Ontology Approximation in Horn Description Logics

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Abstract

We study the approximation of a description logic (DL) ontology in a less expressive DL, focussing on the case of Horn DLs. It is common to construct such approximations in an ad hoc way in practice and the resulting incompleteness is typically neither analyzed nor understood. In this paper, we show how to construct complete approximations. These are typically infinite or of excessive size and thus cannot be used directly in applications, but our results provide an important theoretical foundation that enables informed decisions when constructing incomplete approximations in practice.

1 Introduction

There is a large number of description logics (DLs) that vary considerably regarding their expressive power and computational properties [Baader *et al.*, 2017] and despite prominent standardization efforts, many different DLs continue to be used.[1] As a result, it can be necessary to convert an ontology formulated in a DL $\mathcal{L}_S$, the *source DL*, into a different DL $\mathcal{L}_T$, the *target* DL. For example, this happens in *ontology import* when an engineer who designs an ontology formulated in $\mathcal{L}_T$ wants to reuse content from an existing ontology formulated in $\mathcal{L}_S$. A particularly important case is that $\mathcal{L}_T$ is a fragment of $\mathcal{L}_S$, in which case the described problem is *ontology approximation*, a form of knowledge compilation [Selman and Kautz, 1996; Darwiche and Marquis, 2002].

In this paper, we are interested in approximating an ontology $\mathcal{O}_S$ formulated in a DL $\mathcal{L}_S$ by an ontology $\mathcal{O}_T$ formulated in a fragment $\mathcal{L}_T$ of $\mathcal{L}_S$, aiming to preserve all information from $\mathcal{O}_S$ that is expressible in $\mathcal{L}_T$; this is called a *greatest lower bound* in knowledge compilation [Selman and Kautz, 1996]. Formally, for every $\mathcal{L}_T$ concept inclusion $C \sqsubseteq D$ that is formulated in the signature Σ of $\mathcal{O}_S$, we require that $\mathcal{O}_S \models C \sqsubseteq D$ if and only if $\mathcal{O}_T \models C \sqsubseteq D$, and likewise for role inclusions and any other type of ontology statement supported by $\mathcal{L}_T$. We say that $\mathcal{O}_T$ is *sound* as an approximation if it satisfies the "if" part of this property and *complete* if it satisfies the "only if" part. We consider the case that $\mathcal{O}_T$ must be formulated in Σ (*non-projective* approximation) and the case that additional symbols are admitted (*projective* approximation).

In practice, approximations are often constructed in an ad hoc way that is sound but not complete. In fact, it is common to simply drop all statements from $\mathcal{O}_S$ that are not expressible in $\mathcal{L}_T$, or at least inexpressible parts thereof. This easily leads to incompleteness, as illustrated by the following example extracted from the Galen ontology[2], slightly simplified for presentation purposes. Let PathoPhen stand for "pathological phenomenon", isConOf for "is consequence of", and hasCon for "has consequence". Galen contains the following statements formulated in the DL $\mathcal{ELHI}$, the first three being concept inclusions and the fourth one a role inclusion:

$$\mathtt{Hyperhidrosis} \sqsubseteq \mathtt{PathoPhen} \sqcap \exists \mathtt{hasCon}.\mathtt{ClammySkin}$$
$$\exists \mathtt{isConOf}.\mathtt{PathoPhen} \sqsubseteq \mathtt{PathoPhen}$$
$$\exists \mathtt{hasCon}.\mathtt{PathoPhen} \sqsubseteq \mathtt{PrecipitatingFactor}$$
$$\mathtt{hasCon} \sqsubseteq \mathtt{isConOf}^-.$$

These imply as a consequence

$$\mathtt{Hyperhidrosis} \sqsubseteq \mathtt{PrecipitatingFactor}. \quad (1)$$

Assume that this ontology has to be approximated in the fragment $\mathcal{ELH}$ of $\mathcal{ELHI}$ that does not admit inverse roles. A typical ad hoc approach would be to simply drop the role inclusion in the fourth line, resulting in an incomplete approximation that no longer has Consequence (1). This, however, can easily be fixed by further adding the concept inclusion

$$\mathtt{Hyperhidrosis} \sqcap \exists \mathtt{hasCon}.\top \sqsubseteq \exists \mathtt{hasCon}.\exists \mathtt{isConOf}.\mathtt{Hyperhidrosis}. \quad (2)$$

as a (partial) substitute of the dropped role inclusion.

The aim of this paper is to systematically study the structure of *complete* ontology approximations. There is, however, a major caveat. As we show, complete approximations must be infinite even in rather simple cases. Moreover, while finite approximations exist when the depth of the concept inclusions to be preserved is bounded by a constant, the resulting approximations are still of non-elementary size. There is no miraculous way around these facts and thus the approximations constructed in this paper cannot be directly used in

[1]See, for example, the BioPortal repository at https://bioportal.bioontology.org/.

[2]http://www.opengalen.org/

applications. However, they provide an important theoretical foundation that enable and guide informed decisions when constructing incomplete approximations in practice. In the above example taken from Galen, for instance, concept inclusion (2) is part of the complete approximation proposed in this paper and thus an explicit candidate for inclusion also in approximations constructed in practice.

As the source DL $\mathcal{L}_S$, we consider the expressive Horn DL Horn-$\mathcal{SRIF}$ and fragments thereof. As the target DL $\mathcal{L}_T$, we consider $\mathcal{ELR}_\bot$ and corresponding fragments thereof, where $\mathcal{ELR}_\bot$ denotes the extension of the more widely known DL $\mathcal{ELH}_\bot$ with role inclusions of the form $r_1 \circ \cdots \circ r_n \sqsubseteq r$. Subsumption is EXPTIME-complete in all considered source DLs and PTIME-complete in all cosidered target DLs [Baader *et al.*, 2005; Krötzsch *et al.*, 2013]. While our approximations do *not* aim at efficient reasoning, we thus support ontology designers who build an ontology in a tractable DL and want to import in a well-understood way from an existing ontology formulated in a computationally more expensive DL.

We provide the following results. In Section 3, we construct $\mathcal{ELF}$-to-$\mathcal{EL}$ approximations, thus approximating away functional roles. We then proceed to $\mathcal{ELH}\scriptstyle\mathcal{I}$-to-$\mathcal{ELH}$, approximating away inverse roles, where the $\mathcal{I}$ typeset in small font means that inverse roles are admitted only in role inclusions of the form $r \sqsubseteq s^-$ but not in concept inclusions. This is a very common way to use inverse roles in practice, for example more than 96% of the ontologies in BioPortal that use inverse roles at all use them only in this form and this is similar for other ontology repositories. We next treat $\mathcal{ELH}{\scriptstyle\mathcal{I}}\mathcal{F}_\bot$-to-$\mathcal{ELH}_\bot$ under a certain syntactic assumption that restricts the interplay of role inclusions, functional roles, and inverse roles in $\mathcal{O}_S$. This covers also other relevant subcases such as $\mathcal{ELHF}$-to-$\mathcal{ELH}$, without syntactic restrictions. All approximations constructed in Section 3 are non-projective and also provide finite approximations in the depth bounded case. The completeness proofs are non-trivial and use a version of the chase that is specifically tailored to our approximation schemes.

In Section 4, we present $\mathcal{ELR}{\scriptstyle\mathcal{I}}\mathcal{F}_\bot$-to-$\mathcal{ELR}_\bot$ approximations. The presented approximations are non-projective when $\mathcal{O}_S$ is *inverse closed*, meaning that for every role name r in $\mathcal{O}_S$, there is a role name $\hat{r}$ that is defined via role inclusions to be the inverse of r. This also yields *projective* approximations for the case where inverse closedness is not assumed and for the Horn-$\mathcal{SRIF}$-to-$\mathcal{ELR}_\bot$ case through a well-known normalization procedure. The completeness proof is again non-trivial, but based on a different approach, namely a novel connection between ontology approximation and the axiomatizations of quasi-equations valid in classes of semilattices with operators (SLOs) [Sofronie-Stokkermans, 2013; Sofronie-Stokkermans, 2017; Kikot *et al.*, 2017].

We then proceed to study $\mathcal{ELI}_\bot$-to-$\mathcal{EL}_\bot$ approximations in Section 5. In contrast to the cases considered before, where (after normalization) both $\mathcal{L}_S$ and $\mathcal{L}_T$ are based on the concept language $\mathcal{EL}_\bot$, here the *concept language* of $\mathcal{L}_S$ is different from the one of $\mathcal{L}_T$. We present non-projective approximations for unrestricted ontologies $\mathcal{O}_S$ and for ontologies $\mathcal{O}_S$ which are in the well-known normal form for $\mathcal{ELI}_\bot$ ontologies that avoids syntactic nesting of concepts. The two approximation schemes are remarkably different.

In Section 6, we show that finite approximations are not guaranteed to exist and that there are cases where depth bounded approximations must be non-elementary in size.

Proof details are available in the appendix, which is available at http://www.informatik.uni-bremen.de/tdki/.

Related Work

Approximation in a DL context was first studied in [Selman and Kautz, 1996] where $\mathcal{FL}$ concepts are approximated by $\mathcal{FL}^-$ concepts and in [Brandt *et al.*, 2002] where $\mathcal{ALC}$ concepts are approximated by $\mathcal{ALE}$ concepts. In both cases, the approximation always exists, but ontologies are not considered. An incomplete approach to approximating $\mathcal{SHOIN}$ ontologies in DL-Lite$_\mathcal{F}$ is presented in [Pan and Thomas, 2007] and complete (projective) approximations of $\mathcal{SROIQ}$ ontologies in DL-Lite$_\mathcal{A}$ are given in [Botoeva *et al.*, 2010]. Such approximations are guaranteed to exist due to the limited expressive power of DL-Lite$_\mathcal{A}$. In [Lutz *et al.*, 2012], approximation of $\mathcal{ELU}$ ontologies in terms of $\mathcal{EL}$ ontologies is studied, the main result being that it is EXPTIME-hard and in 2EXPTIME to decide whether a finite complete approximation exists. An incomplete approach to approximating $\mathcal{SROIQ}$ ontologies in $\mathcal{EL}^{++}$ is in [Ren *et al.*, 2010]. There are also approaches towards efficient DL reasoning that involve computing approximations which are intentionally incomplete to avoid compromising efficiency [Schaerf and Cadoli, 1995; Groot *et al.*, 2005; Carral *et al.*, 2014]. Related to approximation is the problem whether a given $\mathcal{L}_S$ ontology can be equivalently rewritten into the fragment $\mathcal{L}_T$ of $\mathcal{L}_S$, either non-projectively [Lutz *et al.*, 2011] or projectively [Konev *et al.*, 2016]; note that this asks whether we have to approximate at all.

2 Preliminaries

Let $\mathsf{N_C}$ and $\mathsf{N_R}$ be disjoint and countably infinite sets of *concept* and *role names*. A *role* is a role name r or an *inverse role* r^-, with r a role name. A *Horn-$\mathcal{SRIF}$ concept inclusion (CI)* is of the form $L \sqsubseteq R$, where L and R are concepts defined by the syntax rules

$$R, R' ::= \top \mid \bot \mid A \mid \neg A \mid R \sqcap R' \mid \neg L \sqcup R \mid \exists \rho.R \mid \forall \rho.R$$
$$L, L' ::= \top \mid \bot \mid A \mid L \sqcap L' \mid L \sqcup L' \mid \exists \rho.L$$

with A ranging over concept names and ρ over roles. The *depth* of a concept R or L is the nesting depth of the constructors $\exists\rho$ and $\forall\rho$. For example, the concept $\exists r.B \sqcap \exists r.\exists s.A$ is of depth 2. A *Horn-$\mathcal{SRIF}$ ontology* $\mathcal{O}$ is a set of

- Horn-$\mathcal{SRIF}$ concept inclusions,
- *functionality assertions* $\mathsf{func}(\rho)$, and
- *role inclusions (RIs)* $\rho_1 \circ \cdots \circ \rho_n \sqsubseteq \rho$.

We adopt the standard restriction that if $n \geq 2$, then neither $\mathcal{O} \models \mathtt{func}(\rho)$ nor $\mathcal{O} \models \mathtt{func}(\rho^-)$. The semantics of Horn-$\mathcal{SRIF}$ is standard, see [Baader *et al.*, 2017]. While ontologies used in practice have to be finite, in this paper we shall frequently consider also infinite ontologies. W.l.o.g,, we generally assume that the $\bot$ concept occurs only in CIs of the form $C \sqsubseteq \bot$, where C does not contain $\bot$.

We briefly introduce the relevant fragments of Horn-$\mathcal{SRIF}$, for details see [Baader *et al.*, 2017]. An $\mathcal{ELI}_\bot$ *concept* is built according to the syntax rule for L above, but omitting disjunction. An $\mathcal{ELRIF}_\bot$ *ontology* is a Horn-$\mathcal{SRIF}$ ontology in which both the left- and right-hand sides of CIs are $\mathcal{ELI}_\bot$ concepts. $\mathcal{ELHIF}_\bot$ is defined likewise, but admitting only *role hierarchies* instead of role inclusions, which take the form $r \sqsubseteq s$. Fragments of $\mathcal{ELRIF}_\bot$ and $\mathcal{ELHIF}_\bot$ can obtained by dropping expressive means that are identified by a standard naming scheme: $\mathcal{H}$ indicates role hierarchies, $\mathcal{R}$ role inclusions, $\mathcal{I}$ inverse roles, $\mathcal{F}$ functionality assertions, and $\cdot_\bot$ the bottom concept. It should thus be understood, for example, what an $\mathcal{ELI}_\bot$ ontology is and what an $\mathcal{EL}$ concept is. Among these DLs, $\mathcal{ELR}_\bot$ is maximal with a tractable subsumption problem. In all of the above DLs not contained in $\mathcal{ELR}_\bot$, subsumption is ExpTime-complete [Baader *et al.*, 2005; Baader *et al.*, 2008].

A *signature* Σ is a set of concept and role names, uniformly referred to as *symbols*. When speaking of $\mathcal{EL}(\Sigma)$ concept, we mean $\mathcal{EL}$ concepts that only use symbols from Σ, and likewise for other DLs. We use $\mathtt{sig}(\mathcal{O})$ to denote the set of symbols used in ontology $\mathcal{O}$.

We now define the central notions of this paper.

Definition 1 *Let $\mathcal{O}_S$ be a Horn-$\mathcal{SRIF}$ ontology with $sig(\mathcal{O}_S) = \Sigma$ and let $\mathcal{L}_T$ be any of the fragments of Horn-$\mathcal{SRIF}$ introduced above. A (potentially infinite) $\mathcal{L}_T$ ontology $\mathcal{O}_T$ is an $\mathcal{L}_T$* approximation of $\mathcal{O}_S$ *if*

$$\mathcal{O}_S \models \alpha \textit{ iff } \mathcal{O}_T \models \alpha$$

for all concept inclusions, role inclusions, and functionality assertions α that fall within $\mathcal{L}_T$ and use only symbols from Σ. We say that $\mathcal{O}_T$ is non-projective *if $\mathtt{sig}(\mathcal{O}_T) \subseteq \Sigma$ and* projective *otherwise.*

*For $\ell \in \mathbb{N} \cup \{\omega\}$, (non-projective and projective) ℓ-*bounded $\mathcal{L}_T$ approximations *are defined in the same way except that only concept inclusions $\alpha = C \sqsubseteq D$ are considered where C and D are of depth bounded by ℓ.*

Note that ω-bounded approximations are identical to unbounded approximations, we use the term only for uniformity. Trivially, infinite (non-projective and projective) approximations always exist: take as $\mathcal{O}_T$ the set of all inclusions and assertions from $\mathcal{L}_T$ that are entailed by $\mathcal{O}_S$. One can show that there are $\mathcal{ELI}$ ontologies $\mathcal{O}_S$ that have a finite projective $\mathcal{EL}$ approximation, but no finite non-projective $\mathcal{EL}$ approximation; details are in the appendix.

Example 1 *Consider the $\mathcal{ELF}$ ontology*

$$\mathcal{O}_S = \{\exists \mathtt{hasSupervisor}.\top \sqsubseteq \mathtt{Employee}, \mathtt{func}(\mathtt{reportsTo})\}.$$

There is no finite $\mathcal{EL}$ approximation since for all $n, m \geq 1$, $\mathcal{O}_S$ entails the $\mathcal{EL}$ concept inclusion

$$\exists \mathtt{reportsTo}.\exists \mathtt{hasSupervisor}^n.\top \sqcap \exists \mathtt{reportsTo}^m.\top \sqsubseteq \exists \mathtt{reportsTo}.(\exists \mathtt{hasSupervisor}^n.\top \sqcap \exists \mathtt{reportsTo}^{m-1}.\top).$$

In practice, it clearly does not make sense to include all these CIs in the approximation. Similarly to the example in the introduction, though, it may pay off to include some *of them. Choosing the right ones requires a careful inspection of the ontology and application at hand.*

With $\mathcal{L}_S$*-to-*$\mathcal{L}_T$ *approximation*, $\mathcal{L}_S$ an ontology language and $\mathcal{L}_T$ a fragment thereof, we mean the task of approximating an $\mathcal{L}_S$ ontology in $\mathcal{L}_T$. We call $\mathcal{L}_S$ the *source DL* and $\mathcal{L}_T$ the *target DL*.

An alternative definition of approximations is obained by dropping the restriction that α can use only symbols from Σ. We do not use that definition because then even in the 1-bounded case, finite approximations might not exist.

Example 2 *Take the $\mathcal{ELI}$ ontology $\mathcal{O}_S = \{\exists r^-.A \sqsubseteq B\}$. Then $\mathcal{O} \models A \sqcap \exists r.X \sqsubseteq \exists r.(B \sqcap X)$ for each of the infinitely many concept names $X \in \mathsf{N_C}$. Thus, every (projective or non-projective) 1-bounded $\mathcal{EL}$ approximation of $\mathcal{O}_S$ must be infinite under the alternative definition of approximation.*

We now make some basic observations regarding approximations. The proof is straightforward.

Lemma 1 *Let $\mathcal{O}_S$ be a Horn-$\mathcal{SRIF}$ ontology with $sig(\mathcal{O}_S) = \Sigma$ and $\mathcal{L}_T$ a fragment of Horn-$\mathcal{SRIF}$. Then*

1. *$\mathcal{O}_T$ is an $\mathcal{L}_T$ approximation of $\mathcal{O}_S$ iff $\mathcal{O}_S \models \mathcal{O}_T$ and for every $\mathcal{L}_T$ ontology $\mathcal{O}$ with $\mathcal{O}_S \models \mathcal{O}$ and $\mathtt{sig}(\mathcal{O}) \subseteq \Sigma$, $\mathcal{O}_T \models \mathcal{O}$;*
2. *$\bigcup_{i \geq 0} \mathcal{O}_\ell$ is an $\mathcal{L}_T$ approximation of $\mathcal{O}_S$ if for all $\ell \geq 0$, $\mathcal{O}_\ell$ is an ℓ-bounded $\mathcal{L}_T$ approximation of $\mathcal{O}_S$; the same is true for projective $\mathcal{L}_T$ approximations provided that $\mathtt{sig}(\mathcal{O}_\ell) \cap \mathtt{sig}(\mathcal{O}_{\ell'}) \subseteq \Sigma$ when $\ell \neq \ell'$.*

Point 1 may be viewed as an alternative definition of (non-projective) approximations. Point 2 is important because it sometimes allows us to concentrate on bounded approximations in proofs. The following is well-known, see for example [Bienvenu *et al.*, 2016].

Lemma 2 *Given a Horn-$\mathcal{SRIF}$ ontology $\mathcal{O}_S$ with $\mathtt{sig}(\mathcal{O}) = \Sigma$, one can construct in polynomial time an $\mathcal{ELRIF}_\bot$ ontology $\mathcal{O}'_S$ with $\mathtt{sig}(\mathcal{O}'_S) \supseteq \Sigma$ that entails the same Horn-$\mathcal{SRIF}(\Sigma)$ concept inclusions, role inclusions, and functionality assertions.*

The construction of the ontology $\mathcal{O}'_S$ from Lemma 2 requires the introduction of fresh concept names. Still, every ℓ-bounded $\mathcal{L}_T$ approximation of $\mathcal{O}'_S$ is a projective ℓ-bounded $\mathcal{L}_T$ approximation of $\mathcal{O}_S$.

3 Depth Bounded Approximation

The goal of this section is to study non-projective approximations in various DLs, both in the unbounded case and in the depth bounded case. We start with approximating away functionality assertions, then inverse roles, and finally their combination (assuming a certain syntactic restriction), also admitting role hierarchies and the bottom concept. This step-by-step approach aims to facilitate presentation and in fact the final theorem in this section subsumes the earlier ones and is the only one that we prove explicitly.

We start with $\mathcal{ELF}$-to-$\mathcal{EL}$ approximation. Let C be an $\mathcal{EL}$ concept and $k \geq 0$. A *leaf occurrence* of a concept name A in C means an occurrence of A inside a conjunction that does not contain conjuncts of the form $\exists r.D$. For example, all occurrences of A in $A \sqcap B$ and in $B \sqcap \exists r.(A \sqcap B)$ are leaf occurrences, but the occurrence of A in $A \sqcap B \sqcap \exists r.\top$ is not. By *decorating C with subconcepts from $\mathcal{O}_S$ at leaves*, we mean to replace any number of leaf occurrences of a concept name A by a concept $A \sqcap D_1 \sqcap \cdots \sqcap D_k$ where $D_1, \ldots, D_k$ are concepts that occur in $\mathcal{O}_S$, possibly as subconcepts.

