{DC}	{DC, PO, PP}	{PP}	$\mathcal{U}$
PPi	{PPi}	{DC, PO, PPi}	{PO, PPi}	{EQ, PO, PP, PPi}	{PPi}

Table VIII: Converse and composition tables for the point calculus RCC-5. Universal relation $\mathcal{U}$ stands for {EQ, DC, PO, PP, PPi}

— The composition operation $\diamond$ is given by a 5×5 table where each cell represents $r \diamond s$, see Table VIII b. For its extension to composite relations, we have, for example:

$$\begin{aligned}\{\texttt{PP},\texttt{PPi}\} \diamond \{\texttt{DC}\} &= \{\texttt{PP}\} \diamond \{\texttt{DC}\} \ \cup\ \{\texttt{PPi}\} \diamond \{\texttt{DC}\} \\ &= \{\texttt{DC}\} \ \cup\ \{\texttt{DC},\texttt{PO},\texttt{PPi}\} \\ &= \{\texttt{DC},\texttt{PO},\texttt{PPi}\}\end{aligned}$$

•

Example A.18. CYC_b too is a binary calculus, with the following components.

— The set of relation symbols is $\mathsf{Rel} = \{\texttt{e},\texttt{o},\texttt{l},\texttt{r}\}$, denoting the relations depicted in Figure 7 c. The $2^4 = 16$ composite relations include, for example, $R_1 = \{\texttt{e},\texttt{l}\}$ ("the orientation y is to the left of, or equal to, x") and $R_2 = \{\texttt{e},\texttt{o}\}$ ("both orientations are equal or opposite to each other").
— The standard interpretation for CYC_b is $\mathsf{Int} = \{\mathcal{U}, \varphi, \pi, \circ\}$ with the following components.
 — The universe $\mathcal{U}$ is the set of all *2D-orientations*, which can equivalently be viewed as either the set of radii of a given fixed circle C, or the set of points on the periphery of C, or the set of directed lines through a given fixed point (the origin of C).
 — The map φ maps, for example, `l` to all pairs (x, y) of directed lines where the angle α from x to y, in counterclockwise fashion, satisfies $0° < \alpha < 180°$. Analogously `o` is mapped to those pairs where that angle is exactly $180°$. Figure 7 c illustrates $\varphi(r)$ for all relation symbols $r =$ `e`, `o`, `l`, `r`.
 — The operations π and $\circ$ are the standard binary converse and composition operations from (2) and (3).
— The converse operation $\smile$ is given by Table IX a. For its extension to composite relations, we have, e.g., $R_1{}^{\smile} = \{\texttt{e},\texttt{l}\}$.

(a)

r	$r^{\smile}$
e	e
o	o
l	r
r	l

(b)

r\s	e	o	l	r
e	{e}	{o}	{l}	{r}
o	{o}	{e}	{r}	{l}
l	{l}	{r}	{l, o, r}	{e, l, r}
r	{r}	{l}	{e, l, r}	{l, o, r}

Table IX: Converse and composition tables for the point calculus CYC_b.

— The composition operation $\diamond$ is given by a 4×4 table where each cell represents $r \diamond s$, see Table IX b. For its extension to composite relations, we have, for example:

$$\begin{aligned} R_1 \diamond R_1 &= \{\mathtt{e}, \mathtt{l}\} \diamond \{\mathtt{e}, \mathtt{l}\} \\ &= \{\mathtt{e}\} \diamond \{\mathtt{e}\} \,\cup\, \{\mathtt{e}\} \diamond \{\mathtt{l}\} \,\cup\, \{\mathtt{l}\} \diamond \{\mathtt{e}\} \,\cup\, \{\mathtt{l}\} \diamond \{\mathtt{l}\} \\ &= \{\mathtt{e}\} \,\cup\, \{\mathtt{l}\} \,\cup\, \{\mathtt{l}\} \,\cup\, \{\mathtt{e}, \mathtt{l}, \mathtt{r}\} \\ &= \{\mathtt{e}, \mathtt{l}, \mathtt{r}\} \end{aligned}$$

•

Example A.19. The converse and permutation operation in PC_1 are both strong because (9) holds for all three relation symbols (e.g., $\varphi(<^{\smile}) = \varphi(>) = > = <^{\smile} = \varphi(<)^{\smile}$), and the binary version of (11), namely

$$\varphi(r_1 \diamond r_2) = \varphi(r_1) \circ \varphi(r_2),$$

holds for all nine pairs of relation symbols (e.g., $\varphi(> \diamond >) = \varphi(>) = > = > \circ > = \varphi(>) \circ \varphi(>)$). •

Example A.20. Consider the variant $\mathsf{PC}_1^{\mathbb{N}}$ of PC_1 that is interpreted over the universe $\mathbb{N}$. It contains the same base relations with the usual interpretation and, obviously, the same converse operation, see Example A.16. However, composition is no longer strong because $< \circ < \subsetneq <$ holds: for "$\subseteq$" observe that, whenever $x < y < z$ for three points $x, y, z \in \mathbb{N}$, it follows that $x < z$; and "$\not\supseteq$" holds because there are points x, z with $x < z$ for which there is no y with $x < y < z$, for example, $x = 0$, $z = 1$. More precisely, the result of the composition $< \diamond <$ should be the relation $<^1 = \{(x, z) \mid x + 1 < z\}$. Since $<^1$ is not expressible by a union of base relations, we cannot endow this calculus with a strong symbolic composition operation. Consequently we have a choice as to the composition result in question. Regardless of that choice, the composition table will incur a *loss of information* because it cannot capture that the pair (x, z) is in $<^1$.

If we opt for weak composition, then Equation (10) requires us to generate the result of $< \diamond <$ from the symbols for exactly those relations that overlap with the domain-level composition $< \circ <$. From the above it is clear that this is exactly $<$. One can now easily check that, for the case of weak composition, we get precisely Table VIII b.

