Game-theoretic Semantics for ATL$^+$ with Applications to Model Checking

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Abstract

We develop a game-theoretic semantics (GTS) for the fragment ATL$^+$ of the Alternating-time Temporal Logic ATL*, essentially extending a recently introduced GTS for ATL. We show that the new game-theoretic semantics is equivalent to the standard compositional semantics of ATL$^+$ (with perfect-recall strategies). Based on the new semantics, we provide an analysis of the memory and time resources needed for model checking ATL$^+$ and show that strategies of the verifier that use only a very limited amount of memory suffice. Furthermore, using the GTS we provide a new algorithm for model checking ATL$^+$ and identify a natural hierarchy of tractable fragments of ATL$^+$ that extend ATL.

1 Introduction

The full Alternating-time Temporal Logic ATL* [3] is one of the main logical systems used for formalising and verifying strategic reasoning about agents in multi-agent systems. It is very expressive, and that expressiveness comes at the high (2-EXPTIME) price of computational complexity of model checking. Its basic fragment ATL (the multi-agent analogue of CTL) has, on the other hand, tractable model checking but its expressiveness is rather limited. In particular, ATL only allows expressing strategic objectives of the type $\langle\!\langle A \rangle\!\rangle\, \Phi$ where Φ is a simple temporal goal involving a single temporal operator applied directly to state sub-formula(e). Thus ATL cannot express multiple simultaneous temporal goals. The intermediate fragment ATL$^+$ naturally emerges as a good alternative, extending ATL so that it is possible to directly express strategic objectives which are Boolean combinations of simple temporal goals. The price for this is the reasonably higher computational complexity of model checking ATL$^+$, viz. PSPACE-completeness [5]. Still, the PSPACE-completeness result alone gives a rather crude estimate of the amount of memory needed for model checking ATL$^+$.

In this paper we take an alternative approach to semantic analysis and model checking of some fragments of ATL*, in particular of ATL$^+$, based not on the standard compositional semantics but on game-theoretic semantics GTS. The main aims and contributions of this paper are three-fold:

1. We introduce an adequate GTS for ATL$^+$ equivalent to the standard compositional semantics.

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2. We propose new model checking algorithms for ATL^+ and some of its fragments, using the GTS developed here, rather than the standard semantics. We also analyse the use of memory resources in ATL^+ via GTS.
3. We apply the GTS-based approach to model checking in order to identify new tractable fragments of ATL^+.

The main part of the paper consists of a detailed presentation and analysis of the new GTS for ATL^+, which we prove to be equivalent to the standard compositional semantics with perfect-recall strategies. While our proposal looks similar in spirit to the recently introduced in [11] GTS for ATL, it is based on a range of new technical ideas and mechanisms needed for the correct evaluation of multiple temporal goals pursued simultaneously by the proponent coalition. We prove—as shown for ATL in [11]—that our GTS for ATL^+ ensures (even on infinite models) that it is always sufficient to construct *finite paths only* when formulae are evaluated. While the proof for ATL^+ is substantially more complex, the technical machinery we use is in many respects lighter, we claim, than the one used in [11].

The approach via GTS enables us, inter alia, to perform a more precise analysis on the memory resources needed in evaluating ATL^+-formulae than the algorithm from [5] which employs a mix of a path construction procedure for checking strategic formulae $\langle\langle A\rangle\rangle\,\Phi$ on one hand, and the standard labelling algorithm on the other hand. Our model checking algorithm for ATL^+ follows uniformly a procedure directly based on GTS and in fact enables us, inter alia, to identify and correct a flaw in the model checking procedure of [5] and some of the claims on which it is based (see Section 5). However, the PSPACE upper bound result of [5] is easily confirmed by our algorithm, and we provide a new simple proof of that result. In addition to new methods, we use some ideas from [5].

As a new complexity result obtained via GTS, we identify in Section 5 a natural hierarchy of fragments of ATL^+ that extend ATL and have a *tractable* model checking. The hierarchy is based on bounding the *Boolean strategic width* (cf. Section 5) of formulae.

We note that a GTS for ATL^+ alternative to ours could be obtained via a GTS for coalgebraic fixed point logic [18, 9, 10], but such a semantics (being designed for more powerful logics) would not directly lead to our GTS that is custom-made for ATL^+ and thereby enables the complexity analysis we require. Also, the alternative approach would not give a semantics where the construction of only finite paths suffices.

The current paper expands the results in [11] in various non-trivial ways. Several new ideas and technical notions, such as the *role of a seeker* and the use of a *truth function*, will be introduced in order to enable the transition from ATL to ATL^+ in the GTS setting. Also, a connection of our GTS with Büchi games will be established; the connection applies trivially also to the games of [11]. Most importantly, we can directly use the new upgraded semantics in a model checking procedure for ATL^+ and the fragments ATL^k. This would not be possible with with the semantics in [11].

We mention here a few other relevant works with respect to ATL and its extensions: [1, 20, 8, 2, 15, 17, 6].

2 Preliminaries

Definition 2.1. *A* ***concurrent game model*** *(*CGM*) is* $\mathcal{M} := (\mathbb{A}\mathrm{gt}, \mathrm{St}, \Pi, \mathrm{Act}, d, o, v)$ *which consists of:*
– The following non-empty sets: ***agents*** $\mathbb{A}\mathrm{gt} = \{a_1, \ldots, a_k\}$*,* ***states*** St*,* ***proposition symbols*** Π*,* ***actions*** Act*;*
– The following functions: an ***action function*** d *(s.t.* $d : \mathbb{A}\mathrm{gt} \times \mathrm{St} \to \mathcal{P}(\mathrm{Act}) \setminus \{\emptyset\}$*)*

which assigns a non-empty set of actions available to each agent at each state; a ***transition function*** *o which assigns an* ***outcome state*** *$o(q, \vec{\alpha})$ to each state $q \in \mathrm{St}$ and action profile (a tuple of actions $\vec{\alpha} = (\alpha_1, \ldots, \alpha_k)$ such that $\alpha_i \in d(a_i, q)$ for each $a_i \in \mathbb{A}\mathrm{gt}$); and finally, a* ***valuation function*** *$v : \Pi \to \mathcal{P}(\mathrm{St})$.*

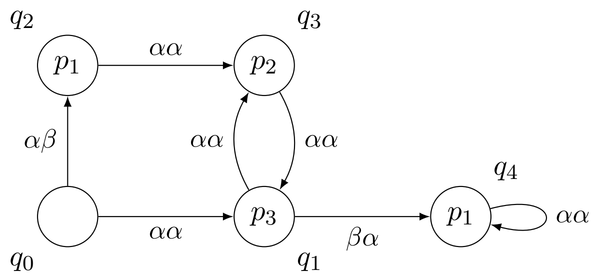
We use symbols $p, p_0, p_1, \ldots$ to denote proposition symbols and $q, q_0, q_1, \ldots$ to denote states. Sets of agents are called ***coalitions****. The complement $\overline{A} = \mathbb{A}\mathrm{gt} \setminus A$ of a coalition A is the* ***opposing coalition*** *of A. The set $\mathsf{action}(A, q)$ of action tuples* available *to coalition A at state $q \in \mathrm{St}$ is defined as*

$$\mathsf{action}(A, q) := \{(\alpha_i)_{a_i \in A} \mid \alpha_i \in d(a_i, q) \textit{ for each } a_i \in A\}.$$

Example 2.2. *Let $\mathcal{M}^* = (\mathbb{A}\mathrm{gt}, \mathrm{St}, \Pi, \mathrm{Act}, d, o, v)$, where:*

$$\begin{aligned}
&\mathbb{A}\mathrm{gt} = \{a_1, a_2\},\ \mathrm{St} = \{q_0, q_1, q_2, q_3, q_4\}\\
&\Pi = \{p_1, p_2, p_3\},\ \mathrm{Act} = \{\alpha, \beta\}\\
&d(a_2, q_0) = d(a_1, q_1) = \{\alpha, \beta\} \textit{ and else } d(a_i, q_i) = \{\alpha\}\\
&o(q_0, \alpha\alpha) = q_1,\ o(q_0, \alpha\beta) = q_2,\ o(q_1, \alpha\alpha) = q_2,\\
&o(q_1, \alpha\beta) = q_3,\ o(q_2, \alpha\alpha) = q_1 \textit{ and } o(q_3, \alpha\alpha) = q_3\\
&v(p_1) = \{q_2, q_4\},\ v(p_2) = \{q_3\} \textit{ and } v(p_3) = \{q_1\}.
\end{aligned}$$

$\mathcal{M}^*$:



Definition 2.3. *Let $\mathcal{M} = (\mathbb{A}\mathrm{gt}, \mathrm{St}, \Pi, \mathrm{Act}, d, o, v)$ be a* CGM*. A* ***path*** *in $\mathcal{M}$ is a sequence $\Lambda : \mathbb{N} \to \mathrm{St}$ of states such that for each $n \in \mathbb{N}$, we have $\Lambda[n{+}1] = o(\Lambda[n], \vec{\alpha})$ for some admissible action profile $\vec{\alpha}$ in $\Lambda[n]$. A* ***finite path*** *(aka* ***history****) is a finite prefix sequence of a path in $\mathcal{M}$. We let $\mathsf{paths}(\mathcal{M})$ denote the set of all paths in $\mathcal{M}$ and $\mathsf{paths}_{\mathit{fin}}(\mathcal{M})$ the set of all finite paths in $\mathcal{M}$.*

A ***positional strategy*** *of an agent $a \in \mathbb{A}\mathrm{gt}$ is a function $s_a : \mathrm{St} \to \mathrm{Act}$ such that $s_a(q) \in d(a, q)$ for each $q \in \mathrm{St}$. A (perfect-recall)* ***strategy*** *of agent $a \in \mathbb{A}\mathrm{gt}$ is a function $s_a : \mathsf{paths}_{\mathit{fin}}(\mathcal{M}) \to \mathrm{Act}$ such that $s_a(\lambda) \in d(a, \lambda[k])$ for each $\lambda \in \mathsf{paths}_{\mathit{fin}}(\mathcal{M})$ where $\lambda[k]$ is the last state in λ. A* ***collective strategy*** *S_A for $A \subseteq \mathbb{A}\mathrm{gt}$ is a tuple of individual strategies, one for each agent in A. We let $\mathsf{paths}(q, S_A)$ denote the set of all paths that can be formed when the agents in A play according to a strategy S_A, beginning from q. In this paper, we only consider the perfect information scenario.*

The syntax of ATL^+ is given by the following grammar.

State formulae: $\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle\!\langle A \rangle\!\rangle\, \Phi$

Path formulae: $\Phi ::= \varphi \mid \neg\Phi \mid \Phi \vee \Phi \mid \mathsf{X}\,\varphi \mid \varphi\, \mathsf{U}\, \varphi$

where $p \in \Pi$. Other Boolean connectives are defined as usual, and furthermore, $\mathsf{F}\,\varphi$, $\mathsf{G}\,\varphi$ and $\varphi\, \mathsf{R}\, \psi$ are abbreviations for $\top\, \mathsf{U}\, \varphi$, $\neg(\top\, \mathsf{U}\, \neg\varphi)$, and $\neg(\neg\varphi\, \mathsf{U}\, \neg\psi)$ respectively. We use Φ and Ψ to denote path formulae only; φ, ψ, χ will be freely used to denote state and path formulae.