Theorem 1 *Let $\mathcal{O}_S$ be an $\mathcal{ELF}$ ontology, $\Sigma = \mathtt{sig}(\mathcal{O}_S)$, and $\ell \in \mathbb{N} \cup \{\omega\}$ a depth bound. Define $\mathcal{O}_T$ to be the $\mathcal{EL}$ ontology that contains:*

1. *all CIs from $\mathcal{O}_S$;*
2. *all CIs $\exists r.C_1 \sqcap \exists r.C_2 \sqsubseteq \exists r.(C_1 \sqcap C_2)$ such that $\mathtt{func}(r) \in \mathcal{O}_S$ and C_1, C_2 are $\mathcal{EL}(\Sigma)$ concepts of depth bounded by $\max\{0, \ell - 1\}$ decorated with subconcepts of $\mathcal{O}_S$ at leaves.*

Then $\mathcal{O}_T$ is an ℓ-bounded approximation of $\mathcal{O}_S$.

Note that the construction of $\mathcal{O}_T$ is entirely syntactic, that is, it involves no reasoning. Due to Point 2, $\mathcal{O}_T$ is infinite when $\ell = \omega$. On the other hand, $\mathcal{O}_T$ is finite when $\ell < \omega$ and thus Theorem 1 proves the existence of finite depth bounded non-projective approximations in the $\mathcal{ELF}$-to-$\mathcal{EL}$ case. It is interesting to remark that Theorem 1 also reproves the upper bound for subsumption in $\mathcal{ELF}$: first extend the ontology so that it suffices to decide subsumption between concept *names*, then compute the 0-bounded approximation which is of single exponential size, and then decide subsumption in $\mathcal{EL}$ in PTIME [Baader *et al.*, 2005]. Note that for $\ell = 0$, the concepts C_1, C_2 in Point 2 of Theorem 1 are simply conjunctions of subconcepts from $\mathcal{O}_S$.

We next consider inverse roles. Here, the most basic case is that inverse roles can occur in role inclusions, but not in concepts. As noted in the introduction, this case actually occurs rather frequently in practice. To indicate the restricted use of inverse roles, we typeset the $\mathcal{I}$ in a smaller font, as in $\mathcal{ELHIF}$ and $\mathcal{ELRIF}$. The most basic case is now that of $\mathcal{ELHI}$-to-$\mathcal{ELH}$ approximation. We assume w.l.o.g. that role inclusions only take the two forms $r \sqsubseteq s$ and $r \sqsubseteq s^-$.

Theorem 2 *Let $\mathcal{O}_S$ be an $\mathcal{ELHI}$ ontology, $\Sigma = \mathtt{sig}(\mathcal{O}_S)$, and $\ell \in \mathbb{N} \cup \{\omega\}$ a depth bound. Define $\mathcal{O}_T$ to be the $\mathcal{ELH}$ ontology that contains, for $\ell' = \max\{0, \ell - 1\}$:*

1. *all CIs from $\mathcal{O}_S$;*
2. *all RIs $r \sqsubseteq s$ such that $\mathcal{O}_S \models r \sqsubseteq s$ and role names r, s occur in $\mathcal{O}_S$;*
3. *all CIs $C_1 \sqcap \exists r.C_2 \sqsubseteq \exists r.(C_2 \sqcap \exists s.C_1)$ such that $\mathcal{O}_S \models r \sqsubseteq s^-$, $\exists s.C_1$ is a subconcept of $\mathcal{O}_S$ or an $\mathcal{EL}(\Sigma)$ concept of depth bounded by ℓ, and C_2 is an $\mathcal{EL}(\Sigma)$ concept of depth bounded by ℓ' decorated with subconcepts of $\mathcal{O}_S$ at leaves.*

Then $\mathcal{O}_T$ is an ℓ-bounded approximation of $\mathcal{O}_S$.

Note that Point 2 is not entirely syntactic, but involves reasoning. It is easy to see and well-known, however, that in $\mathcal{ELI}$ deciding whether $\mathcal{O}_S \models r \sqsubseteq s$ is in PTIME.

We now consider the $\mathcal{ELHIF}_\bot$-to-$\mathcal{ELH}_\bot$ case which combines functional and inverse roles. It turns out that there are subtle interactions between functional, inverse roles, and role hierarchies which we tame by making the following assumption:

(♡) $\mathcal{O}_S \models r \sqsubseteq s^-$ implies that neither $\mathtt{func}(s) \in \mathcal{O}_S$ nor $\mathtt{func}(s^-) \in \mathcal{O}_S$.

The next theorem is the main result of this section.

Theorem 3 *Let $\mathcal{O}_S$ be an $\mathcal{ELHIF}_\bot$ ontology that satisfies (♡), $\Sigma = \mathtt{sig}(\mathcal{O}_S)$, and $\ell \in \mathbb{N} \cup \{\omega\}$ a depth bound. Define $\mathcal{O}_T$ to be the $\mathcal{ELH}_\bot$ ontology that contains, for $\ell' = \max\{0, \ell - 1\}$:*

1. *all CIs from $\mathcal{O}_S$;*
2. *all $r \sqsubseteq s$ such that $\mathcal{O}_S \models r \sqsubseteq s$, r, s role names that occur in $\mathcal{O}_S$;*
3. *all CIs $C_1 \sqcap \exists r.C_2 \sqsubseteq \exists r.(C_2 \sqcap \exists s.C_1)$ such that $\mathcal{O}_S \models r \sqsubseteq s^-$, $\exists s.C_1$ is a subconcept of $\mathcal{O}_S$ or an $\mathcal{EL}(\Sigma)$ concept of depth bounded by ℓ, and C_2 is an $\mathcal{EL}(\Sigma)$ concept of depth bounded by ℓ' decorated with subconcepts of $\mathcal{O}_S$ at leaves;*
4. *all CIs $\exists r_1.C_1 \sqcap \exists r_2.C_2 \sqsubseteq \exists r_1.(C_1 \sqcap C_2)$ such that there is a role name s with $\mathcal{O}_S \models r_1 \sqsubseteq s$, $\mathcal{O}_S \models r_2 \sqsubseteq s$, and $\mathtt{func}(s) \in \mathcal{O}_S$, and C_1, C_2 are $\mathcal{EL}(\Sigma)$ concepts of depth bounded by ℓ' decorated with subconcepts of $\mathcal{O}_S$ at leaves.*

Then $\mathcal{O}_T$ is an ℓ-bounded approximation of $\mathcal{O}_S$.

It is not hard to see that Theorem 3 implies Theorems 1 and 2. It also settles additional approximation cases such as $\mathcal{ELHF}_\bot$-to-$\mathcal{ELH}_\bot$, without syntactic assumptions. Points 2 to 4 require deciding whether $\mathcal{O}_S \models r \sqsubseteq s^{(-)}$, which is EXPTIME-complete in $\mathcal{ELHIF}_\bot$ as can be proved by mutual reduction with subsumption.

It is straightforward to verify that the ontology $\mathcal{O}_T$ constructed in Theorem 3 is sound as an approximation, that is, $\mathcal{O}_S \models \mathcal{O}_T$. Completeness is non-trivial. It is established by first introducing a version of the chase that is closely tailored towards the construction of $\mathcal{O}_T$ given in Theorem 3, then showing that the chase is sound and complete regarding the derivation of $\mathcal{EL}(\Sigma)$ CIs of depth bounded by ℓ, and finally proving that the CIs in $\mathcal{O}_T$ can simulate derivations of the chase. With chase, we mean a rule based approach to constructing (infinite) 'canonical models' [Kontchakov and Zakharyaschev, 2014].

An interesting observation about the proof of Theorem 3 is that it actually yields a more general result than stated in that theorem. Instead of ℓ-bounded approximations, one could define Γ-*bounded approximations* for any set of $\mathcal{EL}_\bot$-concepts Γ closed under subconcepts, that is, only concepts from Γ are considered in concept inclusions α in Definition 1. We then obtain a version of Theorem 3 in which "concept of depth bounded by ℓ or ℓ'" is replaced with "concept from Γ" (decorated with subconcepts of $\mathcal{O}_S$ at leaves as needed). While one could choose for Γ the set of all concepts of depth bounded by some ℓ, other choices of Γ might be natural, too. For example, if one wants to decide subsumption between compound $\mathcal{EL}$ concepts C and D relative to an $\mathcal{ELHI}$ ontology $\mathcal{O}_S$ without resorting to concept names, then one can approximate $\mathcal{O}_S$ in $\mathcal{EL}$ relative to the set Γ of subconcepts of C and D and then check whether $\mathcal{O}_T$ entails $C \sqsubseteq D$. While this is clearly not efficient in practice, it raises the interesting question of how to identify sets Γ that are tailored towards the actual application of an ontology.

We now briefly discuss the case in which the restriction (♡) is dropped. One can prove that we then need to extend Points 1 to 4 of Theorem 3 with the following:

5. all CIs $\exists r.\exists s.C \sqsubseteq C$ such that $\mathcal{O}_S \models r \sqsubseteq s^-$, $\mathtt{func}(s) \in \mathcal{O}_S$, and C is a subconcept of $\mathcal{O}_S$ or an $\mathcal{EL}(\Sigma)$ concept of depth bounded by ℓ;
6. all CIs $\exists s.\exists r.C \sqsubseteq C$ such that $\mathcal{O}_S \models r \sqsubseteq s^-$, $\mathtt{func}(s^-) \in \mathcal{O}_S$, and C is a subconcept of $\mathcal{O}_S$ or an $\mathcal{EL}(\Sigma)$ concept of depth bounded by ℓ.

However, this is still not sufficient to obtain a complete approximation. Consider the $\mathcal{ELHI}$ ontology

$$\mathcal{O}_S = \{A \sqsubseteq \exists r_1.\exists r_2.(B \sqcap \exists s.\top), \\ s \sqsubseteq r_1,\ s \sqsubseteq r_2^-,\ \mathtt{func}(r_1^-),\ \mathtt{func}(r_2^-)\}.$$

It can be verified that $\mathcal{O}_S \models A \sqsubseteq B$. However, it can also be proved that even when $\mathcal{O}_T$ is the set of all statements from Points 1 to 6 with $\ell = 0$, $\mathcal{O}_T \not\models A \sqsubseteq B$. It remains open whether a transparent (non-projective) approximation is possible when (♡) is dropped.

4 Unbounded Approximation

We provide a significant extensions of Theorem 3 for the case of unbounded approximations, using an entirely different strategy for the completeness proof. In particular, we do not assume the (♡) restriction adopted in Theorem 3, admit inverse roles also in concepts, and add general role inclusions both to the source and target DL, that is, we consider $\mathcal{ELRIF}_\bot$-to-$\mathcal{ELR}_\bot$ approximation. There is a small price that we have to pay for this generality: the approximations constructed here are projective as for every role name from the original ontology, they contain a (potentially fresh) role name that represents its inverse. It is remarkable that this rather mild form of projectiveness overcomes several problems from the purely non-projective case.

An $\mathcal{ELRIF}_\bot$ ontology $\mathcal{O}$ is *inverse closed*, that is, for every role name r in $\mathcal{O}$, there is a role name $\hat{r}$ such that $\mathcal{O}$ contains $r \sqsubseteq \hat{r}^-$ and $\hat{r} \sqsubseteq r^-$. We provide non-projective approximations under the assumption that the source ontology is inverse closed. This also yields projective approximations for source ontologies that are not inverse closed because we can first extend $\mathcal{O}_S$ with the required role names $\hat{r}$ and then approximate. Note that in practice, there are relevant examples of ontologies that are inverse closed such as Galen. If our source is inverse closed, we can further assume that there are no other occurrences of inverse roles in $\mathcal{O}_S$, neither in concept inclusions nor in other role inclusions. In other words, our source ontology is formulated in $\mathcal{ELR}\imath\mathcal{F}_\bot$.

Theorem 4 *Let $\mathcal{O}_S$ be an inverse closed $\mathcal{ELR}\imath\mathcal{F}_\bot$ ontology and $\Sigma = \mathsf{sig}(\mathcal{O}_S)$. Define $\mathcal{O}_T$ to be the $\mathcal{ELR}_\bot$ ontology that contains for all $\mathcal{EL}(\Sigma)$ concepts C, D:*

1. *all CIs in $\mathcal{O}_S$;*
2. *all RIs $r \sqsubseteq s$ with $\mathcal{O}_S \models r \sqsubseteq s$, $r, s \in \Sigma$ role names;*
3. *all RIs $r_1 \circ \cdots \circ r_n \sqsubseteq r$ and $\hat{r}_n \circ \cdots \circ \hat{r}_1 \sqsubseteq \hat{r}$ such that $r_1 \circ \cdots \circ r_n \sqsubseteq r \in \mathcal{O}_S$ with $n \geq 2$;*
4. *all CIs $C \sqcap \exists r.D \sqsubseteq \exists r.(D \sqcap \exists \hat{r}.C)$;*
5. *all CIs $\exists r.C \sqcap \exists r.D \sqsubseteq \exists r.(C \sqcap D)$ such that $\mathsf{func}(r) \in \mathcal{O}_S$;*
6. *all CIs $\exists r.\exists \hat{r}.C \sqsubseteq C$ such that $\mathsf{func}(\hat{r}) \in \mathcal{O}_S$.*

Then $\mathcal{O}_T$ is an $\mathcal{ELR}_\bot$ approximation of $\mathcal{O}_S$.

Note that Points 1 to 3 essentially take over the part of $\mathcal{O}_S$ that is expressible in $\mathcal{ELR}_\bot$, Point 4 aims at capturing the consequences of inverse roles, Point 5 at functional roles, and Point 6 at the interaction between functional roles and inverse roles. Points 4 to 6 all introduce infinitely many CIs. Via Lemma 2, Theorem 4 also yields projective approximations for the Horn-$\mathcal{SRIF}$-to-$\mathcal{ELR}_\bot$ case.

The following example shows that the CIs in Point 5 of Theorem 3, which unlike Point 5 of Theorem 4 mix functional roles and role inclusions, are implied by the ontology $\mathcal{O}_T$ constructed in Theorem 4. The example also illustrates the strength of the 'inverse closed' property.

Example 3 *Let $r_1 \sqsubseteq s$, $r_2 \sqsubseteq s$, $\mathtt{func}(s) \in \mathcal{O}_S$, and let C_1, C_2 be $\mathcal{EL}$-concepts. We aim to show that*

$$\mathcal{O}_T \models \exists r_1.C_1 \sqcap \exists r_2.C_2 \sqsubseteq \exists r_1.(C_1 \sqcap C_2).$$

Due to Points 2 and 5 in Theorem 4, it suffices to show that

$$\mathcal{O}_T \models \exists s.C_i \sqcap \exists r_i.\top \sqsubseteq \exists r_i.C_i \ \text{ for } \ i \in \{1, 2\}.$$

This CI, in turn, can be proved from $\mathcal{O}_T$ as follows:

$$\begin{aligned} \mathcal{O}_T \models\ & \exists s.C_i \sqcap \exists r_i.\top \\ & \sqsubseteq \exists r_i.(\top \sqcap \exists \hat{r}_i.\exists s.C_i) \quad \textit{(Point 4)} \\ & \sqsubseteq \exists r_i.(\top \sqcap \exists \hat{s}.\exists s.C_i) \quad (\hat{r}_i \sqsubseteq \hat{s}_i \in \mathcal{O}_T \textit{ by Point 2}) \\ & \sqsubseteq \exists r_i.(\top \sqcap C_i) \quad \textit{(Point 6).} \end{aligned}$$

It is straightforward to show that the ontology $\mathcal{O}_T$ from Theorem 4 is sound as an approximation. To prove completeness, we establish a novel connection between $\mathcal{EL}_\bot$ approximations and axiomatizations of the quasi-equations that are valid in classes of semilattices with operators (SLOs) [Jackson, 2004; Sofronie-Stokkermans, 2017; Kikot *et al.*, 2017]. Roughly speaking, an approximation is obtained from such an axiomatization by instantiating its equations, which correspond (in the sense of modal correspondence theory) to the role inclusions in the original ontology, with $\mathcal{EL}$ concepts.

5 Inverse Roles in Concept Inclusions

As discussed before Theorem 4, the approximations provided by that theorem also cover the case where inverse roles are admitted in concept inclusions. This is achieved, however, by first making the ontology inverse closed and then dropping inverse roles from CIs. Here, we investigate alternative approaches in the basic case of $\mathcal{ELI}_\bot$-to-$\mathcal{EL}_\bot$, both non-projectively and projectively, in the latter case using a well-known normal form for $\mathcal{ELI}_\bot$ ontologies that avoids syntactic nesting [Baader *et al.*, 2017].

A key to constructing non-projective approximations is the observation that concepts of the form $\exists r^-.C$ can be used as a marker that is invisible to $\mathcal{EL}_\bot$.

Example 4 *Let $\mathcal{O}_S = \{A \sqsubseteq \exists s^-.\top, \exists r^-.\exists s^-.\top \sqsubseteq \exists s^-.\top, \exists s^-.\top \sqsubseteq B\}$. Then $\mathcal{O}_S \models C \sqsubseteq C'$ for all $\mathcal{EL}$ concepts C, C' where C' is obtained from C by decorating with*

B any node that is reachable in C from a node decorated with A along an r-path (we view an $\mathcal{EL}$ concept as a tree in the standard way, see for example [Konev et al.*, 2018]).*

We now give a non-projective approximation that captures the effect demonstrated in Example 4. For an $\mathcal{ELI}_\bot$ ontology $\mathcal{O}_S$, let $\mathtt{cl}_{\mathcal{EL}}(\mathcal{O}_S)$ denote the set of all $\mathcal{EL}$ concepts that can be obtained by starting with a subconcept of a concept from $\mathcal{O}_S$ and then replacing every subconcept of the form $\exists r^-.D$ with $\top$. Let C be an $\mathcal{EL}$ concept. An $\mathcal{EL}$ concept C' is a $\mathtt{cl}_{\mathcal{EL}}(\mathcal{O}_S)$ *decoration of* C if it can be obtained from C by conjunctively adding concepts from $\mathtt{cl}_{\mathcal{EL}}(\mathcal{O}_S)$ to a *single* occurrence of a subconcept in C.

Theorem 5 *Let $\mathcal{O}_S$ be an $\mathcal{ELI}_\bot$ ontology and $\Sigma = \mathtt{sig}(\mathcal{O}_S)$. Define $\mathcal{O}_T$ to be the $\mathcal{EL}_\bot$ ontology $\mathcal{O}_T$ that contains for all $\mathcal{EL}(\Sigma)$ concepts C:*

1. *all CIs $C \sqsubseteq C'$ such that $\mathcal{O}_S \models C \sqsubseteq C'$, C' a $\mathtt{cl}_{\mathcal{EL}}(\mathcal{O}_S)$ decoration of C;*
2. *all CIs $C \sqsubseteq \bot$ such that $\mathcal{O}_S \models C \sqsubseteq \bot$.*

Then $\mathcal{O}_T$ is an $\mathcal{EL}_\bot$ approximation of $\mathcal{O}_S$.

We prove completeness by a chase based approach. The CIs in Theorem 5 are rather different from those that we have used in Sections 3 and 4 to deal with inverse roles. Being much less constrained, they provides less guidance for constructing approximations in practice. We next observe that we can get back to the more constrained CI scheme for inverse roles by assuming the source ontology $\mathcal{O}_S$ to be in *normal form*, that is, all CIs in $\mathcal{O}_S$ have one of the forms $\top \sqsubseteq A_1$, $A_1 \sqsubseteq \bot$, $A_1 \sqsubseteq \exists\rho.A_2$, $\exists\rho.A_1 \sqsubseteq B$, and $A_1 \sqcap \cdots \sqcap A_n \sqsubseteq B$ where $A_1,\dots,A_n,B$ range over concept names and ρ ranges over roles. Every $\mathcal{ELI}_\bot$ ontology $\mathcal{O}_S$ can be converted into an $\mathcal{ELI}_\bot$ ontology $\mathcal{O}'_S$ in normal form in linear time such that $\mathcal{O}'_S$ is a conservative extension of $\mathcal{O}_S$ [Baader *et al.*, 2017]. Clearly, any approximation of $\mathcal{O}'_S$ is then a projective approximation of $\mathcal{O}_S$.

Theorem 6 *Let $\mathcal{O}_S$ be an $\mathcal{ELI}_\bot$ ontology in normal form, $\Sigma = \mathtt{sig}(\mathcal{O}_S)$, and $\ell \in \mathbb{N} \cup \{\omega\}$ a depth bound. Define $\mathcal{O}_T$ to be the $\mathcal{EL}_\bot$ ontology $\mathcal{O}_T$ that contains:*

1. *all CIs from $\mathcal{O}_S$ that are of the form $\top \sqsubseteq A$, $A \sqsubseteq \bot$, $\exists r.A \sqsubseteq B$, or $A \sqsubseteq \exists r.B$,;*
2. *all CIs $A_1 \sqcap \cdots \sqcap A_n \sqsubseteq B$ such that $\mathcal{O}_S \models A_1 \sqcap \cdots \sqcap A_n \sqsubseteq B$, $A_1,\dots,A_n,B \in \mathsf{N_C}$ occur in $\mathcal{O}_S$;*
3. *all CIs $A \sqcap \exists r.C \sqsubseteq \exists r.(C \sqcap B)$ such that $\exists r^-.A \sqsubseteq B \in \mathcal{O}_S$ and C is an $\mathcal{EL}(\Sigma)$ concept of depth bounded by $\ell - 1$.*

Then $\mathcal{O}_T$ is an ℓ-bounded $\mathcal{EL}_\bot$ appoximation of $\mathcal{O}_S$.

