Model Comparison Games for Horn Description Logics

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***Abstract*—Horn description logics are syntactically defined fragments of standard description logics that fall within the Horn fragment of first-order logic and for which ontology-mediated query answering is in PTIME for data complexity. They were independently introduced in modal logic to capture the intersection of Horn first-order logic with modal logic. In this paper, we introduce model comparison games for the basic Horn description logic *horn*$\mathcal{ALC}$ (corresponding to the basic Horn modal logic) and use them to obtain an Ehrenfeucht-Fraïssé type definability result and a van Benthem style expressive completeness result for *horn*$\mathcal{ALC}$. We also establish a finite model theory version of the latter. The Ehrenfeucht-Fraïssé type definability result is used to show that checking *horn*$\mathcal{ALC}$ indistinguishability of models is EXPTIME-complete, which is in sharp contrast to $\mathcal{ALC}$ indistinguishability (i.e., bisimulation equivalence) checkable in PTIME. In addition, we explore the behavior of Horn fragments of more expressive description and modal logics by defining a Horn guarded fragment of first-order logic and introducing model comparison games for it.**

I. INTRODUCTION

Description logics (DLs) [1], [2] have been introduced as knowledge representation formalisms supported by efficient reasoning tools. The basic DL, called $\mathcal{ALC}$, is a notational variant of the classical multi-modal logic. In fact, numerous applications have led to the development of a large family of DLs with different features. DLs serve as the logical underpinning of ontologies, finitely axiomatized theories known as TBoxes. Two main reasoning problems have to be solved efficiently for TBoxes, often containing thousands of axioms:

Deduction: does a formula follow from a TBox?

Query answering: is a tuple $\mathbf{d}$ in a database D a certain answer to an ontology-mediated query $(\mathcal{T}, \boldsymbol{q}(\mathbf{x}))$ comprising a TBox $\mathcal{T}$ and a query $\boldsymbol{q}(\mathbf{x})$? In other words, does $\boldsymbol{q}(\mathbf{d})$ follow from $\mathcal{T} \cup D$?

As the data is typically much larger than both TBox and query, the appropriate efficiency measure for ontology-mediated query answering is *data complexity*, under which the database is the only input to the problem, while the TBox and query are regarded as fixed [3]. For $\mathcal{ALC}$, deduction is EXPTIME-complete and ontology-mediated query answering with conjunctive queries is CONP-complete [2].

Horn DLs have been introduced as syntactically defined fragments of standard DLs that fall within the Horn fragment of first-order logic (henceforth Horn FO) and for which ontology-mediated query answering is in PTIME for data complexity [4], [5]. The Horn fragment of $\mathcal{ALC}$ is called *horn*$\mathcal{ALC}$. The modal logic corresponding to *horn*$\mathcal{ALC}$ was introduced independently, actually five years earlier, with the aim of capturing the intersection of Horn FO and modal logic [6].[1] The introduction of Horn DLs had an enormous impact on description logic research and applications: while the weaker Horn DLs of the $\mathcal{EL}$ [7] and *DL-Lite* [8], [9] families gave rise to two Web Ontology Language OWL 2 profiles (trading expressiveness for high efficiency), the more expressive Horn DLs starting at *horn*$\mathcal{ALC}$ have also been used extensively, and investigated in depth for ontology-mediated query answering [10]–[15]. Moreover, despite the fact that deduction in many expressive Horn DLs, including *horn*$\mathcal{ALC}$, is EXPTIME-hard, it turned out that efficient reasoners capable of coping with very large real-world TBoxes could be developed [16], [17]. The complexity of reasoning in various types of Horn DLs has been investigated in [18]. It also turned out that basic questions relevant for ontology-mediated query answering, such as query-inseparability and conservative extensions, query emptiness, and query by example, admit more elegant solutions and are easier to solve computationally for Horn DLs than in the classical case [19]–[22]. The relationship between Horn DLs and PTIME query answering is by now well-understood [23], [24].

In contrast, the model theory for expressive Horn DLs remains largely undeveloped. Even very basic questions such as whether *horn*$\mathcal{ALC}$ indeed captures the intersection of $\mathcal{ALC}$ (or modal logic) and Horn FO are still unanswered. The aim of this paper is to lay foundations for a model-theoretic understanding of Horn DLs (and Horn modal logic) by developing model comparison games and using them to obtain Ehrenfeucht-Fraïssé type definability and van Benthem style expressive completeness results. In a first application of these results, we show that concept learning and model indistinguishability in *horn*$\mathcal{ALC}$ are EXPTIME-complete and that *horn*$\mathcal{ALC}$ does not capture the intersection of $\mathcal{ALC}$ and Horn FO.

[1]The results obtained in this paper could have been presented as a contribution to modal rather than description logic. The only reason why we have chosen the DL environment is that the impact of Horn fragments in description logic has so far been much more significant than in modal logic.

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The original definition of Horn DLs [4] was based on the polarity of concepts, as used in automated theorem proving. The equivalent definition given for modal logic [6] (and also for DLs [23]) is more similar to the classical definition of Horn FO as the closure of Horn clauses $\varphi_1 \wedge \cdots \wedge \varphi_n \rightarrow \varphi_{n+1}$, with atomic φ_i, under $\wedge$, $\exists$, and $\forall$. The obvious analogue of this definition in description (modal) logic is the closure of Horn clauses $A_1 \sqcap \cdots \sqcap A_n \rightarrow A_{n+1}$, with concept names (propositional variables in modal logic) A_i, under $\sqcap$, $\exists R$, $\forall R$ (respectively, $\wedge$, $\Diamond$, $\Box$). However, in contrast to the first-order case, this definition leaves out the substitution instances $C_1 \sqcap \cdots \sqcap C_n \rightarrow C$ with positive existential C_i and Horn C (understood recursively), which have thus been explicitly included in *horn*$\mathcal{ALC}$. It is easy to show that *horn*$\mathcal{ALC}$ is a fragment of Horn FO under the standard translation.

As the first contribution of the paper, we introduce model comparison games for characterizing *horn*$\mathcal{ALC}$. These *Horn simulation games* differ from standard bisimulation or Ehrenfeucht-Fraïssé games in the following respects:

(1) the Horn simulation relations underlying Horn simulation games are non-symmetric (which reflects that Horn languages are not closed under negation);
(2) positions in the games consist of pairs (X, b) with a *set* X of nodes and a node b (which reflects that Horn languages are not closed under disjunction);
(3) Horn simulation games use as a subgame the basic simulation game for checking indistinguishability by positive existential $\mathcal{ALC}$ formulas (which reflects that the left-hand side of *horn*$\mathcal{ALC}$ implications are such).

Both (2) and (3) have important consequences. The latter means that Horn simulation games are *modular* as far as the characterization of the left-hand side of implications is concerned. For example, by dropping the subgames entirely, we characterize the positive fragment of $\mathcal{ALC}$ and by restricting it to depth 0, we characterize the weaker Horn fragment of $\mathcal{ALC}$ discussed above. We will use this modularity to characterize a proper extension, *horn*$\mathcal{ALC}_\nabla$, of *horn*$\mathcal{ALC}$ with the operators $\nabla R.C = \exists R.\top \sqcap \forall R.C$ (or $\nabla p = \Diamond \top \wedge \Box p$ in modal logic) on the left-hand side of *horn*$\mathcal{ALC}$ implications, which also lies in Horn FO.

The consequences of (2) are three-fold. First, using sets rather than nodes in positions implies that the obvious algorithm checking the existence of Horn simulations containing a pair $(\{a\}, b)$ of nodes runs in exponential time. Thus, using Horn simulation games to check whether two nodes a and b satisfy the same *horn*$\mathcal{ALC}$-concepts or whether two models satisfy the same TBox axioms yields exponential time algorithms. We show that this is unavoidable by proving corresponding ExpTime lower bounds. The ExpTime-completeness results are in sharp contrast to the typical complexity of indistinguishability in modal-like languages. For example, as bisimilarity of nodes can be checked in polynomial time, deciding whether two nodes satisfy the same $\mathcal{ALC}$-concepts is in PTime; similarly, since one can check in polynomial time whether there is a standard simulation between two nodes, deciding whether they satisfy the same $\mathcal{EL}$-concepts is in PTime as well. Thus, *horn*$\mathcal{ALC}$ sits between languages for which definability questions are computationally and model-theoretically much more straightforward.

Second, as player 2 does *not* have a winning strategy in position (X, b) in the Horn simulation game if, and only if, there exists a *horn*$\mathcal{ALC}$-concept that is true at all nodes in X but not true at b, our complexity results are directly applicable to the *concept learning by example* (*CBE*) *problem*: given a data set, and sets P and N of positive and negative examples, does there exist a *horn*$\mathcal{ALC}$-concept C separating P from N over the data? The goal of this supervised learning problem is to automatically derive new concept descriptions from labelled data. It has been investigated before in DL [25]–[27] and for many logical languages, in particular in databases [28]–[31]. Horn DLs are of particular interest as target languages for CBE as they can be regarded as 'maximal DLs without disjunction,' and the unlimited use of disjunction in derived concept descriptions is undesirable as it leads to *overfitting*: learnt concepts enumerate the positive examples rather than generalize from the examples [26]. The complexity analysis for Horn simulation games shows that the CBE problem for *horn*$\mathcal{ALC}$ is ExpTime-complete, again in contrast to $\mathcal{ALC}$, where CBE is in PTime. We regard the increased complexity as the price for obtaining proper generalizations.

Finally, the presence of sets in positions of the Horn simulation games has an impact on the standard infinitary saturated model approach to proving van Benthem style expressive completeness results [32], [33]. For example, we aim to prove that an FO-formula with one free variable is equivalent to (the standard translation of) a *horn*$\mathcal{ALC}$-concept just in case it is preserved under Horn simulations. Then, for the infinitary proof method, not only do the structures showing that non-equivalence to a *horn*$\mathcal{ALC}$-concept implies non-preservation under Horn simulations have to satisfy appropriate saturatedness conditions, but also the *substructures* induced by the sets X chosen by the players have to be saturated. However, saturated structures do not enjoy the latter property for arbitrary subsets X of their domain. In fact, it currently seems that the only way to obtain expressive completeness results with an infinitary approach is to restrict the moves of players to 'saturated sets,' say sets definable as the intersection of FO-definable sets.

In this paper, we prove van Benthem style expressive completeness results for *horn*$\mathcal{ALC}$-concepts and TBoxes via Horn simulation games by developing appropriate finitary methods which do not require saturated structures. As a consequence, the results hold both in the classical and the finite model theory setting, and without any restrictions on the moves of players. In fact, we show that preservation under ℓ-round Horn simulation games coincides with preservation under infinitary Horn simulation games for $\mathcal{ALC}$-concepts and TBoxes of nesting depth ℓ. We thus also obtain decidability of the problem whether an $\mathcal{ALC}$-concept or TBox is equivalent to a *horn*$\mathcal{ALC}$-concept or TBox, respectively. The finitary approach to van Benthem style expressive completeness results was first used by Rosen [34] to obtain a bisimulation characterization of $\mathcal{ALC}$ in the finite model theory setting, and has been further developed and applied with great success by Otto et al. [33], [35]–[37]. The lifting of our results from expressive completeness for *horn*$\mathcal{ALC}$ within $\mathcal{ALC}$ to expressive completeness for *horn*$\mathcal{ALC}$ within FO relies on these earlier results.

It is straightforward to extend the Horn simulation games for *horn*$\mathcal{ALC}$ to games providing Ehrenfeucht-Fraïssé type definability results for the Horn fragments of many popular extensions of $\mathcal{ALC}$, such as the extension by inverse roles or the universal role. Instead of going through those extensions step-by-step, however, we consider the guarded fragment, GF, of FO and introduce its Horn fragment, *hornGF*, by generalizing the definition of Horn DLs in the obvious way. We remind the reader that GF has been introduced as an extension of multi-modal logic to predicates of arbitrary arity, which still has many of the fundamental properties of modal and description logics [32], [38]–[41]. Like *horn*$\mathcal{ALC}$, *hornGF* is contained in Horn FO and ontology-mediated query answering using conjunctive queries is in PTIME for data complexity. The latter can be shown by establishing a close relationship between *hornGF* and guarded tuple-generating dependencies (guarded tgds), a popular member of the Datalog$^\pm$ family for which query answering is in PTIME [42], [43]. In fact, guarded tgds can be seen as normal forms for *hornGF*, and deduction and query answering in *hornGF* can both be polynomially reduced to the same problem for guarded tgds, and vice versa. To study the model theory of *hornGF*, we generalize Horn simulations to guarded Horn simulations, and show an Ehrenfeucht-Fraïssé type definability result for *hornGF*. This result is used to prove an EXPTIME upper bound for model indistinguishability in *hornGF* and to explore the expressive power of *hornGF*. In particular, we show that *hornGF* captures more of the intersection of $\mathcal{ALC}$ and Horn FO than *horn*$\mathcal{ALC}$ but does not capture the intersection of GF and Horn FO. We then show expressive completeness of *hornGF*: an FO-formula is equivalent to a *hornGF*-formula just in case it is preserved under guarded Horn simulations. Our proof uses infinitary methods and thus the moves of player 1 are restricted to intersections of FO-definable sets. It remains open whether the expressive completeness holds without this restriction and whether it holds in the finite model theory setting.

The emerging landscape of the fragments of Horn FO and GF we considered in this paper is discussed in the conclusion.

Related Work. Here we briefly review the related work not yet discussed. The definition of Horn simulations is inspired by games used to provide van Benthem style characterizations of concepts in weak DLs such as $\mathcal{FL}^-$ [44]. Van Benthem style characterizations of DLs in the $\mathcal{EL}$ and *DL-Lite* families are given in [45]. Bisimulations have been studied for the guarded fragment and many variations [33], [38], [46]. Bisimulations have also been studied recently for coalgebraic modal logics [47], [48] and fuzzy modal logics [49].

This paper contributes to the model theory of languages obtained by taking the intersection of Horn FO with modal and description logic. Horn FO was originally introduced in classical model theory [50], [51] with the aim of understanding FO-formulas that are preserved under products of models. As it turned out, an FO-formula is equivalent to a Horn formula iff it is preserved under *reduced* products; for details consult [52], [53]. A complicated recursive characterization of FO-sentences preserved under direct products is given in [54].

There have been other attempts to define Horn modal and temporal logics [55]–[58] with a focus on the complexity of reasoning and not on model theory.

II. PRELIMINARIES

Description logics (DLs) are fragments of first-order logic with unary and binary predicates. However, the standard notation for DL 'formulas' is more succinct and does not use individual variables explicitly [1], [2]. Let τ be a vocabulary consisting of unary and binary predicate names only. In DL parlance, they are called *concept names* (denoted A, B, etc.) and *role names* (denoted R, S, etc.), respectively. The $\mathcal{ALC}[\tau]$-*concepts*, C, are defined by the following grammar:

$$C, D \ ::= \ A \mid \top \mid \bot \mid \neg C \mid C \sqcup D \mid C \sqcap D \mid C \rightarrow D \mid \exists R.C \mid \forall R.C,$$

where $A \in \tau$ is unary, $R \in \tau$ binary, $\top$ is the universal and $\bot$ the empty concept. If not relevant, we drop τ and simply talk about $\mathcal{ALC}$-concepts. An $\mathcal{ALC}[\tau]$-*concept inclusion* (or CI) takes the form $C \sqsubseteq D$, where C and D are $\mathcal{ALC}[\tau]$-concepts. An $\mathcal{ALC}[\tau]$-*TBox*, $\mathcal{T}$, is a finite set of $\mathcal{ALC}[\tau]$-CIs.

$\mathcal{ALC}[\tau]$ is interpreted in usual τ-structures

$$\mathfrak{A} = (\text{dom}(\mathfrak{A}), (A^{\mathfrak{A}})_{A \in \tau}, (R^{\mathfrak{A}})_{R \in \tau})$$

with $\text{dom}(\mathfrak{A}) \neq \emptyset$, $A^{\mathfrak{A}} \subseteq \text{dom}(\mathfrak{A})$ and $R^{\mathfrak{A}} \subseteq \text{dom}(\mathfrak{A})^2$. The semantics of $\mathcal{ALC}$ can be defined via the *standard translation* † of $\mathcal{ALC}$-concepts to FO-formulas with one free variable x:

$A^\dagger = A(x), \quad \top^\dagger = (x = x), \quad \bot^\dagger = \neg(x = x),$

†commutes with the Booleans (changing $\sqcap$ to $\wedge$ and $\sqcup$ to $\vee$),

$(\exists R.C)^\dagger = \exists y \, (R(x, y) \wedge C^\dagger[y/x]),$

$(\forall R.C)^\dagger = \forall y \, (R(x, y) \rightarrow C^\dagger[y/x]).$

The *extension* $C^{\mathfrak{A}}$ of a concept C in a structure $\mathfrak{A}$ is defined as

$$C^{\mathfrak{A}} = \{a \in \text{dom}(\mathfrak{A}) \mid \mathfrak{A} \models C^\dagger(a)\},$$

and the CI $C \sqsubseteq D$ is regarded as a shorthand for the FO-sentence $\forall x \, (C^\dagger(x) \rightarrow D^\dagger(x))$. We write $\mathcal{T} \models C \sqsubseteq D$ to say that the CI $C \sqsubseteq D$ *follows from* the TBox $\mathcal{T}$, that is, $C^{\mathfrak{A}} \subseteq D^{\mathfrak{A}}$ holds in every model $\mathfrak{A}$ of $\mathcal{T}$. Concepts C and D are *equivalent* if $\emptyset \models C \sqsubseteq D$ and $\emptyset \models D \sqsubseteq C$.

$\mathcal{ALC}$-concepts that are built from concept names using $\top$, $\sqcap$, $\sqcup$, and $\exists R.C$ only are called $\mathcal{ELU}$-*concepts*; $\mathcal{ELU}$-concepts without $\sqcup$ are called $\mathcal{EL}$-*concepts*. The FO-translation $C^\dagger$ of any $\mathcal{ELU}$-concept C is clearly a positive existential formula.

Definition 1 (**Horn $\mathcal{ALC}$-concept).** We define *horn*$\mathcal{ALC}[\tau]$-*concepts*, H, by the grammar

$$H, H' \ ::= \ \bot \mid \top \mid A \mid H \sqcap H' \mid L \rightarrow H \mid \exists R.H \mid \forall R.H,$$

where L is an $\mathcal{ELU}[\tau]$-concept. A *horn$\mathcal{ALC}$-CI* takes the form $L \sqsubseteq H$. A *horn$\mathcal{ALC}$-TBox* is a finite set of *horn$\mathcal{ALC}$*-CIs.

Our definition of *horn$\mathcal{ALC}$*-concepts is from [23]. We show in the appendix that both polarity-based definition of *horn$\mathcal{ALC}$*-concepts from [4] and Horn *modal* formulas (appropriately adapted to the DL syntax) defined in [6] are equivalent to our definition.

To put *horn$\mathcal{ALC}$*-concepts into the context of classical Horn FO-formulas, we recall that a *basic Horn formula* is a disjunction $\varphi_1 \vee \dots \vee \varphi_n$ of FO-formulas, with at most one of them being an atom and the remaining ones negations of atoms [52]. A *Horn formula* is constructed from basic Horn formulas using $\wedge$, $\exists$, and $\forall$.

Theorem 1. (i) *Every horn$\mathcal{ALC}$-concept is equivalent to a Horn formula with one free variable and every horn$\mathcal{ALC}$-CI is equivalent to a Horn sentence.*

(ii) *There exists an $\mathcal{ALC}$-concept* (*TBox*) *that is equivalent to a Horn formula* (*Horn sentence*)*, but not equivalent to any horn$\mathcal{ALC}$-concept* (*TBox*)*.*

Proof. (i) is proved by a straightforward induction on the construction of *horn$\mathcal{ALC}$*-concepts. To prove (ii), consider first the $\mathcal{ALC}$-concept

$$C_\nabla = (\exists R.\top \sqcap \forall R.A) \to B.$$

It is not hard to check that $C_\nabla^\dagger$ has the same models as

$$\exists y\, R(x,y) \to \exists z\, (R(x,z) \wedge (A(z) \to B(x))),$$

which is equivalent to a Horn formula. Example 1 below shows that C_∇ is not equivalent to any *horn$\mathcal{ALC}$*-concept.