Definition 2.4. *Let $\mathcal{M}$ be a* CGM*.* ***Truth*** *of state and path formulae of* ATL^+ *is defined, respectively, with respect to states $q \in St$ and paths $\Lambda \in \mathsf{paths}(\mathcal{M})$, as follows:*

- $\mathcal{M}, q \models p$ *iff* $q \in v(p)$ *(for $p \in \Pi$).*
- $\mathcal{M}, q \models \neg\varphi$ *iff* $\mathcal{M}, q \not\models \varphi$.
- $\mathcal{M}, q \models \varphi \vee \psi$ *iff* $\mathcal{M}, q \models \varphi$ *or* $\mathcal{M}, q \models \psi$.
- $\mathcal{M}, q \models \langle\!\langle A \rangle\!\rangle\, \Phi$ *iff there exists a (perfect-recall) strategy S_A such that $\mathcal{M}, \Lambda \models \Phi$ for each $\Lambda \in \mathsf{paths}(q, S_A)$.*
- $\mathcal{M}, \Lambda \models \varphi$ *iff* $\mathcal{M}, \Lambda[0] \models \varphi$ *(where φ is a state formula).*
- $\mathcal{M}, \Lambda \models \mathsf{X}\,\varphi$ *iff* $\mathcal{M}, \Lambda[1] \models \varphi$.
- $\mathcal{M}, \Lambda \models \neg\Phi$ *iff* $\mathcal{M}, \Lambda \not\models \Phi$.
- $\mathcal{M}, \Lambda \models \Phi \vee \Psi$ *iff* $\mathcal{M}, \Lambda \models \Phi$ *or* $\mathcal{M}, \Lambda \models \Psi$.
- $\mathcal{M}, \Lambda \models \varphi\, \mathsf{U}\, \psi$ *iff there exists $i \in \mathbb{N}$ such that $\mathcal{M}, \Lambda[i] \models \psi$ and $\mathcal{M}, \Lambda[j] \models \varphi$ for all $j < i$.*

The **set of subformulae**, $\mathrm{SUB}(\varphi)$, of a formula φ is defined as usual. Subformulae with a temporal operator as the main connective will be called **temporal subformulae**, while subformulae with $\langle\!\langle\rangle\!\rangle$ as the main connective are **strategic subformulae**. The subformula Ψ of a formula $\varphi = \langle\!\langle A \rangle\!\rangle\, \Psi$ is called the **temporal objective of** φ. We also define the set $At(\Phi)$ of **relative atoms** of Φ as follows:

- $At(\chi \vee \chi') = At(\chi) \cup At(\chi')$ and $At(\neg\chi) = At(\chi)$.
- $At(\langle\!\langle A \rangle\!\rangle\, \chi) = \{\langle\!\langle A \rangle\!\rangle\, \chi\}$ and $At(p) = \{p\}$ for $p \in \Pi$.
- $At(\chi\, \mathsf{U}\, \chi') = \{\chi\, \mathsf{U}\, \chi'\}$ and $At(\mathsf{X}\, \chi) = \{\mathsf{X}\, \chi\}$.

We say that $\chi \in At(\Phi)$ occurs **positively** (resp. **negatively**) **in** Φ if χ has an occurrence in the scope of an even (resp. odd) number of negations in Φ. We denote by $\mathrm{SUB}_{At}(\Phi)$ the subset of $\mathrm{SUB}(\Phi)$ that contains all the relative atoms of Φ and also all the Boolean combinations χ of these relative atoms such that $\chi \in \mathrm{SUB}(\Phi)$.

Example 2.5. *Let* $\varphi^* := \langle\!\langle a_1 \rangle\!\rangle\, \Psi$*, where*

$$\Psi := (\neg \mathsf{X}\, p_3 \wedge \langle\!\langle a_2 \rangle\!\rangle\, \mathsf{X}\, p_1) \;\vee\; (\mathsf{F}\, p_1 \wedge (\neg p_1)\, \mathsf{U}\, p_2).$$

Note that φ^* is an ATL^+ *formula. Written without using abbreviations, Ψ becomes*

$$\neg(\neg\neg \mathsf{X}\, p_3 \vee \neg\langle\!\langle a_2 \rangle\!\rangle\, \mathsf{X}\, p_1) \vee \neg(\neg(\top\, \mathsf{U}\, p_1) \vee \neg((\neg p_1)\, \mathsf{U}\, p_2)).$$

Here $At(\Psi) = \{\mathsf{X}\, p_3, \langle\!\langle a_2 \rangle\!\rangle\, \mathsf{X}\, p_1, \top\, \mathsf{U}\, p_1, (\neg p_1)\, \mathsf{U}\, p_2\}$, where $\langle\!\langle a_2 \rangle\!\rangle\, \mathsf{X}\, p_1$ is a state formula and the rest are path formulae. The formula $\mathsf{X}\, p_3$ occurs negatively in Ψ and the rest of the formulae in $At(\Psi)$ occur positively in Ψ.

3 Game-theoretic semantics

In this section we define *bounded*, *finitely bounded* and *unbounded evaluation games* for ATL^+. These games give rise to three different semantic systems, namely, the *bounded*, *finitely bounded* and *unbounded* GTS for ATL^+. We use some terminology and notational conventions introduced in [11].

3.1 Evaluation games: informal description

Given a CGM $\mathcal{M}$, a state q_{in} and a formula φ, the **evaluation game** $\mathcal{G}(\mathcal{M}, q_{in}, \varphi)$ is, intuitively, a formal debate between two opponents, **Eloise** (**E**) and **Abelard** (**A**), about whether the formula φ is true at the state q_{in} in the model $\mathcal{M}$. Eloise claims that φ is true, so she (initially) adopts the role of a **verifier** in the game, and Abelard tries to prove the formula false, so he is (initially) the **falsifier**. These roles (verifier, falsifier) can swap in the course of the game when negations are encountered in the formula. If $\mathbf{P} \in \{\mathbf{E}, \mathbf{A}\}$, then $\overline{\mathbf{P}}$ denotes the **opponent** of **P**, i.e., $\overline{\mathbf{P}} \in \{\mathbf{E}, \mathbf{A}\} \setminus \{\mathbf{P}\}$.

We now provide an intuitive account of the *bounded* evaluation game and thus the *bounded* GTS for ATL^+. The intuitions underlying the finitely bounded and unbounded GTS are similar. A reader unfamiliar with the concept of GTS may find it useful to consult, for example, [13] for GTS in general, or [11] for ATL-specific GTS. The particular GTS for ATL^+ presented here follows the general principles of GTS, the main original feature here being the treatment of strategic formulae $\langle\!\langle A \rangle\!\rangle\, \Phi$. We first give an *informal* account of the way such formulae are treated in our evaluation games. Formal definitions and some concrete examples will be given further, beginning from Section 3.2.

The evaluation of ATL^+ formulae of the type $\langle\!\langle A \rangle\!\rangle\, \Phi$ in a given model is based on constructing finite paths in that model. The following two main ideas are central.

Firstly, the path formula Φ in $\langle\!\langle A \rangle\!\rangle\, \Phi$ can be *divided* into *goals for the verifier* (**V**), these being the relative atoms $\psi \in At(\Phi)$ that occur *positively* in Φ, and *goals for the falsifier* ($\overline{\mathbf{V}}$), these being the relative atoms $\psi \in At(\Phi)$ that occur *negatively* in Φ. (Some formulae may be goals for both players.) For simplicity, let us assume for now that Φ is in negation normal form and all the atoms in $At(\Phi)$ are temporal formulae of the type $\mathsf{F}\, p$. Then the verifier's goals are eventuality statements $\mathsf{F}\, p$, while the falsifier's goals are statements $\mathsf{F}\, p'$ that occur negated, and thus correspond to safety statements $\mathsf{G}\, \neg p'$. The verifier wishes to *verify* her/his goals. The falsifier, likewise, wants to *verify* her/his goals, i.e., (s)he wishes to *falsify* the related safety statements.

Secondly, every temporal goal has a unique "finite determination point" on any given path, meaning the following. If a goal $\mathsf{F}\, p$ is true on an infinite path π, then there exists an earliest point q on that path where the fact that $\mathsf{F}\, p$ holds on π becomes *verified* simply because p is true at q. Once $\mathsf{F}\, p$ has been verified, it will remain *true on* π, no matter what happens on the path after q. Similarly, if a statement $\mathsf{G}\, \neg p'$ is false on an infinite path, there is a unique point where $\mathsf{G}\, \neg p'$ first becomes *falsified.* Furthermore, $\mathsf{G}\, \neg p'$ will remain false on the path no matter what happens later on. (Note that there is no analogous finite determination point for ATL^*-formulae such as $\langle\!\langle A \rangle\!\rangle\, \mathsf{G}\, \mathsf{F}\, p$ on a given infinite path.)

Now, the game-theoretic evaluation procedure of an ATL^+-formula $\langle\!\langle A \rangle\!\rangle\, \Phi$ proceeds roughly as follows. The verifier is controlling the agents in the coalition A and the falsifier the agents in the **opposing coalition** $\overline{A} = \mathbb{A}\mathrm{gt} \setminus A$. The players start constructing a path. (Each transition from one state to another is carried out according to the process "$\mathsf{Step\ phase}$" defined formally in Section 3.2.2.) The verifier is first given a change to verify some of her/his goals in Φ. The falsifier tries to prevent this and to possibly verify some her/his own goals instead. During this path construction/verification process, the verifier is said to have the role of the **seeker**. A player is allowed to stay as the seeker for only a finite number of rounds. This is ensured by requiring the seeker to announce an ordinal, called **timer**, [1] before the path construction process begins, and then lower the ordinal each time a new state is reached. The process ends when the ordinal

[1]Note that the term "timer" is used here differently from [11].

becomes zero or when the seeker is satisfied, having verified some of her/his goals. Since ordinals are well-founded, the process must terminate.

After the verifier has ended her/his seeker turn, the falsifier may either end the game or take the role of the seeker. If (s)he decides to become the seeker, then (s)he sets a new timer and the path construction process continues for some finite number of rounds. When the falsifier is satisfied, having verified some of her/his goals, the verifier may again take the seeker's role, and so on. Thus the verifier and falsifier take turns being the seeker, trying to reach (verify) their goals. The number of these alternations is bounded by a **seeker turn counter** which is a finite number that equals the total number of goals in Φ. (The formal description of seeker turn alternation is given in "Deciding whether to continue and adjusting the timer" in Section 3.2.2.)

Each time a goal in Φ becomes verified, this is recorded in a **truth function** T. (The recording of verified goals is described formally in the process "Adjusting the truth function" defined in Section 3.2.2.) The truth function carries the following information at any stage of the game:

- The verifier's goals that have been verified.
- The falsifier's goals that have been verified.
- The goals that are not mentioned above remain **open**.

When neither of the players wants to become the seeker, or when the seeker turn counter becomes zero, the path construction process *ends* and the players play a standard Boolean evaluation game on Φ by using the values given by T; the open goals are then given truth values as follows:

- The verifier's open goals are *(so far) not verified* and thus considered *false.*
- Likewise, the falsifier's open goals are *(so far) not verified* and thus considered *false.* (Recall that the falsifiers goals are negated.)

Next we consider the conditions when a player is "satisfied" with the current status of the truth function T—and thus wants to end the game—and when (s)he is "unsatisfied" and wants to continue the game as the seeker. Note that when the path construction ends, then every goal is given a Boolean truth value based on the truth function T, as described above. With these values, the formula Φ is either true or false. If Φ is true with the current values based on T, then the verifier can win the Boolean game for Φ; dually, if Φ is not true with the values based on T, then the falsifier can win the Boolean game for Φ. Hence the players want to take the role of the seeker in order to modify the truth function T in such a way that the truth of Φ with respect to T changes from false to true (whence $\mathbf{V}$ is satisfied) or from true to false (whence $\overline{\mathbf{V}}$ is satisfied).

The thruth value of Φ with respect to T can keep changing when T is modified, but only a finite number of changes is possible. Indeed, the maximum number of such truth alternations is the total number of goals in Φ.