It is straightforward to verify that $\mathcal{O}_T$ is sound. To prove completeness, we again use a chase based strategy.

6 Size of Approximations

We prove that finite approximations are not guaranteed to exist and that depth bounded approximations can be non-elementary in size. These results hold both for projective and non-projective approximations and for all combinations of source and target DL considered in this paper. The ontologies used to prove these results are simple and show that also for most ontologies that occur in practical applications, neither finite (complete) approximations nor depth bounded (complete) approximations of elementary size can be expected. We focus on the cases $\mathcal{ELIH}$-to-$\mathcal{ELH}$, $\mathcal{ELHF}$-to-$\mathcal{ELH}$, and $\mathcal{ELH}\mathcal{I}$-to-$\mathcal{ELH}$, starting with unbounded approximations.

Theorem 7 *None of the ontologies*

$$\{\exists r^-.A \sqsubseteq B\}, \quad \{\mathsf{func}(r), A \sqsubseteq A\}, \quad \{r \sqsubseteq s^-, A \sqsubseteq A\}$$

has finite projective $\mathcal{ELH}$ approximations.

To get an idea of the proof, consider $\mathcal{O}_S = \{\exists r^-.A \sqsubseteq B\}$ and let $\mathcal{O}_T$ be a projective $\mathcal{ELH}$ approximation of $\mathcal{O}_S$. For all $n \geq 0$, let $C_n = \exists r^n.\top$, where $\exists r^n$ denotes n-fold nesting of an existential restriction, and observe that

$$\mathcal{O}_S \models A \sqcap \exists r.C_n \sqsubseteq \exists r.(B \sqcap C_n).$$

To establish the desired result, we prove that for every $n \geq 0$, there is a subconcept M_n of $\mathcal{O}_T$ such that $\mathcal{O}_T \models M_n \sqsubseteq C_n$ and $\mathcal{O}_T \not\models M_n \sqsubseteq C_m$ for any $m > n$.

We next show that bounded depth approximations can be non-elementary in size. The function $\mathtt{tower} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is defined as $\mathtt{tower}(0,n) := n$ and $\mathtt{tower}(k+1,n) := 2^{\mathtt{tower}(k,n)}$. The *size* of a (finite) ontology is the number of symbols needed to write it, with concept and role names counting as one. We use Γ_n to denote a fixed finite tautological set of $\mathcal{EL}$ concept inclusions that contains the symbols $\Sigma_n = \{r_1, r_2, A_1, \hat{A}_1, \dots, A_n, \hat{A}_n\}$.

Theorem 8 *Let $n \geq 0$ and let $\mathcal{O}_n$ be the union of Γ_n with any of the following sets:*

$$\{\exists r^-.A \sqsubseteq B\}, \quad \{\mathsf{func}(r), A \sqsubseteq A\}, \quad \{r \sqsubseteq s^-, A \sqsubseteq A\}$$

For every $\ell \geq 1$, any ℓ-bounded projective $\mathcal{ELH}$ approximation $\mathcal{O}_T$ of $\mathcal{O}_n$ must be of size at least $\mathtt{tower}(\ell, n)$.

7 Conclusion

It remains an open problem to develop informative non-projective approximations for (unrestricted) $\mathcal{ELHIF}_\bot$-to-$\mathcal{ELH}_\bot$ or even for Horn-$\mathcal{SHIF}$-to-$\mathcal{ELH}_\bot$ and Horn-$\mathcal{SRIF}$-to-$\mathcal{ELR}_\bot$. It would also be interesting to further extend the expressive power of both the source and target DLs. For example, nominals and range restrictions could be added even without compromising tractability of the latter [Baader *et al.*, 2005]. We remark that Theorem 4 can be adapted to the extension $\mathcal{ELR}^{dr}_\bot$ of $\mathcal{ELR}_\bot$ with range restrictions as the target DL by additionally including in $\mathcal{O}_T$ the range restriction $\mathtt{ran}(r) \sqsubseteq \exists\hat{r}.\top$ for every role name r in $\mathcal{O}_S$. Once more, inverse closedness pays off here as a corresponding extension of Theorem 3 appears to be more challenging.

There are many other relevant approximation cases that we did not touch upon, including the approximation of non-Horn DLs such as $\mathcal{ALC}$, $\mathcal{SHIQ}$, and $\mathcal{SROIQ}$ in (tractable and intractable) Horn DLs. It would further be of interest to understand how approximations can be better tailored towards relevant applications, for example in the spirit of choosing a set Γ of relevant concepts as discussed in Section 3.

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A Details for Section 2

We give an example of a $\mathcal{ELI}$ ontology with a finite projective $\mathcal{EL}$ approximation but no finite non-projective $\mathcal{EL}$ approximation.

Example 5 *Consider the $\mathcal{ELI}$ ontology*

$$\mathcal{O}_S = \mathcal{O} \cup \{A \sqsubseteq \exists r.\exists s^-.\top, \exists s^-.\top \sqsubseteq \exists r.\exists s^-.\top\}$$

where

$$\mathcal{O} = \{\exists s.\top \sqsubseteq \exists s.(A \sqcap X), X \sqsubseteq \exists r.(A \sqcap X), X \sqsubseteq \exists s.(A \sqcap X)\}.$$

Then an infinite non-projective $\mathcal{EL}$ approximation of $\mathcal{O}_S$ is $\mathcal{O}_T = \mathcal{O} \cup \{A \sqsubseteq \exists r^i.\top \mid i \geq 0\}$, there is no finite non-projective approximation, and a finite projective $\mathcal{EL}$ approximation for $\mathcal{O}_S$ is $\mathcal{O}_T = \mathcal{O} \cup \{A \sqsubseteq Y, Y \sqsubseteq \exists r.Y\}$.

B Details for Section 3

Theorem 9 *Given an $\mathcal{ELHF}$-ontology $\mathcal{O}$ and roles r, s, it is* EXPTIME*-complete to decide whether $\mathcal{O}_S \models r \sqsubseteq s$.*

Proof. The theorem is proved by mutual reduction with subsumption between concept names A, B relative to an $\mathcal{ELHF}$-ontology $\mathcal{O}$. Such reductions are easy since we have that

1. $\mathcal{O} \models A \sqsubseteq B$ iff $\mathcal{O} \cup \mathcal{O}' \models s_1 \sqsubseteq s_2$ where

   $\mathcal{O}' = \{\exists s_1.\top \sqsubseteq A, B \sqsubseteq \exists s_2.\top, s_1 \sqsubseteq r, s_2 \sqsubseteq r, \texttt{func}(r)\}$,

   s_1, s_2, r fresh role names;

2. $\mathcal{O} \models r \sqsubseteq s$ iff $\mathcal{O} \cup \mathcal{O}' \models A \sqsubseteq B$ where

   $$\mathcal{O}' = \{A \sqsubseteq \exists r.\top, \exists s.\top \sqsubseteq B\},$$

   A, B fresh concept names.

❏

B.1 The Chase

We start with introducing ABoxes, which the chase procedure uses as a data structure. Let $\mathsf{N_I}$ be a countably infinite set of *individual names* disjoint from $\mathsf{N_C}$ and $\mathsf{N_R}$. An *ABox* is a finite set of *concept assertions* $A(a)$, and *role assertions* $r(a, b)$ where $A \in \mathsf{N_C}$, $r \in \mathsf{N_R}$, and $a, b \in \mathsf{N_I}$. We use $\texttt{Ind}(\mathcal{A})$ to denote the set of individual names that occur in the ABox $\mathcal{A}$. An interpretation $\mathcal{I}$ *satisfies* a concept assertion $A(a)$ if $a \in A^{\mathcal{I}}$ and a role assertion $r(a, b)$ if $(a, b) \in r^{\mathcal{I}}$. Note that we adopt the standard names assumption here, which implies the unique name assumption. An interpretation if a *model* of an ABox if it satisfies all assertions in it. For an ontology $\mathcal{O}$, ABox $\mathcal{A}$, $a \in \texttt{Ind}(\mathcal{A})$, and $\mathcal{ELI}$ concept C, we write $\mathcal{A}, \mathcal{O} \models C(a)$ if $a \in C^{\mathcal{I}}$ for every model $\mathcal{I}$ of $\mathcal{A}$ and $\mathcal{O}$. Moreover, $\mathcal{A}$ is *consistent* with $\mathcal{O}$ if $\mathcal{A}$ and $\mathcal{O}$ have common model. We write $\mathcal{A} \models C(a)$ if $a \in C^{\mathcal{I}}$ where $\mathcal{I}$ is $\mathcal{A}$ viewed as an interpretation in the obvious way. An ABox $\mathcal{A}$ is *ditree-shaped* if the directed graph $G_{\mathcal{A}} = (\texttt{Ind}(\mathcal{A}), \{(a, b) \mid r(a, b) \in \mathcal{A}\}$ is a tree; note that multi-edges are admitted.

Let $\mathcal{O}$ be an $\mathcal{ELHIF}_\bot$ ontology. Starting from a ditree-shaped ABox $\mathcal{A}$, the chase exhaustively applies the following rules:

R1 If $\mathcal{A} \models C(a)$ and $C \sqsubseteq D \in \mathcal{O}$ with $D \neq \bot$, then add $D(a)$ to $\mathcal{A}$;

R2 If $r(a, b) \in \mathcal{A}$ and $\mathcal{O} \models r \sqsubseteq s$ with r, s role names, then add $s(a, b)$ to $\mathcal{A}$;

R3 If $r(a, b) \in \mathcal{A}$, $\mathcal{O} \models r \sqsubseteq s^-$, and $\mathcal{A} \models C(a)$ with $\exists s.C$ a subconcept of $\mathcal{O}$ or an $\mathcal{EL}(\Sigma)$ concept of depth bounded by $\ell - 1$,[3] then add $\exists s.C(b)$ to $\mathcal{A}$;

R4 If $r_1(a, b_1), r_2(a, b_2) \in \mathcal{A}$, $\mathcal{A} \models C_1(b_1)$, $\mathcal{A} \models C_2(b_2)$, C_1, C_2 $\mathcal{EL}(\Sigma)$ concepts of depth bounded by $\ell - 1$, $\mathcal{O} \models r_1 \sqsubseteq s$, $\mathcal{O} \models r_2 \sqsubseteq s$, and $\texttt{func}(s) \in \mathcal{O}$, then add $\exists r_1.(C_1 \sqcap C_2)(a)$ to $\mathcal{A}$;

Note that the rule R3 is parameterized by a depth bound ℓ that is assumed to be identical to the depth bound ℓ used in the construction of $\mathcal{O}_T$. It can be verified that when the chase is started on a ditree-shaped ABox, then all ABoxes produced are ditree-shaped; in particular, 'backwards edges' as enforced by role inclusions of the form $r \sqsubseteq s^-$ are not made explicit, but only treated implicitly.

The chase applies the above rules exhaustively in a fair way. We assume that rules are *not* applied when its postcondition is already satisfied. For example, R3 is not applied when $\mathcal{A} \models \exists s.C(b)$. Using database theory parlance, one could say that our chase is *not oblivious*. This has the (undesired) consequence that the result of the chase, obtained in the limit by exhaustive and fair rule application, is not unique as it depends on the order in which rules are applies. However, all possible results are homomorphically equivalent and for the constructions in this paper, it does not matter which of the many possible results we use. For simplicity, we thus pretend that the outcome of the chase is unique and denote it with $\texttt{chase}_{\mathcal{O}}(\mathcal{A})$. The (desired) consequence of not being oblivious is that the (infinite) ABox $\texttt{chase}_{\mathcal{O}}(\mathcal{A})$ has finite outdegree.

The following lemma implies that the chase is (sound and) complete regarding consequences formulated in terms of $\mathcal{EL}$ concepts of depth bounded by ℓ. It is, however, incomplete regarding deeper $\mathcal{EL}$ concepts and regarding consequences formulated in $\mathcal{ELI}$.

Lemma 3 *Let $\mathcal{O}$ be an $\mathcal{ELHIF}_\bot$ ontology and $\mathcal{A}$ a ditree-shaped ABox with root a_0. Then*

1. *$\mathcal{A}$ is inconsistent with $\mathcal{O}$ iff there are $C \sqsubseteq \bot \in \mathcal{O}$ and $a \in \texttt{Ind}(\mathcal{A})$ with $\texttt{chase}_{\mathcal{O}}(\mathcal{A}) \models C(a)$, and*
2. *if $\mathcal{A}$ is consistent with $\mathcal{O}$, then $\mathcal{A}, \mathcal{O} \models C_0(a_0)$ iff $\texttt{chase}_{\mathcal{O}}(\mathcal{A}) \models C_0(a_0)$ for all $\mathcal{EL}$ concepts of depth at most ℓ.*

Proof. We consider both points simultaneously. The "if" directions are straightforward. In fact, it suffices to show that whenever an ABox $\mathcal{A}'$ is obtained from a ditree-shaped ABox $\mathcal{A}$ by application of one of the rules, then every model of $\mathcal{A}$ and $\mathcal{O}$ is also a model of $\mathcal{A}'$ and $\mathcal{O}$. This is straightforward using a case distinction according to which rule is applied and easy semantic arguments.

For the (contrapositive of the) "only if" directions, assume that there are no $C \sqsubseteq \bot \in \mathcal{O}$ and $a \in \texttt{Ind}(\mathcal{A})$ such that $\texttt{chase}_{\mathcal{O}}(\mathcal{A}) \models C(a)$, respectively that $\texttt{chase}_{\mathcal{O}}(\mathcal{A}) \not\models C(a_0)$. We show how to construct a model $\mathcal{J}$ of $\mathcal{A}$

[3] If $\ell = 0$, then there are no concepts of the latter form.

and $\mathcal{O}$ such that for all $a \in \texttt{Ind}(\mathcal{A})$ and $\mathcal{EL}$ concepts C, $\texttt{chase}_\mathcal{O}(\mathcal{A}) \not\models C(a)$ implies $a \notin C^\mathcal{J}$. This implies that $\mathcal{A}$ is consistent with $\mathcal{O}$ (since $\bot$ occurs in $\mathcal{O}$ only in the form $C \sqsubseteq \bot$), respectively that $\mathcal{A}, \mathcal{O} \not\models C(a_0)$.

Let $\sim$ be the smallest equivalence relation on the individuals in $\texttt{chase}_\mathcal{O}(\mathcal{A})$ such that whenever $\texttt{chase}_\mathcal{O}(\mathcal{A})$ contains $r(a, b_1)$ and $r(a, b_2)$ with $\texttt{func}(r)$, then $b_1 \sim b_2$. Clearly, for any equivalence class of $\sim$, there is an individual a such that all individuals of the class are successors of a in the ditree $\texttt{chase}_\mathcal{O}(\mathcal{A})$. We call a the *predecessor* of the class. An individual b is *maximal* if for every b' with $b \sim b'$ and every $\mathcal{EL}$ concept C of depth at most $\ell - 1$ with $\texttt{chase}_\mathcal{O}(\mathcal{A}) \models C(b')$, we have $\texttt{chase}_\mathcal{O}(\mathcal{A}) \models C(b)$.

Claim 1. Every equivalence class of $\sim$ contains a maximal individual.

It is clear that the outdegree of the ditree $\texttt{chase}_\mathcal{O}(\mathcal{A})$ is finite and thus each equivalence class $\{b_1, \dots, b_k\}$ of $\sim$ is finite since all individuals in it have a common predecessor. Assume that the class does not contain a maximal individual. Then there must be b_{i_1}, b_{i_2} in the class and $\mathcal{EL}$ concepts C_1, C_2 of depth at most $\ell - 1$ such that $\texttt{chase}_\mathcal{O}(\mathcal{A}) \models C_1(b_{i_1})$, $\texttt{chase}_\mathcal{O}(\mathcal{A}) \models C_2(b_{i_2})$, and there is no b_{i_3} in the class with $\texttt{chase}_\mathcal{O}(\mathcal{A}) \models C_1(b_{i_3})$ and $\texttt{chase}_\mathcal{O}(\mathcal{A}) \models C_2(b_{i_3})$. But this situation is impossible since R4 was applied exhaustively.

Let $\mathcal{A}_1$ be obtained by closing $\texttt{chase}_\mathcal{O}(\mathcal{A})$ as follows:

($\dagger_1$) whenever $r(a, b) \in \texttt{chase}_\mathcal{O}(\mathcal{A})$ and $\mathcal{O} \models r \sqsubseteq s^-$, then add $s(b, a)$.

Note that while $\texttt{chase}_\mathcal{O}(\mathcal{A})$, $\mathcal{A}_1$, and $\mathcal{A}_2$ only have downwards edges, ($\dagger_3$) only adds upwards edges. Moreover, due to the assumed syntactic restriction ($\heartsuit$), when ($\dagger_3$) adds $s(b, a)$, then $\texttt{func}(s) \notin \mathcal{O}$ and $\texttt{func}(s^-) \notin \mathcal{O}$. We say that ($\dagger_3$) does *not add functional edges* respectively *does not add inverse functional edges*.

Next, let $\mathcal{A}_2$ be obtained from $\mathcal{A}_1$ as follows:

($\dagger_2$) whenever b_1 and b_2 are successors of a with $b_1 \sim b_2$ and $\rho(a, b_2) \in \mathcal{A}_1$, then add $\rho(a, b_1)$.

Note that ($\dagger_2$) may add both upwards and downwards edges, but it does not add functional or inverse functional upwards edges. In fact, let r_{ab_2} be the *primary* role name between a and b_2, that is, $t(a, b_2) \in \mathcal{A}_0$ implies $\mathcal{O} \models r_{ab_2} \sqsubseteq t$. Such a role name must exist by definition of the chase: r_{ab_i} is the role name from the first edge that the chase has introduced between a and b_2 and all remaining edges were added later by R2. Now observe that $r_{ab_2} \sqsubseteq \rho$ and thus if ρ is an inverse role then by ($\heartsuit$) it can neither be functional nor inverse functional.

Finally, let $\mathcal{A}_3$ be obtained from $\mathcal{A}_2$ as follows:

($\dagger_3$) for every individual a and every $\sim$-equivalence class $\{b_1, \dots, b_k\}$ of which a is the predecessor: choose a maximum individual b_i and remove all edges $r(a, b_j)$ and subtrees rooted at b_j, $j \neq i$.

For brevity, let $\mathcal{A}_0 = \texttt{chase}_\mathcal{O}(\mathcal{A})$. We prove the following central claim:

Claim 2. For every $\mathcal{EL}$ concept C that is a subconcept of $\mathcal{O}$ or of depth bounded by ℓ, every $a \in \Delta^\mathcal{I}$, and $i \in \{0, 1, 2\}$, $\mathcal{A}_i \models C(a)$ iff $\mathcal{A}_{i+1} \models C(a)$.

We distinguish the cases $i \in \{0, 1, 2\}$. In all cases, the proof is by induction on the structure of C and the only interesting case is that C is of the form $\exists s.D$.

Case $i = 0$. Since the "only if" direction is clear, we concentrate on "if". Assume that $\mathcal{A}_1 \models \exists s.D(a)$ and let $s(a, b) \in \mathcal{A}_1$ with $b \in \mathcal{A}_1 \models D(b)$. The induction hypothesis yields $\texttt{chase}_\mathcal{O}(\mathcal{A}) \models D(b)$. If $s(a, b) \in \texttt{chase}_\mathcal{O}(\mathcal{A})$, then clearly $\texttt{chase}_\mathcal{O}(\mathcal{A}) \models \exists s.D(a)$. If this is not the case, then $s(a, b)$ was added by ($\dagger_1$). Then b is a predecessor of a and there is $r(b, a) \in \texttt{chase}_\mathcal{O}(\mathcal{A})$ such that $\mathcal{O} \models r \sqsubseteq s^-$. Since $\texttt{chase}_\mathcal{O}(\mathcal{A}) \models D(b)$, R3 was applied resulting in $\texttt{chase}_\mathcal{O}(\mathcal{A}) \models \exists s.D(a)$.

Case $i = 1$. Since the "only if" direction is clear, we concentrate on "if". First assume that ρ is a role name s. Thus assume that $\mathcal{A}_2 \models \exists s.D(a)$ and let $s(a, b) \in \mathcal{A}_2$ with $\mathcal{A}_2 \models D(b)$. The induction hypothesis and Case $i = 0$ yield $\mathcal{A}_0 \models D(b)$. The interesting case is that $s(a, b)$ was added to $\mathcal{A}_2$ by ($\dagger_2$). Then b is a successor of an individual a and a has another successor b_2 such that $s(a, b_2) \in \mathcal{A}_1$ and $b \sim b_2$. Since ($\dagger_1$) adds only upwards edges, $s(a, b_2) \in \mathcal{A}_0$. Since $b \sim b_2$, there are individuals $c_1, \dots, c_k$ and role names $r_1, \dots, r_{k-1}$ such that

- $c_1 = b$ and $c_k = b_2$
- $r_i(a, c_i), r_i(a, c_{i+1}) \in \mathcal{A}_0$ for $1 \leq i < k$
- $\texttt{func}(r_1), \dots, \texttt{func}(r_{k-1}) \in \mathcal{O}$.