Next, consider the $\mathcal{ALC}$-TBox $\mathcal{T}_{horn}$ with the following CIs:

$$E \sqsubseteq A_1 \sqcup A_2 \sqcup \exists R.(\neg B_1 \sqcap \neg B_2), \quad \exists R.(B_1 \sqcap B_2) \sqsubseteq \bot,$$
$$E \sqsubseteq \exists R.\top, \quad \exists R.B_1 \sqsubseteq \exists R.B_2, \quad \exists R.B_2 \sqsubseteq \exists R.B_1.$$

The FO-translations of all of them but the first one are obviously (equivalent to) Horn sentences. We take a conjunction of these translations together with the sentence

$$\forall x \big[E(x) \to \exists y\, (R(x,y) \wedge (B_1(y) \to A_1(x)) \wedge (B_2(y) \to A_2(x)))\big],$$

which is also equivalent to a Horn one. One can check that the resulting sentence is equivalent to $\mathcal{T}_{horn}$. On the other hand, Example 1 below shows that $\mathcal{T}_{horn}$ is not equivalent to any *horn$\mathcal{ALC}$*-TBox. □

Given Theorem 1, a natural question arises whether it is possible to design a syntactic extension of *horn$\mathcal{ALC}$* that captures the intersection of $\mathcal{ALC}$ and Horn FO. We discuss this problem in the conclusion of this paper.

We remind the reader of two usual operations on structures. The *product* $\prod_{i\in I}\mathfrak{A}_i$ of a family of τ-structures $\mathfrak{A}_i$, $i \in I$, is defined as follows: its domain $\mathrm{dom}(\prod_{i\in I}\mathfrak{A}_i)$ is the set of functions $f\colon I \to \bigcup_{i\in I}\mathrm{dom}(\mathfrak{A}_i)$ with $f(i) \in \mathrm{dom}(\mathfrak{A}_i)$, for $i \in I$, and

$$A^{\prod_{i\in I}\mathfrak{A}_i} = \{f \in \mathrm{dom}(\prod_{i\in I}\mathfrak{A}_i) \mid \forall i \in I\, f(i) \in A^{\mathfrak{A}_i}\},$$
$$R^{\prod_{i\in I}\mathfrak{A}_i} = \{(f,g) \in (\mathrm{dom}(\prod_{i\in I}\mathfrak{A}_i))^2 \mid \forall i \in I\, (f(i),g(i)) \in R^{\mathfrak{A}_i}\}.$$

Horn formulas are *preserved under products* in the sense that

$$\forall i \in I\, \mathfrak{A}_i \models \varphi(f_1(i),\dots,f_n(i)) \;\Rightarrow\; \prod_{i\in I}\mathfrak{A}_i \models \varphi(f_1,\dots,f_n)$$

for all Horn formulas φ. Note that an FO-formula is equivalent to a Horn formula iff it is preserved under the more general *reduced* products (modulo filters over I) [52].

The *disjoint union* $\mathfrak{A}$ of a family $\mathfrak{A}_i$, $i \in I$, of structures has domain $\bigcup_{i\in I}\mathrm{dom}(\mathfrak{A}_i)\times\{i\}$ and

$$A^{\mathfrak{A}} = \{(a,i) \mid a \in A^{\mathfrak{A}_i}\},$$
$$R^{\mathfrak{A}} = \{((a,i),(b,i)) \mid (a,b) \in R^{\mathfrak{A}_i}\}.$$

$\mathcal{ALC}$-TBoxes $\mathcal{T}$ are *invariant under disjoint unions*, that is:

$$\forall i \in I\; \mathfrak{A}_i \models \mathcal{T} \quad\Leftrightarrow\quad \mathfrak{A} \models \mathcal{T}.$$

By the *depth* of a concept C we mean the maximal number of nestings of $\exists R$ and $\forall R$ in C. For example, the concepts $\exists R.\exists R.A$ and $\exists R.\forall R.A$ are of depth 2. The *depth* of a TBox is the maximum over the depths of the concepts occurring in it. By a *pointed structure* we mean a pair $\mathfrak{A}, X$ with a structure $\mathfrak{A}$ and a non-empty set $X \subseteq \mathrm{dom}(\mathfrak{A})$. If $X = \{a\}$, we simply write $\mathfrak{A}, a$.

Definition 2 (**DL indistinguishability**)**.** For any DL $\mathcal{L}$, τ-structures $\mathfrak{A}$ and $\mathfrak{B}$, $a \in \mathrm{dom}(\mathfrak{A})$, $X \subseteq \mathrm{dom}(\mathfrak{A})$, $b \in \mathrm{dom}(\mathfrak{B})$, and $\ell < \omega$, we write:

- $\mathfrak{A}, X \leq_{\mathcal{L}}^{(\ell)} \mathfrak{B}, b$ if $X \subseteq C^{\mathfrak{A}}$ implies $b \in C^{\mathfrak{B}}$, for any $\mathcal{L}$-concept C (of depth $\leq \ell$);
- $\mathfrak{A}, a \equiv_{\mathcal{L}}^{(\ell)} \mathfrak{B}, b$ if $\mathfrak{A}, a \leq_{\mathcal{L}}^{(\ell)} \mathfrak{B}, b$ and $\mathfrak{B}, b \leq_{\mathcal{L}}^{(\ell)} \mathfrak{A}, a$;
- $\mathfrak{A} \leq_{\mathcal{L}} \mathfrak{B}$ if $\mathfrak{A} \models C \sqsubseteq D$ implies $\mathfrak{B} \models C \sqsubseteq D$, for any $\mathcal{L}$-CI $C \sqsubseteq D$;
- $\mathfrak{A} \equiv_{\mathcal{L}} \mathfrak{B}$ if $\mathfrak{A} \leq_{\mathcal{L}} \mathfrak{B}$ and $\mathfrak{B} \leq_{\mathcal{L}} \mathfrak{A}$.

We now recall the model comparison games for indistinguishability in $\mathcal{ELU}$ that will be required as part of the model comparison games for *horn$\mathcal{ALC}$*. The $\mathcal{ELU}$ case is rather straightforward and folklore [59], but it will remind the reader of the basics of model comparison games used in this paper.

Definition 3 (**simulation**)**.** A relation $S \subseteq \mathrm{dom}(\mathfrak{A}) \times \mathrm{dom}(\mathfrak{B})$ is a *simulation* between τ-structures $\mathfrak{A}$ and $\mathfrak{B}$ if the following conditions hold:

(*atom$_f$*) for any $A \in \tau$, if $(a,b) \in S$ and $a \in A^{\mathfrak{A}}$, then $b \in A^{\mathfrak{B}}$,

(*forth*) for any $R \in \tau$, if $(a,b) \in S$ and $(a,a') \in R^{\mathfrak{A}}$, then there is b' with $(b,b') \in R^{\mathfrak{B}}$ and $(a',b') \in S$.

We write $\mathfrak{A}, a \preceq_{sim} \mathfrak{B}, b$ if there exists a simulation S between $\mathfrak{A}$ and $\mathfrak{B}$ with $(a,b) \in S$.

Simulations can be equivalently described as games between two players on the disjoint union of $\mathfrak{A}$ and $\mathfrak{B}$. A position in the *simulation game* is a pair of nodes $(a, b) \in \mathrm{dom}(\mathfrak{A}) \times \mathrm{dom}(\mathfrak{B})$, marked by pebbles. The players move, in turns, the pebbles along binary relations in τ. The first player chooses an $R \in \tau$ and moves the pebble in $\mathfrak{A}$ along $R^{\mathfrak{A}}$, the second player must respond in $\mathfrak{B}$ complying with (*atom$_f$*) and (*forth*). The second player wins a game starting at (a, b) if she can always respond to the first player's moves, *ad infinitum*. One can show that the second player has a winning strategy iff $\mathfrak{A}, a \preceq_{sim} \mathfrak{B}, b$.

Besides the infinitary simulation games corresponding to Definition 3, we consider games with a fixed number ℓ of moves. We write $\mathfrak{A}, a \preceq^{\ell}_{sim} \mathfrak{B}, b$ if the second player has a winning strategy in the simulation game with ℓ rounds starting from (a, b). We write $\mathfrak{A}, a \preceq^{\omega}_{sim} \mathfrak{B}, b$ if $\mathfrak{A}, a \preceq^{\ell}_{sim} \mathfrak{B}, b$ for every $\ell < \omega$.

Theorem 2 (Ehrenfeucht-Fraïssé game for $\mathcal{ELU}$). *For any finite vocabulary τ, pointed τ-structures $\mathfrak{A}, a$ and $\mathfrak{B}, b$, and any $\ell < \omega$, we have*

$$\mathfrak{A}, a \leq^{\ell}_{\mathcal{ELU}} \mathfrak{B}, b \quad \text{iff} \quad \mathfrak{A}, a \preceq^{\ell}_{sim} \mathfrak{B}, b.$$

Thus, $\mathfrak{A}, a \leq_{\mathcal{ELU}} \mathfrak{B}, b$ iff $\mathfrak{A}, a \preceq^{\omega}_{sim} \mathfrak{B}, b$. If $\mathfrak{A}$ and $\mathfrak{B}$ are finite, then

$$\mathfrak{A}, a \leq_{\mathcal{ELU}} \mathfrak{B}, b \quad \text{iff} \quad \mathfrak{A}, a \preceq_{sim} \mathfrak{B}, b.$$

In some proofs, we shall also be using bisimulations. Recall that a relation S between $\mathfrak{A}$ and $\mathfrak{B}$ is a *bisimulation* if S is a simulation between $\mathfrak{A}$ and $\mathfrak{B}$, and its inverse is a simulation between $\mathfrak{B}$ and $\mathfrak{A}$. The notion of *ℓ-bisimilarity* is defined by restricting the corresponding *bisimulation game* to ℓ moves. This notion characterizes indistinguishability in $\mathcal{ALC}$: pointed structures $\mathfrak{A}, a$ and $\mathfrak{B}, b$ are ℓ-bisimilar iff $\mathfrak{A}, a \equiv^{\ell}_{\mathcal{ALC}} \mathfrak{B}, b$ [33].

III. Simulations for *horn$\mathcal{ALC}$*

We now define a new model comparison game, the *Horn simulation game*, and prove that it provides an Ehrenfeucht-Fraïssé characterization of the relation $\mathfrak{A}, a \leq_{horn\mathcal{ALC}} \mathfrak{B}, b$. As *horn$\mathcal{ALC}$* is not closed under negation, Horn simulations will be non-symmetric. Moreover, since *horn$\mathcal{ALC}$* is not closed under disjunction, Horn simulations relate non-empty *sets* of elements from $\mathrm{dom}(\mathfrak{A})$ with elements from $\mathrm{dom}(\mathfrak{B})$. hus, rather than characterizing $\mathfrak{A}, a \leq_{horn\mathcal{ALC}} \mathfrak{B}, b$ only, we actually characterize the relation $\mathfrak{A}, X \leq_{horn\mathcal{ALC}} \mathfrak{B}, b$. To define the relation between subsets of $\mathrm{dom}(\mathfrak{A})$ along which the pebble is moved in the Horn simulation game, we set $XR^{\uparrow}Y$, for a binary relation R and sets X, Y, if for any $a \in X$, there exists $b \in Y$ with $(a, b) \in R$, and we set $XR^{\downarrow}Y$ if, for any $b \in Y$, there exists $a \in X$ with $(a, b) \in R$.

Definition 4 (Horn simulation). A *Horn simulation* between τ-structures $\mathfrak{A}$ and $\mathfrak{B}$ is a relation $Z \subseteq \mathcal{P}(\mathrm{dom}(\mathfrak{A})) \times \mathrm{dom}(\mathfrak{B})$ such that $(X, b) \in Z$ implies $X \neq \emptyset$ and the following hold:

(*atom$_h$*) for any $A \in \tau$, if $(X, b) \in Z$ and $X \subseteq A^{\mathfrak{A}}$, then $b \in A^{\mathfrak{B}}$;

(*forth$_h$*) for any $R \in \tau$, if $(X, b) \in Z$ and $XR^{\mathfrak{A}\uparrow}Y$, then there exist $Y' \subseteq Y$ and $b' \in \mathrm{dom}(\mathfrak{B})$ with $(b, b') \in R^{\mathfrak{B}}$ and $(Y', b') \in Z$;

(*back$_h$*) for any $R \in \tau$, if $(X, b) \in Z$ and $(b, b') \in R^{\mathfrak{B}}$, then there is $Y \subseteq \mathrm{dom}(\mathfrak{A})$ with $XR^{\mathfrak{A}\downarrow}Y$ and $(Y, b') \in Z$;

(*sim*) if $(X, b) \in Z$, then $\mathfrak{B}, b \preceq_{sim} \mathfrak{A}, a$ for every $a \in X$.

We write $\mathfrak{A}, X \preceq_{horn} \mathfrak{B}, b$ if there exists a Horn simulation Z between $\mathfrak{A}$ and $\mathfrak{B}$ such that $(X, b) \in Z$.

Condition (*atom$_h$*) ensures that concept names are preserved under Horn simulations, and conditions (*forth$_h$*), (*back$_h$*), and (*sim*) ensure, recursively, the preservation of concepts of the form $\exists R.H$, $\forall R.H$ and $L \to H$, respectively. Note that (*sim*) implies that the converse of (*atom$_h$*) holds as well, and so $\mathfrak{A}, X \preceq_{horn} \mathfrak{B}, b$ entails $X \subseteq A^{\mathfrak{A}}$ iff $b \in A^{\mathfrak{B}}$, for all $A \in \tau$, which reflects that $A \to \bot$ is a *horn$\mathcal{ALC}$*-concept.

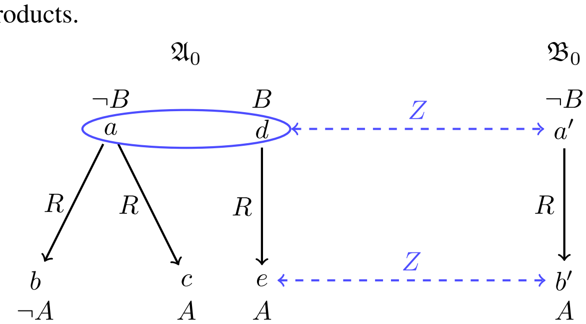
As we intend Horn simulations to characterize *horn$\mathcal{ALC}$*-concepts, which are $\mathcal{ALC}$-concepts, Horn simulations should subsume bisimulations. The following lemma states that this is indeed the case. It also shows that having *sets* as the first component of positions in the Horn simulation games is the defining difference between Horn simulations and bisimulations. The (straightforward) proof is instructive to understand Horn simulations.

Lemma 1. (i) *If Z is a bisimulation between τ-structures $\mathfrak{A}$ and $\mathfrak{B}$, then $\{(\{a\}, b) \mid (a, b) \in Z\}$ is a Horn simulation between $\mathfrak{A}$ and $\mathfrak{B}$.* (ii) *Conversely, if Z is a Horn simulation with a singleton X in every $(X, b) \in Z$, then $\{(a, b) \mid (\{a\}, b) \in Z\}$ is a bisimulation between $\mathfrak{A}$ and $\mathfrak{B}$.*

As the FO-translations of *horn$\mathcal{ALC}$*-concepts are Horn formulas, and the Horn formulas are (almost) characterized as the fragment of FO preserved under products, one could expect products to be closely related to Horn simulations. We now show this to be the case. Consider a family of τ-structures $\mathfrak{A}_i$, $i \in I$, and let $\mathfrak{A}$ be the disjoint union of the $\mathfrak{A}_i$. Define a relation Z between $\mathcal{P}(\mathrm{dom}(\mathfrak{A}))$ and $\prod_{i \in I} \mathfrak{A}_i$ by setting $(Y, f) \in Z$ if $Y \subseteq \mathrm{dom}(\mathfrak{A})$, $f \in \mathrm{dom}(\prod_{i \in I} \mathfrak{A}_i)$, and $\mathrm{dom}(\mathfrak{A}_i) \cap Y = \{f(i)\}$ for all $i \in I$. The proof of the following is again straightforward and instructive.

Lemma 2. *Z is a Horn simulation between $\mathfrak{A}$ and $\prod_{i \in I} \mathfrak{A}_i$.*

The following examples illustrate that Horn simulations can be seen as a proper generalization of both bisimulations and products.

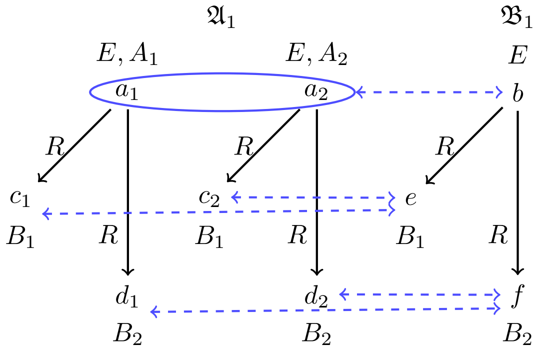


Example 1. (*i*) Let $\mathfrak{A}_0$ and $\mathfrak{B}_0$ be the structures shown above. $Z = \{(\{a,d\},a'),(\{e\},b')\}$ is a Horn simulation between $\mathfrak{A}_0$ and $\mathfrak{B}_0$. For the concept C_∇ from Theorem 1, we have $\{a,d\} \subseteq C_\nabla^{\mathfrak{A}_0}$ but $a' \notin C_\nabla^{\mathfrak{B}_0}$. Thus, by Lemma 3 below, C_∇ is not equivalent to any *horn*$\mathcal{ALC}$-concept.

(*ii*) Let $\mathfrak{A}_1$ and $\mathfrak{B}_1$ be the structures below. Then the relation

$$Z = \{(\{a_1,a_2\},b),(\{c_1\},e),(\{c_2\},e),(\{d_1\},f),(\{d_2\},f)\}$$

is a surjective Horn simulation between $\mathfrak{A}_1$ and $\mathfrak{B}_1$. Moreover, we have $\mathfrak{A}_1 \models \mathcal{T}_{horn}$ but $\mathfrak{B}_1 \not\models \mathcal{T}_{horn}$ for the TBox $\mathcal{T}_{horn}$ from Theorem 1. By Theorem 4 below, $\mathcal{T}_{horn}$ is not equivalent to any *horn*$\mathcal{ALC}$-TBox.



Given the notion of Horn simulation, we next introduce the *Horn simulation game* between τ-structures $\mathfrak{A}$ and $\mathfrak{B}$ in the expected way. In the infinite case, it consists of the following nested games. Using simulation games, one can check whether condition (*sim*) holds for a pair (X,b). Then, in the main Horn simulation game, the second player must respond with pairs (Y',b') for (*forth$_h$*) and sets Y for (*back$_h$*) such that the new position satisfies (*sim*) and the remaining conditions of Definition 4. In the Horn simulation game with ℓ rounds, some care must be taken: as we want to characterize the depth ℓ fragment of *horn*$\mathcal{ALC}$, we have to decompose condition (*sim*). Thus, define pairs (X,b) satisfying (sim^ℓ) as those for which player 2 has a winning strategy for the ℓ-round simulation game for all (b,a) with $a \in X$. Then, inductively, player 2 has a winning strategy in the $(\ell+1)$-round Horn simulation game at position (X,b) if (X,b) satisfies ($sim^{\ell+1}$) and player 2 can react to player 1's first move by choosing Y in such a way that condition (sim^ℓ) holds for the resulting position and she has a winning strategy in the resulting ℓ-round game. A formal definition is given in the appendix. We write $\mathfrak{A},X \preceq^\ell_{horn} \mathfrak{B},b$ if player 2 has a winning strategy in the ℓ-round game.