In the general case, formulae of the type $\varphi \,\mathsf{U}\, \psi$, $\mathsf{X}\,\varphi$ and state formulae φ may also occur in $At(\Phi)$ as goals, and Φ does not have to be in negation normal form. Formulae of the type $\varphi \,\mathsf{U}\, \psi$ can be either verified, by showing that ψ is true, or falsified, by showing that φ is not true at respective states. State formulae φ can only be verified at the initial state and the nexttime formulae $\mathsf{X}\,\varphi$ can only be verified at the second state on the path.

3.2 Evaluation games: formal description

Now we will present the **bounded evaluation game** which uses the **bounded transition game** as a subgame for evaluating strategic subformulae. Interleaved with the definition we will provide, in *italics*, a running example that uses $\mathcal{M}^*$ and φ^* from Examples 2.2 and 2.5 respectively.

3.2.1 Rules of the bounded evaluation game

Let $\mathcal{M} = (\mathbb{A}\mathrm{gt}, \mathrm{St}, \Pi, \mathrm{Act}, d, o, v)$ be a CGM, $q_{in} \in \mathrm{St}$ a state, φ a state formula and $\Gamma > 0$ an ordinal called a **timer bound**. The **Γ-bounded evaluation game** $\mathcal{G}(\mathcal{M}, q_{in}, \varphi, \Gamma)$ between the players **A** and **E** is defined as follows.

A **location** of the game is a tuple $(\mathbf{P}, q, \psi, T)$ where $\mathbf{P} \in \{\mathbf{A}, \mathbf{E}\}$, $q \in \mathrm{St}$ is a state, ψ is a subformula of φ and T is a **truth function**, mapping some subset of $\mathrm{SUB}(\varphi)$ into $\{\top, \bot, \mathsf{open}\}$. ($T$ can also be called a *truth history function.*)

The **initial location** of the game is $(\mathbf{E}, q_{in}, \varphi, T_{in})$, where T_{in} is the empty function. In every location $(\mathbf{P}, q, \psi, T)$, the player $\mathbf{P}$ is called the **verifier** and $\overline{\mathbf{P}}$ the **falsifier** for that location. Intuitively, q is the current state of the game and T encodes truth values of formulae on a path that has been constructed earlier in the game.

Each location is associated with exactly one of the rules **1–6** given below. First we provide the rules for locations $(\mathbf{P}, q, \psi, T)$ where ψ is either a proposition symbol or has a Boolean connective as its main operator:

1. A location $(\mathbf{P}, q, p, T)$, where $p \in \Pi$, is an **ending location** of the evaluation game. If $T \neq \emptyset$, then $\mathbf{P}$ wins the game if $T(p) = \top$ and else $\overline{\mathbf{P}}$ wins. Respectively, if $T = \emptyset$, then $\mathbf{P}$ wins if $q \in v(p)$ and else $\overline{\mathbf{P}}$ wins.

2. From a location $(\mathbf{P}, q, \neg\psi, T)$ the game moves to the location $(\overline{\mathbf{P}}, q, \psi, T)$.

3. In a location $(\mathbf{P}, q, \psi \vee \theta, T)$ the player $\mathbf{P}$ chooses one of the locations $(\mathbf{P}, q, \psi, T)$ and $(\mathbf{P}, q, \theta, T)$, which becomes the next location of the game.

We then define the rules of the evaluation game for locations of type $(\mathbf{P}, q, \langle\!\langle A\rangle\!\rangle\, \Phi, T)$ as follows.

4. Suppose a location $(\mathbf{P}, q, \langle\!\langle A\rangle\!\rangle\, \Phi, T)$ is reached.

- If $T \neq \emptyset$, then this location is an ending location where $\mathbf{P}$ wins if $T(\langle\!\langle A\rangle\!\rangle\, \Phi) = \top$ and else $\overline{\mathbf{P}}$ wins.
- If $T = \emptyset$, then the evaluation game enters a **transition game $\mathbf{g}(\mathbf{P}, q, \langle\!\langle A\rangle\!\rangle\, \Phi, \Gamma)$**. The transition game is a subgame to be defined later on. The transition game eventually reaches an **exit location** $(\mathbf{P}', q', \psi, T')$, and the evaluation game continues from that location. Note that an *exit location* only ends the transition game, so exit locations of transition games and *ending locations* of the evaluation game are different concepts.

The rules for temporal formulae are defined using the truth function T (updated in an earlier transition game) as follows.

5. A location $(\mathbf{P}, q, \varphi \,\mathsf{U}\, \psi, T)$ is an ending location of the evaluation game. $\mathbf{P}$ wins if $T(\varphi \,\mathsf{U}\, \psi) = \top$ and else $\overline{\mathbf{P}}$ wins.

6. Likewise, a location $(\mathbf{P}, q, \mathsf{X}\,\varphi, T)$ is an ending location. $\mathbf{P}$ wins if $T(\mathsf{X}\,\varphi) = \top$ and otherwise $\overline{\mathbf{P}}$ wins.

These are the rules of the evaluation game. We note that the timer bound Γ will be used in the transition games. If $\Gamma = \omega$, we say that the evaluation game is **finitely bounded**.

The initial location of the finitely bounded evaluation game $\mathcal{G}(\mathcal{M}^*, q_0, \varphi^*, \omega)$ (see Examples 2.2 and 2.5) is $(\mathbf{E}, q_0, \langle\!\langle a_1\rangle\!\rangle\, \Psi, \emptyset)$*, from where the transition game* $\boldsymbol{g}(\mathbf{E}, q_0, \langle\!\langle a_1\rangle\!\rangle\, \Psi, \omega)$ *begins.*

3.2.2 Rules of the transition game

Recall that transition games are subgames of evaluation games. Their purpose is to evaluate the truth of strategic subformulae, in a game-like fashion.

Now we give a detailed description of transition games. [2] A **transition game** $\mathbf{g}(\mathbf{V}, q_0, \langle\!\langle A \rangle\!\rangle\, \Phi, \Gamma)$, where $\mathbf{V} \in \{\mathbf{A}, \mathbf{E}\}$, $q_0 \in \mathrm{St}$, $\langle\!\langle A \rangle\!\rangle\, \Phi \in \mathsf{ATL}^+$ and $\Gamma > 0$ is an ordinal, is defined as follows. $\mathbf{V}$ is called **the verifier in the transition game**. The game $\mathbf{g}(\mathbf{V}, q_0, \langle\!\langle A \rangle\!\rangle\, \Phi, \Gamma)$ is based on **configurations**, i.e., tuples $(\mathbf{S}, q, T, n, \gamma, x)$, where the player $\mathbf{S} \in \{\mathbf{E}, \mathbf{A}\}$ is called the **seeker**; q is the **current state**; $T : At(\Phi) \to \{\top, \bot, \mathsf{open}\}$ is a **truth function**; $n \in \mathbb{N}$ is a **seeker turn counter** ($n \leq |At(\Phi)|$); $\gamma < \Gamma$ is an ordinal called **timer**; and $x \in \{\, \mathbf{i}, \mathbf{ii}, \mathbf{iii} \,\}$ is an index showing the current **phase** of the transition game. The game $\mathbf{g}(\mathbf{V}, q_0, \langle\!\langle A \rangle\!\rangle\, \Phi, \Gamma)$ begins at the **initial configuration** $(\mathbf{V}, q_0, T_0, |At(\Phi)|, \Gamma, \mathbf{i})$, with $T_0(\chi) = \mathsf{open}$ for all $\chi \in At(\Phi)$.

The transition game $\boldsymbol{g}(\mathbf{E}, q_0, \langle\!\langle a_1 \rangle\!\rangle\, \Psi, \omega)$ *begins from the initial configuration* $(\mathbf{E}, q_0, T_0, 4, \omega, \mathbf{i})$, *since* $|At(\Psi)| = 4$.

The transition game then proceeds by iterating the following phases **i**, **ii** and **iii** which we *first* describe informally; detailed formal definitions are given afterwards.

- **i.** Adjusting the truth function: In this phase the players make claims on the truth of state formulae at the current state q. If $\mathbf{P}$ makes some claim, then the opponent $\overline{\mathbf{P}}$ may either: 1) accept the claim, whence truth function is updated accordingly, or 2) challenge the claim. In the latter case the *transition game ends* and truth of the claim is verified in a continued evaluation game.
- **ii.** Deciding whether to continue and adjusting the timer: Here the current seeker $\mathbf{S}$ may either continue her/his seeker turn and lower the value of the timer, or end her/his seeker turn. If $\mathbf{S}$ chooses the latter option, then the opponent $\overline{\mathbf{S}}$ of the seeker may either 1) take the role of the seeker and announce a new value for the timer or 2) end the transition game, whence the formula Φ is evaluated based on current values of the truth function.
- **iii.** Step phase: Here the verifier $\mathbf{V}$ chooses actions for the agents in the coalition in A at the current state q. Then $\overline{\mathbf{V}}$ chooses actions for the agents in the opposing coalition $\overline{A}$. After the resulting transition to a new state q' has been made, the game continues again with phase **i**.

We now describe the phases **i**, **ii** and **iii** in detail:

i. Adjusting the truth function.
Suppose the current configuration is $(\mathbf{S}, q, T, n, \gamma, \mathbf{i})$. Then the truth function T is updated by considering, one by one, each formula $\chi \in At(\Phi)$ (in some fixed order). If $T(\chi) \neq \mathsf{open}$, then the value χ cannot be updated. Else the value of χ may be modified according to the rules $\mathbf{A} - \mathbf{C}$ below.

A. *Updating* T *on temporal formulae*: Suppose that we have $\varphi \,\mathsf{U}\, \psi \in At(\Phi)$. Now first the verifier $\mathbf{V}$ may *claim that* ψ *is true* at the current state q. If $\mathbf{V}$ makes this claim, then $\overline{\mathbf{V}}$ chooses either of the following:

- $\overline{\mathbf{V}}$ *accepts* the claim of $\mathbf{V}$, whence the truth function is updated such that $\varphi \,\mathsf{U}\, \psi \mapsto \top$ ($\varphi \,\mathsf{U}\, \psi$ becomes **verified**).
- $\overline{\mathbf{V}}$ *challenges* the claim of $\mathbf{V}$, whence the transition game ends at the **exit location** $(\mathbf{V}, q, \psi, \emptyset)$. (We note that here, and elsewhere, when a transition game ends, the

[2] A transition game for ATL^+ is similar to the 'embedded game' introduced in [11] for the GTS of ATL. The role of the seeker $\mathbf{S}$ here is similar to the role of the controller in that embedded game.

evaluation game will be continued from the related exit location and the evaluation game will *never* return to the same exited transition game any more.)

If $\mathbf{V}$ does not claim that ψ is true at q, then $\overline{\mathbf{V}}$ may make the same claim (that ψ is true at q). If $\overline{\mathbf{V}}$ makes this claim, then the same above two steps concerning *accepting* and *challenging* are followed, but with $\mathbf{V}$ and $\overline{\mathbf{V}}$ swapped everywhere.

Suppose then that neither of the players claims that ψ is true at q. Then first $\mathbf{V}$ can *claim that φ is false* at q. If $\mathbf{V}$ makes this claim, then $\overline{\mathbf{V}}$ chooses either of the following:

- $\overline{\mathbf{V}}$ accepts the claim, whence the truth function is updated such that $\varphi \,\mathsf{U}\, \psi \mapsto \bot$ ($\varphi \,\mathsf{U}\, \psi$ becomes **falsified**).
- $\overline{\mathbf{V}}$ challenges the claim. Then the transition game ends at the exit location $(\mathbf{V}, q, \neg\varphi, \emptyset)$.

If $\mathbf{V}$ does not claim that φ is false at q, then $\overline{\mathbf{V}}$ may make the same claim. If (s)he does, then the same steps as those above are followed, but with $\mathbf{V}$ and $\overline{\mathbf{V}}$ swapped.