For $1 \leq i \leq k$, let r_{ac_i} be the *primary* role name between a and c_i. We must have

- $\mathcal{O} \models r_{ac_i} \sqsubseteq r_i(a, c_i)$ for $1 \leq i < k$
- $\mathcal{O} \models r_{ac_i} \sqsubseteq r_i(a, c_{i-1})$ for $1 < i \leq k$.

We can thus apply R4 $k - 1$ times to obtain an individual b' such that $r_{ac_k}(a, b') \in \mathcal{A}_0$ and $\mathcal{A}_0 \models D(b')$; note in this context that D is of depth at most $\ell - 1$. We already know that $\mathcal{O} \models r_{ac_k} \sqsubseteq s$ and thus R2 yields $s(a, b') \in \mathcal{A}_0$ which implies $\mathcal{A}_0 \models \exists s.D(a)$ and thus $\mathcal{A}_1 \models \exists s.D(a)$ as required. The case where ρ is an inverse role s^- is similar. In fact, the statement "$s(a, b_2) \in \mathcal{A}_0$" is then replaced with "$t(a, b_2) \in \mathcal{A}_0$ for some role name t with $t \sqsubseteq s^-$". We can use the same argument as above and add at the end that ($\dagger_1$) has added $s(b', a)$.

Case $i = 2$. Here, the "if" direction is clear and we concentrate on "only if". Thus assume that $\mathcal{A}_2 \models \exists s.D(a)$ and let $s(a, b) \in \mathcal{A}_2$ with $\mathcal{A}_2 \models D(b)$. The interesting case is when $s(a, b)$ and the subtree below b was removed by ($\dagger_3$). By Cases $i = 0$ and $i = 1$, $\mathcal{A}_0 \models D(b)$. By Claim 1, there is a maximal b' from the equivalence class of b and thus $\mathcal{A}_0 \models D(b')$ implying $\mathcal{A}_2 \models D(b')$. Because of ($\dagger_2$), $s(a, b') \in \mathcal{A}_2$ and thus $\mathcal{A}_2 \models \exists s.D(a)$ as required. This finishes the proof of Claim 2.

Let $\mathcal{I}$ be $\mathcal{A}_3$ viewed as an interpretation. We first argue that $\mathcal{I}$ satisfies all role inclusions in $\mathcal{O}$. Thus let $r \sqsubseteq \rho \in \mathcal{I}$ and $(a, b) \in r^\mathcal{I}$. Then $r(a, b) \in \mathcal{A}_3$. First assume that $r(a, b) \in \mathcal{A}_0$. Then $\rho(a, b) \in \mathcal{A}_0$ by R2 and ($\dagger_1$) and thus $\rho(a, b) \in \mathcal{A}_3$. The case that $r(a, b)$ was added by ($\dagger_1$) also relies on R2 and ($\dagger_1$), and the semantics. Now assume that

$r(a, b)$ was added by ($\dagger_2$). There are two cases. Either b is a successor of a and a has another successor b_2 such that $b \sim b_2$ and $r(a, b_2) \in \mathcal{A}_1$. But then $\rho(a, b_2) \in \mathcal{A}_1$ and thus ($\dagger_2$) adds also $s(a, b)$. Or a is a successor of b and b has another successor a_2 such that $a \sim a_2$ and $r(a, b_2) \in \mathcal{A}_1$. Again, ($\dagger_2$) adds also $s(a, b)$.

We next show that $\mathcal{I}$ satisfies all functionality assertions in $\mathcal{O}$. To see this, assume that $(a, b_1), (a, b_2) \in \rho^{\mathcal{I}}$ and $\mathtt{func}(\rho) \in \mathcal{O}$ where ρ is a role name or the inverse thereof. If ρ is an inverse role, we must have $b_1 = b_2$ as required: since $\mathtt{chase}_{\mathcal{O}}(\mathcal{A})$ is ditree-shaped, for every individual a there is at most one b with $r(b, a) \in \mathtt{chase}_{\mathcal{O}}(\mathcal{A})$ for every role name r. Since ($\dagger_1$) and ($\dagger_2$) do not add inverse functional edges, the same is true for $\mathcal{A}_3$ when $\mathtt{func}(r^-) \in \mathcal{O}$. Now assume that ρ is a role name r. Since ($\dagger_1$) and ($\dagger_2$) do not add functional upwards edges, both edges $r(a, b_1), r(a, b_2)$ must also be in $\mathcal{A}_0$ and must thus be downwards edges. But then ($\dagger_3$) ensures that $b_1 = b_2$.

It follows from Claim 2 that $\mathcal{I}$ satisfies all concept inclusions $C \sqsubseteq D \in \mathcal{O}$ with $D \neq \bot$. In fact, let $a \in C^{\mathcal{I}}$. Claim 2 yields $\mathtt{chase}_{\mathcal{O}}(\mathcal{A}) \models C(a)$, rule R1 gives $\mathtt{chase}_{\mathcal{O}}(\mathcal{A}) \models D(a)$ and applying Claim 2 once more gives $a \in D^{\mathcal{I}}$.

We now finish the proofs of the "only if" directions of Points 1 and 2 of Lemma 3. For Point 1, by assumption we have $\mathtt{chase}_{\mathcal{O}}(\mathcal{A}) \not\models C(a)$ for all $a \in \mathtt{Ind}(\mathcal{A})$ and $C \sqsubseteq \bot \in \mathcal{O}$, and thus Claim 2 implies that all concept inclusions $C \sqsubseteq \bot \in \mathcal{O}$ are satisfied by $\mathcal{I}$. Thus, $\mathcal{I}$ is a model of $\mathcal{O}$, which shows that $\mathcal{A}$ is consistent with $\mathcal{O}$, finishing the argument.

For Point 2, by assumption we have that $\mathcal{A}$ is consistent with $\mathcal{O}$ and that $\mathtt{chase}_{\mathcal{O}}(\mathcal{A}) \not\models C(a_0)$. From the former and the already established "if" direction of Point 1, we get $\mathtt{chase}_{\mathcal{O}}(\mathcal{A}) \not\models C(a)$ for all $a \in \mathtt{Ind}(\mathcal{A})$ and $C \sqsubseteq \bot \in \mathcal{O}$. Thus, $\mathcal{I}$ is again a model of $\mathcal{O}$ and Claim 2 yields $a_0 \notin C_0^{\mathcal{I}}$, thus $\mathcal{A}, \mathcal{O} \not\models C_0(a_0)$ as required. ❑

B.2 Completeness

We now prove the completeness part of Theorem 3, starting with some preliminaries. For an ABox $\mathcal{A}$ and an interpretation $\mathcal{I}$, a function $h : \mathtt{Ind}(\mathcal{A}) \rightarrow \Delta^{\mathcal{I}}$ is a *homomorphism from $\mathcal{A}$ to $\mathcal{I}$* if $h(a) \in A^{\mathcal{I}}$ for every $A(a) \in \mathcal{A}$ and $(h(a), h(b)) \in r^{\mathcal{I}}$ for every $r(a, b) \in \mathcal{A}$. For two ABoxes $\mathcal{A}_1$ and $\mathcal{A}_2$, a function $h : \mathtt{Ind}(\mathcal{A}_1) \rightarrow \mathtt{Ind}(\mathcal{A}_2)$ is a *homomorphism from $\mathcal{A}_1$ to $\mathcal{A}_2$* if $A(h(a)) \in \mathcal{A}_2$ for every $A(a) \in \mathcal{A}_1$ and $r(h(a), h(b)) \in \mathcal{A}_2$ for every $r(a, b) \in \mathcal{A}_1$. We recall that every $\mathcal{EL}$ concept C can be viewed as a ditree-shaped ABox $\mathcal{A}_C$. By convention, we assume that the root individual in such an ABox is a_0. For example, the $\mathcal{EL}$ concept $A \sqcap \exists r.B \sqcap \exists s.\top$ can be viewed as the ABox $\{A(a_0), r(a_0, b_1), B(b_1), s(a_0, b_2)\}$. The following is widely known and straightforward to establish.

Lemma 4 *Let C be an $\mathcal{EL}$ concept, $\mathcal{I}$ an interpretation, and $d \in \Delta^{\mathcal{I}}$. Then $d \in C^{\mathcal{I}}$ iff there is a homomorphism h from $\mathcal{A}_C$ to $\mathcal{I}$ with $h(a_0) = d$.*

The next lemma is the central step in the completeness proof. It says that the chase introduced above can in a sense be simulated by the statements in the ontology $\mathcal{O}_T$.

Lemma 5 *Let C be an $\mathcal{EL}(\Sigma)$ concept of depth bounded by ℓ decorated with subconcepts of $\mathcal{O}_S$ at leaves, $\mathcal{A}_0, \mathcal{A}_1, \ldots$ the sequence of ABoxes constructed by $\mathtt{chase}_{\mathcal{O}_S}(\mathcal{A}_C)$, $\mathcal{I}$ a model of $\mathcal{O}_T$, and $d \in C^{\mathcal{I}}$. Then for every $i \geq 0$, there is a homomorphism h_i from $\mathcal{A}_i$ to $\mathcal{I}$ with $h_i(a_0) = d$.*

Proof. The proof is by an induction on the number of applications of the chase rule R3 used to compute the sequence $\mathcal{A}_0, \ldots, \mathcal{A}_i$. For the induction start, assume that R3 was not applied at all. We show that for all $j \leq i$, there is a homomorphism h_j from $\mathcal{A}_j$ to $\mathcal{I}$ with $h_j(a_0) = d$. For $j = 0$, it suffices to apply Lemma 4. Now assume that h_j has already been constructed, $j < i$. We show how to find h_{j+1}, making a case distinction according to the rule that is applied in order to obtain $\mathcal{A}_{j+1}$ from $\mathcal{A}_j$.

R1. Then $\mathcal{A}_j \models C(a)$, $C \sqsubseteq D \in \mathcal{O}_S$, and $\mathcal{A}_{j+1}$ is obtained from $\mathcal{A}_j$ by adding $D(a)$. We must have $h_j(a) \in C^{\mathcal{I}}$. Since $C \sqsubseteq D$ is a CI in $\mathcal{O}_T$ and $\mathcal{I}$ is a model of $\mathcal{O}_T$, $a \in D^{\mathcal{I}}$. Thus h_j can be extended to a homomorphism from $\mathcal{A}_{j+1}$ to $\mathcal{I}$ in a straightforward way.

R2. Then $r(a, b) \in \mathcal{A}_j$, $\mathcal{O}_S \models r \sqsubseteq s$, and $\mathcal{A}_{j+1}$ is obtained from $\mathcal{A}_j$ by adding $s(a, b)$. We have $r \sqsubseteq s \in \mathcal{O}_T$ and thus $h_{j+1} = h_j$ is a homomorphism from $\mathcal{A}_{j+1}$ to $\mathcal{I}$.

R4. There are $r_1(a, b_1), r_2(a, b_2) \in \mathcal{A}_j$ such that $\mathcal{A}_j \models C_1(b_1)$, $\mathcal{A}_j \models C_2(b_2)$, C_1, C_2 $\mathcal{EL}(\Sigma)$ concepts of depth bounded by $\ell - 1$, $\mathcal{O}_S \models r_1 \sqsubseteq s$, $\mathcal{O}_S \models r_1 \sqsubseteq s$, $\mathtt{func}(s) \in \mathcal{O}_S$, and $\mathcal{A}_{j+1}$ is obtained from $\mathcal{A}_j$ by adding $\exists r_1.C_1 \sqcap C_2(a)$. Clearly, $h_j(a) \in (\exists r_1.C_1 \sqcap \exists r_2.C_2)^{\mathcal{I}}$. By construction, $\mathcal{O}_T$ contains $\exists r_1.C_1 \sqcap \exists r_2.C_2 \sqsubseteq \exists r_1.(C_1 \sqcap C_2)$. Consequently, $h_j(a) \in \exists r_1.(C_1 \sqcap C_2)^{\mathcal{I}}$. We can thus extend h_j to the desired homomorphism h_{j+1} from $\mathcal{A}_{j+1}$ to $\mathcal{I}$ in a straightforward way.

Now for the induction step. Assume that there were $k > 0$ applications of R3 in the sequence $\mathcal{A}_0, \ldots, \mathcal{A}_i$ and that the last such application was used to obtain $\mathcal{A}_{p+1}$ from $\mathcal{A}_p$, $p < i$. By induction hypothesis, we find a homomorphism h_p from $\mathcal{A}_p$ to $\mathcal{I}$ with $h_p(a_0) = d$. We argue that we also find such a homomorphism h_{p+1} from $\mathcal{A}_{p+1}$ to $\mathcal{I}$. We can then proceed as in the induction start to obtain the desired homomorphism from $\mathcal{A}_i$ to $\mathcal{I}$.

Since R3 was applied, there are $r(a, b) \in \mathcal{A}_p$, a role name s, and an $\mathcal{EL}$ concept C' with $\exists s.C'$ a subconcept of $\mathcal{O}_S$ or of depth bounded by ℓ such that $\mathcal{A}_p \models C'(a)$, $\mathcal{O}_S \models r \sqsubseteq s^-$, and $\mathcal{A}_{p+1}$ is obtained from $\mathcal{A}_p$ by adding $\exists s.C'(b)$. Since $\mathcal{A}_p$ is ditree-shaped, a is the predecessor of b. By definition of the chase, there is a primary role name r_{ab} between a and b, as in the proof of Lemma 3: for every $t(a, b) \in \mathcal{A}_p$, we have $\mathcal{O}_S \models r_{ab} \sqsubseteq t$ and thus also $\mathcal{O}_T \models r_{ab} \sqsubseteq t$. It suffices to show that there is a $d_b \in \Delta^{\mathcal{I}}$ such that $(h_p(a), d_b) \in r_{ab}^{\mathcal{I}}$ and a homomorphism h_b from $\mathcal{A}_{p+1}|_b$ to $\mathcal{I}$ with $h_b(b) = d_b$, where $\mathcal{A}_{p+1}|_b$ is the restriction of $\mathcal{A}_{p+1}$ to the subtree rooted at b. In fact, it is then straightforward to combine h_p and h_b into the desired homomorphism h_{p+1}: set $h_{p+1}(c) = h_p(c)$ if c is not in $\mathcal{A}_{p+1}|_b$ and $h_{p+1}(c) = h_b(c)$ otherwise. Observe that $t(a, b) \in \mathcal{A}_{p+1}$ implies $(h(a), d_b) \in t^{\mathcal{I}}$ for all role names t since $(h_p(a), d_b) \in r_{ab}^{\mathcal{I}}$, $\mathcal{I}$ is a model of $\mathcal{O}_T$, and no new edges between a and b have been added in the construction of $\mathcal{A}_{p+1}$ from $\mathcal{A}_p$. In particular, $\mathcal{O}_T \models r_{ab} \sqsubseteq r$.

For brevity, let $\mathcal{A}_b = \mathcal{A}_p|_b$ and set $\texttt{Ind} = \{b\} \cup (\texttt{Ind}(\mathcal{A}_b) \cap \texttt{Ind}(\mathcal{A}_C))$. That is, Ind contains only those individuals from $\mathcal{A}_b$ that were present already in the initial ABox $\mathcal{A}_C$, and if there is no such individual, then $\texttt{Ind} = \{b\}$. An individual $c \in \texttt{Ind}$ is a *fringe individual* if there is some $t(c, c') \in \mathcal{A}_p^b$ with $c' \notin \texttt{Ind}$. Further, let $\mathcal{A}_p^{b-}$ be the restriction of $\mathcal{A}_p^b$ to assertions that only use individuals from Ind extended by adding $E(c)$ whenever E is a subconcept of $\mathcal{O}_S$ and c is a fringe individual such that $\mathcal{A}_p \models E(c)$, and let C_b be this ABox viewed as an $\mathcal{EL}$ concept.

We must have $h_p(b) \in C_b^{\mathcal{I}}$ and thus $h_p(a) \in (C' \sqcap \exists r_{ab}.C_b)^{\mathcal{I}}$. Since $\exists s.C'$ is a subconcept of $\mathcal{O}_S$ or of depth at most ℓ and C_b is an $\mathcal{EL}$ concept of depth at most ℓ' (by construction and because C is of depth bounded by ℓ; recall that $\ell' = \max\{\ell - 1, 0\}$) decorated with subconcepts of $\mathcal{O}_S$ at leaves, $C' \sqcap \exists r_{ab}.C_b \sqsubseteq \exists r_{ab}.(C_b \sqcap \exists s.C') \in \mathcal{O}_T$. Consequently, there is a $d_b \in (C_b \sqcap \exists s.C')^{\mathcal{I}}$ with $(d_a, d_b) \in r_{ab}^{\mathcal{I}}$. Let $\mathcal{B}$ be obtained from $\mathcal{A}_b^-$ by adding $\exists s.C'(b)$, as in the construction of $\mathcal{A}_{p+1}$. By Lemma 4, there is a homomorphism h_b from $\mathcal{B}$ to $\mathcal{I}$ with $h_b(b) = d_b$. It remains to extend h_b from $\mathcal{B}$ to $\mathcal{A}_{p+1}|_b$.

To this end, consider each fringe individual c. Let C_c be the $\mathcal{EL}$ concept that is the conjunction of all subconcepts E of $\mathcal{O}_S$ with $\mathcal{A}_b \models E(c)$. We can extract from the chase sequence $\mathcal{A}_0, \dots, \mathcal{A}_p$ a chase sequence that constructs $\mathcal{A}_p|_c$ starting from $\mathcal{A}_{C_c}$ and uses at most $k-1$ applications of special rules. From the induction hypothesis and since clearly $h_b(c) \in C_c^{\mathcal{I}}$, we thus obtain a homomorphism h_c from $\mathcal{A}_c$ to $\mathcal{I}$ with $h_c(c) = h_b(c)$. It is now straightforward to combine our initial h_b with all the homomorphisms h_c into the desired homomorphism h_b from $\mathcal{A}_{p+1}|_b$ to $\mathcal{I}$. ❏

We are now ready to prove completeness of the approximation constructed in Theorem 3. It is immediate by construction of $\mathcal{O}_T$ that $\mathcal{O}_S \models r \sqsubseteq s$ implies $\mathcal{O}_T \models r \sqsubseteq s$ for all role names $r, s \in \Sigma$. it thus remains to show the following.

Lemma 6 *$\mathcal{O}_S \models C \sqsubseteq D$ implies $\mathcal{O}_T \models C \sqsubseteq D$ for all $\mathcal{EL}(\Sigma)$ concepts C of depth at most ℓ and $\mathcal{EL}_\bot$ concepts D.*

Proof. By Point 2 of Lemma 1 and construction of $\mathcal{O}_T$, it suffices to consider the case $\ell < \omega$. Assume $\mathcal{O}_S \models C \sqsubseteq D$ with C, D as in Lemma 6 and let $\mathcal{I}$ be a model of $\mathcal{O}_T$ with $d \in C^{\mathcal{I}}$. We have to show that $d \in D^{\mathcal{I}}$. First assume that the ABox $\mathcal{A}_C$ is consistent with $\mathcal{O}_S$. By Point 2 of Lemma 3, $\texttt{chase}_{\mathcal{O}_S}(\mathcal{A}_C) \models D(a_0)$. Let $\mathcal{A}_C = \mathcal{A}_0, \mathcal{A}_1, \dots$ be the sequence of ABoxes generated by the chase when started on $\mathcal{A}_C$. Then $\texttt{chase}_{\mathcal{O}_S}(\mathcal{A}_C) \models D(a_0)$ implies that there is an $\mathcal{A}_i$ with $\mathcal{A}_i \models D(a_0)$. An analogue of Lemma 4 for homomorphisms into ABoxes thus yields a homomorphism h from $\mathcal{A}_D$ to $\mathcal{A}_i$ with $h(a_0) = a_0$ (defined in the expected way). By Lemma 5, there further is a homomorphism h' from $\mathcal{A}_i$ to $\mathcal{I}$ with $h(a_0) = d$. Composing these, we obtain a homomorphism from $\mathcal{A}_D$ to $\mathcal{I}$ that maps a_0 to d and applying Lemma 4 yields $d \in D^{\mathcal{I}}$, as required. Now assume that $\mathcal{A}_C$ is inconsistent with $\mathcal{O}_S$. Then by Point 1 of Lemma 3, there are $C' \sqsubseteq \bot \in \mathcal{O}_S$ and $a \in \texttt{Ind}(\mathcal{A})$ with $\texttt{chase}_{\mathcal{O}_S}(\mathcal{A}) \models C'(a)$. We can argue as above that there is a homomorphism from $\mathcal{A}_{C'}$ to $\mathcal{I}$, and thus $C'^{\mathcal{I}} \neq \emptyset$ in contradiction to the facts that $\mathcal{I}$ is a model of $\mathcal{O}_T$ and $C' \sqsubseteq \bot \in \mathcal{O}_T$. ❏

C Details for Section 4

We show that one can obtain approximations from axiomatizations of the quasi-equations valid in classes of bounded semilattices with operators (SLOs). Theorem 4 is an instance of a general result stating that, under certain conditions, by identifying CIs with equations in the theory of SLOs, the substitution instances of the equations used in an axiomatization provide the additional CIs needed to approximate ontologies. This link between approximation and algebra can be used in a number of different ways: (1) existing axiomatization results can be used directly, as a black box, to obtain approximations; (2) if no axiomatization is available yet for the conditions on roles expressed in a DL of interest, the algebraic machinery can be used to determine a new axiomatization and, thereby, the corresponding approximation; (3) 'negative' results from algebra can be used to show that certain natural candidates for approximations do not work.(4) conversely, one can use approximation results to obtain axiomatizations of classes of SLOs. In fact, the direct approximation proofs presented above provide a novel technique for obtaining axiomatizations of classes of SLOs. Note that the link to algebra does *not* provide any depth bounded approximations from axiomatizations.