Theorem 3 (Ehrenfeucht-Fraïssé game for *horn*$\mathcal{ALC}$). *For any finite vocabulary τ, pointed τ-structures $\mathfrak{A},a$ and $\mathfrak{B},b$, and any $\ell < \omega$, we have*

$$\mathfrak{A},a \leq^\ell_{horn\mathcal{ALC}} \mathfrak{B},b \quad \textit{iff} \quad \mathfrak{A},a \preceq^\ell_{horn} \mathfrak{B},b.$$

Thus, $\mathfrak{A},a \leq_{horn\mathcal{ALC}} \mathfrak{B},b$ iff $\mathfrak{A},a \preceq^\omega_{horn} \mathfrak{B},b$. If $\mathfrak{A}$ and $\mathfrak{B}$ are finite, then

$$\mathfrak{A},a \leq_{horn\mathcal{ALC}} \mathfrak{B},b \quad \textit{iff} \quad \mathfrak{A},a \preceq_{horn} \mathfrak{B},b.$$

As discussed above, to prove Theorem 3, we actually show the following stronger statement:

Lemma 3. *For any finite vocabulary τ, pointed τ-structures $\mathfrak{A},X$ and $\mathfrak{B},b$, and any $\ell < \omega$,*

$$\mathfrak{A},X \leq^\ell_{horn\mathcal{ALC}} \mathfrak{B},b \quad \textit{iff} \quad \exists X_0 \subseteq X \; \mathfrak{A},X_0 \preceq^\ell_{horn} \mathfrak{B},b.$$

If $\mathfrak{A}$ and $\mathfrak{B}$ are finite, then

$$\mathfrak{A},X \leq_{horn\mathcal{ALC}} \mathfrak{B},b \quad \textit{iff} \quad \exists X_0 \subseteq X \; \mathfrak{A},X_0 \preceq_{horn} \mathfrak{B},b.$$

Proof. (sketch) The second claim follows directly from the first one, which we prove here. For any $\ell < \omega$ and any pointed τ-structure $\mathfrak{A},a$, let $\lambda_{\mathfrak{A},\ell,a}$ be an $\mathcal{ELU}[\tau]$-concept of depth $\leq \ell$ such that, for any pointed τ-structure $\mathfrak{B},b$,

$$b \in \lambda^{\mathfrak{B}}_{\mathfrak{A},\ell,a} \quad \text{iff} \quad \mathfrak{A},a \preceq^\ell_{sim} \mathfrak{B},b. \tag{1}$$

The existence of $\lambda_{\mathfrak{A},\ell,a}$ follows immediately from the fact that there are only finitely-many non-equivalent $\mathcal{ELU}[\tau]$-concepts of any fixed depth ℓ. Similarly, fix a finite set Horn_ℓ of *horn*$\mathcal{ALC}$-concepts of depth $\leq \ell$ such that every *horn*$\mathcal{ALC}$-concept of depth $\leq \ell$ is equivalent to some concept in Horn_ℓ. For a pointed τ-structure $\mathfrak{A},X$, let $\rho_{\mathfrak{A},\ell,X}$ be the conjunction of all concepts C in Horn_ℓ with $X \subseteq C^{\mathfrak{A}}$. Clearly, we have

$$b \in \rho^{\mathfrak{B}}_{\mathfrak{A},\ell,X} \quad \text{iff} \quad \mathfrak{A},X \leq^\ell_{horn\mathcal{ALC}} \mathfrak{B},b. \tag{2}$$

To prove the implication ($\Rightarrow$) of the first claim, we define relations $Z_\ell \subseteq \mathcal{P}(\text{dom}(\mathfrak{A})) \times \text{dom}(\mathfrak{B})$, $\ell < \omega$, by setting $(X,b) \in Z_\ell$ if $X \neq \emptyset$ and the following two conditions hold:

(i) $b \in \rho^{\mathfrak{B}}_{\mathfrak{A},\ell,X}$,

(ii) $X \subseteq \lambda^{\mathfrak{A}}_{\mathfrak{B},\ell,b}$.

Claim 1. For any $\ell < \omega$, $\emptyset \neq X \subseteq \text{dom}(\mathfrak{A})$ and $b \in \text{dom}(\mathfrak{B})$, if $(X,b) \in Z_\ell$, then $\mathfrak{A},X \preceq^\ell_{horn} \mathfrak{B},b$.

Proof of claim. We proceed by induction on $\ell < \omega$. The basis $\ell = 0$ holds by definition. So suppose that Claim 1 has been proved for ℓ and that $(X,b) \in Z_{\ell+1}$. We show $\mathfrak{A},X \preceq^{\ell+1}_{horn} \mathfrak{B},b$. Condition (*atom$_h$*) holds by definition. For (*forth$_h$*), suppose player 1 moves the pebble to Y with $XR^{\mathfrak{A}\uparrow}Y$. Then $X \subseteq (\exists R.\rho_{\mathfrak{A},\ell,Y})^{\mathfrak{A}}$. By the definition of $Z_{\ell+1}$, $b \in (\exists R.\rho_{\mathfrak{A},\ell,Y})^{\mathfrak{B}}$. Let player 2 respond with b' and Y' such that $(b,b') \in R^{\mathfrak{B}}$, $b' \in \rho^{\mathfrak{B}}_{\mathfrak{A},\ell,Y}$, and

$$Y' = Y \cap \lambda^{\mathfrak{A}}_{\mathfrak{B},\ell,b'}.$$

We show that (Y',b') is as required for (*forth$_h$*). By IH, it suffices to prove that $(Y',b') \in Z_\ell$. To show $Y' \neq \emptyset$, suppose otherwise. Then $Y \subseteq (\lambda_{\mathfrak{B},\ell,b'} \to \bot)^{\mathfrak{A}}$, so $(\lambda_{\mathfrak{B},\ell,b'} \to \bot)$ is equivalent to a conjunct of $\rho_{\mathfrak{A},\ell,Y}$. By the construction of b', we have $b' \in (\lambda_{\mathfrak{B},\ell,b'} \to \bot)^{\mathfrak{B}}$. On the other hand, $b' \in \lambda^{\mathfrak{B}}_{\mathfrak{B},\ell,b'}$, which is impossible.

Condition (i) is proved similarly and condition (ii) holds by the definition of Y'.

For (*back$_h$*), suppose player 1 moves the pebble to b' with $(b,b') \in R^{\mathfrak{B}}$. For every $C \in \text{Horn}_\ell$ with $b' \notin C^{\mathfrak{B}}$, take some $a_C \in X$ and a'_C with $(a_C,a'_C) \in R^{\mathfrak{A}}$ such that $a'_C \in (\lambda_{\mathfrak{B},\ell,b'} \sqcap \neg C)^{\mathfrak{A}}$. They exist since otherwise we would have $X \subseteq (\forall R.(\lambda_{\mathfrak{B},\ell,b'} \to C))^{\mathfrak{A}}$ but $b \notin (\forall R.(\lambda_{\mathfrak{B},\ell,b'} \to C))^{\mathfrak{B}}$, which contradicts the definition of $Z_{\ell+1}$. Now let player 2 respond with the set Y of all such a'_C. Then $XR^{\mathfrak{A}\downarrow}Y$ and

$(Y, b') \in Z_\ell$ as $b' \in \rho^{\mathfrak{B}}_{\mathfrak{A},\ell,Y}$ and $Y \subseteq \lambda^{\mathfrak{B}}_{\mathfrak{B},\ell,b'}$ hold by the construction of Y. By IH, $\mathfrak{A}, Y \preceq^{\ell}_{horn} \mathfrak{B}, b'$, as required.

Finally, ($sim^{\ell+1}$) follows from (1), which completes the proof of Claim 1.

Now assume that $\mathfrak{A}, X \leq^{\ell}_{horn\mathcal{ALC}} \mathfrak{B}, b$. Then it suffices to prove that if $b \in \rho^{\mathfrak{B}}_{\mathfrak{A},\ell,X}$, then there exists $X_0 \subseteq X$ with $(X_0, b) \in Z_\ell$. But for $X_0 = X \cap \lambda^{\mathfrak{A}}_{\mathfrak{B},\ell,b}$ this can be proved in the same way as Claim 1 above.

The proof of the implication ($\Leftarrow$) of the first claim is by induction on $\ell < \omega$. □

Horn simulations can also characterize *horn$\mathcal{ALC}$*-TBoxes. For $\ell < \omega$, we write $\mathfrak{A} \preceq^{(\ell)}_{horn} \mathfrak{B}$ if, for every $b \in \text{dom}(\mathfrak{B})$, there exists $X \subseteq \text{dom}(\mathfrak{A})$ such that $\mathfrak{A}, X \preceq^{(\ell)}_{horn} \mathfrak{B}, b$.

Theorem 4. *For any finite vocabulary τ, τ-structures $\mathfrak{A}$ and $\mathfrak{B}$, and $\ell < \omega$, we have*

$$\mathfrak{A} \leq^{\ell}_{horn\mathcal{ALC}} \mathfrak{B} \quad \textit{iff} \quad \mathfrak{A} \preceq^{\ell}_{horn} \mathfrak{B}.$$

If $\mathfrak{A}$ and $\mathfrak{B}$ are finite, then

$$\mathfrak{A} \leq_{horn\mathcal{ALC}} \mathfrak{B} \quad \textit{iff} \quad \mathfrak{A} \preceq_{horn} \mathfrak{B}.$$

IV. Complexity of Model Indistinguishability

We next study the complexity of deciding the relations $\leq_{horn\mathcal{ALC}}$ and $\equiv_{horn\mathcal{ALC}}$ and their restrictions $\leq^{\ell}_{horn\mathcal{ALC}}$ and $\equiv^{\ell}_{horn\mathcal{ALC}}$ on the level of concepts. The related problems *on the TBox level* (cf. Theorem 4) have the same complexity as shown in the appendix. We refer to the respective decision problems as (ℓ)-*entailment* and (ℓ)-*equivalence*; for instance, entailment is the problem of deciding whether $\mathfrak{A}, a \leq_{horn\mathcal{ALC}} \mathfrak{B}, b$ for input $\mathfrak{A}, \mathfrak{B}, a, b$, and ℓ-equivalence is the problem of deciding whether $\mathfrak{A}, a \equiv^{\ell}_{horn\mathcal{ALC}} \mathfrak{B}, b$ for input $\mathfrak{A}, \mathfrak{B}, a, b, \ell$.

As a second application of the Ehrenfeucht-Fraïssé results we investigate *concept learning by example* (*CBE*). CBE is a supervised learning problem with applications in knowledge engineering for automatically deriving new and potentially interesting concept descriptions from labelled data. Intuitively, given some relational data and sets of positive and negative examples, the goal is to find a concept that generalizes the positive examples, but avoids the negative ones. The associated decision problem is formally defined as follows:

Input: structure $\mathfrak{A}$, positive and negative examples P, N.
Question: is there a *horn$\mathcal{ALC}$*-concept C with $P \subseteq C^{\mathfrak{A}}$ and $N \cap C^{\mathfrak{A}} = \emptyset$?

The connection to Horn simulations is given by Lemma 3: an input $\mathfrak{A}, P, N$ is a yes-instance of CBE iff $\mathfrak{A}, P \not\preceq_{horn} \mathfrak{A}, b$ for all $b \in N$. We denote by ℓ-CBE the variant of CBE restricted to *horn$\mathcal{ALC}$*-concepts of depth ℓ. This variant is important in practice as the user is interested in small separating concepts.

Our main result in this section is the following theorem:

Theorem 5. *Entailment, equivalence, and CBE are* ExpTime-*complete. Moreover, ℓ-entailment, ℓ-equivalence and ℓ-CBE are* ExpTime-*complete if ℓ is given in binary and* PSpace-*complete if ℓ is given in unary.*

It is worth mentioning the striking contrast between this ExpTime result and the fact that the same problems for $\mathcal{ALC}$ are in PTime. On the one hand, the ExpTime lower bounds provide evidence that the use of sets in the notion of Horn simulations is inevitable. On the other hand, observe that CBE for $\mathcal{ALC}$ is in PTime because there is an $\mathcal{ALC}$-concept separating the positive and negative examples iff $\mathfrak{A}, a$ and $\mathfrak{A}, b$ are not bisimilar, for any $a \in P$ and $b \in N$. Consequently, the positive examples can be treated essentially separately and a naive application of this leads to overfitting, that is, the intended generalization of the positive examples is not taking place [26]. Thus, we can regard the ExpTime result for CBE in *horn$\mathcal{ALC}$* as the price for obtaining real generalizations.

To prove Theorem 5, we focus on the unrestricted case. The following lemma gives complexity-theoretic reductions between the mentioned problems and the problem *HornSim* of deciding whether $\mathfrak{A}, X \preceq_{horn} \mathfrak{B}, b$ for input $\mathfrak{A}, \mathfrak{B}, X, b$.

Lemma 4. (1) *CBE $\leq^{P}_{T}$ HornSim;*
(2) *$\overline{HornSim} \leq^{P}_{m}$ CBE;*
(3) *HornSim $\leq^{P}_{m}$ Entailment;*
(4) *Entailment $\leq^{P}_{m}$ HornSim;*
(5) *Equivalence $\leq^{P}_{T}$ Entailment;*
(6) *Entailment $\leq^{P}_{m}$ Equivalence.*

Proof. Here, we only show the most interesting reduction (3).

Let $\mathfrak{A}, \mathfrak{B}, X, b$ be the input to *HornSim*. Define $\mathfrak{A}'$ by adding a new R-predecessor a to all nodes in X. Further, define $\mathfrak{B}'$ by taking the disjoint union of $\mathfrak{A}$ and $\mathfrak{B}$ and adding a new R-predecessor d to b, and making d also a predecessor of all nodes in (the copy of) X. Then we have

$$\mathfrak{A}, X \preceq_{horn} \mathfrak{B}, b \quad \text{iff} \quad \mathfrak{A}', a \preceq_{horn} \mathfrak{B}', d,$$

which is equivalent to $\mathfrak{A}', a \leq_{horn\mathcal{ALC}} \mathfrak{B}', d$ by Theorem 3. □

Thus, to prove Theorem 5, it suffices to show the following:

Lemma 5. *HornSim is* ExpTime-*complete.*

For the upper bound, we observe that the Horn simulation game can be implemented by an alternating Turing machine (ATM) using only polynomial space. For the lower bound, we carefully adapt a strategy from [60] for proving that the simulation problem between two structures $\mathfrak{A}$ and $\mathfrak{B}$ is ExpTime-hard when $\mathfrak{A}$ is given as a *fair concurrent transition system*, that is, a certain synchronized product of structures. More precisely, we reduce the word problem of polynomially space-bounded ATMs. Let M be an $s(n)$-space bounded ATM and w an input of length n. We construct structures $\mathfrak{A}$, $\mathfrak{B}$, $X \subseteq \text{dom}(\mathfrak{A})$, and $b \in \text{dom}(\mathfrak{B})$ such that

$$M \text{ accepts } w \quad \text{iff} \quad \mathfrak{A}, X \preceq_{horn} \mathfrak{B}, b.$$

The structure $\mathfrak{A}$ can be thought of as the disjoint union of $s(n)$ structures $\mathfrak{A}_1, \ldots, \mathfrak{A}_{s(n)}$ and a single copy of $\mathfrak{B}$ (plus some connections from the $\mathfrak{A}_i$ to $\mathfrak{B}$). Intuitively, each sub-structure $\mathfrak{A}_i$ is responsible for tape cell i of one of M's configurations on input w; thus, the domain of each $\mathfrak{A}_i$ consists of the possible contents of a single cell. As usual, the challenge is the synchronization. Here, different tape cells are synchronized via the simulation conditions using different role names: one role name $R_{q,a,i,d}$ for every possible state q of M, current head position i, read symbol a, and branching direction d (we assume that M has binary branching). The extension of such a role name $R_{q,a,i,d}$ in a structure $\mathfrak{A}_j$ is defined in the obvious way, respecting M's transition relation. The set X consists of M's initial configuration on input w.

The role of $\mathfrak{B}$ (as the second structure) is to control M's computation; its domain is independent of M and consists of 20 elements only. It manages both the switch between universal and existential states and the intended acceptance value of the current configuration (1 if the configuration leads to acceptance, 0 otherwise). Universal and existential elements are fully interconnected with the mentioned role names $R_{q,a,i,d}$. The initial element b for the reduction is a universal element (without loss of generality M's initial state is universal), and corresponds to acceptance value 1.

Now, conditions (*atom*$_h$) and (*forth*$_h$) are responsible for simulating M's computation on input w and (*atom*$_h$) ensures that every reached set X in $\mathfrak{A}$ corresponds to a valid configuration. Conditions (*sim*) and (*back*$_h$) do not have a real purpose for the reduction, but need to be reflected in the mentioned inclusion of (a copy of) $\mathfrak{B}$ in $\mathfrak{A}$.

V. Expressive Completeness for *horn*$\mathcal{ALC}$

In this section, we prove that an FO-formula with one free variable is equivalent to a *horn*$\mathcal{ALC}$-concept just in case it is preserved under Horn simulations, and that an FO-sentence is equivalent to a *horn*$\mathcal{ALC}$-TBox just in case it is invariant under disjoint unions and preserved under *global* (that is, surjective) Horn simulations. We prove these results both in the classical setting defined above and in the finite model theory setting where the notions of equivalence, preservation, and invariance are relativized to finite models.

In the concept case, by the van Benthem-Rosen characterization of $\mathcal{ALC}$-concepts as the bisimulation invariant fragment of FO and since preservation under Horn simulations implies invariance under bisimulations (by Lemma 1), it suffices to prove that an $\mathcal{ALC}$-concept is equivalent to a *horn*$\mathcal{ALC}$-concept iff it is preserved under Horn simulations (in the classical and finite model theory setting). We will, therefore, formulate the expressive completeness result within $\mathcal{ALC}$. In the TBox case, it is known that an FO-sentence is equivalent to an $\mathcal{ALC}$-TBox just in case it is invariant under disjoint unions and preserved under bisimulations [35], [45]; thus, again by Lemma 1, it suffices to show that an $\mathcal{ALC}$-TBox is equivalent to a *horn*$\mathcal{ALC}$-TBox iff it is preserved under global Horn simulations. In both proofs, we employ Otto's finitary method [35] and show that, for $\mathcal{ALC}$-concepts (TBoxes) of depth $\leq \ell$, preservation under (global) Horn simulations is equivalent to preservation under (global) ℓ-Horn simulations, which is the same as equivalence to a *horn*$\mathcal{ALC}$-concept (TBox) of depth $\leq \ell$.

We start with the concept case. Say that an $\mathcal{ALC}$-concept C is *preserved under* (ℓ)*-Horn simulations* if, for any pointed structures $\mathfrak{A}, X$ and $\mathfrak{B}, b$, whenever $X \subseteq C^{\mathfrak{A}}$ and $\mathfrak{A}, X \preceq^{(\ell)}_{\mathit{horn}} \mathfrak{B}, b$ then $b \in C^{\mathfrak{B}}$.

Theorem 6 **(expressive completeness: *horn*$\mathcal{ALC}$-concepts).** *Let C be an $\mathcal{ALC}$-concept of depth ℓ. Then the following conditions are equivalent (in the classical and finite model theory setting):*

1. *C is equivalent to a horn$\mathcal{ALC}$-concept,*
2. *C is preserved under Horn simulations,*
3. *C is preserved under ℓ-Horn simulations,*
4. *C is equivalent to a horn$\mathcal{ALC}$-concept of depth $\leq \ell$.*

Proof. (sketch) (1) ⇒ (2) follows from Lemma 3; (4) ⇒ (1) is trivial; (3) ⇒ (4) is straightforward and proved in the appendix. We thus focus on (2) ⇒ (3).