On the contrary, if we do not care about composition being weak, then abstract composition (Inequality (7)) requires us to generate the result of $< \diamond <$ from the symbols for *at least* those relations that overlap with $< \circ <$. This means that we can postulate $< \diamond < = \{<\}$ as before or, for example, $< \diamond < = \{<, =, >\}$.

The difference between weak and abstract composition is that abstract composition allows us to make the composition result arbitrarily *general*, whereas weak composition forces us to take exactly those relations into account that contain possible pairs of (x, z). Weak composition therefore restricts the loss of information to an unavoidable minimum, whereas abstract composition does not provide such a guarantee: the more base relations are included in the composition result, the more information we lose on how x and z are interrelated.

In this connection, it becomes clear why we require composition to be *at least* abstract: without this requirement, we could omit, for example, < from the above composition result. This would result in *adding spurious information* because we would suddenly be able to conclude that the constellation $x < y < z$ is impossible, just because $< \diamond < = \emptyset$. This insight, in turn, is particularly important for ensuring soundness of the most common reasoning algorithm, a-closure, see Section 3.2. •

Example A.21. In PC_1 we may have the two constraints $c_1 = x_1 < x_2$ and $c_2 = x_2 \{<,=\} x_3$. The valuation $\psi : X \to \mathbb{R}$ with $\psi(x_1) = \sqrt{2}$, $\psi(x_2) = 3.14$ and $\psi(x_3) = 42$ satisfies both constraints. If we set $\psi(x_3) = 3.14$, then both constraints remain satisfied by ψ; if we set $\psi(x_3) = 2.718$, then ψ no longer satisfies c_2. •

Example A.22. The QCSP in Figure 3 c based on PC_1 is not path-consistent because $r_{A,C}$ implicitly takes on the universal relation, and thus Equation (14) is violated for $x = A, y = C, z = B$. By contrast, the QCSP in Figure 3 b is path-consistent, which can be verified by considering each permutation of A, B, C in turn. •

Example A.23. Consider the PC_1 QCSP in Figure 3 c. The missing edge between variables A and C indicates an implicit constraint via the universal relation $u = \{<,=,>\}$. Enforcing a-closure as per (16) updates this constraint with $u \cup < \diamond <$, which yields <, resulting in Figure 3 b. Further applications of (16) do not cause any more changes; hence the QCSP in Figure 3 b is algebraically closed. •

Example A.24. Consider the modification PC_1''' based on the binary abstract partition scheme $\mathcal{S}(\mathsf{PC}_1''') = (\{0,1,2\}, \{<,=,>\})$, i.e., the domain now has 3 elements. Then the QCSP containing 4 nodes and the constraints $\{x_0 < x_1,\ x_1 < x_2,\ x_2 < x_3\}$ has the algebraic closure $\{x_i < x_j \mid 0 \leqslant i < j \leqslant 3\}$, which has no solution in the 3-element domain. •

B. OBSERVATIONS FOR SECTION 3.1 IN THE SPECIAL CASE OF BINARY RELATIONS

OBSERVATION B.1. *A binary partition scheme $(\mathcal{U}, \mathcal{R})$ is a binary abstract partition scheme with the following two additional properties.*

(*1*) *$\mathcal{R}$ contains the identity relation* id^2.
(*2*) *For every $r \in \mathcal{R}$, there is some $s \in \mathcal{R}$ such that $r^{\smile} = s$.*

OBSERVATION B.2. *A binary qualitative calculus is a tuple* $(\mathsf{Rel}, \mathsf{Int}, {}^{\smile}, \diamond)$ *with the following properties.*

— Rel *is as in Definition 3.3.*
— $\mathsf{Int} = (\mathcal{U}, \varphi, \pi, \circ)$ *is an* interpretation *with the following properties.*
  — *$\mathcal{U}$ is a universe.*
  — *$\varphi : \mathsf{Rel} \to 2^{\mathcal{U}\times\mathcal{U}}$ is an injective map as in Definition 3.3.*
  — *π is the standard converse operation on binary domain relations from* (2).
  — *$\circ$ is the standard composition operation on binary domain relations from* (3).
— *The* converse operation ${}^{\smile}$ *is a map* ${}^{\smile} : \mathsf{Rel} \to 2^{\mathsf{Rel}}$ *that satisfies*

$$\forall r \in \mathsf{Rel} : \quad \varphi(r^{\smile}) \supseteq \varphi(r)^{\pi} .$$

— *The composition operation $\diamond$ is a map* $\diamond : \mathsf{Rel} \times \mathsf{Rel} \to 2^{\mathsf{Rel}}$ *that satisfies*

$$\forall r, s \in \mathsf{Rel} : \quad \varphi(\diamond(r,s)) \supseteq \circ(\varphi(r), \varphi(s)) .$$

C. ADDITIONAL PROOFS FOR SECTION 3.1

C.1. Proof of Fact 3.5

Fact 3.5. Every strong permutation (composition) is weak, and every weak permutation (composition) is abstract.

PROOF. *"Every strong permutation is weak."* We assume that the permutation $\breve{}$ associated with π is strong, i.e., for all $r \in \mathsf{Rel}$,

$$\varphi(r^{\smile}) = \varphi(r)^{\pi}, \tag{18}$$

and show that $\breve{}$ is weak, i.e., for all $r \in \mathsf{Rel}$:

$$r^{\smile} = \bigcap\{S \subseteq \mathsf{Rel} \mid \varphi(S) \supseteq \varphi(r)^{\pi}\} \tag{19}$$

For "$\subseteq$", it suffices to show that, for every $S \subseteq \mathsf{Rel}$ with $\varphi(S) \supseteq \varphi(r)^{\pi}$, we have $r^{\smile} \subseteq S$. This follows from the inclusion "$\subseteq$" of (18) and the injectivity of φ.

For "$\supseteq$", let $s \in \bigcap\{S \subseteq \mathsf{Rel} \mid \varphi(S) \supseteq \varphi(r)^{\pi}\}$, that is, $s \in S$ for every $S \subseteq \mathsf{Rel}$ with $\varphi(S) \supseteq \varphi(r)^{\pi}\}$. Since $r^{\smile}$ is such an S due to the inclusion "$\supseteq$" of (18), we have $s \in r^{\smile}$.