B. *Updating T on proposition symbols and strategic formulae*: The truth function can be updated on proposition symbols $p \in At(\Phi)$ and strategic formulae $\langle\!\langle A' \rangle\!\rangle\, \Psi \in At(\Phi)$ only when the phase $\mathbf{i}$ is executed for the first time (whence we have $q = q_0$). In this case, given such a formula χ, first $\mathbf{V}$ can claim that χ is true at q. Now, if $\overline{\mathbf{V}}$ accepts this claim, then the truth function is updated s.t. $\chi \mapsto \top$. If $\overline{\mathbf{V}}$ challenges the claim, then the transition game ends at the exit location $(\mathbf{V}, q, \chi, \emptyset)$. If $\mathbf{V}$ does not claim that χ is true at q, then $\overline{\mathbf{V}}$ may make the same claim. If (s)he does, then the same steps are followed, but with $\mathbf{V}$ and $\overline{\mathbf{V}}$ swapped.

C. *Updating T on formulae with X*: The truth function can be updated on formulae of type $\mathsf{X}\,\psi \in At(\Phi)$ only when phase $\mathbf{i}$ is executed for the second time in the transition game (whence q is some successor of q_0). First $\mathbf{V}$ can claim that ψ is true at q. If $\overline{\mathbf{V}}$ accepts this claim, then the truth function is updated s.t. $\mathsf{X}\,\psi \mapsto \top$. If $\overline{\mathbf{V}}$ challenges the claim, then the transition game ends at the exit location $(\mathbf{V}, q, \psi, \emptyset)$. If $\mathbf{V}$ does not claim that ψ is true at q, then $\overline{\mathbf{V}}$ can make the same claim. If (s)he does, the same steps are followed, but with $\mathbf{V}$ and $\overline{\mathbf{V}}$ swapped.

If neither player makes any claim which would update the value of a formula $\chi \in At(\Phi)$, then the value of χ is left open. Once the values of the truth function T have been updated (or left as they are) for all formulae in $At(\Phi)$, a new truth function T' is obtained. The transition game then moves to the new configuration $(\mathbf{S}, q, T', n, \gamma, \mathbf{ii})$.

In the configuration $(\mathbf{E}, q_0, T_0, 4, \omega, \mathbf{i})$ the players begin adjusting T_0 for which initially $T_0(\chi) = \mathsf{open}$ for every $\chi \in At(\Psi)$. Since it is the first round of the transition game, the value of $\mathsf{X}\,p_3$ cannot be modified, but the value of $\langle\!\langle a_2 \rangle\!\rangle\,\mathsf{X}\,p_1$ can be modified. Suppose that Eloise claims that $\langle\!\langle a_2 \rangle\!\rangle\,\mathsf{X}\,p_1$ is true at the current state q_0. Now Abelard could challenge the claim, whence the transition game ends and the evaluation game continues from location $(\mathbf{E}, q_0, \langle\!\langle a_2 \rangle\!\rangle\,\mathsf{X}\,p_1, \emptyset)$ (which leads to a new transition game $\boldsymbol{g}(\mathbf{E}, q_0, \langle\!\langle a_2 \rangle\!\rangle\,\mathsf{X}\,p_1, \omega)$). Suppose Abelard does not challenge the claim, whence $\langle\!\langle a_2 \rangle\!\rangle\,\mathsf{X}\,p_1$ is mapped to $\top$.

Since $\mathsf{F}\,p_1$ and $(\neg p_1)\,\mathsf{U}\,p_2$ occur positively in Φ, Eloise has interest only to verify them and Abelard has interest only to falsify them. Eloise could verify $\mathsf{F}\,p_1$ by claiming that p_1 is true, or verify $(\neg p_1)\,\mathsf{U}\,p_2$ by claiming that p_2 is true. But if Eloise makes either of these claims, then Abelard wins the whole evaluation game by challenging, since $q_0 \notin v(p_1) \cup v(p_2)$. Suppose that Eloise does not make any claims. Now, Abelard could claim that $\neg p_1$ is not true, in order to falsify $(\neg p_1)\,\mathsf{U}\,p_2$. But if he does this, he loses the evaluation game if Eloise challenges, since $q_0 \notin v(p_1)$. Suppose that Abelard does not make any claims, either. Then the transition game proceeds to configuration $(\mathbf{E}, q_0, T, 4, \omega, \mathbf{ii})$, where $T(\langle\!\langle a_2 \rangle\!\rangle\,\mathsf{X}\,p_1) = \top$ and $T(\chi) = \mathsf{open}$ for the other $\chi \in At(\Psi)$.

ii. Deciding whether to continue and adjusting the timer.
Suppose a configuration $(\mathbf{S}, q, T, n, \gamma, \mathbf{ii})$ has been reached. Assume first that $\gamma \neq 0$. Then the seeker $\mathbf{S}$ can choose whether to continue the transition game as the seeker. If yes, then $\mathbf{S}$ chooses some ordinal $\gamma' < \gamma$ and the transition game continues from $(\mathbf{S}, q, T, n, \gamma', \mathbf{iii})$. If $\mathbf{S}$ does not want to continue, or if $\gamma = 0$, then one of the following applies.

a) Assume that $n \neq 0$. Then the player $\overline{\mathbf{S}}$ chooses whether (s)he wishes to continue the transition game. If yes, then $\overline{\mathbf{S}}$ chooses an ordinal $\gamma' < \Gamma$ (note that $\overline{\mathbf{S}}$ may *reset* here the value of timer) and the transition game continues from $(\overline{\mathbf{S}}, q, T, n-1, \gamma', \mathbf{iii})$. Otherwise the transition game ends at the exit location $(\mathbf{V}, q, \Phi, T)$.

b) Assume $n = 0$. Then the transition game ends at the exit location $(\mathbf{V}, q, \Phi, T)$.

In $(\mathbf{E}, q_0, T, 4, \omega, \mathbf{ii})$ Eloise may decide whether to continue the transition game as a seeker. Suppose that Eloise does not continue, whence Abelard may now become a seeker and continue the transition game, or end it. If Abelard ends the transition game, then the evaluation game is continued from $(\mathbf{E}, q_0, \Psi, T)$. But since $T(\mathsf{X}\, p_3) =$ open and $T(\langle\!\langle a_2 \rangle\!\rangle \mathsf{X}\, p_1) = \top$, Eloise can then win the evaluation game by choosing the left disjunct of Ψ. Suppose that Abelard decides to become a seeker, whence he chooses some $m < \omega$ and the next configuration is $(\mathbf{A}, q_0, T, 3, m, \mathbf{iii})$.

iii. Step phase. [3]
Suppose that the configuration is $(\mathbf{S}, q, T, n, \gamma, \mathbf{iii})$.

a) First $\mathbf{V}$ chooses an action $\alpha_i \in d(a_i, q)$ for each $a_i \in A$.

b) Then $\overline{\mathbf{V}}$ chooses an action $\alpha_i \in d(a_i, q)$ for each $a_i \in \overline{A}$.

The resulting action profile produces a **successor state** $q' := o(q, \alpha_1, \ldots, \alpha_k)$. The transition game then moves to the configuration $(\mathbf{S}, q', T, n, \gamma, \mathbf{i})$.

In the configuration $(\mathbf{A}, q_0, T, 3, m, \mathbf{iii})$ Eloise (who is the verifier $\mathbf{V}$) first chooses action for agent a_1, then Abelard chooses action for agent a_2, which produces either successor state q_1 or q_2. Then the transition game continues from the configuration $(\mathbf{A}, q_j, T, 3, m, \mathbf{i})$, where $j \in \{1, 2\}$.

This concludes the definition of the rules for the phases **i, ii** and **iii** in the transition game $\mathbf{g}(\mathbf{V}, q_0, \langle\!\langle A \rangle\!\rangle\, \Phi, \Gamma)$.

Suppose first that the transition game is continued from $(\mathbf{A}, q_2, T, 3, m, \mathbf{i})$. Since it is the second round, Abelard could now try to verify $\mathsf{X}\, p_3$ by claiming that p_3 is true at q_2. However, then Eloise would win by challenging. But if Abelard does not try to verify $\mathsf{X}\, p_3$ now, then the value of $\mathsf{X}\, p_3$ will stay open*. In that case Eloise will win the evaluation game simply by not making any more claims in the transition game.*

Suppose then that the game continued from $(\mathbf{A}, q_1, T, 3, m, \mathbf{i})$. Suppose that Abelard verifies $\mathsf{X}\, p_3$ by claiming that p_3 is true and that Eloise does not challenge. If the transition game would end at $(\mathbf{E}, q_1, \Psi, T')$, where $T'(\mathsf{X}\, p_3) = \top$, Abelard would win. Thus, suppose that Abelard ends his seeker turn and Eloise chooses some finite timer, say 2. At $(\mathbf{E}, q_1, T', 2, 2, \mathbf{iii})$ Eloise can force the resulting state q_3 by choosing α for a_1. At $(\mathbf{E}, q_3, T', 2, 2, \mathbf{i})$ Eloise can verify $(\neg p_1)\, \mathsf{U}\, p_2$ by claiming that p_2 is true at q_2. Furthermore, Eloise can move via q_1 to q_4 and verify $\mathsf{F}\, p_1$ there, before timer reaches 0. When the evaluation game is eventually continued, Eloise wins by choosing the right disjunct of Ψ.

[3]The procedure in this phase is analogous to the *step game*, $\mathsf{step}(\mathbf{V}, A, q)$, which was introduced for the GTS for ATL ([11]).

3.2.3 The unbounded evaluation game

Let $\mathcal{G}(\mathcal{M}, q, \varphi, \Gamma)$ be a Γ-bounded evaluation game. We can define a corresponding **unbounded evaluation game**, $\mathcal{G}(\mathcal{M}, q, \varphi)$, by replacing transition games $\mathbf{g}(\mathbf{V}, q, \langle\!\langle A \rangle\!\rangle \Phi, \Gamma)$ with **unbounded transition games**, $\mathbf{g}(\mathbf{V}, q, \langle\!\langle A \rangle\!\rangle \Phi)$; these are played with the same rules as $\mathbf{g}(\mathbf{P}, q_0, \langle\!\langle A \rangle\!\rangle \Phi, \Gamma)$ except that timers γ are not used in them. Instead, the players can keep the role of a seeker for arbitrarily long and thus the game may last for an infinite number of rounds. In the case of an infinite play, the player who took the last seeker turn loses the entire evaluation game. (Recall that the number of seeker alternations is bounded by the number $|At(\Phi)|$.)

3.3 Defining the game theoretic semantics

In this section we define a game-theoretic semantics (GTS) for ATL^+ by defining truth as an existence of a winning strategy for Eloise in the corresponding evaluation game. Γ-bounded and unbounded evaluation games lead to Γ-bounded and unbounded GTS, respectively.

Remark 3.1. *The description of transition games given here was based on a simplified notion of configurations. The phases **i**–**iii** consist of several "subphases" and there is more information that should be encoded into configurations. The fully extended notion of configuration, should also include the following: In phase **i**, a counter indicating the relative atom currently under consideration by the players; flags for each player indicating whether and what claim (s)he has made on the truth of the current relative atom; a 3-bit flag indicating if it is the first, second or some later round in the transition game. For phase **ii**, a flag whether the current seeker wants to continue, and for phase **iii**, a record of the current choice of actions for the agents in A by $\mathbf{V}$. To keep things simple, we shall not present configurations in a fully extended form like this.*

Hereafter a **position** in an evaluation game will mean either a location of the form $(\mathbf{P}, q, \varphi, T)$ or a configuration in the fully extended form described in the remark above. Note that by this definition, at every position, only one of the players (Abelard or Eloise) has a move to choose. Thus the entire evaluation game—including transition games as subgames—is a turn-based game of perfect information.