A structure $\mathfrak{A}$ is called *tree-shaped* if the directed graph $G_{\mathfrak{A}} = (\mathrm{dom}(\mathfrak{A}), E)$ with $E = \bigcup_{R \in \tau} R^{\mathfrak{A}}$ is a directed tree and $R^{\mathfrak{A}} \cap S^{\mathfrak{A}} = \emptyset$ for all distinct role names R and S. The root of $G_{\mathfrak{A}}$ is called the *root* of $\mathfrak{A}$. The *depth* of $a \in \mathrm{dom}(\mathfrak{A})$ is the length of the path from the root of $\mathfrak{A}$ to a; the root of $\mathfrak{A}$ has depth 0. The disjoint union of tree-shaped structures is a *forest*. Recall that every pointed $\mathfrak{A}, a$ can be unravelled into a tree-shaped structure $\mathfrak{A}^*$ with root a such that $\mathfrak{A}, a$ and $\mathfrak{A}^*, a$ are bisimilar [32]. Note that $\mathfrak{A}^*$ is infinite (even for finite $\mathfrak{A}$) if $G_{\mathfrak{A}}$ contains a cycle. The finite model theory version of Theorem 6 is not affected as one only needs the unravelled tree-shaped structures up to a finite depth ℓ.

Suppose C is an $\mathcal{ALC}$-concept of depth $\leq \ell$ preserved under Horn simulations. Let $\mathfrak{A}, X$ and $\mathfrak{B}, b$ be pointed structures such that $\mathfrak{A}, X \preceq^{\ell}_{\mathit{horn}} \mathfrak{B}, b$ and $X \subseteq C^{\mathfrak{A}}$. We have to show that $b \in C^{\mathfrak{B}}$. For every $a \in X$, take a tree-shaped pointed structure $\mathfrak{A}_a, a$ bisimilar to $\mathfrak{A}, a$. Let $\mathfrak{B}', b$ be a tree-shaped pointed structure bisimilar to $\mathfrak{B}, b$. Then $\mathfrak{A}', X \preceq^{\ell}_{\mathit{horn}} \mathfrak{B}', b$ for the disjoint union $\mathfrak{A}'$ of $\mathfrak{A}_a$, $a \in X$. By bisimulation invariance of $\mathcal{ALC}$-concepts, we have $X \subseteq C^{\mathfrak{A}'}$ and it suffices to prove that $b \in C^{\mathfrak{B}'}$. Remove from $\mathfrak{A}'$ and $\mathfrak{B}'$ all nodes of depth $> \ell$ and denote the resulting structures by $\mathfrak{A}''$ and $\mathfrak{B}''$, respectively. As C is of depth $\leq \ell$, we have $X \subseteq C^{\mathfrak{A}''}$ and it suffices to prove that $b \in C^{\mathfrak{B}''}$. Using $\mathfrak{A}', X \preceq^{\ell}_{\mathit{horn}} \mathfrak{B}', b$, it is straightforward to show that $\mathfrak{A}'', X \preceq_{\mathit{horn}} \mathfrak{B}'', b$. Then $b \in C^{\mathfrak{B}''}$ follows from the preservation of C under Horn simulations.

For the finite model theory setting, observe that $\mathfrak{A}''$ and $\mathfrak{B}''$ are finite if $\mathfrak{A}$ and $\mathfrak{B}$ are finite. □

We now consider the TBox case. We say that an $\mathcal{ALC}$-TBox is *preserved under global* (ℓ)*-Horn simulations* if $\mathfrak{A} \models \mathcal{T}$ and $\mathfrak{A} \preceq^{(\ell)}_{\mathit{horn}} \mathfrak{B}$ imply $\mathfrak{B} \models \mathcal{T}$.

Theorem 7 **(expressive completeness: *horn*$\mathcal{ALC}$-TBoxes).** *For any $\mathcal{ALC}$-TBox $\mathcal{T}$ of depth ℓ, the following conditions are equivalent (in the classical and finite model theory setting):*

1. *$\mathcal{T}$ is equivalent to a horn$\mathcal{ALC}$-TBox;*
2. *$\mathcal{T}$ is preserved under global Horn simulations;*
3. *$\mathcal{T}$ is preserved under global ℓ-Horn simulations;*
4. *$\mathcal{T}$ is equivalent to a horn$\mathcal{ALC}$-TBox of depth $\leq \ell$.*

Proof. (sketch) We use the notation from the previous proof and focus on (2) ⇒ (3), showing (3) ⇒ (4) in the appendix.

We require *injective ℓ-Horn simulations*, which are defined as follows. Let $\mathfrak{A}$ be a forest and $\mathfrak{B}$ a tree-shaped structure. A sequence $H^0, \dots, H^\ell$ of relations between $\mathcal{P}(\text{dom}(\mathfrak{A}))$ and $\text{dom}(\mathfrak{B})$ is called an *injective ℓ-Horn simulation* if for each $(X, b) \in H^i$ all $a \in X$ are of depth i in $\mathfrak{A}$ and b is of depth i in $\mathfrak{B}$, and the following conditions hold:

- if $(X, b) \in H^i$, then $\mathfrak{A}, X \preceq^i_{horn} \mathfrak{B}, b$, for $0 \leq i \leq \ell$;
- if $(X, b) \in H^i$ and $X R^{\mathfrak{A}\uparrow} Y$, then there are $Y' \subseteq Y$ and $b' \in \text{dom}(\mathfrak{B})$ with $(b, b') \in R^{\mathfrak{B}}$ and $(Y', b') \in H^{i+1}$, for all $R \in \tau$ and $0 \leq i < \ell$;
- if $(X, b) \in H^i$ and $(b, b') \in R^{\mathfrak{B}}$, then there exists $Y \subseteq \text{dom}(\mathfrak{A})$ such that $X R^{\mathfrak{A}\downarrow} Y$ and $(Y', b') \in H^{i+1}$, for all $R \in \tau$ and $0 \leq i < \ell$;
- if $(X_0, b), (X_1, b) \in H^i$, then $X_0 = X_1$, for $0 \leq i \leq \ell$.

If $\mathfrak{A}, X \preceq^\ell_{\text{horn}} \mathfrak{B}, b$, we can take, for $a \in X$, a tree-shaped pointed structure $\mathfrak{A}_a, a$ bisimilar to $\mathfrak{A}, a$ and a tree-shaped pointed structure $\mathfrak{B}', b$ bisimilar to $\mathfrak{B}, b$. Then $\mathfrak{A}', X \preceq^\ell_{\text{horn}} \mathfrak{B}', b$ for the disjoint union $\mathfrak{A}'$ of the $\mathfrak{A}_a$, $a \in X$. By duplicating successors in $\mathfrak{B}'$ sufficiently often (possibly exponentially many times), we obtain a tree-shaped pointed structure $\mathfrak{B}'', b$ bisimilar to $\mathfrak{B}', b$ such that there is an injective ℓ-Horn simulation $H^0, \dots, H^\ell$ between $\mathfrak{A}'$ and $\mathfrak{B}''$ with $(X, b) \in H^0$.

Now suppose $\mathcal{T}$ is preserved under global Horn simulations. Let $\mathfrak{B}$ be a structure such that there exists a model $\mathfrak{A}$ of $\mathcal{T}$ with $\mathfrak{A} \preceq^\ell_{horn} \mathfrak{B}$. We have to show that $\mathfrak{B}$ is a model of $\mathcal{T}$. Let $b_0 \in \text{dom}(\mathfrak{B})$ be arbitrary. It suffices to show $b_0 \in (\neg C \sqcup D)^{\mathfrak{B}}$ for all $C \sqsubseteq D \in \mathcal{T}$. Since $\mathfrak{A} \preceq^\ell_{horn} \mathfrak{B}$, there is a set $X \subseteq \text{dom}(\mathfrak{A})$ with $\mathfrak{A}, X \preceq^\ell_{horn} \mathfrak{B}, b_0$. For every $a \in X$, take a tree-shaped pointed structure $\mathfrak{A}_a, a$ bisimilar to $\mathfrak{A}, a$. By the observation above, we can take a tree-shaped pointed interpretation $\mathfrak{B}', b_0$ bisimilar to $\mathfrak{B}, b_0$ and the disjoint union $\mathfrak{A}'$ of the $\mathfrak{A}_a$, $a \in X$, such that there is an injective ℓ-Horn simulation $H^0, \dots, H^\ell$ between $\mathfrak{A}'$ and $\mathfrak{B}'$ with $(X, b_0) \in H^0$. By bisimulation invariance of $\mathcal{ALC}$-concepts, $\mathfrak{A}'$ is a model of $\mathcal{T}$, and so it is enough to show that $b_0 \in (\neg C \sqcup D)^{\mathfrak{B}'}$ for all $C \sqsubseteq D \in \mathcal{T}$.

Let $\mathfrak{B}'|_\ell$ be the structure obtained from $\mathfrak{B}'$ by dropping all nodes of depth $> \ell$. We hook to every leaf $b \in \mathfrak{B}'|_\ell$ of depth ℓ a structure $\mathfrak{B}_b$ so that $\mathfrak{A}', X \preceq_{horn} \mathfrak{B}'', b_0$ for the resulting structure $\mathfrak{B}''$. As $\mathcal{T}$ is preserved under global Horn simulations, $\mathfrak{B}''$ is a model of $\mathcal{T}$. As $\mathcal{T}$ has depth $\leq \ell$, we have $b_0 \in (\neg C \sqcup D)^{\mathfrak{B}'}$ for all $C \sqsubseteq D \in \mathcal{T}$, as required. We come to the construction of the $\mathfrak{B}_b$ for b a leaf of depth ℓ in $\mathfrak{B}'|_\ell$. Since $H^0, \dots, H^\ell$ is injective, there is a unique non-empty $X_b \subseteq \text{dom}(\mathfrak{A}')$ such that $(X_b, b) \in H^\ell$. Observe that from $(X_b, b) \in H^\ell$ it follows that $X_b \subseteq A^{\mathfrak{A}'}$ iff $b \in A^{\mathfrak{B}'}$ for any $A \in \tau$. Let $\mathfrak{A}'_a$ be the tree-shaped substructure of $\mathfrak{A}'$ rooted at a, for $a \in X_b$. Then we hook to b the structure $\mathfrak{B}_b = \prod_{a \in X_b} \mathfrak{A}'_a$ by identifying $(a \mid a \in X_b) \in \prod_{a \in X_b} \mathfrak{A}'_a$ with b. Using Lemma 2, it is readily checked that the resulting structure is as required.

In the finite model theory setting, we consider finite $\mathfrak{A}$ and $\mathfrak{B}$. Then we can assume that the structures $\mathfrak{B}'|_\ell$ and $\mathfrak{A}'|_\ell$ are finite. Now, rather than hooking the (possibly infinite) $\mathfrak{B}_b = \prod_{a \in X_b} \mathfrak{A}'_a$ to every leaf b of depth ℓ in $\mathfrak{B}'|_\ell$ we (*i*) replace all $\mathfrak{A}'_a$ with a of depth ℓ in $\mathfrak{A}'$ by *finite* models $\mathfrak{A}''_a$ of $\mathcal{T}$ satisfying the same subconcepts of $\mathcal{T}$ as $\mathfrak{A}'_a$ in a and (*ii*) hook $\prod_{a \in X_b} \mathfrak{A}''_a$ to b. Then $\mathfrak{A}'', X \preceq_{horn} \mathfrak{B}''', b_0$ for the resulting finite structures $\mathfrak{A}''$ and $\mathfrak{B}'''$, and $\mathfrak{A}''$ is a model of $\mathcal{T}$. □

As there is only a finite number of *horn$\mathcal{ALC}$*-concepts and TBoxes of bounded depth in a finite vocabulary τ, it follows from Theorems 6 and 7 that it is decidable whether an $\mathcal{ALC}$-concept or TBox is equivalent to a *horn$\mathcal{ALC}$*-concept or TBox, respectively.

VI. Horn guarded fragment *hornGF* of FO

We extend *horn$\mathcal{ALC}$* to the Horn fragment, *hornGF*, of the guarded fragment of FO in the obvious way. *hornGF* contains numerous popular Horn DLs including those extending *horn$\mathcal{ALC}$* with inverse roles, the universal role, and role inclusions [4], [5]. We then generalize the Horn simulation games to guarded Horn simulation games for *hornGF* and prove an Ehrenfeucht-Fraïssé type definability theorem and a van Benthem style expressive completeness result for *hornGF*. Applications include an ExpTime upper bound for model indistinguishability.

Let τ be a vocabulary of predicate names R of arbitrary arity $r_R \geq 0$. The *guarded fragment* GF$[\tau]$ of FO is defined by the following rules:

- GF$[\tau]$ contains the constants $\top$ (truth) and $\bot$ (falsehood);
- GF$[\tau]$ contains the *atomic formulas* $R(\mathbf{x})$ and $x = y$ with $R \in \tau$;
- GF$[\tau]$ is closed under the connectives $\wedge$, $\vee$, and $\neg$;
- if $\varphi(\mathbf{xy})$ is in GF$[\tau]$ with free variables among $\mathbf{xy}$ and $G(\mathbf{xy})$ is an atomic formula containing all the variables in $\mathbf{xy}$, then
$$\forall \mathbf{y}\, (G(\mathbf{xy}) \to \varphi(\mathbf{xy})), \qquad \exists \mathbf{y}\, (G(\mathbf{xy}) \wedge \varphi(\mathbf{xy}))$$
are in GF$[\tau]$ (these are called the universal and existential *guarded quantifiers* of GF$[\tau]$).

If the particular vocabulary τ is not relevant, we simply write GF for GF$[\tau]$. The *nesting depth of guarded quantifiers* in a formula φ in GF, or simply the *depth* of φ, is defined as the number of nestings of guarded quantifiers in φ. The formulas of the *positive existential guarded fragment* GF$^\exists[\tau]$ of GF$[\tau]$ are constructed from atomic formulas using $\wedge$, $\vee$, and the guarded existential quantifiers.

Definition 5 (*hornGF*). The fragment *hornGF*$[\tau]$ of GF$[\tau]$ is given by the following grammar:

$$\begin{aligned}\varphi, \varphi' \ ::= \ \bot \ \mid \ \top \ \mid x = y \ \mid \ R(\mathbf{x}) \ \mid \ \varphi \wedge \varphi' \ \mid \ \lambda \to \varphi \\ \mid \ \exists\, \mathbf{y}(G(\mathbf{xy}) \wedge \varphi(\mathbf{xy})) \ \mid \ \forall\, \mathbf{y}(G(\mathbf{xy}) \to \varphi(\mathbf{xy})),\end{aligned}$$

where $R \in \tau$, $G(\mathbf{xy})$ are atomic formulas containing all the variables in $\mathbf{xy}$, and $\lambda \in \text{GF}^\exists[\tau]$.

hornGF is closely related to guarded tuple-generating dependencies (guarded tgds), a member of the Datalog$^\pm$ family of ontology languages for which query answering is in PTime [43]. *Guarded tgds* are FO-formulas of the form $\forall \mathbf{x} \forall \mathbf{y}\, (\psi(\mathbf{xy}) \ \to \ \exists \mathbf{z}\, \varphi(\mathbf{xz}))$ with conjunctions of atoms

$\psi(\mathbf{xy})$ and $\varphi(\mathbf{xz})$ such that ψ contains an atom $G(\mathbf{xy})$ guarding all the variables in $\mathbf{xy}$. Thus, in contrast to *hornGF*, guarded tgds have no quantifier alternation and can be regarded as a normal form for *hornGF*. In the appendix, we give a polynomial time reduction of deduction and query answering in *hornGF* to the respective problems for guarded tgds by introducing fresh predicate names for complex formulas. We also provide a polynomial reduction in the converse direction. We note that satisfiability in *hornGF* has the same complexity as satisfiability in GF [39]: EXPTIME-complete if the arity of predicates is bounded and 2-EXPTIME-complete otherwise.

Example 2. The Horn formulas equivalent to the concept C_∇ and TBox $\mathcal{T}_{horn}$ from the proof of Theorem 1 are in *hornGF*. Thus, there are $\mathcal{ALC}$-concepts and TBoxes that are not equivalent to any *horn*$\mathcal{ALC}$-concepts or TBoxes, respectively, but nevertheless are equivalent to formulas in *hornGF*.

Theorem 8. (i) *Every formula in hornGF is equivalent to a Horn formula.*

(ii) *There exists a sentence in GF—in fact, an $\mathcal{ALC}$-TBox—that is equivalent to a Horn sentence, but not equivalent to any hornGF sentence.*

Proof. The proof of (i) is by a straightforward induction. For (ii), consider the TBox

$$\mathcal{T}_{guard} \;=\; \{\, E \sqsubseteq \exists R.\top \sqcap \exists S.\top,\; E \sqcap \forall R.A \sqcap \forall S.B \sqsubseteq D \,\}.$$

It is equivalent to the Horn sentence

$$\forall x\,(E(x) \to \exists y_1 y_2\,(R(x,y_1) \wedge S(x,y_2) \wedge ((A(y_1) \wedge B(y_2)) \to D(x))))$$

but, as shown in Example 4 below, $\mathcal{T}_{guard}$ is not equivalent to any *hornGF*-sentence. □

Let $\mathfrak{A} = (\mathsf{dom}(\mathfrak{A}), (R^{\mathfrak{A}})_{R\in\tau})$ be a τ-structure. Denote by $\mathbf{a}$ a *tuple* $a_1 \dots a_n$ of elements of $\mathfrak{A}$ and set $[\mathbf{a}] = \{a_1, \dots, a_n\}$. A set $X \subseteq \mathrm{dom}(\mathfrak{A})$ is *guarded* in $\mathfrak{A}$ if X is a singleton or $R^{\mathfrak{A}}(\mathbf{a})$ for some $R \in \tau$ and $X = [\mathbf{a}]$. A tuple $\mathbf{a}$ is *guarded* if $[\mathbf{a}]$ is guarded.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures. If $\mathbf{a}$ and $\mathbf{b}$ are tuples of the same length in $\mathfrak{A}$ and $\mathfrak{B}$, respectively, we use $p\colon \mathbf{a} \mapsto \mathbf{b}$ to denote the map from $[\mathbf{a}]$ to $[\mathbf{b}]$ with $p(a_i) = b_i$. If $\mathbf{a}' = a_{i_1} \dots a_{i_k}$ is a subtuple of $\mathbf{a}$, then $p(\mathbf{a}')$ denotes the subtuple $p(a_{i_1}) \dots p(a_{i_k})$ of $\mathbf{b}$. The map p is a *homomorphism* from $\mathfrak{A}|_{[\mathbf{a}]}$ to $\mathfrak{B}|_{[\mathbf{b}]}$ if $\mathbf{c} \in R^{\mathfrak{A}}$ implies $p(\mathbf{c}) \in R^{\mathfrak{B}}$ for all $R \in \tau$ and $\mathbf{c}$ with $[\mathbf{c}] \subseteq [\mathbf{a}]$.

In this section, by a *pointed structure* we mean a pair $\mathfrak{A}, X$ where $X \subseteq \mathrm{dom}(\mathfrak{A})$ is a nonempty set of *guarded tuples*, all of the same positive length. We again write $\mathfrak{A}, \mathbf{a}$ for $\mathfrak{A}, \{\mathbf{a}\}$. We give the straightforward Ehrenfeucht-Fraïssé type characterization for GF$^\exists$ needed for the charachterization of *hornGF*. It is obtained from the standard guarded bisimulation characterization of GF [33] by replacing partial isomorphisms by homomorphisms and dropping the backward condition.

Definition 6 (guarded simulation). For τ-structures $\mathfrak{A}$ and $\mathfrak{B}$, a set Z of maps from guarded sets in $\mathfrak{A}$ to guarded sets in $\mathfrak{B}$ is called a *guarded simulation* if the following conditions hold for all $p\colon \mathbf{a} \mapsto \mathbf{b}$ in Z:

- (*atom*^{g}) $p : \mathfrak{A}|_{[\mathbf{a}]} \to \mathfrak{B}|_{[\mathbf{b}]}$ is a homomorphism;
- (*forth*^{g}) for every guarded tuple $\mathbf{a}'$ in $\mathfrak{A}$, there exist a guarded tuple $\mathbf{b}'$ in $\mathfrak{B}$ and p' such that $p'\colon \mathbf{a}' \to \mathbf{b}'$ is in Z and $p|_{[\mathbf{a}]\cap[\mathbf{a}']} = p'|_{[\mathbf{a}]\cap[\mathbf{a}']}$.