"Every weak permutation is abstract." Strictly speaking, the phrasing in Definition 3.4 implies this statement. However, it is easy to show the stronger statement that (19) implies

$$\varphi(r^{\smile}) \supseteq \varphi(r)^{\pi}\,.$$

Indeed, this is justified by the following chain of equalities and inclusions.

$$\begin{aligned}\varphi(r^{\smile}) &= \varphi\left(\bigcap\{S \subseteq \mathsf{Rel} \mid \varphi(S) \supseteq \varphi(r)^{\pi}\}\right)\\ &= \bigcap\{\varphi(S) \subseteq \mathsf{Rel} \mid \varphi(S) \supseteq \varphi(r)^{\pi}\}\\ &\supseteq \varphi(r)^{\pi},\end{aligned}$$

where the first equality follows from (19), the second follows from the extension of φ to composite relations as per Definition 3.3, and the final inclusion is an obvious property of sets.

The respective statements about composition are proven analogously. □

C.2. Proof of Fact 3.6

Fact 3.6. Given a qualitative calculus $(\mathsf{Rel}, \mathsf{Int}, {}^{\smile_1}, \ldots, {}^{\smile_k}, \diamond)$ based on the interpretation $\mathsf{Int} = (\mathcal{U}, \varphi, \cdot^{\pi_1}, \ldots, \cdot^{\pi_k}, \circ)$, the following hold.

For all relations $R \subseteq \mathsf{Rel}$ and $i = 1, \ldots, k$:

$$\varphi(R^{\smile_i}) \supseteq \varphi(R)^{\pi_i} \tag{20}$$

For all relations $R_1, \ldots, R_m \subseteq \mathsf{Rel}$:

$$\varphi(\diamond(R_1, \ldots, R_m)) \supseteq \circ(\varphi(R_1), \ldots, \varphi(R_m)) \tag{21}$$

If ${}^{\smile_i}$ is a weak permutation, then, for all $R \subseteq \mathsf{Rel}$:

$$R^{\smile_i} = \bigcap\{S \subseteq \mathsf{Rel} \mid \varphi(S) \supseteq \varphi(R)^{\pi_i}\} \tag{22}$$

If ${}^{\smile_i}$ is a strong permutation, then, for all $R \subseteq \mathsf{Rel}$:

$$\varphi(R^{\smile_i}) = \varphi(R)^{\pi_i} \tag{23}$$

If $\diamond$ is a weak composition, then, for all $R_1, \ldots, R_m \subseteq \mathsf{Rel}$:

$$\diamond\,(R_1, \ldots, R_m) = \bigcap\{S \subseteq \mathsf{Rel} \mid \varphi(S) \supseteq \circ(\varphi(R_1), \ldots, \varphi(R_m))\} \tag{24}$$

If $\diamond$ is a strong composition, then, for all $R_1, \ldots, R_m \subseteq \mathsf{Rel}$:

$$\varphi(\diamond(R_1, \ldots, R_m)) = \circ(\varphi(R_1), \ldots, \varphi(R_m)) \tag{25}$$

PROOF. For (20), consider

$$\begin{aligned}
\varphi(R^{\smile i}) &= \bigcup_{r \in R} \varphi(r^{\smile i}) && \text{definition of } \varphi(R^{\smile i})\\
&\supseteq \bigcup_{r \in R} \varphi(r)^{\pi_i} && \text{property (6)}\\
&= \left(\bigcup_{r \in R} \varphi(r)\right)^{\pi_i} && \text{distributivity in set theory}\\
&= \varphi(R)^{\pi_i} && \text{definition of } \varphi(R).
\end{aligned}$$

For (21), consider

$$\begin{aligned}
\varphi(\diamond(R_1, \ldots, R_m)) &= \bigcup_{r_1 \in R_1} \cdots \bigcup_{r_m \in R_m} \varphi(\diamond(r_1, \ldots, r_m)) && \text{definition of } \varphi(\diamond(R_1, \ldots, R_m))\\
&\supseteq \bigcup_{r_1 \in R_1} \cdots \bigcup_{r_m \in R_m} \circ(\varphi(r_1), \ldots, \varphi(r_m)) && \text{property (7)}\\
&= \circ\left(\bigcup_{r_1 \in R_1} \varphi(r_1), \ldots, \bigcup_{r_m \in R_m} \varphi(r_m)\right) && \text{distributivity in set theory}\\
&= \circ(\varphi(R_1), \ldots, \varphi(R_m)) && \text{definition of } \varphi(R_i)
\end{aligned}$$

Properties (23) and (25) are proven using (9) and (11) in the same way as we have just proven (20) and (21) using (6) and (7).

For (22), let $R = \{r_1, \ldots, r_n\}$ for some $n \geqslant 1$ and $r_1, \ldots, r_n \in \mathsf{Rel}$. Due to Definition 3.4 (8), we have that

$$r_j^{\smile i} = \bigcap\{S \subseteq \mathsf{Rel} \mid \varphi(S) \supseteq \varphi(r_i)^{\pi_i}\}$$

for every $j = 1, \ldots, n$. Let $S_{j1}, \ldots, S_{jm_j}$ be the S over which the above intersection ranges, i.e.,

$$r_j^{\smile i} = \bigcap_{h=1}^{m_j} S_{jh}\,.$$

Due to Definition 3.3, we have that

$$R^{\smile i} \;=\; \bigcup_{j=1}^{n} r_j^{\smile i} \;=\; \bigcup_{j=1}^{n} \bigcap_{h=1}^{m_j} S_{jh} \;=\; \bigcap_{h_1=1}^{m_1} \cdots \bigcap_{h_n=1}^{m_n} \bigcup_{j=1}^{n} S_{jh_j}\,,$$

where the last equality is due to the distributivity of intersection over union. Now (22) follows if we show that, for every $S \in \mathsf{Rel}$, the following are equivalent.