By the **game tree**, $T_{\mathcal{G}}$, of an evaluation game $\mathcal{G}$, we mean the tree whose nodes correspond to all positions arising in $\mathcal{G}$, and whose every branch corresponds to a possible play of $\mathcal{G}$ (including transition games as subgames). Note that some of these plays may be infinite, but only because an embedded transition game does not terminate, in which case a winner in the entire evaluation game is uniquely assigned according to the rules in Section 3.2.3.

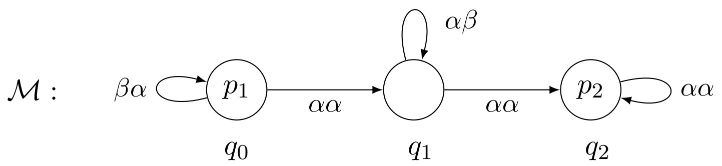
The formal definitions of players' memory-based strategies in the evaluation games games are defined as expected, based on histories of positions. As usual, a strategy for a player $\mathbf{P}$ is called **winning** if, following that strategy, $\mathbf{P}$ is guaranteed to win regardless of how $\overline{\mathbf{P}}$ plays. A strategy is **positional** if it is only based on the current position.

Definition 3.2. *Let $\mathcal{M}$ be a* CGM*, $q \in \mathrm{St}$, $\varphi \in \mathsf{ATL}^+$ and Γ an ordinal. **Truth of** φ **according to** Γ**-bounded** ($\Vdash_\Gamma$) and **unbounded** ($\Vdash$) GTS is defined as follows:*

$$\mathcal{M}, q \Vdash_\Gamma \varphi \text{ (resp. } \mathcal{M}, q \Vdash \varphi\text{) iff Eloise has a positional winning strategy in } \mathcal{G}(\mathcal{M}, q, \varphi, \Gamma) \text{ (resp. } \mathcal{G}(\mathcal{M}, q, \varphi)\text{).}$$

We will show later that evaluation games are determined with positional strategies. Hence, if we allow perfect-recall strategies in the truth definition above, we would obtain equivalent semantics.

Example 3.3. *Consider the following* CGM



$\mathcal{M} = (\mathbb{A}\mathrm{gt}, \mathrm{St}, \Pi, \mathrm{Act}, d, o, v)$, *where:*

$$\mathbb{A}\mathrm{gt} = \{1,2\},\ \mathrm{St} = \{q_0, q_1, q_2\},\ \Pi = \{p_1, p_2\},\ \mathrm{Act} = \{\alpha, \beta\}$$
$$d(1, q_0) = d(2, q_1) = \{\alpha, \beta\};\ d(a, q_i) = \{\alpha\} \text{ in all other cases;}$$
$$o(q_0, sm) = q_0,\ o(q_0, mm) = q_1,\ o(q_1, mm) = o(q_2, mm) = q_2$$
$$v(p_1) = \{q_0\} \text{ and } v(p_2) = \{q_2\}.$$

Let $\varphi := \langle\!\langle\{a_2\}\rangle\!\rangle (\mathsf{G}\, p_1 \vee \mathsf{F}\, p_2)$ (note that $\mathsf{G}\, p_1 = \neg\mathsf{F}\,\neg p_1$). We describe a winning strategy for Eloise in the unbounded evaluation game $\mathcal{G}(\mathcal{M}, q_0, \varphi)$. Eloise immediately ends her seeker's turn and does not make any claims while being at q_0. If Abelard makes any claims at q_0, then she challenges those claims. If Abelard ends the transition game at q_0, then Eloise wins the Evaluation game by choosing $\neg\mathsf{F}\,\neg p_1$, since now the value of $\mathsf{F}\,\neg p_1$ has to be open*. Suppose that Abelard forces a transition to q_1 by choosing α for agent a_1. If he claims that $\neg p_1$ is true at q_1, Eloise does not challenge. If Abelard ends his seeker turn at q_1, then Eloise becomes seeker. At q_1 she forces a transition to q_2, by choosing α for agent a_2. Then she verifies $\mathsf{F}\, p_2$ by claiming that p_2 is true at q_2. If the transition game ends at q_2, then she wins by choosing $\mathsf{F}\, p_2$, whose value is now $\top$. Note that by following this strategy, Eloise cannot stay as a seeker for infinitely many rounds.*

3.4 Regular strategies

In this subsection we will define a concept of a regular strategy which will be important for the proofs later in this paper. We only define this concept for Eloise in and only for the transition games in which Eloise is the verifier. This suffices for our needs, but this definition—and the related proof—could easily be generalized for both players and all kinds of transition games.

Definition 3.4. *We say that a Eloise's strategy τ for a transition game $\boldsymbol{g}(\mathbf{E}, q, \langle\!\langle A\rangle\!\rangle\, \Phi)$ is **regular**, if the following properties hold:*

i) *τ instructs Eloise to challenge all the claims made by Abelard. (That is because Eloise makes all the valid verifications/falsifications by herself).*

ii) *τ instructs to Eloise to try to end the game (by ending her seeker turn or by not taking a new seeker turn) always when the truth function T has winning values for Eloise (that is, she would win from the exit location if Abelard would not want to continue as a seeker).*

iii) *Actions chosen by τ (for the agents in A) are independent of the current seeker $\mathbf{S}$ and seeker turn counter $n \in \mathbb{N}$ in configurations.*

Note that the conditions i)-iii) together imply that *all* the actions chosen by a regular strategy are independent of the current seeker $\mathbf{S}$ and seeker turn counter $n \in \mathbb{N}$ in configurations. Hence the actions chosen by a regular strategy depend only[4] on the

[4]The parameter x and all the other information that is should be encoded in the configurations (see Remark 3.1) are only for used for describing the current subphase of the game. Hence it is easy to see players' strategies cannot depend on these parameters.

pairs (q, T), where q is the current state and T is the current truth function. Also note that since, by i), Eloise makes all the valid verifications and falsifications, the truth function T is determined by the path that has been formed by the transition game.

The following proposition shows that from now on we may assume all winning strategies to be regular. Since regular strategies depend only on the states and the truth function, the extra parameters $\mathbf{S}$ and n cannot be used for signaling any information.

Claim I. *If Eloise has a winning strategy in a transition game $\boldsymbol{g}(\mathbf{E}, q, \langle\!\langle A\rangle\!\rangle\, \Phi)$, then she has a regular winning strategy in that game.*

Proof. Suppose that Eloise has winning strategy τ in a transition game $\mathbf{g}(\mathbf{E}, q, \langle\!\langle A\rangle\!\rangle\, \Phi)$. We first note that, for checking the regularity conditions i)-iii), it suffices the we only consider the configurations that can be reached with Eloise strategy. This is because we can choose arbitrary actions for all the other configurations in order to satisfy the regularity conditions.

We make the strategy τ regular by doing the following modifications (in the given order).

i) Let $c = (\mathbf{S}, q, T, \mathbf{i})$ be configuration which can be reached with τ and in which τ does not instruct Eloise to challenge some claim φ that Abelard can make (if Abelard claims that some formula ψ is false, then here $\varphi = \neg\psi$). If $(\mathbf{A}, q, \varphi)$ is a winning location for Eloise, then we can redefine τ such that it instructs Eloise to challenge the claim φ. Suppose then that $(\mathbf{A}, q, \varphi)$ is not a winning location for Eloise, whence by determinacy, $(\mathbf{E}, q, \varphi)$ is a winning location for Eloise. Then we redefine τ in such a way that Eloise makes the claim φ by herself. If Abelard then challenges this claim, the exit position will be winning location for Eloise.

   We do this modification for all configurations for which τ violates the regularity property i).

ii) Let $c = (\mathbf{P}, q, T, n, \mathbf{ii})$ be a configuration that can be reached with τ such that $(\mathbf{E}, q, \Phi, T)$ is is a winning location for Eloise, but τ does not instruct Eloise to try to end the game at c. We redefine τ to instruct Eloise to try to end the game at c. If Abelard also wants to end the game, then Eloise wins. And if Abelard does not want to end the game, then the game continues from a configuration c' that must be a winning configuration for Eloise. We can thus modify τ in such way that it is a winning strategy from c'. We can do this by maintaining the regularity conditions i) and ii)—we simply do the same modifications as above for all new configurations that violate the regularity.

   By doing the the procedure above for all configurations for which τ violates the regularity property ii), τ now satisfies the properties i) and ii).

iii) Suppose that $c = (\mathbf{A}, q, T, n, \mathbf{iii})$ is a configuration such that T is not winning for Eloise. We now redefine τ at c to make the same choice as for $c' = (\mathbf{E}, q, T, n-1, \mathbf{iii})$. Since Abelard could end his seeker turn at $(\mathbf{A}, q, T, n, \mathbf{ii})$, the configuration c' must be winning for Eloise. We can thus modify τ in such way that it is a winning strategy from c. We can do this by maintaining the regularity conditions i) and ii) by doing the modifications above, if necessary.

   We then do the following procedure for every integer $n \leq |At(\Phi)|$, beginning from $n = |At(\Phi)|$. Let $c_n = (\mathbf{P}, q, T, n, \mathbf{iii})$ be a configuration that can be reached with τ. Let $n' \leq |At(\Phi)|$ be the largest integer such that $c_{n'} = (\mathbf{P}, q, T, n', \mathbf{iii})$ can be reached with τ. We redefine τ at c_n in such a way that it will select the same actions as at $c_{n'}$. We continue this modification in such a way that, when playing from c_n, we can only reach configurations of the same form as those that can be reached from $c_{n'}$, the only difference being the value of seeker alternation counter. Now all the exit positions that can be reached by using τ from c_n must be winning positions for Eloise. Since the truth function can be updated at most $|At(\Phi)|$ many times and, by condition ii), T gets updated after every seeker alternation, it is impossible that Eloise would lose the game because the seeker turn counter would become zero.

After doing this procedure down to an integer $n' \leq |At(\Phi)|$, then τ is independent of seeker turn counter n for every $n \geq n'$. By applying the procedure for n' we only modify actions for configurations whose seeker turn counter is less or equal to n'. Hence, by applying the procedure for all $n' \leq |At(\Phi)|$, we finally obtain a winning strategy that is completely independent of the seeker turn counter. Also note that, by this applying this procedure, we also maintain the regularity conditions i) and ii) for τ.

For proving that the actions chosen by τ for A are now independent of *both* the seeker $\mathbf{S}$ and the seeker turn counter n, suppose that τ assigns different actions for A in configurations $c = (\mathbf{P}, q, T, n, \mathbf{iii})$ and $c' = (\mathbf{P}', q, T, n', \mathbf{iii})$ such that $c \neq c'$ and both c and c' can be reached with τ. Since τ is independent of the seeker turn counter, we must have $\mathbf{P} \neq \mathbf{P}'$. By symmetry we may assume that $\mathbf{P} = \mathbf{E}$ and $\mathbf{P}' = \mathbf{A}$.

Suppose first that $(\mathbf{E}, q, \Phi, T)$ is a winning position for Eloise. Now, by the condition ii), τ instructs Eloise to end her seeker turn at $(\mathbf{E}, q, T, n, \mathbf{ii})$, whence the configuration c cannot be reached with τ and thus may be ignored. Suppose then that $(\mathbf{E}, q, \Phi, T)$ is not a winning position for Eloise. Recall that we have defined τ to make the same choice at c' as at the configuration $c'' = (\mathbf{E}, q, T, n' - 1, \mathbf{iii})$. But this is impossible since τ is independent of the seeker turn counter and that is the only parameter that separates the configurations c and c''.

By doing all the modifications above, τ becomes a regular strategy. Since it maintains as a winning strategy for Eloise even after these modifications, Eloise thus has a regular winning strategy. □

4 Results on evaluation games

4.1 Positional determinacy

Here we show that both bounded and unbounded evaluation games are determined, and that the winner in either of them has a positional strategy.