We write $\mathfrak{A}, \mathbf{a} \preceq_{gsim} \mathfrak{B}, \mathbf{b}$ if there exists a guarded simulation between $\mathfrak{A}$ and $\mathfrak{B}$ containing $p\colon \mathbf{a} \mapsto \mathbf{b}$.

In the same way as for $\mathcal{ELU}$, one can capture guarded simulations by *guarded simulation games* between two players such that player 2 has a winning strategy (can respond to any move of player 1) iff $\mathfrak{A}, \mathbf{a} \preceq_{gsim} \mathfrak{B}, \mathbf{b}$. We write $\mathfrak{A}, \mathbf{a} \preceq^{\ell}_{gsim} \mathfrak{B}, \mathbf{b}$ if player 2 has a winning strategy for the guarded simulation game with ℓ rounds.

We write $\mathfrak{A}, \mathbf{a} \leq^{(\ell)}_{\mathrm{GF}^\exists} \mathfrak{B}, \mathbf{b}$ if, for all formulas λ in GF$^\exists$ (of depth $\leq \ell$), $\mathfrak{A} \models \lambda(\mathbf{a})$ implies $\mathfrak{B} \models \lambda(\mathbf{b})$.

Theorem 9 (Ehrenfeucht-Fraïssé game for GF$^\exists$). *For any finite vocabulary τ, pointed τ-structures $\mathfrak{A}, \mathbf{a}$ and $\mathfrak{B}, \mathbf{b}$, and any $\ell < \omega$, we have*[2]

$$\mathfrak{A}, \mathbf{a} \leq^{\ell}_{GF^\exists} \mathfrak{B}, \mathbf{b} \quad \textit{iff} \quad \mathfrak{A}, \mathbf{a} \preceq^{\ell}_{gsim} \mathfrak{B}, \mathbf{b}.$$

If $\mathfrak{A}$ and $\mathfrak{B}$ are finite, then

$$\mathfrak{A}, \mathbf{a} \leq_{GF^\exists} \mathfrak{B}, \mathbf{b} \quad \textit{iff} \quad \mathfrak{A}, \mathbf{a} \preceq_{gsim} \mathfrak{B}, \mathbf{b}.$$

A. Simulations for hornGF

We introduce guarded Horn simulation games and prove an Ehrenfeucht-Fraïssé type definability result for *hornGF*. A *link* between structures $\mathfrak{A}$ and $\mathfrak{B}$ is a pair $(P, \mathbf{b})$ with $\mathbf{b}$ a guarded tuple in $\mathfrak{B}$ and P a nonempty set of mappings $p\colon \mathbf{b} \mapsto p(\mathbf{b})$ such that each p is a homomorphism from $\mathfrak{B}|_{[\mathbf{b}]}$ to $\mathfrak{A}|_{[p(\mathbf{b})]}$. We denote by $P[\mathbf{b}]$ the set $\{p(\mathbf{b}) \mid p \in P\}$ and define the analogue of $XR^{\uparrow}Y$ for guarded Horn simulations. If $(P, \mathbf{b})$ is a link between $\mathfrak{A}$ and $\mathfrak{B}$, and $\mathfrak{A}, Y$ a pointed structure, $R(\mathbf{x}_0\mathbf{y})$ an atomic formula of the same arity as the tuples in Y, and $\mathbf{b}_0$ a subtuple of $\mathbf{b}$ of the same length as $\mathbf{x}_0$, then we say that Y *is an* $R(\mathbf{b}_0\mathbf{y})$*-successor of* $(P, \mathbf{b})$ when, for any $p \in P$, there exists a tuple $\mathbf{a}$ with $p(\mathbf{b}_0)\mathbf{a} \in Y$ and $\mathfrak{A} \models R(p(\mathbf{b}_0)\mathbf{a})$.

Definition 7 (guarded Horn simulation). A *guarded Horn simulation* between structures $\mathfrak{A}$ and $\mathfrak{B}$ is a set Z of links between $\mathfrak{A}$ and $\mathfrak{B}$ such that for all $(P, \mathbf{b}) \in Z$, we have:

- (*atom*^g_h) for all atomic formulas $R(\mathbf{x})$ and tuples $\mathbf{b}'$ with $[\mathbf{b}'] \subseteq [\mathbf{b}]$, if $\mathfrak{A} \models R(p(\mathbf{b}'))$ for all $p \in P$, then $\mathfrak{B} \models R(\mathbf{b}')$;
- (*forth*^g_h) for all sets Y of guarded tuples and atomic formulas $R(\mathbf{x}_0\mathbf{y})$, if Y is an $R(\mathbf{b}_0, \mathbf{y})$-successor of $(P, \mathbf{b})$, then there exists $(P', \mathbf{b}_0\mathbf{b}') \in Z$ such that $P'[\mathbf{b}_0\mathbf{b}'] \subseteq Y$;
- (*back*^g_h) for every guarded tuple $\mathbf{b}'$ in $\mathfrak{B}$, there exists a link $(P', \mathbf{b}')$ in Z such that, for any $p' \in P'$, there exists $p \in P$ with $p|_{[\mathbf{b}]\cap[\mathbf{b}']} = p'|_{[\mathbf{b}]\cap[\mathbf{b}']}$;

[2] Here and in what follows the assumption that the tuples considered in Ehrenfeucht-Fraïssé characterizations are guarded is not essential. It is straightforward to modify the model comparison games in such a way that the characterizations hold for arbitrary tuples.

(sim^g_h) there exists a guarded simulation between $(\mathfrak{B}, \mathbf{b})$ and $(\mathfrak{A}, p(\mathbf{b}))$ for every $p \in P$.

We write $\mathfrak{A}, X \preceq_{ghsim} \mathfrak{B}, \mathbf{b}$ if there exists a guarded Horn simulation Z between $\mathfrak{A}$ and $\mathfrak{B}$ such that $X = P[\mathbf{b}]$ for some P with $(P, \mathbf{b}) \in Z$.

Lemma 1 linking Horn simulations with bisimulations can be lifted to the guarded case. In fact, any guarded bisimulation Z between $\mathfrak{A}$ and $\mathfrak{B}$ defines a guarded Horn simulation $Z' = \{(\{p^{-1}\}, \mathbf{b}) \mid p\colon \mathbf{a} \mapsto \mathbf{b} \in Z\}$ and if Z is a guarded Horn simulation with singleton P for every $(P, \mathbf{b}) \in Z$, then $\{p^{-1} \mid (\{p\}, \mathbf{b}) \in Z\}$ is a guarded bisimulation (notice that, by ($atom^g_h$) and (sim^g_h), p is a partial isomorphism if $P = \{p\}$).

On the other hand, as the moves of player 1 are no longer restricted to those along $R^{\mathfrak{A}}$, $R \in \tau$, the relationship to products is subtler than in the Horn simulation case (Lemma 2).

Example 3. For $i = 1, 2$, let $\mathfrak{A}_i = (\{a_i\}, A_1^{\mathfrak{A}_i}, A_2^{\mathfrak{A}_i})$, where $A_1^{\mathfrak{A}_1} = \{a_1\}$, $A_2^{\mathfrak{A}_2} = \{a_2\}$, and $A_2^{\mathfrak{A}_1} = A_1^{\mathfrak{A}_2} = \emptyset$. Then $Z = \{(\{a_1, a_2\}, (a_1, a_2))\}$ is a Horn simulation between the disjoint union $\mathfrak{A}$ of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ and the product $\mathfrak{A}_1 \times \mathfrak{A}_2$, but it is not a guarded Horn simulation as $Y = \{a_1\}$ is a $(y = y)$-successor of $(\{a_1, a_2\}, (a_1, a_2))$ (with empty $\mathbf{b}_0$) for which there is no link with Y as the first component in Z. Clearly, it is also not possible to expand Z to a guarded Horn simulation. We will revisit the relationship to products below.

For $\ell < \omega$, we define the relations $\mathfrak{A}, X \preceq^{\ell}_{ghsim} \mathfrak{B}, \mathbf{b}$ in the obvious way following the definition Horn simulation games with ℓ rounds. We write $\mathfrak{A}, X \leq^{(\ell)}_{hornGF} \mathfrak{B}, \mathbf{b}$ if for all formulas φ in *hornGF* (of depth $\leq \ell$) the following holds: if $\mathfrak{A} \models \varphi(\mathbf{a})$ for all $\mathbf{a} \in X$, then $\mathfrak{B} \models \varphi(\mathbf{b})$.

Theorem 10 (Ehrenfeucht-Fraïssé game for *hornGF*). *For any finite vocabulary τ, pointed τ-structures $\mathfrak{A}, \mathbf{a}$ and $\mathfrak{B}, \mathbf{b}$, and any $\ell < \omega$, we have*

$$\mathfrak{A}, \mathbf{a} \leq^{\ell}_{hornGF} \mathfrak{B}, \mathbf{b} \quad \text{iff} \quad \mathfrak{A}, \mathbf{a} \preceq^{\ell}_{ghsim} \mathfrak{B}, \mathbf{b}.$$

If $\mathfrak{A}$ and $\mathfrak{B}$ are finite, then

$$\mathfrak{A}, \mathbf{a} \leq_{hornGF} \mathfrak{B}, \mathbf{b} \quad \text{iff} \quad \mathfrak{A}, \mathbf{a} \preceq_{ghsim} \mathfrak{B}, \mathbf{b}.$$

The proof of Theorem 10 is similar to that of Theorem 3 and given in the appendix. In particular, one has to prove again a stronger version where the tuple $\mathbf{a}$ is replaced by a set X of tuples. The existence of a winning strategy for player 2 in the guarded Horn simulation game is decidable in exponential time. Thus, it follows from Theorem 10 that entailment and equivalence in *hornGF* are decidable in ExpTime.

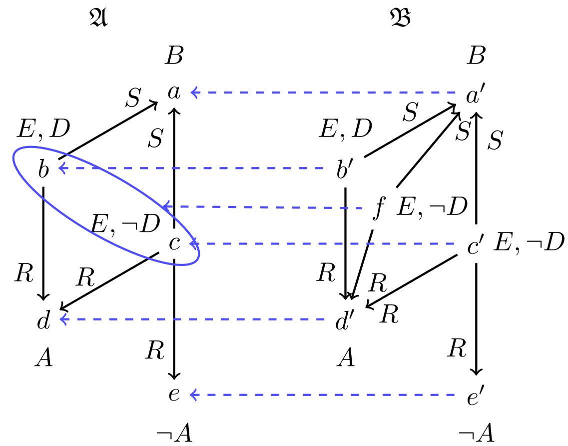
Theorem 11. *In hornGF, entailment, equivalence, and CBE are in* ExpTime. *Moreover, ℓ-entailment, ℓ-equivalence, and ℓ-CBE are in* ExpTime *for binary encoding of ℓ and in* PSpace *for unary encoding.*

In contrast to *horn$\mathcal{ALC}$*, it remains open whether the ExpTime upper bound is tight. Using guarded Horn simulations, we now show that the TBox $\mathcal{T}_{guard}$ from the proof of Theorem 8 is not equivalent to any *hornGF*-sentence.

Example 4. Let $\mathfrak{A}$ and $\mathfrak{B}$ be the structures below. $\mathfrak{B}$ is a copy of $\mathfrak{A}$ with the extra node f; $\mathfrak{B}$ refutes $\mathcal{T}_{guard}$ in f, but $\mathfrak{A}$ is a model of $\mathcal{T}_{guard}$. A guarded Horn simulation Z between $\mathfrak{A}$ and $\mathfrak{B}$ is given by adding to the set of singleton links

$$\{(\{u\}, u') \mid u \in \text{dom}(\mathfrak{A})\} \cup \{(\{uv\}, u'v') \mid (u, v) \in R^{\mathfrak{A}} \cup S^{\mathfrak{A}}\}$$

the links $(\{b, c\}, f)$, $(\{bd, cd\}, fd')$, and $(\{ba, ca\}, fa')$.



B. Expressive Completeness for hornGF

Our next aim is to show that an FO-formula φ is equivalent to a *hornGF*-formula just in case it is preserved under guarded Horn simulations. This statement needs qualification, however, in two respects. First, our infinitary proof goes through only if we require the sets Y where player 1 moves to in condition ($forth^g_h$) to be intersections of FO-definable sets. Second, in contrast to *horn$\mathcal{ALC}$*, the language *hornGF* admits equality guards and is not local in the sense that the truth of a *hornGF*-formula $\varphi(\mathbf{x})$ in $\mathfrak{A}, \mathbf{a}$ is not determined by some neighbourhood of $\mathbf{a}$ in the Gaifman graph of $\mathfrak{A}$. As a consequence, GF-sentences such as $\varphi = \forall x\, A_1(x) \vee \forall x\, A_2(x)$ (with omitted equality guards) are not equivalent to any *hornGF*-sentence but preserved under guarded Horn simulations.

To deal with this issue, we lift the definition of guarded Horn simulations from single structures to families of structures. Let $\mathfrak{A}_i$, $i \in I$, be a family of disjoint structures. A set X of tuples in $\bigcup_{i \in I} \text{dom}(\mathfrak{A}_i)$ *intersects with all* $\mathfrak{A}_i$, $i \in I$, if X contains at least one tuple from each $\text{dom}(\mathfrak{A}_i)$. For an open formula $\varphi(\mathbf{x})$, we write $(\mathfrak{A}_i \mid i \in I) \models \varphi(\mathbf{a})$ if $\mathbf{a}$ is a nonempty tuple in some $\text{dom}(\mathfrak{A}_i)$ and $\mathfrak{A}_i \models \varphi(\mathbf{a})$. For closed φ, we write $(\mathfrak{A}_i \mid i \in I) \models \varphi$ if $\mathfrak{A}_i \models \varphi$ for all $i \in I$. A set X of tuples is *FO$^\infty$-definable* in $(\mathfrak{A}_i \mid i \in I)$ if there is a set $\Gamma(\mathbf{x})$ of FO-formulas with $X = \{\mathbf{a} \mid \forall \varphi \in \Gamma(\mathbf{x})\ (\mathfrak{A}_i \mid i \in I) \models \varphi(\mathbf{a})\}$.

Definition 8 (generalized guarded Horn simulation). Let $\mathfrak{A}_i$, $i \in I$, be a family of disjoint structures, $\mathfrak{A}$ the disjoint union of $\mathfrak{A}_i$, $i \in I$, and $\mathfrak{B}$ a structure. A set Z of links between $\mathfrak{A}$ and $\mathfrak{B}$ is a *generalized guarded Horn simulation between* $(\mathfrak{A}_i \mid i \in I)$ *and* $\mathfrak{B}$ if all $(P, \mathbf{b}) \in Z$ satisfy conditions ($atom^g_h$) and ($back^g_h$) from Definition 7 and

($forth^{gg}_h$) for all sets Y of guarded tuples in $\mathfrak{A}$ and atomic formulas $R(\mathbf{x}_0\mathbf{y})$, if Y is an $R(\mathbf{b}_0\mathbf{y})$-successor of $(P, \mathbf{b})$ in $\mathfrak{A}$ and

– $\mathbf{b}_0$ is not empty or

– $\mathbf{b}_0$ is empty and Y intersects with all $\mathfrak{A}_i$, $i \in I$,

then there is $(P', \mathbf{b}_0\mathbf{b}') \in Z$ with $P'[\mathbf{b}_0\mathbf{b}'] \subseteq Y$;

(sim_h^{gg}) there exists a guarded simulation between $(\mathfrak{B}, \mathbf{b})$ and $(\mathfrak{A}_i, p(\mathbf{b}))$ for every $p \in P$ and $p(\mathbf{b})$ in $\mathrm{dom}(\mathfrak{A}_i)$.

Z is *FO-restricted* if ($forth_h^{gg}$) holds for all FO^∞-definable Y. We write $(\mathfrak{A}_i \mid i \in I), X \preceq_{ghorn}^{FO} \mathfrak{B}, \mathbf{b}$ if there exists an FO-restricted generalized guarded Horn simulation Z between $(\mathfrak{A}_i \mid i \in I)$ and $\mathfrak{B}$ such that $X = P[\mathbf{b}]$ for some $(P, \mathbf{b}) \in Z$.

Note that, as we modified ($forth_h^g$), the set Z in Example 3 is a generalized guarded Horn simulation. In fact, now Lemma 2 can be lifted to the guarded case: if $\mathfrak{A}_i$, $i \in I$, is a family of structures, the set of all $(P, f_1 \dots f_n)$ with $f_1 \dots f_n$ a guarded tuple in $\prod_{i \in I} \mathfrak{A}_i$ and $p \in P$ just in case there exists $i \in I$ such that $p(f_j) = f_j(i)$, $1 \leq j \leq n$, is a generalized guarded Horn simulation between the disjoint union of the $\mathfrak{A}_i$ and $\prod_{i \in I} \mathfrak{A}_i$.

A formula $\varphi(\mathbf{x})$ is *preserved under FO-restricted generalized guarded Horn simulations* if $(\mathfrak{A}_i \mid i \in I) \models \varphi(\mathbf{a})$ for all $\mathbf{a} \in X$ and $(\mathfrak{A}_i \mid i \in I), X \preceq_{ghorn}^{FO} \mathfrak{B}, \mathbf{b}$ imply $\mathfrak{B} \models \varphi(\mathbf{b})$.

Theorem 12 (expressive completeness: *hornGF*). *An FO-formula is equivalent to a hornGF-formula iff it is preserved under FO-restricted generalized guarded Horn simulations.*

Proof. (sketch) The implication $(\Rightarrow)$ is straightforward. Conversely, suppose $\varphi(\mathbf{x}_0)$ is preserved under FO-restricted generalized guarded Horn simulations. Let $\mathrm{cons}(\varphi)$ be the set of all $\psi(\mathbf{x}_0)$ in *hornGF* entailed by $\varphi(\mathbf{x}_0)$. By compactness, it suffices to show $\mathrm{cons}(\varphi) \models \varphi$. Let $\mathfrak{B}$ be an ω-saturated [52] model satisfying $\mathrm{cons}(\varphi)(\mathbf{b}_0)$ for some tuple $\mathbf{b}_0$ in $\mathrm{dom}(\mathfrak{B})$. We show $\mathfrak{B} \models \varphi(\mathbf{b}_0)$. For any tuple $\mathbf{b}$ and tuple $\mathbf{x}$ of variables of the same length as $\mathbf{b}$, we denote by $\lambda_{\mathfrak{B},\mathbf{b}}(\mathbf{x})$ the set of guarded existential positive $\lambda(\mathbf{x})$ with $\mathfrak{B} \models \lambda(\mathbf{b})$. Let $\mathcal{C}$ be the set of all sets $\Gamma(\mathbf{x}_0)$ of FO-formulas with $\mathfrak{B} \models \Gamma(\mathbf{b}_0)$ and such that $\Gamma(\mathbf{x}_0) \cup \{\varphi(\mathbf{x}_0)\}$ is satisfiable and take, for any $\Gamma(\mathbf{x}_0) \in \mathcal{C}$, an ω-saturated structure $\mathfrak{A}_\Gamma$ and tuple $\mathbf{a}_\Gamma$ with $\mathfrak{A}_\Gamma \models (\Gamma \cup \{\varphi\})(\mathbf{a}_\Gamma)$. Let $\mathfrak{A}$ be the disjoint union of $(\mathfrak{A}_\Gamma \mid \Gamma \in \mathcal{C})$ and let Z be the set of pairs $(X, \mathbf{b})$ such that

(a) for any $\psi(\mathbf{x}) \in$ *hornGF*, if $(\mathfrak{A}_\Gamma \mid \Gamma \in \mathcal{C}) \models \psi(\mathbf{a})$ for all $\mathbf{a} \in X$, then $\mathfrak{B} \models \psi(\mathbf{b})$;

(b) there exists a set $\Phi(\mathbf{x}) \supseteq \lambda_{\mathfrak{B},\mathbf{b}}$ of FO-formulas such that X is the set of all tuples $\mathbf{a}$ with $(\mathfrak{A}_\Gamma \mid \Gamma \in \mathcal{C}) \models \Phi(\mathbf{a})$.