This section is structured as follows. After introducing the relevant algebraic notation, we prove a general result linking approximations to *complex* equational theories of SLOs, where an equational theory Ax of SLOs is complex if every SLO validating Ax can be respresented by subsets (complexes) of an interpretation validating Ax. This link is proved for equational theories of SLOs corresponding to arbitrary first-order conditions on roles. We then prove that any set P of functionality assertions and role inclusions that is inverse closed corresponds to a complex equational theory and apply this result to prove Theorem 4.

We introduce the relevant notation for semilattices with operators. A *bounded semilattice with monotone operators (SLO)* is an algebraic structure

$$\mathfrak{A} = (A, \wedge^{\mathfrak{A}}, \bot^{\mathfrak{A}}, \top^{\mathfrak{A}}, (\Diamond_r^{\mathfrak{A}} \mid r \in \mathcal{R}))$$

such that $(A, \wedge^{\mathfrak{A}}, \bot^{\mathfrak{A}}, \top^{\mathfrak{A}})$ is a *bounded semilattice* satisfying the equations

$$\forall x\ (x \wedge^{\mathfrak{A}} x \approx x) \quad (3)$$

$$\forall x\, \forall y\ (x \wedge^{\mathfrak{A}} y \approx y \wedge^{\mathfrak{A}} x) \quad (4)$$

$$\forall x\, \forall y\, \forall z\ (x \wedge^{\mathfrak{A}} (y \wedge^{\mathfrak{A}} z)) \approx (x \wedge^{\mathfrak{A}} y) \wedge^{\mathfrak{A}} z) \quad (5)$$

$$\forall x\ (x \wedge^{\mathfrak{A}} \top^{\mathfrak{A}} \approx x), \quad \forall x\ (x \wedge^{\mathfrak{A}} \bot^{\mathfrak{A}} \approx \bot^{\mathfrak{A}}) \quad (6)$$

and $\mathcal{R}$ is a set of role names such that the unary operators $\Diamond_r^{\mathfrak{A}}$, $r \in \mathcal{R}$, satisfy the equation

$$\forall x\, \forall y\ (\Diamond_r^{\mathfrak{A}}(x \wedge^{\mathfrak{A}} y) \wedge^{\mathfrak{A}} \Diamond_r^{\mathfrak{A}} y) \approx \Diamond_r^{\mathfrak{A}}(x \wedge^{\mathfrak{A}} y) \quad (7)$$

$$\Diamond_r^{\mathfrak{A}} \bot^{\mathfrak{A}} \approx \bot^{\mathfrak{A}} \quad (8)$$

In a SLO $\mathfrak{A}$, the partial order $\leq^{\mathfrak{A}}$ is defined as usual by taking $a \leq^{\mathfrak{A}} b$ iff $a \wedge^{\mathfrak{A}} b = a$, for all a, b in A. It is readily seen that

$\Diamond_r^{\mathfrak{A}}$ is *monotone* with respect to $\leq^{\mathfrak{A}}$: if $a \leq^{\mathfrak{A}} b$ then $\Diamond_r^{\mathfrak{A}} a \leq^{\mathfrak{A}} \Diamond_r^{\mathfrak{A}} b$, for all a, b in A, and that $\Diamond_r^{\mathfrak{A}} \top^{\mathfrak{A}} = \top^{\mathfrak{A}}$. *SLO terms* τ over $\mathcal{R}$ are constructed from variables using the connectives $\wedge$, $\bot$, $\top$, and $\Diamond_r$, $r \in \mathcal{R}$, in the obvious way:

$$\tau, \sigma \quad := \quad x \mid \bot \mid \top \mid \tau \wedge \sigma \mid \Diamond_r \tau$$

where x ranges over a countably infinite set of variables. A *SLO equation* takes the form $\sigma \approx \tau$, where σ, τ are SLO terms; a *SLO quasi-equation* takes the form $\alpha_1 \wedge \cdots \wedge \alpha_n \rightarrow \alpha$, where $\alpha_1, \ldots, \alpha_n, \alpha$ are SLO equations. For SLO terms σ and τ we use $\sigma \leq \tau$ as a shorthand for the equation $\sigma \wedge \tau \approx \sigma$. A *valuation* v in a SLO $\mathfrak{A}$ is a mapping from the set of variables into A. The *value* $v(\tau)$ of a SLO term τ in $\mathfrak{A}$ is defined by induction over the construction of τ by setting $v(\bot) = \bot^{\mathfrak{A}}$, $v(\top) = \top^{\mathfrak{A}}$, $v(\sigma \wedge \tau) = v(\sigma) \wedge v(\tau)$, and $v(\Diamond_r \tau) = \Diamond_r^{\mathfrak{A}} v(\tau)$, for all role names $r \in \mathcal{R}$. An equation $\sigma \approx \tau$ is *true under* v *in* $\mathfrak{A}$ if $v(\sigma) = v(\tau)$. An equation $\sigma \approx \tau$ is *valid in* $\mathfrak{A}$, in symbols $\mathfrak{A} \models \sigma \approx \tau$, if $\sigma \approx \tau$ is true under all valuations in $\mathfrak{A}$. A quasi-equation $\rho = \alpha_1 \wedge \cdots \wedge \alpha_n \rightarrow \alpha$ is *valid in* $\mathfrak{A}$ if α is true under all valuations under which $\alpha_1, \ldots, \alpha_n$ are true. Equational theories of SLOs have been investigated in [Jackson, 2004; Beklemishev, 2014; Sofronie-Stokkermans, 2017; Kikot *et al.*, 2017].

Every SLO term τ defines an $\mathcal{EL}_\bot$ concept τ^C by replacing every variable x with a concept name A_x, and the connectives $\wedge$ and $\Diamond_r$ with $\sqcap$ and $\exists r$, respectively. For example, $(\Diamond_r x \wedge \top)^C = \exists r.A_x \sqcap \top$. Then any SLO equation $\alpha = (\sigma \leq \tau)$ defines the CI $\alpha^C = \sigma^C \sqsubseteq \tau^C$. We denote by $\cdot^T$ the obvious converse of $\cdot^S$ associating with every $\mathcal{EL}_\bot$ concept C (CI α) a SLO term C^T (SLO equation α^T, respectively). For example, $(\exists r.\exists r.A_x \sqsubseteq \exists r.A_x)^T = \Diamond_r \Diamond_r x \leq \Diamond_r x$.

We are in the position now to formulate the fundamental equivalence of $\mathcal{EL}_\bot$ TBox reasoning and the validity of SLO quasi-equations [Sofronie-Stokkermans, 2017; Kikot *et al.*, 2017].

Theorem 10 *For any $\mathcal{EL}_\bot$ ontology $\mathcal{O}$ and CI $C \sqsubseteq D$: $\mathcal{O} \models C \sqsubseteq D$ iff $\bigwedge_{\alpha \in \mathcal{O}} \alpha^T \rightarrow (C \sqsubseteq D)^T$ is valid in all SLOs.*

Theorem 10 and the correspondence between axiomatizations and approximations we are after are proved by showing that every SLO (validating a set Ax of equations) can be represented by the set of subsets of a DL interpretation (satisfying a role constraint P corresponding to Ax). In detail, every interpretation $\mathcal{I}$ defines a SLO $\mathcal{I}^+$ over any set $\mathcal{R}$ of role names by setting [Goldblatt, 1989]:

$$\mathcal{I}^+ = (2^{\Delta^\mathcal{I}}, \wedge^{\mathcal{I}^+}, \bot^{\mathcal{I}^+}, \top^{\mathcal{I}^+}, (\Diamond_r^{\mathcal{I}^+} \mid r \in \mathcal{R})),$$

where for $X, Y \subseteq \Delta^\mathcal{I}$:

$$\begin{array}{rcl} X \wedge^{\mathcal{I}^+} Y & = & X \cap Y \\ \top^{\mathcal{I}^+} & = & \Delta^\mathcal{I} \\ \bot^{\mathcal{I}^+} & = & \emptyset \\ \Diamond_r^{\mathcal{I}^+} X & = & \{d \in \Delta^\mathcal{I} \mid \exists d' \in X \; (d, d') \in r^\mathcal{I}\} \end{array}$$

Observe that the definition of the SLO $\mathcal{I}^+$ does not depend on the interpretation of concept names in $\mathcal{I}$. Therefore, we mostly define the SLO $\mathcal{F}^+$ for *frames* $\mathcal{F}$, interpretations in which $A^\mathcal{F} = \emptyset$ for all concept names A. One can apply algebraic notation to interpretations in a straightforward way. For example, we say that a SLO equation or quasi-equation is *valid* in $\mathcal{I}$ if it is valid in the algebra $\mathcal{I}^+$. It is known that natural contraints on the interpretation of roles can be captured by the validity of SLO equations. For example, if r, s are role names, then r is included in s in an interpretation $\mathcal{I}$ iff the equation $\Diamond_r x \leq \Diamond_s x$ is valid in $\mathcal{I}$. Formally, call a set P of first-order sentences using role names as binary predicates a *role constraint*. Then we say that a role constraint P *corresponds* to a set Ax of SLO equations if any interpretation $\mathcal{I}$ satisfies P iff $\mathcal{I}^+ \models \alpha$, for all $\alpha \in \mathsf{Ax}$. The table below gives a sample set of role constraints and the corresponding SLO equations. These correspondences are well known from correspondence theory in modal logic and are, in particular, instances of the correspondence part of Sahlqvist's Theorem [van Benthem, 1984; Blackburn *et al.*, 2001]. We refer the reader to [Sofronie-Stokkermans, 2017; Kikot *et al.*, 2017] for more examples.

Role constraint	Equation
$\mathsf{func}(r)$	$\Diamond_r x \wedge \Diamond_r y \leq \Diamond_r (x \wedge y)$
$r \sqsubseteq s^-$	$x \wedge \Diamond_r y \leq \Diamond_r (y \wedge \Diamond_s x)$
$r_1 \circ \cdots \circ r_n \sqsubseteq r$	$\Diamond_{r_1} \cdots \Diamond_{r_n} x \leq \Diamond_r x$

The following example illustrates how we use these correspondences to determine approximations.

Example 6 *Consider a $\mathcal{ELF}_\bot$ ontology $\mathcal{O}_S$ containing as its only role assertion $\mathsf{func}(r)$. Thus, $\mathcal{O}_S = \mathcal{O} \cup \{\mathsf{func}(r)\}$, for a set $\mathcal{O}$ of CIs. Set $\Sigma = \mathsf{sig}(\mathcal{O}_S)$. To approximate $\mathcal{O}_S$ by a $\mathcal{EL}_\bot$ ontology, we take the equation $\alpha = (\Diamond_r x \wedge \Diamond_r y \leq \Diamond_r (x \wedge y))$ corresponding to $\mathsf{func}(r)$ and replace in $\mathcal{O}_S$ the functionality assertion $\mathsf{func}(r)$ by the set of all CIs obtained from α^C by substituting A_x and A_y by arbitrary $\mathcal{EL}_\bot(\Sigma)$ concepts. Thus, we form the set α^Σ of all CIs $\exists r.C \sqcap \exists r.D \sqsubseteq \exists r.(C \sqcap D)$, where C, D are $\mathcal{EL}_\bot(\Sigma)$ concepts and claim that $\mathcal{O}_T = \mathcal{O} \cup \alpha^\Sigma$ is an $\mathcal{EL}_\bot$ approximation of $\mathcal{O}_S$.*

It is easy to see that correpondence of $\mathsf{func}(r)$ to α entails that $\mathcal{O}_S \models \mathcal{O}_T$. The converse direction ($\mathcal{O}_S \models C \sqsubseteq D$ implies $\mathcal{O}_T \models C \sqsubseteq D$ for all $\mathcal{EL}_\bot(\Sigma)$ CIs $C \sqsubseteq D$), however, does not *follow from correspondence and requires significantly more work - which we discuss next.*

We develop a necessary and sufficient condition for when equations Ax corresponding to a role constraint P provide a $\mathcal{EL}_\bot$ approximation of ontologies of the form $\mathcal{O} \cup P$, where $\mathcal{O}$ is a set of $\mathcal{EL}_\bot$ CIs.

A *homomorphism* h between SLOs $\mathfrak{A}_1$ and $\mathfrak{A}_2$ is a mapping h from the domain A_1 of $\mathfrak{A}_1$ to the domain A_2 of $\mathfrak{A}_2$ preserving all operations, for example $h(\Diamond_r^{\mathfrak{A}_1} a) = \Diamond_r^{\mathfrak{A}_2} h(a)$ for all $a \in A_1$ and $r \in \mathcal{R}$. An *embedding* is an injective homomorphism. A set Ax of SLO equations is *complex* if for every SLO $\mathfrak{A}$ validating Ax there exists a frame $\mathcal{F}$ validating Ax such that $\mathfrak{A}$ can be embedded into $\mathcal{F}^+$. Thus, if Ax is complex, then every SLO $\mathfrak{A}$ validating Ax can be regarded as a system of sets (aka complexes) over a frame validating Ax. Call Ax *quasi-equation complete* if a quasi-equation is valid in all SLOs validating Ax just in case it is valid in all SLOs of the form $\mathcal{I}^+$ validating Ax. It can be proved that a set Ax of SLO equations is complex iff it is quasi-equation complete [Kikot *et al.*, 2017]. Theorem 10 can be proved by showing that the *empty set* of SLO equations is complex. In other words, the *empty* role constraint P

corresponds to the *empty* complex set Ax of equations. For an equation $\alpha = (\sigma \leq \tau)$ we denote by α^Σ the set of all CIs obtained from α^C by uniformly substituting every A_x in α^C by any $\mathcal{EL}_\bot(\Sigma)$ concept D. Let Ax^Σ denote the union of all α^Σ, $\alpha \in \mathsf{Ax}$. We are now in the position to formulate the announced criterion for approximations. Observe that the approximations only depend on the role constraint of the ontology and not on its CIs.

Theorem 11 *[Approximation/Axiomatization] Let P be a role constraint and Ax a set of SLO equations corresponding to P. Then the following conditions are equivalent:*

1. *$\mathcal{O} \cup \mathsf{Ax}^\Sigma$ is a $\mathcal{EL}_\bot$ approximation of $\mathcal{O} \cup P$, for all $\mathcal{EL}_\bot$ ontologies $\mathcal{O}$ with $\Sigma = \mathsf{sig}(\mathcal{O})$;*
2. *Ax is complex;*
3. *Ax is quasi-equation complete.*

Proof. We first show that (2) implies (1). Assume Ax is complex and let $\mathcal{O}$ be an $\mathcal{EL}_\bot$ ontology and with $\Sigma = \mathsf{sig}(\mathcal{O} \cup P)$. We have to show that for all $\mathcal{EL}_\bot(\Sigma)$ concepts C, D: $\mathcal{O} \cup P \models C \sqsubseteq D$ iff $\mathcal{O} \cup \mathsf{Ax}^\Sigma \models C \sqsubseteq D$. The direction ($\Leftarrow$) follows from the observation that every interpretation satisfying P validates Ax and, therefore, satisfies all CIs in Ax^Σ. Conversely, assume that $\mathcal{O} \cup \mathsf{Ax}^\Sigma \not\models C \sqsubseteq D$. Take an interpretation $\mathcal{I}$ satisfying $\mathcal{O} \cup \mathsf{Ax}^\Sigma$ such that $\mathcal{I} \not\models C \sqsubseteq D$. Let $\mathcal{R}$ be the set of role names in Σ. Define the SLO $\mathfrak{A} = (A, \wedge^\mathfrak{A}, \bot^\mathfrak{A}, \top^\mathfrak{A}, (\Diamond_r^\mathfrak{A} \mid r \in \mathcal{R}))$ as the restriction of the SLO $\mathcal{I}^+$ to $\{C^\mathcal{I} \mid \mathsf{sig}(C) \subseteq \Sigma\}$. In more detail,

$$\begin{array}{rcl} A &=& \{C^\mathcal{I} \mid \mathsf{sig}(C) \subseteq \Sigma\} \\ X \wedge^\mathfrak{A} Y &=& X \cap Y \\ \top^\mathfrak{A} &=& \top^\mathcal{I} \\ \bot^\mathfrak{A} &=& \emptyset \\ \Diamond_r^\mathfrak{A} X &=& \{d \in \Delta^\mathcal{I} \mid \exists d' \in X \ (d, d') \in r^\mathcal{I}\} \end{array}$$

Then $\mathfrak{A}$ validates Ax: to see this let v be a valuation in $\mathfrak{A}$. By definition, for every variables x there exists a $\mathcal{EL}_\bot(\Sigma)$ concept C_x with $v(x) = C_x^\mathcal{I}$. Let $\sigma \leq \tau \in \mathsf{Ax}$. Obtain σ^s and τ^s from σ^C and τ^C by substituting every A_x by C_x. Then $\sigma^s \sqsubseteq \tau^s \in \mathsf{Ax}^\Sigma$. Thus $\mathcal{I} \models \sigma^s \sqsubseteq \tau^s$ and so $\mathfrak{A} \models_v \sigma \leq \tau$, as required.

As Ax is complex, there exists a frame $\mathcal{G}$ validating Ax such that there is an embedding h from $\mathfrak{A}$ into $\mathcal{G}^+$. Extend $\mathcal{G}$ to an interpretation $\mathcal{J}$ by setting $A^\mathcal{J} = h(A^\mathcal{I})$ for every concept name A. Then $\mathcal{J}$ is a model of $\mathcal{O}$ validating Ax and refuting $C \sqsubseteq D$. Thus, $\mathcal{J}$ is a model of $\mathcal{O}$ and P and refuting $C \sqsubseteq D$.

We now show that (1) implies (3). Assume Ax is not quasi-equation complete. Take $\mathfrak{A}$ validating Ax and a quasi-equation $\alpha_1 \wedge \cdots \wedge \alpha_n \rightarrow \alpha$ such that

(a) $\mathfrak{A} \not\models \alpha_1 \wedge \cdots \wedge \alpha_n \rightarrow \alpha$;

(b) $\mathcal{I} \models \alpha_1 \wedge \cdots \wedge \alpha_n \rightarrow \alpha$ for all $\mathcal{I}$ validating Ax.

We may assume that all variables and $\Diamond_r$ used in α are used in some α_i. Let v be a valuation in $\mathfrak{A}$ such that $\alpha_1, \ldots, \alpha_n$ are true in $\mathfrak{A}$ under v and α is refuted in $\mathfrak{A}$ under v. Take a frame $\mathcal{F}$ such that $\mathfrak{A}$ is embedded into $\mathcal{F}^+$ via an injective homomorphism h. Define a model $\mathcal{J}$ by expanding $\mathcal{F}$ by setting $A_x^\mathcal{J} = h(v(x))$, for all variables x in $\alpha_1, \ldots, \alpha_n, \alpha$. Let $\mathcal{O} = \{\alpha_1^T, \ldots, \alpha_n^T\}$ and $\Sigma = \mathsf{sig}(\mathcal{O} \cup P)$. Then $\mathcal{I}$ is a model of Ax^Σ (since $\mathfrak{A}$ validates Ax) and $\mathcal{I} \not\models \alpha^T$. It follows that $\mathcal{O} \cup \mathsf{Ax}^\Sigma \not\models \alpha^T$. However, by Point (b), $\mathcal{O} \cup P \models \alpha^T$.

The equivalence (2) $\Leftrightarrow$ (3) is proved in [Kikot *et al.*, 2017]. ❑

We now exhibit role constraints given by role assertions and inclusions and corresponding axioms that are complex. An application of Theorem 11 then provides the desired approximations. Let P be a set of role assertions and inclusions. Recall that we call P *inverse closed* if for every role name r in P there is a role name $\hat{r}$ such that $r \sqsubseteq \hat{r}^-, \hat{r} \sqsubseteq r^- \in P$ and there are no additional occurrences of inverse roles in P. Recall that we assume that P is *safe* in the sense that for any assertion $\rho_1 \circ \cdots \circ \rho_n \sqsubseteq \rho \in P$ with $n \geq 2$ neither $P \models \mathsf{func}(\rho)$ nor $P \models \mathsf{func}(\rho^-)$ holds.