Proposition 4.1. *Bounded evaluation games are determined and the winner has a positional winning strategy.*

Proof. (Sketch) Since ordinals are well-founded and they must decreased during transition games, it is easy to see that the game tree is well-founded. Thus positional determinacy follows essentially by backward induction. □

Proposition 4.2. *Unbounded evaluation games are determined and the winner has a positional winning strategy.*

Proof. We will show that unbounded evaluation games are essentially Büchi-games (see, e.g., [16]). We first discuss the case where the underlying CGM $\mathcal{M}$ is finite. We follow the technicalities for Büchi-games from [7].

Take a triple $(\mathcal{M}, q, \varphi)$, where $\mathcal{M}$ is a finite CGM, q a state of $\mathcal{M}$, and φ a formula of ATL. We will convert this triple into a Büchi game BG such that $\mathcal{M}, q \models \varphi$ iff player 2 has a winning strategy in BG from a certain position of BG determined by the state q. The required Büchi game BG corresponds almost exactly to the unbounded evaluation game $\mathcal{G}(\mathcal{M}, q, \varphi)$. The set of states of BG is the finite set of positions in $\mathcal{G}(\mathcal{M}, q, \varphi)$. The states of BG assigned to player 1 (resp., player 2) of BG are the positions where Abelard (resp., Eloise) is to move. The edges of the binary transition relation E of BG correspond to the changes of positions in $\mathcal{G}(\mathcal{M}, q, \varphi)$. Also, E is defined such that ending locations in the evaluation game connect (only) to themselves via E. This ensures that every state of BG has a successor state.

We set a *co-Büchi-objective* such that an infinite play of BG is winning for player 2 iff the set of states visited infinitely often is a subset of the following states of BG:

1. States of BG corresponding to configurations of the transition games where Abelard is the seeker.
2. States of BG corresponding to such ending locations in the game $\mathcal{G}(\mathcal{M}, q, \varphi)$ where Eloise has already won.

Clearly, Eloise (Abelard) has a positional winning strategy in the evaluation game starting at a position *pos* of the evaluation game iff the player 2 (player 1) of BG has a positional winning strategy from the state of BG corresponding to *pos*. Finite Büchi games enjoy positional determinacy (see e.g. [7]), which completes the case of finite CGM*s*. For infinite CGM*s*, the argument is the same but requires positional determinacy of Büchi games on infinite game graphs. This is well-known and follows easily from Theorem 4.3 of [12]. □

We say that Eloise (Abelard) has a *winning strategy in a transition game*, if she (he) can force that game to end at a position where she (he) has a winning strategy in the evaluation game that continues. By the previous propositions, both bounded and unbounded transition games are positionally determined. By the determinacy, we have the following consequence: If Eloise (Abelard) has a perfect recall strategy in a bounded or unbounded evaluation game (or transition game), then she (he) has a positional winning strategy in that game.

4.2 Finding stable timer bounds

We then consider a "semi-bounded" variant of the transition game in which one of the players must use timers when being the seeker and the other player is allowed to play without timers. We say that a timer bound $\Gamma > 0$ is **stable** for an unbounded transition game $\mathbf{g}(\mathbf{V}, q_0, \langle\!\langle A \rangle\!\rangle\, \Phi)$ if the player who has a winning strategy in $\mathbf{g}(\mathbf{V}, q_0, \langle\!\langle A \rangle\!\rangle\, \Phi)$ can in fact win the game using timers below Γ. If $c = (\mathbf{S}, q, T, n, x)$ is a configuration in an unbounded transition game and γ is an ordinal, we use the following notation $c[\gamma] := (\mathbf{S}, q, T, n, \gamma, x)$.

We first identify stable timer bounds for *finite* models.

Proposition 4.3. *Let $\mathcal{M}$ be a finite* CGM*, $q_0 \in \mathrm{St}$ a state and $\Phi \in \mathsf{ATL}^+$ a path formula. Then $k := |\mathrm{St}| \cdot |At(\Phi)|$ is a stable timer bound for $\boldsymbol{g}(\mathbf{V}, q_0, \langle\!\langle A \rangle\!\rangle\, \Phi)$.*

Proof. (Sketch) Let $c = (\mathbf{E}, q, T, n, x)$ be a configuration (for an *unbounded* game, so no timer is listed). Suppose that $(\mathbf{V}, q, \Phi, T)$ is not a winning location for Eloise. Then she wants to stay as the seeker until the truth function is modified to T' so that T' makes Φ true. Since T is updated state-wise, it is not beneficial for Eloise to go in loops such that T is not updated. Hence, if Eloise has a winning strategy from c, then she has a winning strategy in which T is updated at least once every $|\mathrm{St}|$ rounds. Since T can be updated at most $|At(\Phi)|$ times, we see that a timer greater than $k := |\mathrm{St}| \cdot |At(\Phi)|$ is not needed. □

Corollary 4.4. *If $\mathcal{M}$ is a finite* CGM*, the unbounded* GTS *is equivalent on $\mathcal{M}$ to the* $(|\mathrm{St}| \cdot |\varphi|)$*-bounded* GTS*.*

In order to find stable timer bounds for infinite models, we give the following definition (cf. Def 4.12 in [11]).

Definition 4.5. *Let $\mathcal{M}$ be a* CGM *and let $q \in \mathrm{St}$. We define the* ***branching degree of*** *q, $\mathsf{BD}(q)$, as the cardinality of the set of outcome states from q: $\mathsf{BD}(q) := \mathsf{card}(\{o(q, \vec{\alpha}) \mid \vec{\alpha} \in \mathsf{action}(\mathbb{A}\mathrm{gt}, q)\})$. We define the* ***regular branching bound of*** *$\mathcal{M}$ as the smallest cardinal $\mathsf{RBB}(\mathcal{M})$ for which the following conditions hold:*

1. $\mathsf{RBB}(\mathcal{M}) > \mathsf{BD}(q)$ *for every* $q \in \mathrm{St}$.
2. $\mathsf{RBB}(\mathcal{M})$ *is infinite.*
3. $\mathsf{RBB}(\mathcal{M})$ *is a regular cardinal.*

Note that $\mathsf{RBB}(\mathcal{M}) = \omega$ if and only if $\mathcal{M}$ is ***image finite****.*

Proposition 4.6. *Let $\mathcal{M}$ be a* finite CGM*, $q_0 \in \mathrm{St}$ and $\Phi \in \mathsf{ATL}^+$ a path formula. Then $\mathsf{RBB}(\mathcal{M})$ is a stable timer bound for $\boldsymbol{g}(\mathbf{V}, q_0, \langle\!\langle A \rangle\!\rangle\, \Phi)$.*

Proof. Suppose first that Eloise has a winning strategy τ in $\mathbf{g}(\mathcal{M}, q_0, \langle\!\langle A \rangle\!\rangle\, \Phi)$. Let c be any configuration of the form $c = (\mathbf{P}, \mathbf{A}, q, T, n, \mathbf{ii})$ such that

- c can be reached with τ.
- If Abelard decides to quit seeking at c, then τ instructs Eloise to become seeker.

We need to find an ordinal $\gamma_0 < \mathsf{RBB}(\mathcal{M})$ for Eloise to announce if she needs to become seeker at c and supplement τ with instructions on lowering the ordinal after every transition while she is a seeker. We will use the instructions given by τ for verifications and choices for actions.

Suppose that Abelard quits seeking at c. Let $T_{\mathbf{g},c}$ be the tree that is formed by all of those paths of confiqurations, starting from c, in which Eloise stays as the seeker and plays according to τ. Since τ is a winning strategy, every path in $T_{\mathbf{g},c}$ must be finite, and thus $T_{\mathbf{g},c}$ is well-founded. We prove the following claim by well-founded induction on $T_{\mathbf{g},c}$:

$$\text{For every } c' \in T_{\mathbf{g},c}, \text{ there is an ordinal } \gamma < \mathsf{RBB}(\mathcal{M}) \\ \text{s.t. } c'[\gamma] \text{ is a winning position for Eloise.}$$

We can choose $\gamma = 0$ for every leaf on $T_{\mathbf{g},c}$. Suppose then that c' is not a leaf. By the induction hypothesis, the claim holds for every configuration that can be reach with a transition from c'. We can now define γ to be the *successor of the supremum* of these ordinals. Since $\mathsf{RBB}(\mathcal{M})$ is regular, we have $\gamma < \mathsf{RBB}(\mathcal{M})$. Hence there is $\gamma_0 < \mathsf{RBB}(\mathcal{M})$ such the $c[\gamma_0]$ is a winning configuration for Eloise. □

Corollary 4.7. *Suppose that $\Gamma \geq \mathsf{RBB}(\mathcal{M})$. Then the unbounded* GTS *is equivalent on $\mathcal{M}$ to the Γ-bounded* GTS*.*

Proof. Suppose first that $\mathcal{M}, q \Vdash \varphi$. By Proposition 4.6 Eloise can win the evaluation game using timers smaller than Γ when being the seeker. Hence clearly $\mathcal{M}, q \Vdash_\Gamma \varphi$.

Suppose then $\mathcal{M}, q \not\Vdash \varphi$. By Proposition 4.2, Abelard has a winning strategy in $\mathcal{G}(\mathcal{M}, q, \varphi)$. Thus, by Proposition 4.6, Abelard can win $\mathcal{G}(\mathcal{M}, q, \varphi)$ using timers smaller than Γ when being the seeker. Hence Abelard clearly has a winning strategy in $\mathcal{G}(\mathcal{M}, q, \varphi, \Gamma)$ and thus $\mathcal{M}, q \not\Vdash_\Gamma \varphi$. □

Consequently, finite timers suffice in image finite models. However, the finitely bounded GTS ($\Gamma = \omega$) is not generally equivalent to the unbounded GTS (see Example 3.7 in [11]).

4.3 GTS vs compositional semantics for ATL$^+$

We now define a so-called **finite path semantics**, to be used later. See [5] for a similar definition. We define the **length** $lgt(\lambda)$ of a finite path λ as the number of transitions in λ (whence the last state of λ is $\lambda[lgt(\lambda)]$.) If λ is a prefix sequence of λ', we write $\lambda \preceq \lambda'$.

Definition 4.8. *Let $\mathcal{M}$ be a* CGM *and $\lambda \in \mathsf{paths}_{fin}(\mathcal{M})$. **Truth** of an* ATL$^+$ *path formula Φ on λ is defined as follows:*

- $\mathcal{M}, \Lambda \models \varphi$ *iff* $\mathcal{M}, \Lambda[0] \models \varphi$ *(where φ is a state formula).*
- $\mathcal{M}, \Lambda \models \mathsf{X}\,\varphi$ *iff* $lgt(\lambda) \geq 1$ *and* $\mathcal{M}, \Lambda[1] \models \varphi$.
- $\mathcal{M}, \Lambda \models \neg\Phi$ *iff* $\mathcal{M}, \Lambda \not\models \Phi$.
- $\mathcal{M}, \Lambda \models \Phi \vee \Psi$ *iff* $\mathcal{M}, \Lambda \models \Phi$ *or* $\mathcal{M}, \Lambda \models \Psi$.
- $\mathcal{M}, \Lambda \models \varphi\,\mathsf{U}\,\psi$ *iff there exists some $i \leq lgt(\lambda)$ such that $\mathcal{M}, \Lambda[i] \models \psi$ and $\mathcal{M}, \Lambda[j] \models \varphi$ for all $j < i$.*

Definition 4.9. *Let $\mathcal{M}$ be a* CGM*, $\lambda \in \mathsf{paths}(\mathcal{M})$ and Φ a path formula of* ATL$^+$*. An index $i \geq 1$ is a **truth swap point** of Φ on λ if either of the following holds:*

1. $\mathcal{M}, \lambda[i-1] \not\models \Phi$ *and* $\mathcal{M}, \lambda[i] \models \Phi$.
2. $\mathcal{M}, \lambda[i-1] \models \Phi$ *and* $\mathcal{M}, \lambda[i] \not\models \Phi$.