(1) $\varphi(S) \supseteq \varphi(R)^{\pi_i}$

(2) there are $S_1, \ldots, S_n$ with $S = S_1 \cup \cdots \cup S_n$ and $\varphi(S_j) \supseteq \varphi(r_j)^{\pi_i}$ for every $j = 1, \ldots, n$.

For "$1 \Rightarrow 2$", assume $\varphi(S) \supseteq \varphi(R)^{\pi_i}$, i.e., $\varphi(S) \supseteq \bigcup_{j=1}^{n} \varphi(r_j)^{\pi_i}$ (Definition 3.3). If we further assume that $S = \{s_1, \ldots, s_\ell\}$, which implies that $\varphi(S) \supseteq \bigcup_{h=1}^{\ell} \varphi(s_h)$ (Definition 3.3), then we can choose $S_j = \{s_h \mid \varphi(s_h) \cap \varphi(r_j)^{\pi_i} \neq \emptyset\}$ for every $j = 1, \ldots, n$. Because C is based on JEPD relations, we have that $\varphi(S_j) \supseteq \varphi(r_j)^{\pi_i}$.

For "$2 \Rightarrow 1$", let $S = S_1 \cup \cdots \cup S_n$ and $\varphi(S_j) \supseteq \varphi(r_j)^{\pi_i}$ for every $j = 1, \ldots, n$. Due to Definition 3.3 and because C is based on JEPD relations, we have that $\varphi(S) = \bigcup_{j=1}^{n} \varphi(S_j)$. Hence, $\varphi(S) \supseteq \bigcup_{j=1}^{n} \varphi(r_j)^{\pi_i}$ via the assumption, and $\varphi(S) \supseteq \varphi(R)^{\pi_i}$ due to Definition 3.3.

(24) is proven analogously. □

D. EXPRESSIVITY RELATIONS BETWEEN CALCULI, FIGURE 5

We give additional proof sketches for expressivity relations presented in Figure 5. Recall that we say a calculus is of equivalent expressivity as another calculus if every QCSP instance of the first can be simulated by a propositional formulae of constraints in the second.

THEOREM D.1. *Temporal calculi PC, IA, SIC, DIA, GenInt and spatial calculi BA, CDC, and CI form a cluster of expressivity.*

PROOF SKETCH. Temporal point- and interval-based calculi (semi-intervals in case of SIC) represent ordering relations which can all be translated into Boolean formulae of PC relations among interval start and end point. Solutions for QCSPs over these temporal calculi in the cluster can easily be obtained from their corresponding PC formulae by instantiating intervals from their start and and points.

The spatial calculus BA is an independent product IA×IA easily expressible using propositional BA formulae, analogously is CDC expressible as product PC×PC. CI represents a cyclic order (e.g., intervals of longitude). These relations can be simulated with PC by instantiating an lower and upper limit points p_- and p_+ and splitting all intervals containing either p_- or p_+ to continue from the opposite border. □

THEOREM D.2. *VR relations can be expressed using LR constraints.*

PROOF SKETCH. VR expresses visibility of convex objects in the plane using ternary relations. Visibility relations can be represented based on the relative position of tangent points of the base entities, e.g., visibility between two objects is not affected if and only if a third object discrete from the first two does not intersect with the four-sided polygon obtained by connecting the upper and lower tangent points of the two objects. Overlap between polygonal contours can easily be written using LR constraints, e.g., a point is outside a convex polygon if it is located to the right hand side of at least one edge of the polygon, assuming the polygon edges to be ordered in counter-clockwise manner. The construction is then performed for every visibility relation, instantiating lower and upper tangent points individually for every pair of VR entities. The VR entities which are regions are then represented only by their set of tangent points which can be enforced to be arranged along a convex-shaped contour. Additional details are provided by Wolter and Lee [2016]. □

THEOREM D.3. *Calculi TPCC, OPRA, EOPRA, 1-, and 2-cross constitute a cluster of equal expressive power for Boolean combinations of constraints.*

PROOF SKETCH. This group of calculi considers locations of points in the Euclidean plane. We first consider equivalence of OPRA, 1-, and 2-cross and later address TPCC and EOPRA which augment the first group by additional distance concepts. All calculi from the first group employ a partition scheme that is based on relations that specifies directions to points relative to some entity-specific orientation (either by reference to another entity in case of 1-, and 2-cross or as intrinsic part of the base entity in case of OPRA). Directions measured in radians are represented by membership in a finite and JEPD set of intervals partitioning $(0, 2\pi]$, using solely rational ratios of π as boundaries. By geometric construction one can obtain any of these direction intervals (i.e., sectors) of these calculi from a any partition scheme for point location that is able to express superposition of points, a statement that two line segments connecting three points A, B, C meet in a right angle, i.e., $\angle(A, B, C) = \frac{\pi}{2}$ as well as a statement saying that a point is located directly in front of some point P with respect to "front" orientation of P All the named calculi meet these conditions and allow for the following construction: Let P be the entity for which we seek to construct direction intervals in form of a sector. First, enforce four points A, B, C, D to form a rectangle with A in superposition with P and C in front of P. Next we construct E to be positioned on the intersection of $\overline{AC}$ and $\overline{BD}$ which meet in a right angle. Doing so we have constructed a square. Repeating the construction we can construct a grid from which we can derive the desired angular sectors.

Now we show that OPRA, TPCC, and EOPRA have the same expressivity. EOPRA augments OPRA by a relative distance concept in the same way TPCC augments 1-cross. Constructions translating EOPRA to OPRA are very similar to translating TPCC to 1-cross, so we only consider the first case. Distance classes in the calculi OPRA and TPCC are named "close", "same", and "far" and are defined by comparison of the Euclidean distance between two entities with an object-specific threshold distance. This means that the statement "A is close to B" is independent from "B is close to A". These distance constraints can be simulated in OPRA by introducing *border* points for each entity along the "same" distance, one for every pair of entities. To this end we have to enforce that all border points are in the same distance to their corresponding entity. This can be accomplished by OPRA constraints by first constructing a bisector for a pair of border points (as done in the construction above) and, second, enforcing a right angle between the line connecting two border points with the bisector. Additional details are provided by Wolter and Lee [2016]. □

E. ADDITIONAL COMPLEXITY PROOFS FOR TABLE IV

Fact E.1. Consistency of QCSPs for DRA-conn can be decided in time $\mathcal{O}(n^3)$.