Theorem 12 *Let P be an inverse closed and safe set of functionality assertions and RIs and let Ax contain the following equations, for all role names r, s in P:*

1. $\Diamond_r x \leq \Diamond_s x$ *if* $P \models r \sqsubseteq s$*;*
2. $\Diamond_r x \wedge \Diamond_r y \leq \Diamond_r (x \wedge y)$ *if* $\mathsf{func}(r) \in P$*;*
3. $x \wedge \Diamond_r y \leq \Diamond_r (y \wedge \Diamond_{\hat{r}} x)$*;*
4. $\Diamond_r \Diamond_{\hat{r}} x \leq x$ *if* $\mathsf{func}(\hat{r}) \in P$*;*
5. $\Diamond_{r_1} \cdots \Diamond_{r_n} x \leq \Diamond_r x$ *and* $\Diamond_{\hat{r}_n} \cdots \Diamond_{\hat{r}_1} x \leq \Diamond_{\hat{r}} x$ *if* $r_1 \circ \cdots \circ r_n \sqsubseteq r \in P$.

Then P corresponds to Ax and Ax is complex.

Proof. Correpondence is straightforward, so we focus on proving that Ax is complex. We start by introducing some equations implied by Ax. We set $P \models^* \mathsf{func}(r)$ if there exists s such that $P \models r \sqsubseteq s$ and $P \models \mathsf{func}(s)$.

(A) $\mathsf{Ax} \models \Diamond_r x \wedge \Diamond_s \top \leq \Diamond_s x$ if $P \models^* \mathsf{func}(r)$ and $P \models s \sqsubseteq r$. To see this, assume $P \models r \sqsubseteq u$ and $\mathsf{func}(u) \in P$. Then

$$\mathsf{Ax} \models \Diamond_r x \wedge \Diamond_s \top \leq \Diamond_u x \wedge \Diamond_s \top$$

by the equations under Point (1.). We have by the equations under Point (3.)

$$\mathsf{Ax} \models \Diamond_u x \wedge \Diamond_s \top \leq \Diamond_s (\top \wedge \Diamond_{\hat{s}} \Diamond_u x)$$

Thus,

$$\mathsf{Ax} \models \Diamond_u x \wedge \Diamond_s \top \leq \Diamond_s \Diamond_{\hat{s}} \Diamond_u x$$

By the equations under Point (1.)

$$\mathsf{Ax} \models \Diamond_u x \wedge \Diamond_s \top \leq \Diamond_s \Diamond_{\hat{u}} \Diamond_u x$$

By the equations under Point (4.)

$$\mathsf{Ax} \models \Diamond_u x \wedge \Diamond_s \top \leq \Diamond_s x$$

We obtain

$$\mathsf{Ax} \models \Diamond_r x \wedge \Diamond_s \top \leq \Diamond_s x,$$

as required.

(B) $\mathsf{Ax} \models \Diamond_r x \wedge \Diamond_r y \leq \Diamond_r (x \wedge y)$ if $P \models^* \mathsf{func}(r)$. To see this, assume $P \models r \sqsubseteq u$ and $\mathsf{func}(u) \in P$. Then, by the equations under Point (1.)

$$\mathsf{Ax} \models \Diamond_r x \wedge \Diamond_r y \leq \Diamond_u x \wedge \Diamond_u y$$

By the equations under Point (2.)

$$\mathsf{Ax} \models \Diamond_u x \wedge \Diamond_u y \leq \Diamond_u (x \wedge y)$$

Thus, by (A),

$$\mathsf{Ax} \models \Diamond_r x \wedge \Diamond_r y \leq \Diamond_r (x \wedge y)$$

(C) $\mathsf{Ax} \models \Diamond_r \Diamond_{\hat{r}} x \leq x$ if $P \models^* \mathsf{func}(\hat{r})$. This follows from the equations under Point (1.) and (4.).

We introduce the notion of a filter that is used in the proof. A *filter* F in a SLO $\mathfrak{A}$ is a subset of A such that $\top^{\mathfrak{A}} \in F$, $\bot^{\mathfrak{A}} \notin F$, and for all $a, b \in A$: $b \in F$ if $a \in F$ and $a \leq^{\mathcal{A}} b$, and $a \wedge b \in F$ if $a, b \in F$.

Now assume $\mathfrak{A}$ is given. We define a frame $\mathcal{F}$ such that there is an embedding from $\mathfrak{A}$ to $\mathcal{F}^+$. Let $\Delta^{\mathfrak{F}}$ be the set of all filters in $\mathfrak{A}$. For any role name r, the definition of $r^{\mathcal{F}}$ depends on whether r and/or $\hat{r}$ are functional:

1. if $P \models^* \mathsf{func}(r)$ and $P \models^* \mathsf{func}(\hat{r})$, then set $(F_1, F_2) \in r^{\mathcal{F}}$ if for all $a \in A$ the following hold: (i) $\Diamond_r a \in F_1$ iff $a \in F_2$ and (ii) $\Diamond_{\hat{r}} a \in F_2$ iff $a \in F_1$. $\hat{r}^{\mathcal{F}}$ is defined as the inverse of $r^{\mathcal{F}}$.
2. if $P \models^* \mathsf{func}(r)$ and $P \not\models^* \mathsf{func}(\hat{r})$, then set $(F_1, F_2) \in r^{\mathcal{F}}$ if for all $a \in A$ the following hold: (i) $\Diamond_r a \in F_1$ iff $a \in F_2$ and (ii) $\Diamond_{\hat{r}} a \in F_2$ if $a \in F_1$. $\hat{r}^{\mathcal{F}}$ is defined as the inverse of $r^{\mathcal{F}}$.
3. if $P \not\models^* \mathsf{func}(r)$ and $P \not\models^* \mathsf{func}(\hat{r})$, then set $(F_1, F_2) \in r^{\mathcal{F}}$ if for all $a \in A$ the following hold: (i) $\Diamond_r a \in F_1$ if $a \in F_2$ and (ii) $\Diamond_{\hat{r}} a \in F_2$ if $a \in F_1$. $\hat{r}^{\mathcal{F}}$ is defined as the inverse of $r^{\mathcal{F}}$.

This finishes the definition of $\mathcal{F}$. We first show that $\mathcal{F}$ satisfies P.

- Assume $r \sqsubseteq s \in P$. We have to check that $r \sqsubseteq s$ is satisfied in $\mathcal{F}$. If neither $P \models^* \mathsf{func}(s)$ nor $P \models^* \mathsf{func}(\hat{s})$, then $r^{\mathcal{F}} \subseteq s^{\mathcal{F}}$ follows directly from the definition and the equations under Point (1.) of Theorem 12.

  Now assume that $P \models^* \mathsf{func}(s)$. Then $P \models^* \mathsf{func}(r)$. Assume $(F_1, F_2) \in r^{\mathcal{F}}$. We have to show $(F_1, F_2) \in s^{\mathcal{F}}$. Thus, we first have to show that $\Diamond_s a \in F_1$ iff $a \in F_2$. If $a \in F_2$, then $\Diamond_r a \in F_1$. Then $\Diamond_s a \in F_1$ by the equations under Point (1) of Theorem 12, as required. If $\Diamond_s a \in F_1$, then $\Diamond_r a \in F_1$ since $\Diamond_r \top \in F_1$ (by the equations under (A)). Then $a \in F_2$, as required. Next we make a case distinction: if $P \not\models^* \mathsf{func}(\hat{s})$, then we have to show that $\Diamond_{\hat{r}} a \in F_2$ if $a \in F_1$. But this follows from $P \models \hat{r} \sqsubseteq \hat{s}$ and the equations under Point (1). If $P \models^* \mathsf{func}(\hat{s})$, then we have to show that $\Diamond_{\hat{s}} a \in F_2$ iff $a \in F_1$. This can be proved again using the equations under Point (1) and (A).

  The case $P \models^* \mathsf{func}(\hat{s})$ is considered in the same way.
- Assume $\mathsf{func}(r) \in P$. Then functionality of $r^{\mathcal{F}}$ follows directly from the definition.
- Assume $r_1 \circ \cdots \circ r_n \sqsubseteq r \in P$. Let $(F_1, F_2) \in (r_1 \circ \cdots \circ r_n)^{\mathcal{I}}$. As P is safe, we have to show that if $\Diamond_r a \in F_1$, then $a \in F_2$ and if $\Diamond_{\hat{r}} a \in F_2$, then $a \in F_1$. Both can be proved in a straightforward way using the equations under Point (5) of Theorem 12.

It remains to construct an embedding h from $\mathfrak{A}$ into $\mathcal{F}^+$. Define h by setting

$$h(a) := \{F \in \Delta^{\mathcal{F}} \mid a \in F\},$$

for all $a \in A$. It is straightforward to show that h is an injective mapping with

- $h(\top^{\mathfrak{A}}) = \Delta^{\mathcal{F}}$;
- $h(\bot^{\mathfrak{A}}) = \emptyset$;
- $h(a \wedge^{\mathfrak{A}} b) = h(a) \cap h(b)$.

It thus remains to prove that

$$h(\Diamond_r^{\mathfrak{A}} a) = \Diamond_r^{\mathcal{F}^+} h(a)$$

for all role names r. We first assume that $P \models^* \mathsf{func}(r)$ and $P \models^* \mathsf{func}(\hat{r})$. Assume a filter F is given. Suppose $\Diamond_r a_0 \in F$. We have to show the existence of a filter F' with $a_0 \in F'$ such that $(F, F') \in r^{\mathcal{F}}$. Consider

$$X = \{a \mid \Diamond_r a \in F\} \cup \{\Diamond_{\hat{r}} b \mid b \in F\}$$

and

$$Y = \{a \mid \Diamond_r a \notin F\} \cup \{\Diamond_{\hat{r}} b \mid b \notin F\}$$

It suffices to show the existence of a filter F' containing X with an empty intersection with Y. To this end it suffices to prove that there is no finite conjunction c of members of X such that $c \leq e$ for some $e \in Y$. Assume an arbitrary such c is given. It takes the form

$$c = a_1 \wedge \cdots \wedge a_n \wedge \Diamond_{\hat{r}} b_1 \wedge \cdots \Diamond_{\hat{r}} b_m$$

with $\Diamond_r a_1, \ldots, \Diamond_r a_n \in F$ and $b_1, \ldots, b_m \in F$. Then, by the axioms under Point (B), we have $\Diamond_r (a_1 \wedge \cdots \wedge a_n) \in F$. We also have $b_1 \wedge \cdots \wedge b_m \in F$. Thus we may assume that

$$c = a \wedge \Diamond_{\hat{r}} b$$

for some a with $\Diamond_r a \in F$ and $b \in F$. For a proof by contraction first assume that

$$a \wedge \Diamond_{\hat{r}} b \leq a'$$

for some a' with $\Diamond_r a' \notin F$. Then

$$\Diamond_r (a \wedge \Diamond_{\hat{r}} b) \leq \Diamond_r a'$$

by monotonicity of $\Diamond_r$. But by the equations under Point (3.) of Theorem 12,

$$b \wedge \Diamond_r a \leq \Diamond_r (a \wedge \Diamond_{\hat{r}} b)$$

Then from $b \wedge \Diamond_r a \in F$ (since $b, \Diamond_r a \in F$) and $b \wedge \Diamond_r a \leq \Diamond_r a'$ we obtain $\Diamond_r a' \in F$ and we have derived a contradiction.

Now assume

$$a \wedge \Diamond_{\hat{r}} b \leq \Diamond_{\hat{r}} b'$$

for some $b' \notin F$. Then, by the equations under Point (3.) of Theorem 12 and Point (C) and monotonicity of $\Diamond_r$,

$$b \wedge \Diamond_r a \leq \Diamond_r (a \wedge \Diamond_{\hat{r}} b) \leq \Diamond_r \Diamond_{\hat{r}} b' \leq b'$$

which contradicts the assumptions that $b, \Diamond_r a \in F$ and $b' \notin F$.

Now assume that $P \models^* \mathsf{func}(r)$ and $P \not\models^* \mathsf{func}(\hat{r})$. Assume a filter F is given. Suppose first $\Diamond_r a_0 \in F$. We have to show the existence of a filter F' with $a_0 \in F'$ such that $(F, F') \in r^{\mathcal{F}}$. Consider

$$X = \{a \mid \Diamond_r a \in F\} \cup \{\Diamond_{\hat{r}} b \mid b \in F\}$$

and

$$Y = \{a \mid \Diamond_r a \notin F\}$$

It suffices to show the existence of a filter F' containing X with an empty intersection with Y. To this end it suffices to prove that there is no finite conjunction c of members of X such that $c \leq e$ for some $e \in Y$. Assume an arbitrary such c is given. As shown above, we may assume that

$$c = a \wedge \Diamond_{\hat{r}} b$$

for some a with $\Diamond_r a \in F$ and $b \in F$. Now one can prove that

$$a \wedge \Diamond_{\hat{r}} b \leq a'$$

for some a' with $\Diamond_r a' \notin F$ leads to a contradiction in exactly the same way as above.

Suppose now that $\Diamond_{\hat{r}} a_0 \in F$. We have to show the existence of a filter F' with $a_0 \in F'$ such that $(F', F) \in r^{\mathcal{F}}$. Consider

$$X = \{\Diamond_r a \mid a \in F\} \cup \{a_0\}$$

and

$$Y = \{\Diamond_r a \mid a \notin F\} \cup \{b \mid \Diamond_{\hat{r}} b \notin F\}$$

It suffices to show the existence of a filter F' containing X with an empty intersection with Y. To this end it suffices to prove that there is no finite conjunction c of members of X such that $c \leq e$ for some $e \in Y$. We may assume that $c = a_0 \wedge \Diamond_r a$ for some $a \in F$. Assume that

$$a_0 \wedge \Diamond_r a \leq \Diamond_r a'$$

for some $a' \notin F$. Then by the equations under Point (3.) of Theorem 12 and (C)

$$a \wedge \Diamond_{r^*} a_0 \leq \Diamond_{\hat{r}}(a_0 \wedge \Diamond_r a) \leq \Diamond_{\hat{r}} \Diamond_r a' \leq a'$$

which contradicts the assumption that $a, \Diamond_{\hat{r}} a \in F$.

Assume that

$$a_0 \wedge \Diamond_r a \leq b$$

for some b with $\Diamond_{\hat{r}} b \notin F$. Then by the equations under Point (3.) of Theorem 12

$$a \wedge \Diamond_{\hat{r}} a_0 \leq \Diamond_{\hat{r}}(a_0 \wedge \Diamond_r a) \leq \Diamond_{\hat{r}} b$$

which again contradicts the assumption that $a, \Diamond_{\hat{r}} a \in F$.

The remaining case in which $P \not\models^* \mathsf{func}(r)$ and $P \not\models^* \mathsf{func}(\hat{r})$ is similar and omitted. ❏

Theorems 12 and 11 provide us with an $\mathcal{EL}_\bot$ approximation $\mathcal{O}_T$ of any inverse closed $\mathcal{ELRIF}_\bot$ ontology $\mathcal{O}_S$.

Theorem 13 *Let $\mathcal{O}_S$ be an inverse closed $\mathcal{ELRIF}_\bot$ ontology and $\Sigma = \mathsf{sig}(\mathcal{O}_S)$. Define $\mathcal{O}_T$ as the $\mathcal{EL}_\bot$ ontology containing for all $\mathcal{EL}(\Sigma)$ concepts C, D and role names $r, s \in \Sigma$:*

1. *all CIs in $\mathcal{O}_S$;*
2. *$\exists r.C \sqsubseteq \exists s.C$ if $\mathcal{O}_S \models r \sqsubseteq s$;*
3. *$\exists r_1.\cdots\exists r_n.C \sqsubseteq \exists r.C$ and $\exists \hat{r}_n.\cdots\exists \hat{r}_1.C \sqsubseteq \exists \hat{r}.C$, if $r_1 \circ \cdots \circ r_n \sqsubseteq r \in \mathcal{O}_S$ with $n \geq 2$;*
4. *$C \sqcap \exists r.D \sqsubseteq \exists r.(D \sqcap \exists \hat{r}.C)$;*
5. *$\exists r.C \sqcap \exists r.D \sqsubseteq \exists r.(C \sqcap D)$ if $\mathsf{func}(r) \in \mathcal{O}_S$;*
6. *$\exists r.\exists \hat{r}.C \sqsubseteq C$ if $\mathsf{func}(\hat{r}) \in \mathcal{O}_S$.*

Then $\mathcal{O}_T$ is an $\mathcal{EL}_\bot$ approximation of $\mathcal{O}_S$.

Theorem 4 is an immediate consequence of Theorem 13: if $\mathcal{O}_S$ is the $\mathcal{EL}_\bot$ approximation given by Theorem 13 and $\mathcal{O}'_S$ is the $\mathcal{ELR}_\bot$ ontology given in Theorem 4, then $\mathcal{O}'_S \models \mathcal{O}_S$ since

$$\{r \sqsubseteq s\} \models \exists r.C \sqsubseteq \exists s.C$$

and

$$\{r_1 \circ \cdots \circ r_n \sqsubseteq r\} \models \exists r_1.\cdots\exists r_n.C \sqsubseteq \exists r.C$$

and

$$\{r_1 \circ \cdots \circ r_n \sqsubseteq r\} \models \exists \hat{r}_n.\cdots\exists \hat{r}_1.C \sqsubseteq \exists \hat{r}.C.$$

The following examples show that correspondence between role assertions and SLO axioms alone does not imply that the axioms are complex and, therefore, cannot be used to obtain $\mathcal{EL}_\bot$ approximations. They also show that approximations are not compositional.

Example 7 Recall that the role assertion $P_0 = \{r \sqsubseteq s^-\}$ corresponds to the axiom $\alpha_0 = (x \wedge \Diamond_r y \leq \Diamond_r(y \wedge \Diamond_s x))$. Using the technique of the proof of Theorem 12, it is straightforward to show that α_0 is complex. Thus, Theorem 11 provides a $\mathcal{EL}_\bot$ approximation of any ontology $\mathcal{O} \cup P_0$ with $\mathcal{O}$ a set of $\mathcal{EL}_\bot$ inclusions. This is not the case if one admits two such role assertions. To show this, consider

$$P_1 = \{r_1 \sqsubseteq r_2^-, r_2 \sqsubseteq r_3^-\}$$

and let

$$\mathcal{O}_1 = \{A \sqsubseteq \exists r_1.B\} \cup P_1$$

Then P_1 corresponds to

$$\mathsf{Ax}_1 = \{x \wedge \Diamond_{r_1} y \leq \Diamond_{r_1}(y \wedge \Diamond_{r_2} x), x \wedge \Diamond_{r_2} y \leq \Diamond_{r_2}(y \wedge \Diamond_{r_3} x)\}$$

However, $\mathcal{O}'_1 = \{A \sqsubseteq \exists r_1.B\} \cup \mathsf{Ax}_1^{\{r_1,r_2,r_3,A,B\}}$ is not a $\mathcal{EL}_\bot$ approximation of $\mathcal{O}_1$. To prove this, observe that

$$\mathcal{O}_1 \models A \sqsubseteq r_3.B.$$

We show that $\mathcal{O}' \not\models A \sqsubseteq \exists r_3.B$. Define an interpretation $\mathcal{I}$ by setting

- $\Delta^{\mathcal{I}} = \{a, b, c, d\}$;
- $r_1^{\mathcal{I}} = \{(a, b), (c, d)\}$;
- $r_2^{\mathcal{I}} = \{(b, c), (d, c)\}$;

- $r_3^{\mathcal{I}} = \{(c,d)\}$;
- $A^{\mathcal{I}} = \{a,c\}$, $B^{\mathcal{I}} = \{b,d\}$.

Then $\mathcal{I}$ is a model of $\mathcal{O}_1'$ but $a \in A^{\mathcal{I}} \setminus (\exists r_3.B)^{\mathcal{I}}$.

Observe that one obtains a $\mathcal{EL}_\bot$ approximation of $\mathcal{O}_1$ by adding the axiom $\Diamond_{r_1} x \leq \Diamond_{r_3} x$ corresponding to $r_1 \sqsubseteq r_3$ to Ax_1.

The next example also refutes compositionality of approximations. In this case for combinations of RIs with functionality assertions.

Example 8 Let

$$P_2 = \{r \sqsubseteq s^-, \mathsf{func}(s)\}$$

Both role assertions in P_2 correspond to complex axioms, namely

$$\alpha_2 = (x \wedge \Diamond_r y \leq \Diamond_r (y \wedge \Diamond_s x))$$

and

$$\alpha_3 = (\Diamond_s x \wedge \Diamond_s y \leq \Diamond_s (x \wedge y)),$$

respectively. $\mathsf{Ax}_2 = \{\alpha_2, \alpha_3\}$ is not complex, however, and

$$\mathcal{O}_2' = \{\top \sqsubseteq \exists r.\top, \top \sqsubseteq \exists s.A\} \cup \mathsf{Ax}_2^{\{r,s,A\}}$$

is not a $\mathcal{EL}_\bot$ approximation of

$$\mathcal{O}_2 = \{\top \sqsubseteq \exists r.\top, \top \sqsubseteq \exists s.A\} \cup P_2$$

To show this, first observe that $\mathcal{O}_2 \models \top \sqsubseteq A$. To prove this, let $\mathcal{I}$ be a model of $\mathcal{O}_2$ and assume $(d,d') \in r^{\mathcal{I}}$. Then $(d',d) \in s^{\mathcal{I}}$ and from functionality of s and $\top \sqsubseteq \exists s.A$ we obtain $d \in A^{\mathcal{I}}$. On the other hand, $\mathcal{O}_2' \not\models \top \sqsubseteq A$: consider the interpretation $\mathcal{I}$ with domain $\Delta^{\mathcal{I}} = \{d,d'\}$, $r^{\mathcal{I}} = s^{\mathcal{I}} = \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ and $A^{\mathcal{I}} = \{d\}$. Then $\mathcal{I}$ is a model of $\mathcal{O}_2'$ but $d' \notin A^{\mathcal{I}}$.