*We define the **truth swap number** of Φ on λ, $TSN(\Phi, \lambda)$ as follows:*

$$TSN(\Phi, \lambda) := \mathsf{card}(\{i \mid i \text{ is a truth swap point of } \Phi \text{ on } \lambda\}).$$

The claims of the following lemma are easy to prove. Similar observations have been made in [5].

Lemma 4.10. *Let $\mathcal{M}$ be a* CGM*, $\lambda \in \mathsf{paths}(\mathcal{M})$ and Φ a path formula of* ATL$^+$*. Now the following claims hold:*

1. $TSN(\Phi, \lambda) \leq |\{\Psi \in At(\Phi) \mid \Psi \text{ is temporal subformula}\}|$.
2. $\mathcal{M}, \lambda \models \Phi$ *iff there is some $k \in \mathbb{N}$ s.t. $\mathcal{M}, \lambda_0 \models \Phi$ for every finite $\lambda_0 \preceq \lambda$ for which $lgt(\lambda_0) \geq k$.*

Theorem 4.11. *The unbounded* GTS *is equivalent to the standard (perfect-recall) compositional semantics of* ATL$^+$*.*

Proof. We prove by induction on ATL$^+$ state formulae φ that for any CGM $\mathcal{M}$ and a state q in $\mathcal{M}$:

$$\mathcal{M}, q \models \varphi \;\text{ iff }\; \text{Eloise has a winning strategy in } \mathcal{G}(\mathcal{M}, q, \varphi).$$

If φ is a proposition symbol, the claim holds trivially.

Let $\varphi = \neg\psi$ and suppose first that $\mathcal{M}, q \models \neg\psi$, i.e. $\mathcal{M}, q \not\models \psi$. By the induction hypothesis Eloise does not have a winning strategy in $\mathcal{G}(\mathcal{M}, q, \psi)$. Since evaluation games are determined, Abelard has a winning strategy in $\mathcal{G}(\mathcal{M}, q, \psi)$. Thus, Eloise has a winning strategy in $\mathcal{G}(\mathcal{M}, q, \neg\psi)$.

Suppose then that Eloise has a winning strategy in the evaluation game $\mathcal{G}(\mathcal{M}, q, \neg\psi)$. Then Eloise cannot have a winning strategy in $\mathcal{G}(\mathcal{M}, q, \psi)$. Hence, by the induction hypothesis $\mathcal{M}, q \not\models \psi$, i.e. $\mathcal{M}, q \models \neg\psi$.

Let $\varphi = \psi \vee \theta$ and suppose first that $\mathcal{M}, q \models \psi \vee \theta$, i.e. $\mathcal{M}, q \models \psi$ or $\mathcal{M}, q \models \theta$. Suppose first that $\mathcal{M}, q \models \psi$, whence by the induction hypothesis Eloise has a winning strategy in $\mathcal{G}(\mathcal{M}, q, \psi)$. Now Eloise can win $\mathcal{G}(\mathcal{M}, q, \psi \vee \theta)$ by choosing ψ on the first move. The case when $\mathcal{M}, q \models \theta$ is analoguos. Suppose now that Eloise has a winning strategy in the evaluation game $\mathcal{G}(\mathcal{M}, q, \psi \vee \theta)$. Let $\chi \in \{\psi, \theta\}$ be disjunct that Eloise chooses when following her winning strategy. Now Eloise must have a winning strategy in $\mathcal{G}(\mathcal{M}, q, \chi)$ and thus by the induction hypothesis $\mathcal{M}, q \models \chi$. Therefore $\mathcal{M}, q \models \psi \vee \theta$.

Now, consider $\varphi = \langle\!\langle A \rangle\!\rangle\, \Phi$. It suffices to show that Eloise has winning strategy in the (unbounded) transition game $\mathbf{g}(\mathbf{E}, q, \langle\!\langle A \rangle\!\rangle\, \Phi)$ if and only if the coalition A has a (perfect recall) strategy S_A s.t. $\mathcal{M}, \lambda \models \Phi$ for every $\lambda \in \mathsf{paths}(q, S_A)$.

Suppose firts that $\mathbf{E}$ has a winning strategy τ in the transition game $\mathbf{g}(\mathbf{E}, q, \langle\!\langle A \rangle\!\rangle\, \Phi)$. By Claim I we may assume that τ is regular. Let $T_{\mathbf{g}}$ be the game tree that is formed by all of those configurations that can be encountered with τ. We define S_A by using the actions according to τ for every *finite path of states* that occurs in consecutive configurations in $T_{\mathbf{g}}$. The actions for all other finite paths are irrelevant.

In order to show that S_A is well-defined this way, let Λ, Λ' be finite branches of configurations in $T_{\mathbf{g}}$ such that the *states* occurring in configurations of Λ and Λ' are in the same order. Let $c = (\mathbf{P}, q, T, n, \mathbf{iii})$ and $c' = (\mathbf{P}', q, T', n', \mathbf{iii})$ be the last configurations in Λ and Λ', respectively. It suffices to show that τ assigns the same actions for A in both c and c'. Since Λ and Λ' have visited the same states, by regularity condition i), we must have $T = T'$. Therefore, by regularity condition iii), τ assigns the same actions for c and c'.

Let $\lambda \in \mathsf{paths}(q, S_A)$, whence states in λ occur in some infinite tuple of configurations in $T_{\mathbf{g}}$. In the (infinite) play of $\mathbf{g}(\mathbf{E}, q, \langle\!\langle A \rangle\!\rangle\, \Phi)$, that corresponds to λ, Eloise does only finitely many verifications and cannot stay as a seeker for infinitely many rounds (since τ is a winning strategy). Let $k \in \mathbb{N}$ be such that Eloise neither does any further verifications nor becomes a seeker after the state $\lambda[k]$. Let $\lambda_0 \preceq \lambda$ be a finite path s.t. $|\lambda_0| \geq k$.

We can show by induction on the formulae in $\mathrm{SUB}_{At}(\Phi)$ that if a position of the form $(\mathbf{P}, \lambda_0[l], \Psi, T)$, where $\Psi \in \mathrm{SUB}_{At}(\Phi)$, can be reached by using τ, then the following holds:

$$\mathcal{M}, \lambda_0 \models \Psi \;\text{ iff } \mathbf{P} = \mathbf{E}.$$

- The cases $\Psi = \varphi$ and $\Psi = \mathsf{X}\, \varphi$ are easy to prove.
- Let $\Psi = \psi \,\mathsf{U}\, \theta$ and suppose first that $\mathbf{P} = \mathbf{E}$. Since τ is a winning strategy, there must be $i \leq k$ s.t. Eloise verifies $\psi \,\mathsf{U}\, \theta$ at $\lambda_0[i]$. Now Abelard may have objected, whence the evaluation game would have continued from the position $(\mathbf{E}, \lambda_0[i], \theta, T)$. By the (outer) induction hypothesis $\mathcal{M}, \lambda[i] \models \theta$. Let then $j < i$. Now Abelard could have attempted to falsify ψ at $\lambda[j]$, whence Eloise must have objected since τ is a winning strategy. Then the evaluation game would have continued from the position $(\mathbf{E}, \lambda[j], \psi, T)$ and thus by the (outer) induction hypothesis $\mathcal{M}, \lambda[j] \models \psi$. Thus we have shown that $\mathcal{M}, \lambda_0 \models \psi \,\mathsf{U}\, \theta$.

Suppose now that $\mathbf{P} = \mathbf{A}$. We also suppose, for the sake of contradiction, that $\mathcal{M}, \lambda_0 \models \psi \,\mathsf{U}\, \theta$. Now there is $i \leq k$ s.t. $\mathcal{M}, \lambda_0 \models \theta$. If Abelard would have verified θ at $\lambda_0[i]$, then Eloise would have lost by the (outer) induction hypothesis. Hence Eloise should have falsified $\psi \,\mathsf{U}\, \theta$ at some state $\lambda_0[j]$, where $j < i$. But then by the (outer) induction hypothesis we must have $\mathcal{M}, \lambda_0[j] \not\models \psi$, which is a contradiction.

- Suppose that $\Psi = \neg\Theta$. The next position of the evaluation game is $(\overline{\mathbf{P}}, \lambda[l], \Theta, T)$ and thus by the (inner) induction hypothesis, $\mathcal{M}, \lambda_0 \not\models \Theta$ iff $\overline{\mathbf{P}} = \mathbf{A}$. Hence we have $\mathcal{M}, \lambda_0 \models \neg\Theta$ iff $\mathbf{P} = \mathbf{E}$
- The case $\Psi = \Theta_1 \vee \Theta_2$ is similar to the previous case.

Abelard is the seeker at the last state $\lambda_0[m]$ of λ_0 and may attempt to end the transition game at $\lambda_0[m]$. By our assumption Eloise does not become a seeker and thus the evaluation game is continued from $(\mathbf{E}, \lambda_0[m], \Phi, T)$ for some T. By the induction above, we must have $\mathcal{M}, \lambda_0 \models \Phi$. Hence by Lemma 4.10 we have $\mathcal{M}, \lambda \models \Phi$.

Suppose now that there is a joint (perfect recall) strategy S_A s.t. $\mathcal{M}, \lambda \models \Phi$ for every $\lambda \in \mathsf{paths}(q, S_A)$. We define a *perfect recall* strategy τ for Eloise in the following way: Suppose that game is at some configuration c that is reached with a finite path λ_0 such that q_0 is the last state of λ_0.

- If some $\mathcal{M}, q_0 \models \theta$ for some $\psi \,\mathsf{U}\, \theta \in At(\Phi)$, then Eloise claims that θ is true.
- If some $\mathcal{M}, q_0 \not\models \psi$ for some $\psi \,\mathsf{U}\, \theta \in At(\Phi)$, then Eloise claims that ψ is false.
- Suppose that $q_0 = \lambda[0]$ and $\psi \in At(\Phi)$ is a state formula. If $\mathcal{M}, q_0 \models \psi$, then Eloise claims that ψ is true.
- Suppose $q_0 = \lambda[1]$ and $\mathsf{X}\,\psi \in At(\Phi)$. If $\mathcal{M}, q_0 \models \psi$, then Eloise claims that $\mathsf{X}\,\psi$ is true.
- If Abelard makes any claim on the truth of formulae, Eloise always challenges those claims. (Note here that Abelard's claim must be false—according to the compositional truth condition—since else Eloise would already have made the same claim by herself.)
- If Eloise is the seeker in c and $\mathcal{M}, \lambda_0 \models \Phi$, then Eloise decides to end seeking.
- If Abelard ends the seeking at c and $\mathcal{M}, \lambda_0 \not\models \Phi$, then Eloise decides to become seeker. Else Eloise ends the transition game at c.
- If Eloise needs to choose actions for agents in coalition A at c, she chooses them according to $S_A(\lambda_0)$.

We show by (co)-induction on the configurations of the transition game $\mathbf{g}(\mathbf{E}, q, \langle\!\langle A \rangle\!\rangle\, \Phi)$, that when Eloise uses τ she cannot end up in a losing ending position.