PROOF. The DRA-conn calculus is an abstraction of the more fine-grained dipole calculi, only retaining connectivity relations of line segments. Connectivity is represented by equality relations between positions of a dipole's start or end point. For checking consistency of a set of DRA-conn constraints, the clusters of equally positioned points need to be constructed. This can easily be done with the algebraic closure algorithm. Since the effect of a disjunctive relation in DRA-conn with respect to single point equality is identical to absence of the constraint, reasoning with partial atomic QCSPs is equivalent in complexity to reasoning with general QCSPs with DRA-conn. □

Fact E.2. Consistency of atomic QCSPs for EIA can be decided in polynomial time.

PROOF. As described by Zhang and Renz [2014], extended interval algebra constraints can be translated to INDU constraint networks, and those can be decided in polynomial time [Balbiani et al. 2006]. EIA describes relative ordering with respect to interval start, end, and center point. Consequently, for every single variable in a given EIA network, the translation introduces three variables representing an interval and its two halves, together with the obvious constraints between them. □

Fact E.3. The tractable subset of GenInt consisting of all strongly pre-convex general relations covers less than 1‰ of all relations for the case of 3-intervals.

PROOF. Generalized intervals [Condotta 2000] generalize IA relations to tuples of intervals. Relations between a p- and and a q-tuple, *general relations*, are represented in a $p \times q$ matrix of IA relations. A strongly pre-convex general relation is a matrix where all entries are strongly preconvex. Since the strongly pre-convex relations are a subset of pre-convex relations and only some 10% of all IA relations are pre-convex, at most a fraction of $0.1^{p \cdot q}$ of all general relations is strongly pre-convex, which is far less than 1‰ if $p = q = 3$. Even if we could take the matrix entries from a tractable subset of, say, 20% of IA, we would still get $0.2^{p \cdot q} \ll 1‰$ tractable relations. □

Fact E.4. Deciding consistency of atomic QCSPs for $\mathsf{OM\text{-}3D}$ is NP-hard and can be reduced to solving multivariate polynomial equalities.

PROOF. $\mathsf{OM\text{-}3D}$ generalizes the $\mathsf{double\text{-}cross}$ calculus from 2D arrangement to 3D arrangement, containing the 2D case as a sub-algebra. Since base relations of the 2D case are already NP-hard [Wolter and Lee 2010], so is $\mathsf{OM\text{-}3D}$. All base relations for the 3D case can be modeled by multivariate polynomial equalities similar to the 2D case. □

Fact E.5. Consistency of QCSPs with convex relations for STAR_m and STAR_m^r can be decided in polynomial time.

PROOF. STAR_m defines $4m$ relations (line segments and sectors); STAR_m^r defines $2m$ relations which are all sectors. Tractability of convex relations follows from the observation that these can be represented by half-plane intersections using linear inequalities, systems of which can be decided in polynomial time using linear programming techniques. □

While the number of all relations in $\mathsf{STAR}_m^{(r)}$ grows exponentially with m, there are only m convex relations that include $1, \ldots, m$ relations, i.e., $\mathcal{O}(m^2)$ convex relations. The percentage of convex relations thus decreases with increasing values of m.

F. ADDITIONAL PROOFS FOR SECTION 4.1

F.1. R_6 and R_{6l} from Table VI are equivalent given R_7 and R_9

We only show that R_6 implies R_{6l}; the converse direction is analogous. We first establish that $\mathsf{id}^\smile = \mathsf{id}$.

$$\begin{aligned}
\mathsf{id}^\smile &= \mathsf{id}^\smile \diamond \mathsf{id} && (\mathsf{R}_6)\\
&= \mathsf{id}^\smile \diamond (\mathsf{id}^\smile)^\smile && (\mathsf{R}_7)\\
&= (\mathsf{id}^\smile \diamond \mathsf{id})^\smile && (\mathsf{R}_9)\\
&= (\mathsf{id}^\smile)^\smile && (\mathsf{R}_6)\\
&= \mathsf{id} && (\mathsf{R}_7)
\end{aligned}$$

Now we use this lemma to establish R_{6l}.

$$\begin{aligned}
\mathsf{id} \diamond r &= (\mathsf{id}^\smile)^\smile \diamond (r^\smile)^\smile && (\mathsf{R}_7)\\
&= (r^\smile \diamond \mathsf{id}^\smile)^\smile && (\mathsf{R}_9)\\
&= (r^\smile \diamond \mathsf{id})^\smile && (\text{Lemma})\\
&= (r^\smile)^\smile && (\mathsf{R}_6)\\
&= r && (\mathsf{R}_7)
\end{aligned}$$

□

F.2. Proof of Fact 4.2

Fact 4.2. Every qualitative calculus (Def. 3.3) satisfies R_1–R_3, R_5, $\mathsf{R}_7^{\supseteq}$, R_8, $\mathsf{W}^{\supseteq}$, $\mathsf{S}^{\supseteq}$ for all (atomic and composite) relations. This axiom set is maximal: each of the remaining axioms in Table VI is not satisfied by some qualitative calculus.

PROOF. Axioms R_1–R_3 are always satisfied because they are a characterization of a Boolean algebra; and the set operations on the relations form a Boolean algebra because φ maps base relations to a set of JEPD relations and complex relations to sets of interpretations of base relations.

The definition of the converse and composition operations for non-base relations in Definition 3.3 ensures that Axioms R_5 and R_8 hold.

Axiom $\mathsf{R}_7^{\supseteq}$ always holds due to JEPD and the converse being weak: For every $r \in \mathsf{Rel}$, we have that

$$\varphi(r^{\smile\smile}) \supseteq \varphi(r^{\smile})^{\smile} \supseteq \varphi(r)^{\smile\smile} = \varphi(r),$$

where the first inclusion is due to Fact 3.6 (12) with $R = r^{\smile}$, the second inclusion is due to Definition 3.3 (6) for r, and the equation is due to the properties of binary relations over the universe $\mathcal{U}$. Since the $\varphi(r)$ are a set of JEPD relations, $r^{\smile\smile} \supseteq r$ follows. This reasoning carries over to arbitrary relations.