Each $(X, \mathbf{b}) \in Z$ can be regarded as a link $(P, \mathbf{b})$ with $X = P[\mathbf{b}]$. As we work with ω-saturated structures, one can show that Z is an FO-restricted generalized guarded Horn simulation between $(\mathfrak{A}_\Gamma \mid \Gamma \in \mathcal{C})$ and $\mathfrak{B}$. Since $\varphi(\mathbf{x}_0)$ is preserved under generalized guarded Horn simulations, $\mathfrak{B} \models \varphi(\mathbf{b}_0)$. ☐

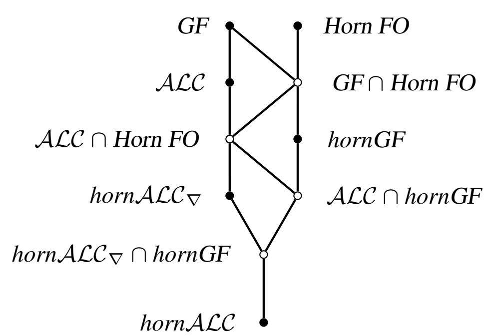
VII. Conclusion and Outlook

We have introduced model comparison games for *hornALC* and *hornGF* and obtained Ehrenfeucht-Fraïssé type definability and van Benthem style expressive completeness results. For *hornALC*, our results are 'complete': the characterizations hold in both classical and finite model theory settings without any restrictions on the players' moves, and the straightforward ExpTime upper bound for checking indistinguishability of models and concept learnability using our model comparison games is tight. For more expressive *hornGF*, it remains open whether the characterization holds in the setting of finite model theory and whether the moves of the players have to be restricted to 'saturated' sets in the expressive completeness result. In this case, it is also open whether the ExpTime upper bound for checking indistinguishability of models is tight.

A different line of open research problems arises from the fact that *hornALC* and *hornGF* do not capture the intersections of $\mathcal{ALC}$ (respectively, GF) and Horn FO. It is thus an open problem to find out whether there exists a 'neat' syntactic definition of the intersection of $\mathcal{ALC}$ and Horn FO such that, if an $\mathcal{ALC}$-concept or TBox is equivalent to a Horn FO formula, then it is equivalent to a concept, or, respectively, TBox, satisfying this definition. The analogous question is also open for *hornGF* and Horn FO. The proofs of Theorems 1 and 8 suggest the following syntactic extension of *hornALC*.

Example 5. Denote by $\mathcal{ELU}_\nabla$ the extension of $\mathcal{ELU}$ with the ∇-*operator* defined as $\nabla R.C = \exists R.\top \sqcap \forall R.C$ and let *hornALC*$_\nabla$ be defined in the same way as *hornALC* (Definition 1) with the exception that now L is an $\mathcal{ELU}_\nabla$-concept. Then the concept C_∇ from the proof of Theorem 1 (i) and the TBox $\mathcal{T}_{guard}$ from the proof of Theorem 8 are clearly a *hornALC*$_\nabla$-concept and TBox, respectively. So *hornALC*$_\nabla$ captures more from the intersection of $\mathcal{ALC}$ and Horn FO than *hornALC*. One can also show by an inductive argument that all *hornALC*$_\nabla$-concepts and TBoxes are indeed equivalent to Horn FO formulas (details are in the appendix). However, again this language does not fully capture the intersection in question as the TBox $\mathcal{T}_{horn}$ from the proof of Theorem 1 (ii) is not equivalent to any *hornALC*$_\nabla$-TBox. This can be shown by introducing model comparison games for *hornALC*$_\nabla$ (obtained by replacing the simulation game for $\mathcal{ELU}$ with a game capturing $\mathcal{ELU}_\nabla$) and showing that the Horn simulation from Example 1 (ii) preserves *hornALC*$_\nabla$.

Taking into account the examples given in this paper, we arrive at the following lattice of languages and their intersections (modulo equivalence) where all inclusions are proper:



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APPENDIX

We show that the definition of *horn*$\mathcal{ALC}$-concepts used in this paper is equivalent to syntactically different definitions of Horn-$\mathcal{ALC}$ concepts and Horn modal formulas given in the literature.

A definition for Horn DLs based on polarity is given in [4]. This definition can be restricted to the $\mathcal{ALC}$ case as follows. We say that an $\mathcal{ALC}$-concept C is a *p-horn*$\mathcal{ALC}$*-concept* if $\mathsf{pol}(C) \leq 1$, where $\mathsf{pol}(C) = \mathsf{pl}^+(C)$ with pl^+ defined as shown in the table below:

C	pl^+	pl^-
$\top$	0	0
$\bot$	0	0
A	1	0
$\neg C'$	$\mathsf{pl}^-(C')$	$\mathsf{pl}^+(C')$
$\bigsqcap_i C_i$	$\mathsf{max}_i \mathsf{pl}^+(C_i)$	$\sum_i \mathsf{pl}^-(C_i)$
$\bigsqcup_i C_i$	$\sum_i \mathsf{pl}^+(C_i)$	$\mathsf{max}_i \mathsf{pl}^-(C_i)$
$\exists R.C'$	$\mathsf{max}\{1, \mathsf{pl}^+(C')\}$	$\mathsf{pl}^-(C')$
$\forall R.C'$	$\mathsf{pl}^+(C')$	$\mathsf{max}\{1, \mathsf{pl}^-(C')\}$

Theorem 13. *Every horn*$\mathcal{ALC}$*-concept is equivalent to a p-horn*$\mathcal{ALC}$*-concept, and vice versa.*

Proof. It is readily checked by induction on the construction of a *horn*$\mathcal{ALC}$-concept C that $\mathsf{pol}(C) \leq 1$, and so C is also a *p-horn*$\mathcal{ALC}$-concept.

For the converse direction, we define a translation sH of *p-horn*$\mathcal{ALC}$-concepts to equivalent *horn*$\mathcal{ALC}$-concepts. To ease the proof, we assume the *p-horn*$\mathcal{ALC}$-concepts to be in negation normal form (NNF). Under this assumption, the definition of *p-horn*$\mathcal{ALC}$-concepts can be simplified. Namely, an $\mathcal{ALC}$ concept C in NNF is a *p-horn*$\mathcal{ALC}$-concept if $\mathsf{pol}(C) \leq 1$, where $\mathsf{pol}(C)$ is defined as follows:

$$\mathsf{pol}(C) = \begin{cases} 0 & \text{if } C = \top \mid \bot \mid \neg A \\ 1 & \text{if } C = A \\ \mathsf{max}_i \mathsf{pol}(C_i) & \text{if } C = C_1 \sqcap \ldots \sqcap C_n \\ \sum_i \mathsf{pol}(C_i) & \text{if } C = C_1 \sqcup \ldots \sqcup C_n \\ \mathsf{max}\{1, \mathsf{pol}(C')\} & \text{if } C = \exists R.C' \\ \mathsf{pol}(C') & \text{if } C = \forall R.C' \end{cases}$$

Claim. Any *p-horn*$\mathcal{ALC}$-concept $C = C_1 \sqcup \ldots \sqcup C_n$ in NNF is equivalent to a concept of the form $L \to D$, where L is an $\mathcal{ELU}$-concept, and either $D = \bot$ or $D = C_j$, for some j, $1 \leq j \leq n$, and $\mathsf{pol}(C_j) = 1$.

Proof of claim. As $\mathsf{pol}(C) \leq 1$, there exists at most one disjunct C_j, $1 \leq j \leq n$, with $\mathsf{pol}(C_j) = 1$. We define a *p-horn*$\mathcal{ALC}$-concept $C_\sqcup \sqcup D$ equivalent to C in the following way. If there exists a disjunct C_j of C with $\mathsf{pol}(C_j) = 1$, then $C_\sqcup = C_1 \sqcup \ldots \sqcup C_{j-1} \sqcup C_{j+1} \sqcup \ldots \sqcup C_n$ and $D = C_j$;

otherwise $C_\sqcup = C$ and $D = \bot$. It follows that $\mathsf{pol}(C_\sqcup) = 0$, which implies that $C_\sqcup$ is an $\mathcal{ALC}$-concept built using $\top$, $\bot$, $\neg A$, $\sqcup$, $\sqcap$ and $\forall R$ only. Let L be the $\mathcal{ELU}$-concept defined as $\mathrm{NNF}(\neg C_\sqcup)$. Then C is equivalent to $L \to D$, as required.

In the following definition of the translation sH, we use L_C and D_C for a *p-horn$\mathcal{ALC}$*-concept $C = C_1 \sqcup \ldots \sqcup C_n$ to denote the equivalent concept $L_C \to D_C$:

$$\mathsf{sH}(C) = \begin{cases} C & \text{if } C ::= \top \mid \bot \mid A \mid \neg A \\ \bigsqcap_{i=1}^{n} \mathsf{sH}(C_i) & \text{if } C = C_1 \sqcap \ldots \sqcap C_n \\ L_C \to \mathsf{sH}(D_C) & \text{if } C = C_1 \sqcup \ldots \sqcup C_n \\ \exists R.\mathsf{sH}(C') & \text{if } C = \exists R.C' \\ \forall R.\mathsf{sH}(C') & \text{if } C = \forall R.C' \end{cases}$$

It is readily seen that $\mathsf{sH}(C)$ is a *horn$\mathcal{ALC}$*-concept equivalent to C. □

Horn modal formulas have been defined in various ways in the literature [6], [55]–[58], and not all of the definitions are equivalent. Sturm [6] defines Horn modal formulas with n-ary modal operators. We show that, when restricted to unary modal operators, his definition is equivalent to the definition of *horn$\mathcal{ALC}$*-concepts given in this paper. We rephrase the definition in [6] in the DL terms as follows. Let $\mathcal{H}_b$ be defined by the grammar

$$H, H' ::= \bot \mid \neg A \mid H \sqcap H' \mid H \sqcup H' \mid \forall R.H,$$

where A is a concept name and R a role name. Then the set $\mathcal{H}$ of *s-horn$\mathcal{ALC}$-concepts* is the smallest set containing $\mathcal{H}_b \cup N_C$, closed under $\sqcap$, $\exists R$, $\forall R$ and such that whenever $C, C' \in \mathcal{H}$ and $C \in \mathcal{H}_b$ or $C' \in \mathcal{H}_b$, then $C \sqcup C' \in \mathcal{H}$. The set $\mathcal{H}_b$ can be seen as the set containing the negation of $\mathcal{ELU}$-concepts, and so the equivalence with our definition can be obtained by an argument analogous to the one in the proof of Theorem 13.

The remaining notions of Horn modal formulas [55]–[58] are rather different from our definition of *horn$\mathcal{ALC}$*-concepts. To illustrate, we focus on Nguyen's definition [55], rephrasing it in the DL parlance. A *n-horn$\mathcal{ALC}$-concept* G is defined by the following grammar:

$$P, P' ::= \top \mid \bot \mid A \mid P \sqcap P' \mid P \sqcup P' \mid \exists R.P \mid \forall R.P$$
$$G, G' ::= A \mid \neg P \mid G \sqcap G' \mid \exists R.G \mid \forall R.G \mid P \to G$$

The crucial difference between *n-horn$\mathcal{ALC}$*-concepts and *horn$\mathcal{ALC}$*-concepts lies in the definition of P, which allows universal role restrictions $\forall R.P$. It is not hard to see that the *n-horn$\mathcal{ALC}$*-concept $\forall R.A \to B$ is not preserved under products, and so is not equivalent to any Horn FO formula.

DEFINITIONS AND PROOFS FOR SECTION: SIMULATIONS FOR *horn$\mathcal{ALC}$*

We first give a rigorous definition of the relation $\mathfrak{A}, X \preceq^\ell_{horn} \mathfrak{B}, b$ and then supply the missing details of the proof of Lemma 3.

Definition 9 (**ℓ-Horn simulation).** Let $\mathfrak{A}$ and $\mathfrak{B}$ be τ-structures. Define relations $\preceq^\ell_{horn}$, $\ell < \omega$, between pointed structures $\mathfrak{A}, X$ and $\mathfrak{B}, b$ by induction:

- $\mathfrak{A}, X \preceq^0_{horn} \mathfrak{B}, b$ if $X \neq \emptyset$ and $X \subseteq A^{\mathfrak{A}}$ implies $b \in A^{\mathfrak{B}}$, for all $A \in \tau$, and $\mathfrak{B}, b \preceq^0_{sim} \mathfrak{A}, a$ for all $a \in X$.
- $\mathfrak{A}, X \preceq^{\ell+1}_{horn} \mathfrak{B}, b$ if the following conditions hold:
  - $\mathfrak{A}, X \preceq^0_{horn} \mathfrak{B}, b$;
  - if $XR^{\mathfrak{A}\uparrow}Y$, then there exist $Y' \subseteq Y$ and $b' \in \mathrm{dom}(\mathfrak{B})$ such that $(b, b') \in R^{\mathfrak{B}}$ and $\mathfrak{A}, Y' \preceq^\ell_{horn} \mathfrak{B}, b'$, for all $R \in \tau$;
  - if $(b, b') \in R^{\mathfrak{B}}$, then there exists $Y \subseteq \mathrm{dom}(\mathfrak{A})$ with $XR^{\mathfrak{A}\downarrow}Y$ and $\mathfrak{A}, Y \preceq^\ell_{horn} \mathfrak{B}, b'$, for all $R \in \tau$;
  - $\mathfrak{B}, b \preceq^{\ell+1}_{sim} \mathfrak{A}, a$ for all $a \in X$.

Lemma 3 For any finite vocabulary τ, pointed τ-structures $\mathfrak{A}, X$ and $\mathfrak{B}, b$, and any $\ell < \omega$,

$$\mathfrak{A}, X \leq^\ell_{horn\mathcal{ALC}} \mathfrak{B}, b \quad \text{iff} \quad \exists X_0 \subseteq X \; \mathfrak{A}, X_0 \preceq^\ell_{horn} \mathfrak{B}, b.$$

If $\mathfrak{A}$ and $\mathfrak{B}$ are finite, then

$$\mathfrak{A}, X \leq_{horn\mathcal{ALC}} \mathfrak{B}, b \quad \text{iff} \quad \exists X_0 \subseteq X \; \mathfrak{A}, X_0 \preceq_{horn} \mathfrak{B}, b.$$

Proof. It remains to prove the direction from right to left of the first claim. Thus we prove the following.

Claim 1. For any $\ell < \omega$, $X \subseteq \mathrm{dom}(\mathfrak{A})$ and $b \in \mathrm{dom}(\mathfrak{B})$, if $\mathfrak{A}, X \preceq^\ell_{horn} \mathfrak{B}, b$, then $X \subseteq C^{\mathfrak{A}}$ implies $b \in C^{\mathfrak{B}}$ for every $C \in \mathrm{Horn}_\ell$.

Proof of claim. For $\ell = 0$, Claim 1 is trivial. Suppose it has been proved for ℓ and show that it holds for $\ell + 1$ by induction on the construction of C. Thus, suppose that Claim 1 has been proved for $C', C_1, C_2 \in \mathrm{Horn}_\ell$, and that $C \in \mathrm{Horn}_{\ell+1}$ is of the form $C = \forall R.C'$, $C = \exists R.C'$, $C = C_1 \sqcap C_2$, or $C = L \to C'$ with L a $\mathcal{ELU}$ concept of depth $\leq \ell + 1$. Suppose also that $\mathfrak{A}, X \preceq^{\ell+1}_{\mathrm{horn}} \mathfrak{B}, b$ and $X \subseteq C^{\mathfrak{A}}$.

Case $C = \forall R.C'$. Suppose $b \notin (\forall R.C')^{\mathfrak{B}}$. Choose b' with $(b, b') \in R^{\mathfrak{B}}$ and $b' \notin C'^{\mathfrak{B}}$. By (*back$_h$*), there is $Y \subseteq \mathrm{dom}(\mathfrak{A})$ with $XR^{\mathfrak{A}\downarrow}Y$ and $\mathfrak{A}, Y \preceq^\ell_{horn} \mathfrak{B}, b'$. We have $C' \in \mathrm{Horn}_\ell$, and so, by IH for ℓ, there exists $a' \in Y$ with $a' \notin C'^{\mathfrak{A}}$. Then there exists $a \in X$ with $a \notin (\forall R.C')^{\mathfrak{A}}$, which is impossible.

Case $C = \exists R.C'$. Suppose $b \notin (\exists R.C')^{\mathfrak{B}}$. We have $XR^{\mathfrak{A}\uparrow}Y$ for $Y = C'^{\mathfrak{A}}$. By (*forth$_h$*) for Y, there exist $Y' \subseteq Y$ and $b' \in \mathrm{dom}(\mathfrak{B})$ with $(b, b') \in R^{\mathfrak{B}}$ and $\mathfrak{A}, Y \preceq^\ell_{horn} \mathfrak{B}, b'$. Since $C' \in \mathrm{Horn}_\ell$ and by IH, $b' \in C'^{\mathfrak{B}}$. But then $b \in (\exists R.C')^{\mathfrak{B}}$, contrary to our assumption.

Case $C = (L \to C')$. Suppose $b \notin (L \to C')^{\mathfrak{B}}$. Then $b \in L^{\mathfrak{B}}$ and $b \notin C'^{\mathfrak{B}}$. By Theorem 2, $X \subseteq L^{\mathfrak{A}}$, and by IH, there exists $a \in X$ with $a \notin C'^{\mathfrak{A}}$. Then $X \not\subseteq (L \to C')^{\mathfrak{A}}$, which is a contradiction.

The remaining case $C = C_1 \sqcap C_2$ is easy and left to the reader. □

PROOFS FOR SECTION IV

Lemma 4.

1. $\overline{\textit{CBE}} \leq^P_T$ *HornSim;*
2. $\overline{\textit{HornSim}} \leq^P_m$ *CBE;*
3. *HornSim* $\leq^P_m$ *Entailment;*

(4) *Entailment* $\leq_m^P$ *HornSim;*
(5) *Equivalence* $\leq_T^P$ *Entailment;*
(6) *Entailment* $\leq_m^P$ *Equivalence.*

Proof. For (1), observe that the following are equivalent by Lemma 3 for all $\mathfrak{A}, P, N$:

- there is some *horn$\mathcal{ALC}$*-concept C such that $P \subseteq C^{\mathfrak{A}}$ and $C^{\mathfrak{A}} \cap N = \emptyset$,
- $\mathfrak{A}, P \not\preceq_{horn} \mathfrak{A}, b$ for all $b \in N$.

To see "⇒", let C be such a concept. By Lemma 3, we have have $\mathfrak{A}, P \not\preceq_{horn} \mathfrak{A}, b$, for all $b \in N$.

Conversely, suppose $\mathfrak{A}, P \not\preceq_{horn} \mathfrak{A}, b$ for all $b \in N$. By Lemma 3, we have $\mathfrak{A}, P \not\leq_{horn\mathcal{ALC}} \mathfrak{A}, b$, for all $b \in N$. Thus, there is a *horn$\mathcal{ALC}$*-concept C_b such that $P \subseteq C_b^{\mathfrak{A}}$ and $b \notin C_b^{\mathfrak{A}}$, for all $b \in N$. Let C be the conjunction of the concepts C_b, for all $b \in N$. Clearly, $P \subseteq C^{\mathfrak{A}}$ and $C^{\mathfrak{A}} \cap N = \emptyset$.