One can define a $\mathcal{EL}_\bot$ approximation of $\mathcal{O}_2$ by adding the axiom $\Diamond_r \Diamond_s x \leq x$ (corresponding to the equations under Point (4.) of Theorem 12) to Ax_2.

Finally, we give an example illustrating why the equations under Point (4.) of Theorem 12 are needed.

Example 9 Let

$$P_3 = \{t \sqsubseteq r, \mathsf{func}(r)\}$$

Both role assertions in P_2 correspond to complex axioms, namely

$$\alpha_4 = (\Diamond_t x \leq \Diamond_r x)$$

and

$$\alpha_5 = (\Diamond_r x \wedge \Diamond_r y \leq \Diamond_r (x \wedge y)),$$

respectively. $\mathsf{Ax}_3 = \{\alpha_4, \alpha_5\}$ is not complex, however, and

$$\mathcal{O}_3' = \{\exists r.A \sqcap \exists t.\top \sqsubseteq \exists t.A\} \cup \mathsf{Ax}_3^{\{t,r,A\}}$$

is not a $\mathcal{EL}_\bot$ approximation of

$$\mathcal{O}_3 = \{\exists r.A \sqcap \exists t.\top \sqsubseteq \exists t.A\} \cup P_3$$

To show this, first observe that $\mathcal{O}_3 \models \exists r.A \sqcap \exists t.\top \sqsubseteq \exists t.A$. On the other hand, $\mathcal{O}_3' \not\models \exists r.A \sqcap \exists t.\top \sqsubseteq \exists t.A$. To see this, consider the interpretation $\mathcal{I}$ defined by setting

- $\Delta^{\mathcal{I}} = \{a,b,c\}$;
- $r^{\mathcal{I}} = \{(a,b)\}$, $t^{\mathcal{I}} = \{(a,c)\}$, $A^{\mathcal{I}} = \{b\}$.

Then $\mathcal{I}$ is a model of $\mathcal{O}_3'$ but $a \notin (\exists t.A)^{\mathcal{I}}$.

D Details for Section 5

In Section 5, we introduced two $\mathcal{ELI}_\bot$-to-$\mathcal{EL}_\bot$ approximations. In this section, we deliver further details for both cases.

D.1 The Chase (non-projective)

Let $\mathcal{O}$ be an $\mathcal{ELI}$ ontology and $\Sigma = \mathtt{sig}(\mathcal{O})$. We again assume that $\bot$ occurs only in CIs of the form $C \sqsubseteq \bot$. Starting from an ABox $\mathcal{A}$, the chase exhaustively applies the following rules:

R1 If $\mathcal{A} \models C(a)$ and $C \sqsubseteq D \in \mathcal{O}_S$ with $D \neq \bot$, then add $D(a)$ to $\mathcal{A}$.

The chase applies this rule exhaustively in a fair way. When the resulting sequence of ABoxes is $\mathcal{A} = \mathcal{A}_0, \mathcal{A}_1, \ldots$, we use $\mathtt{chase}_{\mathcal{O}}(\mathcal{A})$ to denote $\bigcup_{i \geq 0} \mathcal{A}_i$. For simplicity, we assume here that the chase is oblivious in the sense that the rule applies even when the consequence $\mathcal{A} \models D(a)$ already holds. Consequently, $\mathtt{chase}_{\mathcal{O}}(\mathcal{A})$ is uniquely defined. The following is easy to establish, details are omitted.

Lemma 7 *Let $\mathcal{O}$ be an $\mathcal{ELI}$ ontology and $\mathcal{A}$ an ABox. Then*

1. *$\mathcal{A}$ is inconsistent with $\mathcal{O}$ iff there are $C \sqsubseteq \bot \in \mathcal{O}$ and $a \in \mathtt{Ind}(\mathcal{A})$ with $\mathtt{chase}_{\mathcal{O}}(\mathcal{A}) \models C(a)$;*
2. *if $\mathcal{A}$ is consistent with $\mathcal{O}$, then $\mathcal{A}, \mathcal{O} \models C(a)$ iff $\mathtt{chase}_{\mathcal{O}}(\mathcal{A}) \models C(a)$ for all $\mathcal{ELI}$ concepts C and $a \in \mathtt{Ind}(\mathcal{A})$.*

We are going to use the chase on ditree-shaped ABoxes. When started on such an ABox, all generated ABoxes are tree-shaped, that is, the undirected graph $G^u_{\mathcal{A}} = (\mathtt{Ind}(\mathcal{A}), \{\{a,b\} \mid r(a,b) \in \mathcal{A}\}$ is a tree (possibly with multi-edges); they are not guaranteed to be ditree-shaped.

D.2 Completeness (non-projective)

We prove the completeness part of Theorem 5. It is not difficult to prove that since $\mathcal{O}_S$ is formulated in $\mathcal{ELI}$, for any role inclusion $r \sqsubseteq s$, $\mathcal{O}_S \models r \sqsubseteq s$ implies $\mathcal{O}_S^{\mathcal{H}} \models r \sqsubseteq s$ where $\mathcal{O}_S^{\mathcal{H}}$ is the set of role inclusions from $\mathcal{O}_S$. Since $\mathcal{O}_S^{\mathcal{H}} \subseteq \mathcal{O}_T$, we have $\mathcal{O}_S \models r \sqsubseteq s$ iff $\mathcal{O}_T \models r \sqsubseteq s$ and it remains to deal with concept inclusions.

For a tree-shaped ABox $\mathcal{A}$, we use $\mathcal{A}^{\downarrow}$ to denote the restriction of $\mathcal{A}$ to assertions in which all individuals a are reachable from the root of $\mathcal{A}$ along a directed role path, that is, $\mathcal{A}$ contains assertions $r_0(a_0,a_1), \ldots, r_{n-1}(a_{n-1},a_n)$ where a_0 is the root of $\mathcal{A}$ and $a_n = a$.

Lemma 8 *Let C be an $\mathcal{EL}(\Sigma)$ concept, $\mathcal{A}_0, \mathcal{A}_1, \ldots$ the sequence of ABoxes constructed by $\mathtt{chase}_{\mathcal{O}_S}(\mathcal{A}_C)$, $\mathcal{I}$ a model of $\mathcal{O}_T$, and $d \in C^{\mathcal{I}}$. Then for every $i \geq 0$, there is a homomorphism h_i from $\mathcal{A}_i^{\downarrow}$ to $\mathcal{I}$ with $h_i(a_0) = d$.*

Proof. The proof is by induction on i. The induction start is immediate by Lemma 4 and since $\mathcal{A}_0 = \mathcal{A}_C$. For the induction step, we make a case distinction according to the chase rule applied to obtain $\mathcal{A}_{i+1}$ from $\mathcal{A}_i$.

R1. Assume the rule was applied to individual $a \in \mathtt{Ind}(\mathcal{A}_i)$ and CI $E \sqsubseteq F \in \mathcal{O}_S$. Let $F^{\mathcal{EL}}$ be the result of replacing in F every subconcept $\exists r^-.G$ with $\top$. Moreover, let $\mathcal{A}_i^{\downarrow,-}$ be $\mathcal{A}_i^{\downarrow}$ after removal of all role edges that have been added by an application of rule R2, and likewise for $\mathcal{A}_{i+1}^{\downarrow,-}$.

Further, let $C_i^{\downarrow,-}$ be $\mathcal{A}_i^{\downarrow,-}$ viewed as an $\mathcal{EL}$ concept, and likewise for $C_{i+1}^{\downarrow,-}$. Note that $\mathcal{A}_{i+1}^{\downarrow}$ is obtained from $\mathcal{A}_i^{\downarrow}$ by adding $F^{\mathcal{EL}}(a)$ and thus $C_{i+1}^{\downarrow,-}$ is a $\mathtt{cl}_{\mathcal{EL}}(\mathcal{O}_S)$ decoration of $C_i^{\downarrow,-}$. From Point 2 of Lemma 7, we obtain $\mathcal{O}_S \models C \sqsubseteq C_{i+1}^{\downarrow,-}$; note that this (trivially) holds also when $\mathcal{A}_C$ is inconsistent with $\mathcal{O}_S$. Since $\emptyset \models C_i^{\downarrow,-} \sqsubseteq C$, this implies $\mathcal{O}_S \models C_i^{\downarrow,-} \sqsubseteq C_{i+1}^{\downarrow,-}$. As a consequence and since $C_{i+1}^{\downarrow,-}$ is a $\mathtt{cl}_{\mathcal{EL}}(\mathcal{O}_S)$ decoration of $C_i^{\downarrow,-}$, we must have $C_i^{\downarrow,-} \sqsubseteq C_{i+1}^{\downarrow,-} \in \mathcal{O}_T$. Thus, $d \in (C_{i+1}^{\downarrow,-})^{\mathcal{I}}$ and by Lemma 4 we find a homomorphism h_{i+1} from $\mathcal{A}_{i+1}^{\downarrow,-}$ to $\mathcal{I}$ with $h(a_0) = d$. Since $\mathcal{I}$ is a model of $\mathcal{O}$ and $\mathcal{O}_T$ contains the same role inclusions as $\mathcal{O}_S$, h_i must also be a homomorphism from $\mathcal{A}_{i+1}^{\downarrow}$ to $\mathcal{I}$, as required. ❑

Lemma 9 *Let C be an $\mathcal{EL}(\Sigma)$ concept and D an $\mathcal{EL}_\bot$ concept. Then $\mathcal{O}_S \models C \sqsubseteq D$ implies $\mathcal{O}_T \models C \sqsubseteq D$.*

Proof. Let $\mathcal{I}$ be a model of $\mathcal{O}_T$ with $d \in C^{\mathcal{I}}$. We have to show that $d \in D^{\mathcal{I}}$. Let $\mathcal{A}_0, \mathcal{A}_1, \ldots$ be the sequences of ABoxes constructed by the chase started on $\mathcal{A}_C$. First assume that $\mathcal{A}_C$ is consistent with $\mathcal{O}_S$. Then from Point 2 of Lemma 7, we obtain $\mathtt{chase}_{\mathcal{O}_S}(\mathcal{A}_C) \models D(a_0)$. Thus, there is a k with $\mathcal{A}_k \models D(a_0)$. Since D is an $\mathcal{EL}$ concept, this implies $\mathcal{A}_k^{\downarrow} \models D(a_0)$. By Lemma 4, there is thus a homomorphism h_0 from $\mathcal{A}_D$ to $\mathcal{A}_k^{\downarrow}$ with $h_0(a_0) = a_0$. Lemma 8 yields a homomorphism h from $\mathcal{A}_k^{\downarrow}$ to $\mathcal{I}$ with $h(a_0) = d$. Composing h_0 with h and applying Lemma 4 yields $d \in D^{\mathcal{I}}$ as required. Now assume that $\mathcal{A}_C$ is inconsistent with $\mathcal{O}_S$. Then $\mathcal{O}_S \models C \sqsubseteq \bot$ and thus $C \sqsubseteq \bot \in \mathcal{O}_T$, in contradiction to $\mathcal{I}$ being a model of $\mathcal{O}_T$ with $C^{\mathcal{I}} \neq \emptyset$. ❑

D.3 The Chase (projective)

Let $\mathcal{O}$ be an $\mathcal{ELI}_\bot$ ontology in normal form. Starting from an ABox $\mathcal{A}$, the chase exhaustively applies the following rules, constructing in the limit an extended and potentially infinite ABox:

- R1 If $A_1(a), \ldots, A_n(a) \in \mathcal{A}$ and $\mathcal{O} \models A_1 \sqcap \cdots \sqcap A_n \sqsubseteq B$, then add $B(a)$ to $\mathcal{A}$;
- R2 If $r(a,b), A(b) \in \mathcal{A}$ and $\exists r.A \sqsubseteq B \in \mathcal{O}$, then add $B(a)$ to $\mathcal{A}$;
- R3 If $r(b,a), A(b) \in \mathcal{A}$ and $\exists r^-.A \sqsubseteq B \in \mathcal{O}$, then add $B(a)$ to $\mathcal{A}$;
- R4 If $A(a) \in \mathcal{A}$ and $A \sqsubseteq \exists r.B \in \mathcal{O}$, then add $r(a,b)$ and $B(b)$ to $\mathcal{A}$, with b fresh.

We again use $\mathtt{chase}_{\mathcal{O}}(\mathcal{A})$ to denote the result of applying the chase of ABox $\mathcal{A}$, which is unique since rule application is oblivious. Note that this chase is the standard chase for $\mathcal{ELI}$ except that no new successors are introduced to witness existential restrictions on inverse roles. This is compensated by the semantic entailment in rule R1, which is in line with Point 1 in Theorem 6. We remark that, when applied to a ditree-shaped ABox, all ABoxes produced by the chase are ditree-shaped, possibly with multi-edges. We assume that $\bot$ occurs in $\mathcal{O}$ only in CIs of the form $C \sqsubseteq \bot$ with C an $\mathcal{EL}$ concept.

Lemma 10 *Let $\mathcal{O}$ be an $\mathcal{ELI}_\bot$ ontology and $\mathcal{A}$ a ditree-shaped ABox with root a_0. Then*

1. *$\mathcal{A}$ is inconsistent with $\mathcal{O}$ iff there are $C \sqsubseteq \bot \in \mathcal{O}$ and $a \in \mathtt{Ind}(\mathcal{A})$ with $\mathtt{chase}_{\mathcal{O}}(\mathcal{A}) \models C(a)$, and*
2. *if $\mathcal{A}$ is consistent with $\mathcal{O}$, then $\mathcal{A}, \mathcal{O} \models C_0(a_0)$ iff $\mathtt{chase}_{\mathcal{O}}(\mathcal{A}) \models C_0(a_0)$ for all $\mathcal{EL}$ concepts of depth at most ℓ.*

Proof. We prove both point simultaneously, starting with the "if" directions. Let $\mathcal{A}_0, \mathcal{A}_1, \ldots$ be the sequence of ABoxes produced by the chase and let $\mathcal{I}$ be a model of $\mathcal{A}$ and $\mathcal{O}$. We show the following.

Claim For all $i \geq 0$, there is a homomorphism from $\mathcal{A}_i$ to $\mathcal{I}$ with $h(a_0) = a_0$.

This establishes Point 1 because if $C \sqsubseteq \bot \in \mathcal{O}$ and $\mathtt{chase}_{\mathcal{O}}(\mathcal{A}) \models C(a)$, then there is a k such that $\mathcal{A}_k \models C(a)$. The existence of h_k implies that $C^{\mathcal{I}} \neq \emptyset$ (via Lemma 4), in contradiction to $\mathcal{I}$ being a model of $\mathcal{A}$ and $\mathcal{O}$. Thus, $\mathcal{A}$ is inconsistent with $\mathcal{O}$.

It also establishes Point 2. In fact, if $\mathtt{chase}_{\mathcal{O}}(\mathcal{A}) \models C(a_0)$, then there is a k such that $\mathcal{A}_k \models C(a_0)$ and h_k shows that $d \in C^{\mathcal{I}}$ as required.

It thus remains to prove the claim. The case $i = 0$ is trivial as the desired homomorphism is simply the identity on $\mathtt{Ind}(\mathcal{A})$. For the case $i > 0$, we make a case distinction according the the chase rule applied in order to obtain $\mathcal{A}_{i+1}$ from $\mathcal{A}_i$.

R1. If this rule was applied to obtain $\mathcal{A}_{i+1}$ from $\mathcal{A}_i$, then there are $A_1(a), \ldots, A_n(a) \in \mathcal{A}_i$ such that $A_1 \sqcap \ldots \sqcap A_n \sqsubseteq B \in \mathcal{O}_S$. $\mathcal{A}_{i+1}$ is obtained from $\mathcal{A}_i$ by adding $B(a)$. Because h_i is the homomorphism from $\mathcal{A}_i$ to $\mathcal{I}$, we must have $h_i(a) \in (A_1 \sqcap \cdots \sqcap A_n)^{\mathcal{I}}$. Since $\mathcal{I}$ is a model of $\mathcal{A}$ and $\mathcal{O}$, this yields $h_{i+1}(a) \in B^{\mathcal{I}}$. Consequently h_i is also a homomorphism from $\mathcal{A}_{i+1}$ to $\mathcal{I}$ and we can set $h_{i+1} = h_i$.

R2. Then there are $r(a,b), A(b) \in \mathcal{A}_i$ and $\exists r.A \sqsubseteq B \in \mathcal{O}$. $\mathcal{A}_{i+1}$ is obtained from $\mathcal{A}_i$ by adding $B(a)$. Because h_i is the homomorphism from $\mathcal{A}_i$ to $\mathcal{I}$, we must have $h_i(b) \in A^{\mathcal{I}}$ and $(h_i(a), h_i(b)) \in r^{\mathcal{I}}$. Since $\mathcal{I}$ is a model of $\mathcal{A}$ and $\mathcal{O}$, $h_{i+1}(a) \in B^{\mathcal{I}}$. Consequently, h_i is also a homomorphism from $\mathcal{A}_{i+1}$ to $\mathcal{I}$ and we can set $h_{i+1} = h_i$.

R3. If this rule was applied to obtain $\mathcal{A}_{i+1}$ from $\mathcal{A}_i$, then there are $r(b,a), A(b) \in \mathcal{A}_i$ and $\exists r^-.A \sqsubseteq B \in \mathcal{O}$. $\mathcal{A}_{i+1}$ is obtained from $\mathcal{A}_i$ by adding $B(a)$. Because h_i is the homomorphism from $\mathcal{A}_i$ to $\mathcal{I}$, we must have $h_i(b) \in A^{\mathcal{I}}$ and $(h_i(b), h_i(a)) \in r^{\mathcal{I}}$. Since $\mathcal{I}$ is a model of $\mathcal{A}$ and $\mathcal{O}$, this yields $h_{i+1}(a) \in B^{\mathcal{I}}$. Consequently, h_i is also a homomorphism from $\mathcal{A}_{i+1}$ to $\mathcal{I}$ and we can set $h_{i+1} = h_i$.

R4. Then there is an $A(a) \in \mathcal{A}_i$ such that $A \sqsubseteq \exists r.B \in \mathcal{O}$ and $\mathcal{A}_{i+1}$ is obtained from $\mathcal{A}_i$ by adding $B(b)$ and $r(a,b)$, b fresh. Because h_i is the homomorphism from $\mathcal{A}_i$ to $\mathcal{I}$, we must have $h_i(a) \in A^{\mathcal{I}}$. Since $\mathcal{I}$ is a model of $\mathcal{A}$ and $\mathcal{O}$, there is an $e \in B^{\mathcal{I}}$ such that $(h_i(a), e) \in r^{\mathcal{I}}$. Consequently $h_{i+1} = h_1 \cup \{b \mapsto e\}$ is a homomorphism from $\mathcal{A}_{i+1}$ to $\mathcal{I}$.

For the (contrapositive of the) "only if" directions, assume that there are no $C \sqsubseteq \bot \in \mathcal{O}$ and $a \in \mathtt{Ind}(\mathcal{A})$ such that $\mathtt{chase}_{\mathcal{O}}(\mathcal{A}) \models C(a)$, respectively that $\mathtt{chase}_{\mathcal{O}}(\mathcal{A}) \not\models C(a_0)$. We show how to construct a model $\mathcal{J}$ of $\mathcal{A}$ and $\mathcal{O}$ such that for all $a \in \mathtt{Ind}(\mathcal{A})$ and $\mathcal{EL}$ concepts C, $\mathtt{chase}_{\mathcal{O}}(\mathcal{A}) \not\models C(a)$ implies $a \notin C^{\mathcal{J}}$. This implies that $\mathcal{A}$ is consistent with $\mathcal{O}$ (since $\bot$ occurs in $\mathcal{O}$ only in the form $C \sqsubseteq \bot$), respectively that $\mathcal{A}, \mathcal{O} \not\models C(a_0)$.

For brevity, let $\mathtt{Ind}$ denote the set of individual names in the (potentially infinite) ABox $\mathtt{chase}_{\mathcal{O}}(\mathcal{A})$. Further let $\mathcal{I}$ be $\mathtt{chase}_{\mathcal{O}}(\mathcal{A})$ viewed as an interpretation. Because of rule R1, for each $a \in \mathtt{Ind}$ we find a model $\mathcal{I}_a$ of $\mathcal{O}$ and a $d_a \in \Delta^{\mathcal{I}_a}$ such that for all concept names A, $a \in A^{\mathcal{I}}$ iff $d_a \in A^{\mathcal{I}_a}$. We can further assume that the domains of all interpretations $\mathcal{I}_a$ are mutually disjoint, and that they are also disjoint from the domain of $\mathcal{I}$. Let $\mathcal{J}$ be the interpretation obtained as follows:

1. take the disjoint union of $\mathcal{I}$ and all the $\mathcal{I}_a$;
2. for every $a \in \mathtt{Ind}$, every role name r, and every $(e, d_a) \in r^{\mathcal{I}_a}$, add (e, a) to $r^{\mathcal{J}}$.