Axioms $\mathsf{W}^{\supseteq}$ and $\mathsf{S}^{\supseteq}$ always hold due to JEPD and the composition being weak: For every $r \in \mathsf{Rel}$, we have that

$$\varphi((r \diamond 1) \diamond 1) \supseteq \varphi(r \diamond 1) \circ \varphi(1) = \varphi(r \diamond 1) \circ (\mathcal{U} \times \mathcal{U}) \supseteq \varphi(r \diamond 1),$$

where the first inclusion is due to to Fact 3.6 (13) with $R = r \diamond 1$ and $S = 1$, and the last inclusion is due to the fact that $R \circ (\mathcal{U} \times \mathcal{U}) \supseteq R$ for any binary relation $R \subseteq \mathcal{U} \times \mathcal{U}$. Since the $\varphi(r)$ are a set of JEPD relations, $(r \diamond 1) \diamond 1 \supseteq r \diamond 1$ follows. Again, this reasoning carries over to arbitrary relations.

Axioms $\mathsf{R}_6^{\subseteq}$, $\mathsf{R}_{6l}^{\subseteq}$, $\mathsf{R}_7^{\subseteq}$ are violated by the following calculus. Let $\mathsf{Rel} = \{r_1, r_2\}$, $\mathcal{U} = \{0, 1\}$, $\mathsf{id} = r_1$, $1 = \{r_1, r_2\}$ with:

$$\begin{array}{lll}
\varphi(r_1) = \{(0,0),(0,1)\} & r_1^{\smile} = 1 & r_1 \diamond r_1 = 1 \\
\varphi(r_2) = \{(1,0),(1,1)\} & r_2^{\smile} = 1 & r_1 \diamond r_2 = r_1 \\
 & & r_2 \diamond r_1 = 1 \\
 & & r_2 \diamond r_2 = r_2
\end{array}$$

This calculus satisfies the conditions in Definition 3.3 but violates Axioms $\mathsf{R}_6^{\subseteq}$, $\mathsf{R}_{6l}^{\subseteq}$, $\mathsf{R}_7^{\subseteq}$:

$$\begin{array}{ll}
\mathsf{R}_6^{\subseteq} & r_1 \diamond \mathsf{id} = 1 \not\subseteq r_1 \\
\mathsf{R}_{6l}^{\subseteq} & \mathsf{id} \diamond r_1 = 1 \not\subseteq r_1 \\
\mathsf{R}_7^{\subseteq} & r_1^{\smile\smile} = 1 \not\subseteq r_1
\end{array}$$

Axioms $\mathsf{W}^{\subseteq}$, $\mathsf{S}^{\subseteq}$, $\mathsf{R}_4^{\subseteq}$, $\mathsf{R}_4^{\supseteq}$, $\mathsf{R}_6^{\supseteq}$, $\mathsf{R}_{6l}^{\supseteq}$, $\mathsf{R}_9^{\subseteq}$, $\mathsf{R}_9^{\supseteq}$, $\mathsf{R}_{10}^{\subseteq}$, $\mathsf{R}_{10}^{\supseteq}$, $\mathsf{PL}^{\Rightarrow}$, $\mathsf{PL}^{\Leftarrow}$ are violated by the following calculus. Let $\mathsf{Rel} = \{r_1, r_2, r_3, r_4\}$, $\mathcal{U} = \{0, 1\}$, $\mathsf{id} = r_1$, $1 = \{r_1, r_2\}$ with:

$$\begin{array}{ll}
\varphi(r_1) = \{(0,0)\} & r_1^{\smile} = r_1 \\
\varphi(r_2) = \{(1,1)\} & r_2^{\smile} = r_2 \\
\varphi(r_3) = \{(0,1)\} & r_3^{\smile} = r_4 \\
\varphi(r_4) = \{(1,0)\} & r_4^{\smile} = r_3
\end{array}$$

right operand / left operand $\diamond$	r_1	r_2	r_3	r_4
r_1	r_1	$\emptyset$	r_3	$\emptyset$
r_2	$\emptyset$	r_3	$\emptyset$	r_4
r_3	$\emptyset$	r_3	$\emptyset$	r_1, r_4
r_4	r_1, r_4	$\emptyset$	r_2	$\emptyset$

This calculus satisfies the conditions from Definition 3.3 but violates Axioms $\mathsf{W}^{\subseteq}$, $\mathsf{S}^{\subseteq}$, $\mathsf{R}_4^{\subseteq}$, $\mathsf{R}_4^{\supseteq}$, $\mathsf{R}_6^{\supseteq}$, $\mathsf{R}_{6l}^{\supseteq}$, $\mathsf{R}_9^{\subseteq}$, $\mathsf{R}_9^{\supseteq}$, $\mathsf{R}_{10}^{\subseteq}$, $\mathsf{R}_{10}^{\supseteq}$, $\mathsf{PL}^{\Rightarrow}$, $\mathsf{PL}^{\Leftarrow}$:

$$\mathsf{W}^{\subseteq}, \mathsf{S}^{\subseteq}: \quad (r_1 \diamond 1) \diamond 1 = 1 \not\subseteq \{r_1, r_3, r_4\} = r_1 \diamond 1$$

$$\mathsf{R}_4^{\subseteq}: \quad (r_1 \diamond r_3) \diamond r_4 = r_3 \diamond r_4 = \{r_1, r_4\} \not\subseteq r_1 = r_1 \diamond \{r_1, r_4\} = r_1 \diamond (r_3 \diamond r_4)$$

$$\mathsf{R}_4^{\supseteq}: \quad (r_4 \diamond r_3) \diamond r_4 = r_2 \diamond r_4 = r_4 \not\supseteq \{r_1, r_4\} = r_4 \diamond \{r_1, r_4\} = r_4 \diamond (r_3 \diamond r_4)$$