For (2), observe that $\mathfrak{A}, X \preceq_{horn} \mathfrak{B}, b$ iff in the disjoint union $\mathfrak{A} \uplus \mathfrak{B}$ of $\mathfrak{A}$ and $\mathfrak{B}$ the positive examples $P = X$ cannot be distinguished from the negative examples $N = \{b\}$.

For (3), let $\mathfrak{A}, \mathfrak{B}, X, b$ be an input to HornSim. Define $\mathfrak{A}'$ by adding a new R-predecessor a to all nodes in X. Further, define $\mathfrak{B}'$ by taking the disjoint union of $\mathfrak{A}$ and $\mathfrak{B}$ and adding a new R-predecessor d to b, and making d also a predecessor of all nodes in (the copies of) X. Then we have:

$$\mathfrak{A}, X \preceq_{horn} \mathfrak{B}, b \quad \text{iff} \quad \mathfrak{A}', a \preceq_{horn} \mathfrak{B}', d,$$

and the latter is equivalent to $\mathfrak{A}', a \leq_{horn\mathcal{ALC}} \mathfrak{B}', d$ by Theorem 3.

Theorem 3 implies (4).

For (5), observe that equivalence is just mutual entailment.

For (6), let $\mathfrak{A}, \mathfrak{B}, a, b$ be an instance of entailment. Construct a new structure $\mathfrak{A}'$ by adding two fresh elements a' and b' to the disjoint union $\mathfrak{A} \cup \mathfrak{B}$ of $\mathfrak{A}$ and $\mathfrak{B}$ and making a' an R-predecessor of both a and b, and b' an R-predecessor of b. It is routine to verify that:

$$\mathfrak{A}, a \leq_{horn\mathcal{ALC}} \mathfrak{B}, b \quad \text{iff} \quad \mathfrak{A}', a' \equiv_{horn\mathcal{ALC}} \mathfrak{A}', b'.$$

□

Lemma 5. *HornSim is* ExpTime*-complete.*

Proof. For the upper bound, we provide an alternating algorithm which essentially implements the Horn simulation game and requires only polynomial space. Let (X_0, b_0) be the input. The algorithm proceeds in rounds and maintains a pair (X, d) with $X \subseteq \text{dom}(\mathfrak{A})$ and $d \in \text{dom}(\mathfrak{B})$. At pair (X, d), the algorithm proceeds as follows:

- For every $R \in \tau$, and every Y with $XR^{\mathfrak{A}\uparrow}Y$, guess non-empty $Y' \subseteq Y$ and d' with $(d, d') \in R^{\mathfrak{B}}$ and proceed with the pair (Y', d').
- For every $R \in \tau$, and every d' with $(d, d') \in R^{\mathfrak{B}}$, guess non-empty Y with $XR^{\mathfrak{A}\downarrow}Y$ and proceed with the pair (Y, d').

Note the similarity of the two points above with properties *(forth$_h$)* and *(back$_h$)*, respectively. The algorithm rejects if (X, d) does not satisfy *(atom$_h$)* or *(sim)* at some stage, or it fails guessing Y' or Y, respectively, in the two points above; it accepts after $2^{|\text{dom}(\mathfrak{A})|} \cdot |\text{dom}(\mathfrak{B})|$ rounds. It remains to observe that the algorithm obviously requires only polynomial space and that both *(atom$_h$)* and *(sim)* can be checked in polynomial time.

For the lower bound, we reduce the word problem for polynomially space-bounded, alternating Turing machines (ATMs), similar to [60]. An ATM is a tuple $M = (Q_e, Q_u, \Sigma, \Gamma, q_0, \mapsto, F_{rej}, F_{acc})$ where Σ is the input alphabet, Γ is the tape alphabet, Q_e, Q_u, F_{rej}, and F_{acc} are pairwise disjoint sets of existential, universal, rejecting, and accepting states, respectively. We denote the set of all states with Q, the set of all rejecting and accepting states with F, and assume that the initial state q_0 is universal, that is, $q_0 \in Q_u$. The transition relation $\mapsto \subseteq Q \times \Gamma \times Q \times \Gamma \times \{L, R, H\}$ has binary branching degree for all $q \in Q_e \cup Q_u$, that is, every pair (q, a) has precisely two successors. Accordingly, we use $(q, a) \mapsto \langle (q_l, b_l, d_l), (q_r, b_r, d_r) \rangle$ to indicate that after reading symbol a in state q, the TM can branch left with (q_l, b_l, d_l) and right with (q_r, b_r, d_r). We further assume that, for all $q \in F$ and $a \in \Gamma$, we have $(q, a) \mapsto (q, a, H)$, that is, once a final state $q \in F$ is reached, M loops in the same configuration.

Acceptance of such TMs is defined as follows. The possible computations of M on some input word w induce an AND-OR graph whose nodes are M's configurations and whose edges correspond to successor configurations. With each node in the graph, we associate an acceptance value as follows:

- accepting (respectively, rejecting) configurations are associated with value 1 (respectively, 0);
- a universal configuration is associated with the minimum value associated with one of its successor configurations;
- an existential configuration is associated with the maximum value associated with one of its successor configurations.

It is well-known that we can assume that in M from every possible start configuration, we always reach final configurations, so the above is well-defined. The ATM M accepts an input word w iff the initial configuration is associated with value 1.

For the reduction, let M be an $s(n)$-space bounded ATM, for some polynomial $s(n)$ and w some input word of length n. We construct structures $\mathfrak{A}$, $\mathfrak{B}$, a set $X \subseteq \text{dom}(\mathfrak{A})$, and $b \in \text{dom}(\mathfrak{B})$ such that

$$M \text{ accepts } w \quad \text{iff} \quad \mathfrak{A}, X \preceq_{horn} \mathfrak{B}, b.$$

The structure $\mathfrak{A}$ can be thought of as a disjoint union of $s(n)$ structures $\mathfrak{A}_1, \ldots, \mathfrak{A}_{s(n)}$ and a single copy of $\mathfrak{B}$. Intuitively, each sub-structure $\mathfrak{A}_i$ is responsible for a tape cell of one of M's configurations on input w, and different tape cells are synchronized via the simulation conditions using different role names. We include a copy $\mathfrak{B}$ in $\mathfrak{A}$ due to technical reasons made clear below. The role of $\mathfrak{B}$ (as the second structure) is to control M's computation by enforcing the right association of values with configurations in the AND-OR graph; it is essentially independent of M.

The vocabulary consists of the following symbols:

- concept names $U_0, U_1, U_\swarrow, U_\searrow, V_1, \ldots, V_{s(n)}$, and
- role names R_{id} and $R_{q,a,i,d}$ for every $q \in Q$, $a \in \Gamma$, $i \in \{1, \ldots, s(n)\}$, and $d \in \{\swarrow, \searrow\}$.

We start with $\mathfrak{B}$. Its domain $\mathrm{dom}(\mathfrak{B})$ consists of 20 elements: 16 internal elements of shape $(*, l, r, val, d)$, where $* \in \{\wedge, \vee\}$, $l, r, val \in \{0, 1\}$ satisfy $l * r \equiv val$, and $d \in \{\swarrow, \searrow\}$, and final elements (val, d) with $val \in \{0, 1\}$ and $d \in \{\swarrow, \searrow\}$. More precisely, we have for all $d \in \{\swarrow, \searrow\}$ the following elements:

$$(\wedge,0,0,0,d),(\wedge,0,1,0,d),(\wedge,1,0,0,d),(\wedge,1,1,1,d),$$
$$(\vee,0,0,0,d),(\vee,0,1,1,d),(\vee,1,0,1,d),(\vee,1,1,1,d),$$
$$(0,d),(1,d).$$

Note that elements of the shape $(*, l, r, val, d)$ match the truth table entries for the operator $*$ in the sense that $l * r \equiv val$. We refer with *universal* and *existential* elements to elements of the shape $(\wedge, \cdots)$ and $(\vee, \cdots)$, respectively.

For the concept names, we take

$$U_\swarrow^{\mathfrak{B}} = \{(*,l,r,val,d) \in B \mid d = \swarrow\} \cup \{(0,\swarrow),(1,\swarrow)\},$$
$$U_\searrow^{\mathfrak{B}} = \{(*,l,r,val,d) \in B \mid d = \searrow\} \cup \{(0,\searrow),(1,\searrow)\},$$
$$U_0^{\mathfrak{B}} = \{(0,\swarrow),(0,\searrow)\},$$
$$U_1^{\mathfrak{B}} = \{(1,\swarrow),(1,\searrow)\},$$
$$V_i^{\mathfrak{B}} = \emptyset \qquad \text{for all } i \in \{1,\ldots,s(n)\}.$$

For the role names of shape $R_{q,a,i,d}$, we take

- $((*, l, r, val, d), (*', l', r', val', d')) \in R^{\mathfrak{B}}_{q,a,i,d}$ iff $* \neq *'$ and either $val' = l$ and $d' = \swarrow$ or $val' = r$ and $d' = \searrow$.

That is, we switch between existential and universal elements and require that the next value val' coincides with the current l or r depending on the branch. Everything is independent of the values of q, a, i, but depends on d. We further have

- $((*, l, r, val, d), (val', d')) \in R^{\mathfrak{B}}_{q,a,i,d}$ iff either $val' = l$ and $d' = \swarrow$ or $val' = r$ and $d' = \searrow$.

Finally, for the role name R_{id}, $R^{\mathfrak{B}}_{id}$ is the identity on the final elements, that is, $\{(b, b) \mid b \in \mathrm{dom}(\mathfrak{B})$ of shape $(val, d)\}$.

We continue with structures $\mathfrak{A}_i$, for each $i \in \{1, \ldots, s(n)\}$. The domain $\mathrm{dom}(\mathfrak{A}_i)$ of each $\mathfrak{A}_i$ consists of all possible contents of a cell, extended with a direction, that is,

$$\mathrm{dom}(\mathfrak{A}_i) = (\Gamma \cup (Q \times \Gamma)) \times \{\swarrow, \searrow\}.$$

For the concept names, we take:

$$U_\swarrow^{\mathfrak{A}_i} = (\Gamma \cup (Q \times \Gamma)) \times \{\swarrow\},$$
$$U_\searrow^{\mathfrak{A}_i} = (\Gamma \cup (Q \times \Gamma)) \times \{\searrow\},$$
$$U_0^{\mathfrak{A}_i} = F_{rej} \times \Gamma \times \{\swarrow, \searrow\},$$
$$U_1^{\mathfrak{A}_i} = F_{acc} \times \Gamma \times \{\swarrow, \searrow\},$$
$$V_j^{\mathfrak{A}_i} = \begin{cases} \mathrm{dom}(\mathfrak{A}_i) & \text{if } i \neq j \\ \emptyset & \text{otherwise} \end{cases} \qquad \text{for all } j \in \{1,\ldots,s(n)\}.$$

For a role name $R_{q,a,j,d}$, intuitively $R^{\mathfrak{A}_i}_{q,a,j,d}$ contains all pairs (γ, γ') such that there is configuration α in state q, with head position j, reading a, (and previous direction d), and a successor configuration α' of α, in which cell i with content γ has been updated to γ'. More precisely, we include in $R^{\mathfrak{A}_i}_{q,a,j,d}$ for all $(q, a) \to \langle (q_l, b_l, d_l), (q_r, b_r, d_r) \rangle$:

- if $d_l = L$, the following pairs:
  - if $i = j$, then $((q, a, d), (b_l, \swarrow))$,
  - if $i = j + 1$, then $((b, d), (q_l, b, \swarrow))$ for all $b \in \Gamma$,
  - if $i \notin \{j, j + 1\}$, then $((b, d), (b, \swarrow))$ for all $b \in \Gamma$;
- if $d_l = H$, the following pairs:
  - if $i = j$, then $((q, a, d), (q_l, b_l, \swarrow))$,
  - if $i \neq j$, then $((b, d), (b, \swarrow))$, for all $b \in \Gamma$;
- if $d_l = R$, the following pairs:
  - if $i = j$, then $((q, a, d), (b_l, \swarrow))$,
  - if $i = j - 1$, then $((b, d), (q_l, b, \swarrow))$ for all $b \in \Gamma$,
  - if $i \notin \{j, j - 1\}$, then $((b, d), (b, \swarrow))$ for all $b \in \Gamma$;
- the cases for $d_r = L$, $d_r = H$, and $d_r = R$ are obtained by replacing above $\swarrow$, b_l, and q_l with $\searrow$, b_r, and q_r, respectively.

Finally, for the role name R_{id}, $R^{\mathfrak{A}_i}_{id}$ is the identity relation on the set $\Gamma \cup (F \times \Gamma)$.

The structure $\mathfrak{A}$ is now constructed by first taking the disjoint union of all $\mathfrak{A}_i$, $i \in \{1, \ldots, s(n)\}$ and $\mathfrak{B}$, and then adding connections between elements from $\mathfrak{A}_i$ and $\mathfrak{B}$. For all $c \in \mathrm{dom}(\mathfrak{A}_i)$, let us denote with (c, i) the corresponding copy of c in $\mathrm{dom}(\mathfrak{A})$. We add the following connections:

- $((a, d, i), b) \in R^{\mathfrak{A}}$, for every $(a, d, i) \in \mathrm{dom}(\mathfrak{A})$, $b \in \mathrm{dom}(\mathfrak{B})$, and every role name R;
- $((q, a, d, i), b) \in R^{\mathfrak{A}}$, for every for every role name R and every $(q, a, d, i) \in \mathrm{dom}(\mathfrak{A})$ and $b \in \mathrm{dom}(\mathfrak{B})$ such that either $q \in Q_e$ and b is universal or $q \in Q_u$ and b is existential.

Let us point out two important properties of the structure $\mathfrak{A}$. First, by the connections of the $\mathfrak{A}_i$ with the copy of $\mathfrak{B}$ in $\mathfrak{A}$, we have that:

(P1) $\mathfrak{B}, b \preceq_{sim} \mathfrak{A}, x$ for every universal element $b \in \mathrm{dom}(\mathfrak{B})$ and every element x of shape $(a, d, i) \in \mathrm{dom}(\mathfrak{A})$ and every element $x = (q, a, d, i) \in \mathrm{dom}(\mathfrak{A})$ with $q \in Q_u$, and similarly, $\mathfrak{B}, b \preceq_{sim} \mathfrak{A}, x$ for every existential element $b \in \mathrm{dom}(\mathfrak{B})$ and every x of shape $(a, d, i) \in \mathrm{dom}(\mathfrak{A})$ and every element $x = (q, a, d, i) \in \mathrm{dom}(\mathfrak{A})$ with $q \in Q_e$. Even more, the witnessing simulation is of the form $\{((a, d, i), b)\} \cup \{(b', b') \mid b' \in \mathrm{dom}(\mathfrak{B})\}$.

The second property concerns the synchronization of the different $\mathfrak{A}_i$. For formulating the property, it is convenient to associate with a direction $d \in \{\swarrow, \searrow\}$ and some configuration α of M a set $X_{\alpha,d} \subseteq \mathrm{dom}(\mathfrak{A})$ in the following natural way. If α is the configuration $b_1 \cdots b_{s(n)}$, then $X_{\alpha,d}$ is the set

$$X_{\alpha,d} = \{(b_i, d, i) \mid i \in \{1, \ldots, s(n)\}\}.$$

We now have the following property:

(P2) For every configuration α in state q with symbol a at head position i, there are precisely two role names R such that every $(c, j) \in X_{\alpha,d}$ (for any d) has an R-successor in $\mathfrak{A}$, namely $R_{q,a,i,\swarrow}$ and $R_{q,a,i,\searrow}$. Moreover, if we move jointly to these successors, we arrive at $X_{\alpha_l,\swarrow}$ and $X_{\alpha_r,\searrow}$, respectively, where α_l, α_r are the two successor configurations of α according to $\mapsto$.

Let α_0 be M's initial configuration on input w and $\hat{b}$ the element $(\wedge, 1, 1, 1, \swarrow)$ in $\mathfrak{B}$. Based on the insights (P1) and (P2) given above, we can verify correctness of the reduction.

Claim. M accepts w iff $\mathfrak{A}, X_{\alpha_0,\swarrow} \preceq_{horn} \mathfrak{B}, \hat{b}$.

Proof of the Claim. "⇒" We define a Horn simulation Z guided by the AND-OR graph induced by the computation of M on input w. Let $\mathcal{C}$ be the set of all configurations of M and denote with $\ell(\alpha)$ the associated acceptance value of a configuration $\alpha \in \mathcal{C}$. Define Z as follows:

- $X_{\alpha,d}Z(*, l, r, val, d)$ for all $d \in \{\swarrow, \searrow\}$ such that:
  - either α is universal and $* = \wedge$ or α is existential and $* = \vee$,
  - $val = \ell(\alpha)$ and α has a left successor configuration α_l with $l = \ell(\alpha_l)$ and a right successor configuration α_r with $r = \ell(\alpha_r)$.
- $X_{\alpha,d}Z(val, d)$ for all $d \in \{\swarrow, \searrow\}$ such that α is a final configuration with $val = \ell(\alpha)$.
- $\{b'\}Zb'$ for all $b' \in \text{dom}(\mathfrak{B})$,

Note that we clearly have $X_{\alpha_0,\swarrow}Z\hat{b}$. It thus remains to verify that Z is indeed a Horn simulation, that is, it satisfies Conditions *(atom$_h$)*, *(sim)*, *(forth$_h$)*, and *(back$_h$)*.

- Condition *(atom$_h$)* is satisfied due to the definition of $U^{\mathfrak{A}}$ for the concept names U.
- Condition *(forth$_h$)* is a consequence of (P1) above.
- For Condition *(forth$_h$)*, consider a pair $X_{\alpha,d}Z(*, l, r, val, d)$, that is, α is not a final configuration; the other cases are similar. Further assume that $X_{\alpha,d}(R^{\mathfrak{A}})^{\uparrow}Y$. By (P2), we know that R is of shape $R_{q,a,i,d'}$, $d' \in \{\swarrow, \searrow\}$, where q is the state of α, i is its head position, and a is the symbol of the head. Moreover, (P2) implies that $Y = X_{\alpha_l,\swarrow}$ or $Y = X_{\alpha_r,\searrow}$ (depending on d'), where α_l, α_r are left and right successor configurations of α. It remains to note that, by definition of Z, there is some $b' \in \text{dom}(\mathfrak{B})$ such that YZb'.
- Condition *(back$_h$)* is also a consequence of (P1). Indeed, let $X_{\alpha,d}Z(*, l, r, val, d)$ and b some R-successor of $(*, l, r, val, d)$ in $\mathfrak{B}$. By (P1), b is an R-successor x' of some (actually: all) $x \in X_{\alpha,d}$. By definition of Z, we have $\{b\}Zb$.

"⇐" Let Z be a Horn simulation with $X_{\alpha_0,\swarrow}Z\hat{b}$. We show that M accepts w by constructing the (relevant subset of the) AND-OR graph induced by M's computation on input w. We first show the following for all $X_{\alpha,d}Zb$:

1) α is an accepting (rejecting) configuration iff b is an accepting (rejecting) element in $\mathfrak{B}$.
2) if α is not final, then also $X_{\alpha_l,\swarrow}Zb_l$ and $X_{\alpha_r,\searrow}Zb_r$ for some b_l and b_r, where α_l, α_r are the left and right successor configurations of α.

It follows from the definition of $R^{\mathfrak{A}}_{id}$ and $R^{\mathfrak{B}}_{id}$ and the fact that Z satisfies Conditions *(forth$_h$)* and *(back$_h$)*, that α is a final configuration iff b is a final element. Property 1) then follows from Condition *(sim)* and the definition of $U^{\mathfrak{B}}_{0/1}$ for final elements.

For Property 2), let Y be some set with $X_{\alpha,d}(R^{\mathfrak{A}})^{\uparrow}Y$ for some R. By (P2), we know that R is of shape $R_{q,a,i,d'}$, $d' \in \{\swarrow, \searrow\}$, where q is the state of α, i is its head position, and a is the symbol of the head. Moreover, $Y = X_{\alpha_l,\swarrow}$ or $Y = X_{\alpha_r,\searrow}$ (depending on d'). By Condition *(forth$_h$)*, there is a non-empty $Y' \subseteq Y$ and some b' with $(b, b') \in R^{\mathfrak{B}}$ such that $Y'Zb'$. By Condition *(atom$_h$)* and the definition of $V^{\mathfrak{A}}_i$ and $V^{\mathfrak{B}}_i$, we have $Y' = Y$.