- Let $c = (\mathbf{E}, \mathbf{S}, q', T, n, \mathbf{i})$. Since the verifications and objections are made according to the compositional semantics on the current state, Eloise has a winning strategy from any possible exit position by the (outer) induction hypothesis.
- Let $c = (\mathbf{E}, \mathbf{S}, q', T, n, \mathbf{ii})$. By Lemma 4.10 and the definition of τ, the transition game can only end when $\mathcal{M}, \lambda_0 \models \Phi$. Hence from the exit position $(\mathbf{E}, q', \Phi, T)$, Eloise can play in such way that for any position $(\mathbf{P}, q', \Psi, T)$, that is reached, the following condition holds:
$$\mathcal{M}, \lambda_0 \models \Psi \;\text{ iff } \mathbf{P} = \mathbf{E},$$
where Ψ is a subformula of Φ such that there is $\varphi \in At(\Phi)$ which is a subformula of Ψ. Eventually, a location of the form $(\mathbf{P}, q', \varphi, T)$ is reached, where $\varphi \in At(\Phi)$. Since the verifications by τ are made according to the compositional truth of the relational atoms of Φ, it is quite obvious to see that $(\mathbf{P}, q', \varphi, T)$ is a winning position for Eloise.

- Let $c = (\mathbf{E}, \mathbf{S}, q', T, n, \mathbf{iii})$. This configuration does not lead to any exit locations.

Since Eloise chooses actions for agents in A according to S_A, every path of *states* that is formed with τ is a prefix sequence of some path $\lambda \in \mathsf{paths}(q, S_A)$. Since $\mathcal{M}, \lambda \models \Phi$ for every $\lambda \in \mathsf{paths}(q, S_A)$, by Lemma 4.10, and the definition of τ, Eloise cannot stay as a seeker forever when playing with τ. If Abelard stays as a seeker forever, then Eloise wins. Hence τ is a (perfect recall) winning strategy for Eloise. Since unbounded transition games are positionally determined, there is also *positional* winning strategy τ' for Eloise. □

By combining Theorem 4.11 and Corollary 4.7, we immediately obtain the following corollary:

Corollary 4.12. *If* $\Gamma \geq \mathsf{RBB}(\mathcal{M})$*, then the* Γ*-bounded* GTS *is equivalent on* $\mathcal{M}$ *with the standard (perfect recall) compositional semantics of* ATL^+*.*

5 Model checking ATL$^+$ using GTS

In this section we apply our GTS to model checking problems for ATL^+ and its fragments.

5.1 Revisiting the PSPACE upper bound proof

As mentioned earlier, the PSPACE upper bound proof for the model checking of ATL^+ in [5] contains a flaw. Indeed, the claim of Theorem 4 in [5] is incorrect and a counterexample to it can be extracted from our Example 3.3, where $\mathcal{M}, q_0 \models \varphi$ for $\varphi = \langle\langle\{a_2\}\rangle\rangle\, (\mathsf{G}\, p_1 \vee \mathsf{F}\, p_2)$. In the notation of [5], since $|St_{\mathcal{M}}| = 3$ and $\mathcal{APF}(\varphi) = 2$, by the claim there must be a 6-witness strategy for the agent 2 for $(\mathcal{M}, q_0, \mathsf{G}\, p_1 \vee \mathsf{F}\, p_2)$. However, this is not the case, since the player 1 can choose to play at q_0 4 times β, and then α. Then $\mathcal{M}, \lambda \not\models^6 (\mathsf{G}\, p_1 \vee \mathsf{F}\, p_2)$ on any resulting path λ.

The reason for the problem indicated above is that compositional semantics easily ignores the role and power of the falsifier (Abelard) in the formula evaluation process. Still, using the GTS introduced above, we will demonstrate in a simple way that the upper bound result is indeed correct.

The input to the model checking problem of ATL^+ is an ATL^+ formula, a finite CGM $\mathcal{M}$ and a state q in $\mathcal{M}$. We assume that $\mathcal{M}$ is encoded in the standard, explicit way (cf. [3, 5]) that provides a full explicit description of the transition function o. We need not assume any bound on the number of agents or proposition symbols in the input. We do not assume any bounds on the number of proposition symbols or agents in the input. We only consider here the semantics of ATL^+ based on perfect information and perfect-recall strategies.

Theorem 5.1 ([5])**.** *The* ATL^+ *model checking problem is* PSPACE*-complete.*

Proof. We get the lower bound directly from [5], so we only prove the upper bound here. By Theorem 4.11 and Proposition 4.3, if $\mathcal{M}$ is a finite CGM, we have $\mathcal{M}, q \models \varphi$ iff Eloise has a positional winning strategy in $\mathcal{G}(\mathcal{M}, q, \varphi, N)$ with $N = |\,\mathrm{St}\,| \cdot |\varphi|$. It is routine to construct an alternating Turing machine TM that simulates $\mathcal{G}(\mathcal{M}, q, \varphi, N)$ such that the positions for Eloise correspond to existential states of TM and Abelard's positions to universal states. Due to the timer bound N, the machine runs in polynomial time. It is clear that if Eloise has a (positional or not) winning strategy in the evaluation game, then TM accepts. Conversely, if TM accepts, we can read a non-positional winning strategy for Eloise from the the computation tree (with only one successful move for existential states recorded everywhere) which demonstrates that TM accepts. By Proposition 4.1, Eloise thus also has a positional winning strategy in the evaluation game. Since APTIME = PSPACE, the claim follows. □

5.2 A hierarchy of tractable fragments of ATL^+

We now identify a natural hierarchy of tractable fragments of ATL^+. Let k be a positive integer. Define ATL^k to be the fragment of ATL^+ where all formulae $\langle\langle A\rangle\rangle\,\Phi$ have the property that $|At(\Phi)| \leq k$. Note that ATL^1 is essentially the same as ATL (with Release). Note also that the number of non-equivalent formulae of ATL^k is not bounded for any k *even in the special case where the number of propositions and actions is constant*, because nesting of strategic operators $\langle\langle A\rangle\rangle$ is not limited. Still, we will show that the model checking problem for ATL^k is PTIME-complete for any fixed k. Again we assume that CGMs are encoded explicitly and impose no restrictions on the number of allowed propositions or actions in input formulae. Using the GTS, we can prove the following.

Theorem 5.2. *For any fixed $k \in \mathbb{N}$, the model checking problem for ATL^k is* PTIME-*complete.*

Proof. (Sketch) The claim is well-known for ATL (see [3]), so we have the lower bound for free for any k. One possible proof strategy for the upper bound would involve using alternating LOGSPACE-machines, but here we argue via Büchi-games instead.

Consider a triple $(\mathcal{M}, q, \varphi)$, where φ is formula of ATL^k. By the proof of Proposition 4.2, there exists a Büchi game BG such that Eloise wins the unbounded evaluation game $\mathcal{G}(\mathcal{M}, q, \varphi)$ if the player 2 wins BG from the state of BG that corresponds to the beginning position of the evaluation game. We then observe that since we are considering ATL^k for a *fixed* k, the domain size of each truth function T used in the evaluation game is at most k, and thus the number of positions in $\mathcal{G}(\mathcal{M}, q, \varphi)$ is *polynomial* in the size of the input $(\mathcal{M}, q, \varphi)$. (Check Remark 3.1 for all the information that should be encoded in a position in *bounded* evaluation games; here we only use the simpler unbounded games.) Thus also the size of BG is polynomial in the input size.

We note that in order to avoid blow-ups, it is essential that the maximum domain size k of truth functions T is fixed. We also note—as mentioned already in [3]—that the number of transitions in $\mathcal{M}$ is not bounded by the square of the number of states of $\mathcal{M}$. In fact, already because we impose no limit on the number of actions (other than finiteness) in $\mathcal{M}$, the number of transitions in relation to states is arbitrary. However, this is no problem to us since an explicit encoding of $\mathcal{M}$—which lists all transitions explicitly—is part of the input to the model checking problem. Since Büchi games can be solved in PTIME, the claim follows. □

5.3 Bounded memory semantics for ATL^k

Here we show that to capture the compositional (perfect-recall) semantics for ATL^k, it suffices to consider agents' strategies that use only a limited amount of memory.

Strategies with bounded memory for ATL^* can be naturally defined using *finite state transducers.* (For a transdu-cer-based definition of bounded memory strategies, see e.g. [19], and see [4] for more on this topic.) Using such strategies, an agent's moves are determined both by the current state in the model and by the current state (*memory cell*) of the agent's transducer. Then transitions take place both in the model and in the state of the transducer. (Thus, such strategies are positional with respect to the product of the two state spaces.) In the compositional m-**bounded memory**

semantics ($\models^m$) for ATL^+, agents are allowed to use at most m memory cells, i.e., strategies defined by transducers with at most m states.

Since the use of the truth function T in our GTS is analogous to the use of memory cells in m-bounded memory semantics, we obtain the following result.

Theorem 5.3. *For* ATL^k*, the unbounded* GTS *is equivalent to the* m*-bounded memory semantics for* $m = 3^k - 2^k$

Proof. Let $m := 3^k - 2^k$ and $\varphi \in \mathsf{ATL}^k$. We show that

$$\mathcal{M}, q \Vdash \varphi \text{ iff } \mathcal{M}, q \models^m \varphi.$$

The implication from right to left is immediate by Theorem 4.11. We prove the other direction by induction on φ. The only interesting case is when $\varphi = \langle\!\langle A \rangle\!\rangle \, \Phi$. Suppose that Eloise has a winning strategy in $\mathbf{g}(\mathbf{E}, q, \langle\!\langle A \rangle\!\rangle \, \Phi)$. By Claim I we may assume that τ is regular.

We define a memory transducer $\mathcal{T}$ that Eloise can use to define strategies for all agents in A. We fix the set of states C of $\mathcal{T}$ to be the set of all truth functions T for $At(\Phi)$ such that $T(\chi) = \mathsf{open}$ for at least one $\chi \in At(\Phi)$. Since $T(\chi) \in \{\mathsf{open}, \top, \bot\}$, we have $|C| \leq 3^k - 2^k = m$. The initial state of $\mathcal{T}$ is T_0 where $T_0(\chi) = \mathsf{open}$ for every $\chi \in At(\Phi)$. The transitions in $\mathcal{T}$ are defined according to how Eloise updates the truth function T during the transition game. However, when the truth function T becomes fully updated (i.e. $T(\chi) \neq \mathsf{open}$ for every $\chi \in At(\Phi)$), then no further transitions are made, because in this case all the relative atoms have been verified/falsified and the truth of Φ on the path is fixed.

Now, the strategy for each agent $a \in A$ is defined positionally on $C \times \mathrm{St}$ as follows: At a state T of $\mathcal{T}$ and state $q \in \mathcal{M}$, the agent a follows the action prescribed by Eloise's winning strategy for the corresponding step phase in the transition game. Note that the strategy for A is now well-defined since τ is regular and thus depends only on the current state and the current truth function.

It is now easy to show that $\mathcal{M}, \lambda \models^m \Phi$ for any path λ that is consistent with the resulting collective strategy for the coalition A. □

By Theorem 4.11, we obtain the following corollary.

Corollary 5.4. *For* ATL^k*, the perfect recall compositional semantics is equivalent to the* $(3^k - 2^k)$*-bounded memory semantics.*

This extends the known fact that positional strategies (using 1 memory cell) suffice for the semantics of ATL (which is essentially the same as ATL^1).

For a better bound of the required memory it would be sufficient to modify slightly the transition games and consider a bound k not on the number of all relative atoms in strategic subformulae, but only of the temporal objectives occurring in them.

Conclusion

The GTS for ATL^+ developed here has both conceptual and technical importance, as it explains better how the memory-based strategies in the compositional semantics can be generated, and thus also provides better insight on the algorithmic aspects of that semantics. A natural extension of the present work is to develop GTS for the full ATL^*.

Acknowledgements

The work of Valentin Goranko was supported by a research grant 2015-04388 of the Swedish Research Council. The work of Antti Kuusisto was supported by the ERC grant 647289 "CODA." We thank the referees for valuable remarks.

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