$$\mathsf{R}_6^{\supseteq}: \quad r_2 \diamond \mathsf{id} = r_2 \diamond r_1 = \emptyset \not\supseteq r_2$$

$$\mathsf{R}_{6l}^{\supseteq}: \quad \mathsf{id} \diamond r_2 = r_1 \diamond r_2 = \emptyset \not\supseteq r_2$$

$$\mathsf{R}_9^{\subseteq}, \mathsf{R}_9^{\supseteq}: \quad (r_3 \diamond r_4)^{\smile} = \{r_1, r_4\}^{\smile} = \{r_1, r_3\} \overset{\not\subseteq}{\not\supseteq} \{r_1, r_4\} = r_3 \diamond r_4 = r_4^{\smile} \diamond r_3^{\smile}$$

$$\mathsf{R}_{10}^{\subseteq}, \mathsf{R}_{10}^{\supseteq}: \quad r_3^{\smile} \diamond \overline{r_3 \diamond r_1} = r_4 \diamond \emptyset = r_4 \diamond 1 = \{r_1, r_2, r_4\} \overset{\not\subseteq}{\not\supseteq} \{r_2, r_3, r_4\} = \overline{r_1}$$

$$\mathsf{PL}^{\Rightarrow}: \quad (r_1 \diamond r_4) \cap r_1^{\smile} = \emptyset \cap r_1 = \emptyset \text{ but } (r_4 \diamond r_1) \cap r_1^{\smile} = \{r_4, r_1\} \cap r_1 = r_4 \neq \emptyset$$

$$\mathsf{PL}^{\Leftarrow}: \quad (r_4 \diamond r_1) \cap r_1^{\smile} = \{r_4, r_1\} \cap r_1 = r_1 \neq \emptyset \text{ but } (r_1 \diamond r_1) \cap r_4^{\smile} = r_1 \cap r_3 = \emptyset$$

□

Remark F.1. Of course, there are calculi that satisfy only the weak conditions from Definition 3.3 but are a relation algebra, for example the following. Let $\mathsf{Rel} = \{r_0, r_1\}$, $\mathcal{U} = \{0, 1\}$, $\mathsf{id} = r_1$, $1 = \{r_1, r_2\}$ with:

$\varphi(r_1) = \{(0,0),(0,1)\}$	$r_1^{\smile} = r_2$	$r_1 \diamond r_1 = r_1$
$\varphi(r_2) = \{(1,0),(1,1)\}$	$r_2^{\smile} = r_1$	$r_1 \diamond r_2 = 1$
		$r_2 \diamond r_1 = 1$
		$r_2 \diamond r_2 = r_2$

G. DETAILED DESCRIPTION OF THE TEST RESULTS IN SECTION 4.4

The results of the analysis are summarized in Table X. A part of the calculi have already been tested by Mossakowski [2007], using a different CASL specification based on an equivalent axiomatization from [Ligozat and Renz 2004]. He comprehensively reports on the outcome of these tests, and on errors discovered in published composition tables. We now list counterexamples for the cases where axioms are violated.

cCDR

— R_6 is violated for all base relations but one.
— R_{6l} is violated for only 209 base relations.
— R_7 is violated for 214 base relations.
— R_9 is violated for 5,607 pairs of base relations. Counterexample:

$$(\mathsf{S} \diamond \mathsf{S})^{\smile} \neq \mathsf{S}^{\smile} \diamond \mathsf{S}^{\smile}$$

— R_{10} is violated for 41,834 pairs of base relations. Counterexample:

$$\mathsf{S}^{\smile} \diamond \overline{\mathsf{S} \diamond \mathsf{S}} \not\subseteq \overline{\mathsf{S}}$$

— PL is violated for 22,976 triples of base relations. Counterexample:

$$(\text{W-NW-N-NE-E} \diamond \text{NW-N-NE}) \cap \text{B-S}^{\smile} = \{\} \neq \{\text{B}\} = (\text{NW-N-NE} \diamond \text{B-S}) \cap \text{W-NW-N-NE-E}^{\smile}$$

— R_4 is violated for 2,936,946 triples of base relations. Counterexample:

$$\begin{aligned} &\text{W-NW-N-NE-E-SE} \diamond (\text{W-NW-N-NE-E-SE} \diamond \text{W-NW-N-NE-E}) \\ &\quad \neq (\text{W-NW-N-NE-E-SE} \diamond \text{W-NW-N-NE-E-SE}) \diamond \text{W-NW-N-NE-E} \end{aligned}$$

Calculus	Tests[a]	R_4	S	W	R_6	R_{6l}	R_7	R_9	PL	R_{10}
BA_n, $n \leqslant 2$	HS	✓	✓	✓	✓	✓	✓	✓	✓	✓
CDC	MHS	✓	✓	✓	✓	✓	✓	✓	✓	✓
CYC_b	HS	✓	✓	✓	✓	✓	✓	✓	✓	✓
DRA_{fp}, DRA-conn	HS	✓	✓	✓	✓	✓	✓	✓	✓	✓
IA	MHS	✓	✓	✓	✓	✓	✓	✓	✓	✓
PC_1	HS	✓	✓	✓	✓	✓	✓	✓	✓	✓
RCC-5, DepCalc	MHS	✓	✓	✓	✓	✓	✓	✓	✓	✓
RCC-8, 9-int	MHS	✓	✓	✓	✓	✓	✓	✓	✓	✓
STAR_4^r	HS	✓	✓	✓	✓	✓	✓	✓	✓	✓
DRA_f	MHS	19	✓	✓	✓	✓	✓	✓	✓	✓
INDU	MHS	12	✓	✓	✓	✓	✓	✓	✓	✓
OPRA_n, $n \leqslant 8$	MHS	21–91[b]	✓	✓	✓	✓	✓	✓	✓	✓
QTC-Bxx	MHS	✓	✓	✓	89–100		✓	✓	✓	✓
QTC-C21	HS	55	✓	✓	99	99	✓	2	<1	1
QTC-C22	HS	79	✓	✓	99	99	✓	3	<1	1
RCD	HS	✓	✓	✓	97	92	89	66	7	52
cCDR	HS	28	17	✓	99	99	98	12	<1	88

[a]calculus was tested by: M = [Mossakowski 2007], H = *Hets*, S = *SparQ*

[b]21%, 69%, 78%, 83%, 86%, 88%, 90%, 91% for OPRA_n, $n = 1, \ldots, 8$

Table X: Overview of calculi tested and their properties. The symbol "✓" means that the axiom is satisfied; otherwise the percentage of counterexamples (relations, pairs or triples violating the axiom) is given.