It can be verified that, as required, $\mathtt{chase}_{\mathcal{O}}(\mathcal{A}) \not\models C(a)$ implies $a \notin C^{\mathcal{J}}$ for all $a \in \mathtt{Ind}(\mathcal{A})$ and $\mathcal{EL}$ concepts C. In fact, it suffices to observe that we have only added new incoming edges to elements from $\Delta^{\mathcal{I}}$ but no outgoing ones.

By definition, $\mathcal{J}$ is a model of $\mathcal{A}$. To show that it is also a model of $\mathcal{O}$, we make a case distinction on the types of CIs in $\mathcal{O}$:

$A_1 \sqcap \cdots \sqcap A_n \sqsubseteq B \in \mathcal{O}$. Let $a \in (A_1 \sqcap \cdots \sqcap A_n)^{\mathcal{J}}$. By construction of $\mathcal{J}$, $A_1(a), \ldots, A_n(a) \in \mathtt{chase}_{\mathcal{O}}(\mathcal{A})$. Thus, R1 yields $B(a) \in \mathtt{chase}_{\mathcal{O}}(\mathcal{A})$ and $a \in B^{\mathcal{J}}$ by construction of $\mathcal{J}$.

$\exists r.A \sqsubseteq B \in \mathcal{O}$. Let $b \in A^{\mathcal{J}}$ and $(a, b) \in r^{\mathcal{J}}$. By construction of $\mathcal{J}$, $A(b)$ and $r(a, b) \in \mathtt{chase}_{\mathcal{O}}(\mathcal{A})$. Thus, R2 yields $B(a) \in \mathtt{chase}_{\mathcal{O}}(\mathcal{A})$ and $a \in B^{\mathcal{J}}$ by construction of $\mathcal{J}$.

$A \sqsubseteq \exists r.B \in \mathcal{O}$. Let $a \in A^{\mathcal{J}}$. By construction of $\mathcal{J}$, $A(a) \in \mathtt{chase}_{\mathcal{O}}(\mathcal{A})$. Thus, R4 yields $r(a, b)$ and $B(b) \in \mathtt{chase}_{\mathcal{O}}(\mathcal{A})$ and $b \in B^{\mathcal{J}}$ by construction of $\mathcal{J}$.

$\exists r^-.A \sqsubseteq B \in \mathcal{O}$. Let $b \in A^{\mathcal{J}}$ and $(b, a) \in r^{\mathcal{J}}$. By construction of $\mathcal{J}$, $A(b)$ and $r(b, a) \in \mathtt{chase}_{\mathcal{O}}(\mathcal{A})$. Thus, R3 yields $B(a) \in \mathtt{chase}_{\mathcal{O}}(\mathcal{A})$ and $a \in B^{\mathcal{J}}$ by construction of $\mathcal{J}$.

$A \sqsubseteq \exists r^-.B \in \mathcal{O}$. Let $a \in A^{\mathcal{J}}$. Because of R1 we can find a model $\mathcal{I}_a$ with $(b, d_a) \in r^{\mathcal{I}_a}$. By point 2 of the construction of $\mathcal{J}$ we get $(b, a) \in r^{\mathcal{J}}, b \in B^{\mathcal{J}}$.

❑

D.4 Completeness (projective)

We prove the completeness part of Theorem 6, in analogy with the completeness proof for Theorem 3. Thus the following is crucial.

Lemma 11 *Let C be an $\mathcal{EL}(\Sigma)$ concept of depth bounded by ℓ, $\mathcal{A}_0, \mathcal{A}_1, \ldots$ the sequence of ABoxes constructed by $\mathtt{chase}_{\mathcal{O}_S}(\mathcal{A}_C)$, $\mathcal{I}$ a model of $\mathcal{O}_T$, and $d \in C^{\mathcal{I}}$. Then for every $i \geq 0$, there is a homomorphism h_i from $\mathcal{A}_i$ to $\mathcal{I}$ with $h_i(a_0) = d$.*

Proof. The proof is by an induction on the number of applications of rule R3 used to compute the sequence $\mathcal{A}_0, \ldots, \mathcal{A}_i$. For the induction start, assume that R3 was not applied at all. We show that for all $j \leq i$, there is a homomorphism h_j from $\mathcal{A}_j$ to $\mathcal{I}$ with $h_j(a_0) = d$. For $j = 0$, it suffices to apply Lemma 4. Now assume that h_j has already been constructed, $j < i$. We show how to find h_{j+1}, making a case distinction according to the rule that is applied in order to obtain $\mathcal{A}_{j+1}$ from $\mathcal{A}_j$.

R1. Then there are $A_1(a), \ldots, A_n(a) \in \mathcal{A}_j$ such that $A_1 \sqcap \ldots \sqcap A_n \sqsubseteq B \in \mathcal{O}_S$ and $\mathcal{A}_{j+1}$ is obtained from $\mathcal{A}_j$ by adding $B(a)$. We must have $h_j(a) \in (A_1 \sqcap \cdots \sqcap A_n)^{\mathcal{I}}$. By construction of $\mathcal{O}_T$, $A_1 \sqcap \ldots \sqcap A_n \sqsubseteq B$ is a CI in $\mathcal{O}_T$. Since $\mathcal{I}$ is a model of $\mathcal{O}_T$, $a \in B^{\mathcal{I}}$. Thus h_j is also a homomorphism from $\mathcal{A}_{j+1}$ to $\mathcal{I}$.

R2. Then there are $r(a, b), A(b) \in \mathcal{A}_j$ such that $\exists r.A \sqsubseteq B \in \mathcal{O}$ and and $\mathcal{A}_{j+1}$ is obtained from $\mathcal{A}_j$ by adding $B(a)$. We must have $h_j(b) \in A^{\mathcal{I}}$ and $(h_j(a), h_j(b)) \in r^{\mathcal{I}}$. By construction of $\mathcal{O}_T$, $\exists r.A \sqsubseteq B$ is a CI in $\mathcal{O}_T$. Since $\mathcal{I}$ is a model of $\mathcal{O}_T$, $a \in B^{\mathcal{I}}$. Thus h_j is also a homomorphism from $\mathcal{A}_{j+1}$ to $\mathcal{I}$.

R4. Then there is an $A(a) \in \mathcal{A}_j$ such that $A \sqsubseteq \exists r.B \in \mathcal{O}$ and $\mathcal{A}_{j+1}$ is obtained from $\mathcal{A}_j$ by adding $B(b)$ and $r(a, b)$, b fresh. We must have $h_j(a) \in A^{\mathcal{I}}$. By construction of $\mathcal{O}_T$, $A \sqsubseteq \exists r.B$ is a CI in $\mathcal{O}_T$. Consequently and since $\mathcal{I}$ is a model of $\mathcal{O}_T$, there is an $e \in B^{\mathcal{I}}$ such that $(h_j(a), e) \in r^{\mathcal{I}}$. Let h_{j+1} be the extension of h_j obtained by setting $h_{j+1}(b) = e$. It can be verified that h_{j+1} is a homomorphism from $\mathcal{A}_{j+1}$ to $\mathcal{I}$.

Now for the induction step. Assume that there were $k > 0$ applications of R3 in the sequence $\mathcal{A}_0, \ldots, \mathcal{A}_i$ and that the last such application was used to obtain $\mathcal{A}_{\ell+1}$ from $\mathcal{A}_\ell$, $\ell < i$. By induction hypothesis, we find a homomorphism h_ℓ from $\mathcal{A}_\ell$ to $\mathcal{I}$ with $h_\ell(a_0) = d$. We argue that we also find such a homomorphism $h_{\ell+1}$ from $\mathcal{A}_{\ell+1}$ to $\mathcal{I}$. We can then proceed as in the induction start to obtain the desired homomorphism from $\mathcal{A}_i$ to $\mathcal{I}$.

R3. There are $r(b, a), A(b) \in \mathcal{A}_\ell$ such that $\exists r^-.A \sqsubseteq B \in \mathcal{O}$ and $\mathcal{A}_{\ell+1}$ is obtained from $\mathcal{A}_\ell$ by adding $B(a)$. Since $\mathcal{A}_\ell$ is ditree-shaped, b is the predecessor of a in the tree $G_{\mathcal{A}_i}$. We must have $h_\ell(b) \in A^{\mathcal{I}}$ and $(h_\ell(b), h_\ell(a)) \in r^{\mathcal{I}}$. Let $\mathcal{A}_a$ be the ditree-shaped ABox in $\mathcal{A}_\ell$ rooted at a and set $\mathtt{Ind} = \{a\} \cup (\mathtt{Ind}(\mathcal{A}_a) \cap \mathtt{Ind}(\mathcal{A}_C))$. That is, $\mathtt{Ind}$ contains only those individuals in $\mathcal{A}_a$ that were present already in the initial ABox $\mathcal{A}_C$, and if there is no such individual, then $\mathtt{Ind} = \{a\}$. An individual $c \in \mathtt{Ind}$ is a *fringe individual* if there is some $r(c, c') \in \mathcal{A}_b$ with $c' \notin \mathtt{Ind}$. Further, let $\mathcal{A}_a^-$ be the restriction of $\mathcal{A}_a$ to assertions that only use individuals from $\mathtt{Ind}$ and let C_a be this ABox viewed as an $\mathcal{EL}$ concept.

We must have $h_\ell(a) \in C_a^{\mathcal{I}}$. By construction of $\mathcal{O}_T$ and since C_a is of depth at most $\ell - 1$ (because C is of depth bounded by ℓ), $A \sqcap \exists r.C_a \sqsubseteq \exists r.(C_a \sqcap B) \in \mathcal{O}_T$. Consequently, there is an $e \in (C_a \sqcap B)^{\mathcal{I}}$ with $(h_\ell(b), e) \in r^{\mathcal{I}}$. By Lemma 4, there is a homomorphism h_a from $\mathcal{A}_a^-$ to $\mathcal{I}$ with $h_a(a) = e$.

Now consider each fringe individual c. Let $\mathcal{A}_c$ be the ditree-shaped ABox in $\mathcal{A}_a$ rooted at c and let C_c be the $\mathcal{EL}$

concept that is the conjunction of all concept names A with $A(c) \in \mathcal{A}_a$. Since $\mathcal{O}_S$ is in normal form, we can extract from the chase sequence $\mathcal{A}_0, \dots, \mathcal{A}_\ell$ a chase sequence that constructs $\mathcal{A}_c$ starting from $\mathcal{A}_{C_c}$ and uses at most $k-1$ applications of R3. From the induction hypothesis and since clearly $h_a(c) \in C_c^{\mathcal{I}}$, we thus obtain a homomorphism h_c from $\mathcal{A}_c$ to $\mathcal{I}$ with $h_c(c) = h_a(c)$. We obtain the desired homomorphism $h_{\ell+1}$ from $\mathcal{A}_{\ell+1}$ to $\mathcal{I}$ by combining all these homomorphisms, that is,

$$h_{\ell+1}(c) = \begin{cases} h_\ell(c) & \text{if } c \notin \mathtt{Ind}(\mathcal{A}_a) \\ h_a(c) & \text{if } c \in \mathtt{Ind}(\mathcal{A}_a^-) \\ h_{c'}(c) & \text{if } c \in \mathtt{Ind}(\mathcal{A}_{c'}). \end{cases}$$

❑

The proof of the following lemma is now exactly identical to the proof of Lemma 6.

Lemma 12 *Let C be an $\mathcal{EL}(\Sigma)$ concept of depth bounded by ℓ and D an $\mathcal{EL}$ concept. Then $\mathcal{O}_S \models C \sqsubseteq D$ implies $\mathcal{O}_T \models C \sqsubseteq D$.*

E Details for Section 6

We require the following lemma from [Lutz and Wolter, 2010].

Lemma 13 *Asssume $\mathcal{O}$ is a $\mathcal{ELH}$ ontology and $\mathcal{O} \models C \sqsubseteq \exists r.D$. Then one of the following holds:*

1. *there exists a top-level conjunct $\exists s.C'$ of C such that $\mathcal{O} \models s \sqsubseteq r$ and $\mathcal{O} \models C' \sqsubseteq D$; or*
2. *there exists a subconcept M of $\mathcal{O}$ such that $\mathcal{O} \models C \sqsubseteq \exists r.M$ and $\mathcal{O} \models M \sqsubseteq D$.*

We use Lemma 13 to establish the results on non-finite and non-elementary approximations.

Theorem 7 *None of the ontologies*

$$\{\exists r^-.A \sqsubseteq B\}, \quad \{\mathsf{func}(r), A \sqsubseteq A\}, \quad \{r \sqsubseteq s^-, A \sqsubseteq A\}$$

has finite projective $\mathcal{ELH}$ approximations.

Proof. We start with $\mathcal{O}_S = \{\exists r^-.A \sqsubseteq B\}$. Let $\mathcal{O}_T$ be a projective $\mathcal{ELH}$ approximation of $\mathcal{O}_S$. For all $n \geq 0$, let $C_n = \exists r^n.\top$, where $\exists r^n$ denotes n-fold nesting of an existential restriction, and observe that

$$\mathcal{O}_S \models A \sqcap \exists r.C_n \sqsubseteq \exists r.(B \sqcap C_n).$$

To establish the desired result, it suffices to argue that for every $n \geq 0$, there is a subconcept M_n of $\mathcal{O}_T$ such that $\mathcal{O}_T \models M_n \sqsubseteq C_n$ and $\mathcal{O}_T \not\models M_n \sqsubseteq C_m$ for any $m > n$. First note that $\mathcal{O}_T \not\models C_n \sqsubseteq B$ for any $n \geq 0$ because the same is true for $\mathcal{O}_S$. By Lemma 13, to obtain $\mathcal{O}_T \models A \sqcap \exists r.C_n \sqsubseteq \exists r.(B \sqcap C_n)$, there must exist a subconcept M of $\mathcal{O}_T$ such that

- $\mathcal{O}_T \models A \sqcap \exists r.C_n \sqsubseteq \exists r.M$ and
- $\mathcal{O}_T \models M \sqsubseteq B \sqcap C_n$.

We aim to use M as M_n. Assume to the contrary of what remains to be shown that $\mathcal{O}_T \models M \sqsubseteq C_m$ for some $m > n$. Then $\mathcal{O}_T \models A \sqcap \exists r.C_n \sqsubseteq \exists r.(B \sqcap C_m)$, which contradicts the fact that this CI is not entailed by $\mathcal{O}_S$.

We now consider $\mathcal{O}_S = \{\mathsf{func}(r), A \sqsubseteq A\}$. Let $\mathcal{O}_T$ be a projective $\mathcal{ELH}$ approximation of $\mathcal{O}_S$. For all $n \geq 0$, let again $C_n = \exists r^n.\top$, and observe that

$$\mathcal{O}_S \models \exists r.C_n \sqcap \exists r.A \sqsubseteq \exists r.(C_n \sqcap A).$$

Using Lemma 13, we establish that for every $n \geq 0$, there is a subconcept M_n of $\mathcal{O}_T$ such that $\mathcal{O}_T \models M_n \sqsubseteq (C_n \sqcap A)$ and $\mathcal{O}_T \not\models M_n \sqsubseteq C_m$ for any $m > n$. First note that

- $\mathcal{O}_T \not\models C_n \sqsubseteq C_n \sqcap A$ for any $n \geq 0$ and
- $\mathcal{O}_T \not\models A \sqsubseteq C_n \sqcap A$

because the same is true for $\mathcal{O}_S$. To obtain $\mathcal{O}_T \models \exists r.C_n \sqcap \exists r.A \sqsubseteq \exists r.(C_n \sqcap A)$, there must exist a subconcept M of $\mathcal{O}_T$ such that $\mathcal{O}_T \models \exists r.C_n \sqcap \exists r.A \sqsubseteq \exists r.M$ and $\mathcal{O}_T \models M \sqsubseteq C_n \sqcap A$. We use M as M_n. Assume to the contrary of what remains to be shown that $\mathcal{O}_T \models M \sqsubseteq C_m$ for some $m > n$. Then $\mathcal{O}_T \models \exists r.C_n \sqcap \exists r.A \sqsubseteq \exists r.(C_m \sqcap A)$, which contradicts the fact that this CI is not entailed by $\mathcal{O}_S$.

We now consider $\mathcal{O}_S = \{r \sqsubseteq s^-, A \sqsubseteq A\}$. Let $\mathcal{O}_T$ be a projective $\mathcal{ELH}$ approximation of $\mathcal{O}_S$. For all $n \geq 0$, let again $C_n = \exists r^n.\top$, and observe that

$$\mathcal{O}_S \models A \sqcap \exists r.C_n \sqsubseteq \exists r.(C_n \sqcap \exists s.A).$$

Using Lemma 13, we establish that for every $n \geq 0$, there is a subconcept M_n of $\mathcal{O}_T$ such that $\mathcal{O}_T \models M_n \sqsubseteq C_n \sqcap \exists s.A$ and $\mathcal{O}_T \not\models M_n \sqsubseteq C_m$ for any $m > n$. First note that $\mathcal{O}_T \not\models C_n \sqsubseteq \exists s.A$ for any $n \geq 0$ because the same is true for $\mathcal{O}_S$. To obtain $\mathcal{O}_T \models A \sqcap \exists r.C_n \sqsubseteq \exists r.(C_n \sqcap \exists s.A)$, there must exist a subconcept M of $\mathcal{O}_T$ such that $\mathcal{O}_T \models A \sqcap \exists r.C_n \sqsubseteq \exists r.M$ and $\mathcal{O}_T \models M \sqsubseteq C_n \sqcap \exists s.A$. We use M as M_n. Assume to the contrary of what remains to be shown that $\mathcal{O}_T \models M \sqsubseteq C_m$ for some $m > n$. Then $\mathcal{O}_T \models A \sqcap \exists r.C_n \sqsubseteq \exists r.(C_m \sqcap \exists s.A)$, which contradicts the fact that this CI is not entailed by $\mathcal{O}_S$. ❑

Theorem 8 *Let $n \geq 0$ and let $\mathcal{O}_n$ be the union of Γ_n with any of the following sets:*

$$\{\exists r^-.A \sqsubseteq B\}, \quad \{\mathsf{func}(r), A \sqsubseteq A\}, \quad \{r \sqsubseteq s^-, A \sqsubseteq A\}$$

For every $\ell \geq 1$, any ℓ-bounded projective $\mathcal{ELH}$ approximation $\mathcal{O}_T$ of $\mathcal{O}_n$ must be of size at least $\mathtt{tower}(\ell, n)$.

Proof. We start with $\mathcal{O}_S = \Gamma_n \cup \{\exists r^-.A \sqsubseteq B\}$. The proof idea is very similar to the proof of Theorem 7. Assume a depth bound $\ell \geq 1$ is given. Take any set Ω of mutually incomparable $\mathcal{EL}(\Sigma_n)$ concepts of depth at most $\ell - 1$ such that Ω has size $\mathtt{tower}(\ell, n)$, where concepts C_1, C_2 are called *incomparable* if neither $\mathcal{O}_S \models C_1 \sqsubseteq C_2$ nor $\mathcal{O}_S \models C_2 \sqsubseteq C_1$. It is straightforward to construct such a set Ω. Then it suffices to show that for every $C \in \Omega$ there exists a subconcept M_C of $\mathcal{O}_T$ such that $\mathcal{O}_T \models M_C \sqsubseteq C$ and $\mathcal{O}_T \not\models M_C \sqsubseteq C'$ for any $C' \in \Omega$ with $C' \neq C$. Assume $C \in \Omega$ is given. Then

$$\mathcal{O}_S \models A \sqcap \exists r.C \sqsubseteq \exists r.(B \sqcap C)$$

Observe that $\mathcal{O}_T \not\models C \sqsubseteq B$. Thus, similarly to the proof above one can show that to obtain $\mathcal{O}_T \models A \sqcap \exists r.C \sqsubseteq \exists r.(B \sqcap C)$ there must exist a subconcept M_C of $\mathcal{O}_T$ such that

- $\mathcal{O}_T \models A \sqcap \exists r.C \sqsubseteq \exists r.M_C$;
- $\mathcal{O}_T \models M_C \sqsubseteq B \sqcap C$.

Observe that $\mathcal{O}_T \not\models M_C \sqsubseteq C'$ for any $C' \in \Omega \setminus \{C\}$ because $\mathcal{O}_S \not\models A \sqcap \exists s.C \sqsubseteq \exists s.(B \sqcap C')$ for any such C'. Thus, M_C is as required.

The proofs for $\mathcal{O}_n = \Gamma_n \cup \{\mathsf{func}(r), A \sqsubseteq A\}$ and $\mathcal{O}'_n = \Gamma_n \cup \{r \sqsubseteq s^-, A \sqsubseteq A\}$ combine the sketch presented for $\Gamma_n \cup \{\exists r^-.A \sqsubseteq B\}$ with the proof idea from Theorem 7. Thus, one considers the same set Ω and then shows that any ℓ-bounded $\mathcal{ELH}$ approximation of $\mathcal{O}_n$ entails all CIs

$$\exists r.A \sqcap \exists r.C \sqsubseteq \exists r.(A \sqcap C)$$

with $C \in \Omega$ and so is of size at least $\mathtt{tower}(\ell, n)$, and that any ℓ-bounded $\mathcal{ELH}$ approximation of $\mathcal{O}'_n$ entails all CIs

$$A \sqcap \exists r.C \sqsubseteq \exists r.(C \sqcap \exists s.A)$$

with $C \in \Omega$ and so is of size at least $\mathtt{tower}(\ell, n)$. ❑