— S is violated for 38 base relations. Counterexample:

$$(\mathsf{B\text{-}S\text{-}W\text{-}NW} \diamond 1) \diamond 1 \neq \mathsf{B\text{-}S\text{-}W\text{-}NW} \diamond 1$$

DRA

— DRA_c violates R_4 for 704 triples of base relations. Counterexample:

$$\mathsf{rrrl} \diamond (\mathsf{rrrl} \diamond \mathsf{llrl}) \neq (\mathsf{rrrl} \diamond \mathsf{rrrl}) \diamond \mathsf{llrl}$$

— DRA_f violates R_4 for 71,424 triples of base relations, with the same counterexample, or with the one reported by Moratz et al. [2011], who attribute the violation of associativity to the composition operation being weak and illustrate this by the example $\mathsf{bfii} \diamond \mathsf{lllb} = \mathsf{llll}$.

— DRA_{fp} and DRA-conn satisfy all axioms.

INDU

R_4 is violated by 1,880 triples of base relations. The violation of associativity has already been reported and attributed to the absence of strong composition in [Balbiani et al. 2006]: e.g.,

$$\mathsf{bi}^{>} \diamond (\mathsf{mi}^{>} \diamond \mathsf{m}^{>}) \neq (\mathsf{bi}^{>} \diamond \mathsf{mi}^{>}) \diamond \mathsf{m}^{>}.$$

MC-4

MC-4 is not based on a partition scheme because the relation cg ("congruent"), which behaves in the context of the other three relations as if it were an identity relation, is coarser than id^2. Furthermore, MC-4 is still an abstract partition scheme and thus fits into our general notion of a calculus.

For testing purposes, we have implemented an artificial variant of MC-4 where we divided the cg relation into id^2 and the difference of cg and id^2. That calculus too is a relation algebra.

OPRA_n, $n \leqslant 8$

R_4 is violated by

1,664 triples for OPRA_1,	e.g., $3_3 \diamond (3_2 \diamond 0_3) \neq (3_3 \diamond 3_2) \diamond 0_3$
257,024 triples for OPRA_2,	e.g., $7_7 \diamond (7_7 \diamond 6_7) \neq (7_7 \diamond 7_7) \diamond 6_7$
2,963,952 triples for OPRA_3,	e.g., $11_{11} \diamond (11_{11} \diamond 11_{10}) \neq (11_{11} \diamond 11_{11}) \diamond 11_{10}$
16,711,680 triples for OPRA_4,	e.g., $15_{15} \diamond (15_{15} \diamond 15_{15}) \neq (15_{15} \diamond 15_{15}) \diamond 15_{15}$
63,840,000 triples for OPRA_5,	e.g., $19_{19} \diamond (19_{19} \diamond 19_{19}) \neq (19_{19} \diamond 19_{19}) \diamond 19_{19}$
190,771,200 triples for OPRA_6,	e.g., $23_{23} \diamond (23_{23} \diamond 23_{23}) \neq (23_{23} \diamond 23_{23}) \diamond 23_{23}$
481,275,648 triples for OPRA_7,	e.g., $27_{27} \diamond (27_{27} \diamond 27_{27}) \neq (27_{27} \diamond 27_{27}) \diamond 27_{27}$
1,072,693,248 triples for OPRA_8,	e.g., $31_{31} \diamond (31_{31} \diamond 31_{31}) \neq (31_{31} \diamond 31_{31}) \diamond 31_{31}$

QTC

- QTC-B11, -B12, -C21, -C22 violate R_6 and R_{6l} for all base relations but one; QTC-B21, -B22 do so for all base relations. After introducing a new id relation and making the relations JEPD, QTC-B11 and -B12 satisfy all axioms [Mossakowski 2007].
- QTC-C21 (81 base relations) violates R_4 for 292,424 triples, R_9 for 160 pairs, R_{10} for 80 pairs, and PL for 1056 triples.[5]
- QTC-C22 (209 base relations) violates R_9 for 1248 pairs, R_{10} for 624 pairs, PL for 12,768 triples, and R_4 for 7,201,800 triples, see also footnote 5.

RCD

- R_6 is violated for all base relations but one.
- R_{6l} is violated for only 33 base relations.
- R_7 is violated for 32 base relations.
- R_9 is violated for 855 pairs. Counterexample:

$$(\mathsf{B} \diamond \mathsf{S{:}SW})^{\smile} \neq \mathsf{S{:}SW}^{\smile} \diamond \mathsf{B}^{\smile}$$

- R_{10} is violated for 671 pairs. Counterexample:

$$\mathsf{B}^{\smile} \diamond \overline{\mathsf{B} \diamond \mathsf{S{:}SW}} \not\subseteq \overline{\mathsf{S{:}SW}}$$

- PL is violated for 3424 triples. Counterexample:

$$(\mathsf{B} \diamond \mathsf{N}) \cap \mathsf{B{:}W}^{\smile} = \emptyset \not\Leftrightarrow (\mathsf{N} \diamond \mathsf{B{:}W}) \cap \mathsf{B}^{\smile} = \emptyset$$

[5]Note that, for calculi that violate R_9, the equivalence between PL and R_{10} is no longer ensured, hence the mentioning of both of them. Furthermore, R_{10} is the only axiom that should be tested for all relations, but we have only tested it for all base relations. Therefore, there could be more violations than the four listed. The same cautions apply to QTC-C22.