Now, $X_{\alpha_0,\swarrow}Z\hat{b}$ and Property 2) imply that for every configuration α reachable from α_0 we have $X_\alpha Zb$ for some b. We claim that in this case the value val of b is the value associated to the configuration α in the AND-OR graph. Indeed, for final configurations this is a consequence of Property 1) above. For the remaining configurations, this is a consequence of the definition of $R^{\mathfrak{B}}_{q,a,i,d}$. It remains to note that this implies that α_0 is associated with value 1 since the value val in $\hat{b}$ is 1. This finishes the proof of the Claim.

Since the construction of $\mathfrak{A}$ and $\mathfrak{B}$ can be carried out in polynomial time, this establishes EXPTIME-hardness of HornSim. ☐

For the restricted problems ℓ-entailment, ℓ-equivalence, and ℓ-HornSim, we prove the following Lemma analogously to Lemma 4.

Lemma 6. *There are the following reductions:*

(1) *$\overline{\ell\text{-CBE}} \leq^P_T \ell$-HornSim;*
(2) *$\overline{\ell\text{-HornSim}} \leq^P_m \ell$-CBE;*
(3) *ℓ-HornSim $\leq^P_m$ ℓ-Entailment;*
(4) *ℓ-Entailment $\leq^P_m$ ℓ-HornSim;*
(5) *ℓ-Equivalence $\leq^P_T$ ℓ-Entailment;*
(6) *ℓ-Entailment $\leq^P_m$ $(\ell + 1)$-Equivalence.*

Thus, it suffices to establish the following lemma for the complexity of ℓ-HornSim to finish the proof of Theorem 5.

Lemma 7. *ℓ-HornSim is* PSPACE*-complete for unary encoding of ℓ and* EXPTIME*-complete for binary encoding.*

Proof. For the upper bounds, observe that we can use the alternating algorithm given in the proof of Lemma 5 and run it for ℓ rounds. It remains to observe that it is an alternating, polynomially time bounded procedure in case of unary encoding, and an alternating, polynomially space bounded procedure in case of binary encoding. The PSPACE and EXPTIME upper bounds follow.

For the EXPTIME-lower bound, we take the same reduction as in the proof of Theorem 5, but add an input ℓ specifying the maximum time until a final state is reached. It is well-known that we can assume without loss of generality that this happens after $2^{O(s(n))}$ steps. Binary encoding of ℓ yields the result.

The PSPACE lower bound follows a similar strategy. We reduce the word problem for polynomially *time* (instead of space) bounded ATMs. There is a fixed such ATM M with a PSPACE-hard word problem; let $p(n)$ be the polynomial bound on the time of M. The reduction is now as in Lemma 5 except that we replace $s(n)$ with $p(n)$, and on input w, we additionally set $\ell = p(|w|)$. □

We finish this section with TBox distinguishability. Let us denote with *TBox entailment* and *TBox equivalence*, the problems whether $\mathfrak{A} \leq_{\mathit{horn}\mathcal{ALC}} \mathfrak{B}$ and $\mathfrak{A} \equiv_{\mathit{horn}\mathcal{ALC}} \mathfrak{B}$, respectively, for given structures $\mathfrak{A}, \mathfrak{B}$. Moreover, denote with *GHornSim* the problem whether $\mathfrak{A} \preceq_{\mathit{horn}} \mathfrak{B}$. Finally, we refer with TBox ℓ-entailment and TBox ℓ-equivalence to the restricted versions.

Theorem 14. *TBox entailment and equivalence are* EXPTIME-*complete. Moreover, TBox ℓ-entailment and TBox ℓ-equivalence are* EXPTIME-*complete for binary encoding of ℓ and* PSPACE-*complete for unary encoding.*

As we have the following relationships, it suffices to prove Lemma 9 below.

Lemma 8. (1) *TBox entailment is equivalent to GHornSim;*
(2) *TBox ℓ-entailment is equivalent to ℓ-GHornSim;*
(3) *TBox equivalence $\leq^P_T$ TBox entailment;*
(4) *TBox ℓ-equivalence $\leq^P_T$ TBox ℓ-entailment;*
(5) *TBox entailment $\leq^P_m$ TBox equivalence.*
(6) *TBox ℓ-entailment $\leq^P_m$ TBox ℓ-equivalence.*

Proof. Properties (1) and (2) are a consequence of Theorem 4. Properties (3) and (4) is due to the fact that equivalence is just mutual entailment. Properties (5) and (6) follow from the fact that $\mathfrak{A} \leq^{(\ell)}_{\mathit{horn}\mathcal{ALC}} \mathfrak{B}$ iff $\mathfrak{A} \cup \mathfrak{B} \equiv^{(\ell)}_{\mathit{horn}\mathcal{ALC}} \mathfrak{B}$, for the disjoint union $\mathfrak{A} \cup \mathfrak{B}$ of $\mathfrak{A}$ and $\mathfrak{B}$. □

Lemma 9. *GHornSim is* EXPTIME-*complete, and ℓ-GHornSim is* EXPTIME-*complete if ℓ is given in binary, and* PSPACE-*complete if ℓ is given in unary.*

Proof. The upper bound follows from the definition of $\mathfrak{A} \preceq_{\mathit{horn}} \mathfrak{B}$ and the complexity of Horn simulations and restricted Horn simulations, see Lemma 5 and 7, respectively.

For the EXPTIME-lower bound for GHornSim, observe that the proof of Lemma 5 is easily adapted so as to show EXPTIME-hardness also in this case. Indeed, we can mark $\hat{b}$ and all elements in X_{α_0} with a fresh concept name S^*. Now, for every element $d \in \mathrm{dom}(\mathfrak{B}) \setminus \{\hat{b}\}$ there is a copy d' of d in $\mathfrak{A}$, for which trivially $\mathfrak{A}, d' \leq \mathfrak{B}, d$. It remains to note that, by construction, the only candidate set for $\hat{b}$ is in fact X_{α_0}.

Finally, EXPTIME and PSPACE-hardness for the restricted problem is obtained from the above result in the same way as in Lemma 7. □

PROOFS FOR SECTION: EXPRESSIVE COMPLETENESS FOR *horn*$\mathcal{ALC}$

We provide proofs for the implications (3) ⇒ (4) of Theorems 6 and 7.

Lemma 10. *If an $\mathcal{ALC}$-concept C of depth $\leq \ell$ is preserved under ℓ-Horn simulations, then C is equivalent to a horn$\mathcal{ALC}$-concept of depth $\leq \ell$ (also in the setting of finite model theory).*

Proof. Let C be an $\mathcal{ALC}$ concept of depth $\leq \ell$ that is preserved under ℓ-Horn simulations. We use the set Horn_ℓ of concepts and the concept $\rho_{\mathfrak{B},\ell,b}$ defined in the proof of Lemma 3. Let D be the conjunction of all $H \in \mathrm{Horn}_\ell$ with $\emptyset \models C \sqsubseteq H$. It suffices to show that $\emptyset \models D \sqsubseteq C$ establishing that D and C are equivalent. To this aim, observe that D is equivalent to $\rho_{\mathfrak{A}_u,\ell,X}$, where $\mathfrak{A}_u$ is the disjoint union of all finite structures (up to isomorphisms) and $X = C^{\mathfrak{A}_u}$. To see this, suppose first that $H \in \mathrm{Horn}_\ell$ and $\emptyset \models C \sqsubseteq H$. Then $X = C^{\mathfrak{A}} \subseteq H^{\mathfrak{A}}$ for all structures $\mathfrak{A}$. Thus, H is a conjunct of $\rho_{\mathfrak{A}_u,\ell,X}$. Conversely, suppose H is a conjunct of $\rho_{\mathfrak{A}_u,\ell,X}$. Then $X \subseteq H^{\mathfrak{A}_u}$. Thus, $C^{\mathfrak{A}_u} \subseteq H^{\mathfrak{A}_u}$. But then $\emptyset \models C \sqsubseteq H$ since the latter is equivalent to the condition that $C^{\mathfrak{A}} \subseteq H^{\mathfrak{A}}$ for all *finite* structures $\mathfrak{A}$ and $\mathcal{ALC}$ has the finite model property.

Now suppose $\mathfrak{B}, b$ is a pointed structure with $b \in D^{\mathfrak{B}}$. Then $b \in \rho^{\mathfrak{B}}_{\mathfrak{A}_u,\ell,X}$. By Lemma 3, there exists $X_0 \subseteq X$ such that $\mathfrak{A}_u, X_0 \preceq^{\ell}_{\mathit{horn}} \mathfrak{B}, b$. Then $b \in C^{\mathfrak{B}}$ follows from the assumption that C is preserved under ℓ-Horn simulations. The finite model theory version follows directly using the finite model property of $\mathcal{ALC}$. □

Lemma 11. *Let $\mathcal{T}$ be an $\mathcal{ALC}$ TBox of depth ℓ preserved under global ℓ-Horn simulations. Then $\mathcal{T}$ is equivalent to a horn$\mathcal{ALC}$ TBox of depth $\leq \ell$ (also in the setting of finite model theory).*

Proof. Let $\mathcal{T}'$ be the set of *horn*$\mathcal{ALC}$ CIs $\top \sqsubseteq H$ with $H \in \mathrm{Horn}_\ell$ such that $\mathcal{T} \models \top \sqsubseteq H$. We show that $\mathcal{T}' \models \mathcal{T}$. Take a model $\mathfrak{B}$ of $\mathcal{T}'$. Take for every *horn*$\mathcal{ALC}$ CI $\top \sqsubseteq H$ of depth $\leq \ell$ and $b \in \mathrm{dom}(\mathfrak{B}) \setminus H^{\mathfrak{B}}$ a model $\mathfrak{A}_{b,H}$ of $\mathcal{T}$ and $a_{b,H} \in \mathrm{dom}(\mathfrak{A}_{b,H})$ with $a_{b,H} \notin H^{\mathfrak{A}_{b,H}}$. Such a model exists since $\mathcal{T}' \not\models \top \to H$ and so, by the definition of $\mathcal{T}'$, $\mathcal{T} \not\models \top \to H$.

Let $\mathfrak{A}$ be the disjoint union of all $\mathfrak{A}_{b,H}$ and let X_b be the set of all $a_{b,H}$. Then $X_b \subseteq H^{\mathfrak{A}}$ implies $b \in H^{\mathfrak{B}}$, for every *horn*$\mathcal{ALC}$-concept H of depth $\leq \ell$. Thus, by Lemma 3, there exists $Y_b \subseteq X_b$ such that $\mathfrak{A}, Y_b \preceq^{\ell}_{\mathrm{horn}} \mathfrak{B}, b$. Then $\mathfrak{B}$ is model of $\mathcal{T}$ since $\mathfrak{A}$ is a model of $\mathcal{T}$ and $\mathcal{T}$ is preserved under global ℓ-Horn simulations.

In the finite model theory setting it suffices to consider finite models $\mathfrak{B}$ of $\mathcal{T}'$ since $\mathcal{ALC}$ has the finite model property for TBox reasoning. For the same reason one can always choose finite $\mathfrak{A}_{b,H}$. □

PROOFS FOR SECTION ON GUARDED FRAGMENT

A. Basic Properties of hornGF

Theorem 15. *Satisfiability in hornGF$[\tau]$ is* EXPTIME-*complete if the arity of predicate names in τ is bounded, and* 2-EXPTIME-*complete if not.*

Proof. The upper bounds follow from the guarded fragment GF [39]. The lower bound in the bounded arity case is inherited from *horn*$\mathcal{ALC}$ [18]. For the case of unbounded arity, the lower bound is obtained as a straightforward adaptation of the 2-EXPTIME lower bound for GF in [39]. □

Let us fix a vocabulary τ. A *tuple-generating dependency (tgd) over* τ is an FO formula of the form

$$\forall \mathbf{x} \forall \mathbf{y}\, (\psi(\mathbf{x}, \mathbf{y}) \rightarrow \exists \mathbf{z}\, \varphi(\mathbf{x}, \mathbf{z}))$$

where $\psi(\mathbf{x}, \mathbf{y})$ and $\varphi(\mathbf{x}, \mathbf{z})$ are conjunctions of atoms over τ and are called *body* and *head* of the tgd, respectively. A tgd is *guarded* if there is an atom in the body $\psi(\mathbf{x}, \mathbf{y})$ which contains all variables $\mathbf{x} \cup \mathbf{y}$. We allow as special cases empty body and head and denote this as $\top \rightarrow \exists \mathbf{z}\, \varphi(\mathbf{z})$ and $\forall \mathbf{x} \forall \mathbf{y}\, \psi(\mathbf{x}, \mathbf{y}) \rightarrow \bot$, respectively.

The following theorem captures the relation between *hornGF* and guarded tgds; essentially, satisfiability and query answering can be interreduced in polynomial time.

Lemma 12. *For every set* Σ *of guarded tgds over some vocabulary* τ*, we can compute in polynomial time a hornGF formula* φ_Σ *over a larger vocabulary* $\tau' \supseteq \tau$ *such that* Σ *and* φ_Σ *are equisatisfiable and, for every conjunctive query* q *and database* D *over* τ*, we have* $\Sigma \cup D \models q$ *iff* $\{\varphi_\Sigma\} \cup D \models q$*.*

Conversely, every hornGF formula φ *over* τ *can be translated in polynomial time into a set of guarded tgds* Σ_φ *over a larger vocabulary* $\tau' \supseteq \tau$ *such that* φ *and* Σ_φ *are equisatisfiable and, for every conjunctive query* q *and database* D *over* τ*, we have* $\{\varphi\} \cup D \models q$ *iff* $\Sigma_\varphi \cup D \models q$*.*

Proof. The first statement follows from the arguments in [42]. We repeat the idea for the sake of completeness. Every guarded tgd $\forall \mathbf{x} \forall \mathbf{y}\, (\psi(\mathbf{x}, \mathbf{y}) \rightarrow \exists \mathbf{z}\, \varphi(\mathbf{x}, \mathbf{z}))$ in Σ is replaced by two guarded tgds

$$\forall \mathbf{x} \forall \mathbf{y}\, (\psi(\mathbf{x}, \mathbf{y}) \rightarrow \exists \mathbf{z}\, R(\mathbf{x}, \mathbf{z})),$$
$$\forall \mathbf{x} \forall \mathbf{z}\, (R(\mathbf{x}, \mathbf{z}) \rightarrow \varphi(\mathbf{x}, \mathbf{z})),$$

for a fresh predicate name R of appropriate arity. Obviously, the resulting set is a set of *hornGF* sentences; their conjunction satisfies the requirements of the statement.

Conversely, we give a translation of *hornGF* sentences into guarded tgds which can be used to reduce satisfiability and query answering. Let φ_0 be a *hornGF* sentence. Let us denote with $\varphi[\mathbf{y}]$ the fact that a formula φ has precisely free variables $\mathbf{y}$. Further, if φ is a subformula of φ_0, the *guard of* φ *in* φ_0 is the atom $G(\mathbf{xy})$ that guards all free variables of φ; in case such an atom does not exist, we assume that the guard is $\top$. Introduce a fresh predicate name R_φ (of matching arity) for every complex subformula φ of φ_0, and set $R_\alpha = \alpha$ whenever α is $\bot$, $\top$, or an atomic formula.

Now, let Σ_0 be the set consisting of the following sentences:

- $\top \rightarrow R_{\varphi_0}$,
- for every subformula $\psi[\mathbf{x}] = \varphi[\mathbf{y}] \wedge \varphi'[\mathbf{y}']$ of φ_0, include

$$\forall \mathbf{x}\, (R_\psi(\mathbf{x}) \rightarrow R_\varphi(\mathbf{y})), \text{ and}$$
$$\forall \mathbf{x}\, (R_\psi(\mathbf{x}) \rightarrow R_{\varphi'}(\mathbf{y}')),$$

- for every subformula $\psi[\mathbf{x}] = \exists \mathbf{y} (G(\mathbf{x}, \mathbf{y}) \wedge \varphi[\mathbf{z}])$ of φ_0, include

$$\forall \mathbf{x}\, (R_\psi(\mathbf{x}) \rightarrow \exists \mathbf{y}\, R(\mathbf{x}, \mathbf{y})),$$
$$\forall \mathbf{x} \forall \mathbf{y}\, (R(\mathbf{x}, \mathbf{y}) \rightarrow G(\mathbf{x}, \mathbf{y})), \text{ and}$$
$$\forall \mathbf{x} \forall \mathbf{y}\, (R(\mathbf{x}, \mathbf{y}) \rightarrow R_\varphi(\mathbf{z})),$$

  where R is a fresh predicate name of appropriate arity,
- for every subformula $\psi[\mathbf{x}] = \forall \mathbf{y} (G(\mathbf{xy}) \rightarrow \varphi[\mathbf{z}])$ of φ_0, include

$$\forall \mathbf{x} \forall \mathbf{y}\, (G(\mathbf{xy}) \wedge R_\psi(\mathbf{x}) \rightarrow R_\varphi(\mathbf{z})),$$

- for every subformula $\psi[\mathbf{x}] = \lambda[\mathbf{y}] \rightarrow \varphi[\mathbf{y}']$ of φ_0, include

$$\forall \mathbf{x}\, (R_\psi(\mathbf{x}) \wedge R_\lambda(\mathbf{y}) \rightarrow R_\varphi(\mathbf{y}')).$$

  Moreover, for every subformula $\lambda'[\mathbf{v}] = \theta[\mathbf{z}] \wedge \theta'[\mathbf{z}']$ of λ with guard $G(\mathbf{z}_0)$, include

$$\forall \mathbf{x}\, (G(\mathbf{z}_0) \wedge R_\theta(\mathbf{z}) \wedge R_{\theta'}(\mathbf{z}') \rightarrow R_{\lambda'}(\mathbf{v})),$$

  for every subformula $\lambda'[\mathbf{v}] = \theta[\mathbf{z}] \vee \theta'[\mathbf{z}']$ of λ with guard $G(\mathbf{z}_0)$, include

$$\forall \mathbf{x}\, (G(\mathbf{z}_0) \wedge R_\theta(\mathbf{z}) \rightarrow R_{\lambda'}(\mathbf{v})), \text{ and}$$
$$\forall \mathbf{x}\, (G(\mathbf{z}_0) \wedge R_{\theta'}(\mathbf{z}') \rightarrow R_{\lambda'}(\mathbf{v})),$$

  for every subformula $\lambda'[\mathbf{v}] = \exists \mathbf{z}\, H(\mathbf{v}, \mathbf{z}) \wedge \theta[\mathbf{z}']$ of λ with guard $G(\mathbf{z}_0)$, include

$$\forall \mathbf{v} \forall \mathbf{z}\, (H(\mathbf{v}, \mathbf{z}) \wedge R_\theta(\mathbf{z}') \rightarrow R(\mathbf{v})), \text{ and}$$
$$\forall \mathbf{z}_0\, (G(\mathbf{z}_0) \wedge R(\mathbf{v}) \rightarrow R_{\lambda'}(\mathbf{v})),$$

  for some fresh predicate name R of appropriate arity.

It is routine to verify that

(†) Σ_0 is satisfiable iff φ_0 is satisfiable and query answering relative to φ_0 is the same as query answering relative to Σ_0 (over databases in the vocabulary of φ_0).

Notice that the sentences in Σ_0 are guarded tgds whenever, φ_0 is equality-free, that is, it does not contain atoms of the form $x = y$. Obtain Σ_1 from Σ_0 by removing all occurrences of $x = y$ on the left-hand side of some rule and, in such a case, replacing all occurrences of y with x in the rule. Obviously, Σ_1 still satisfies (†). To remove equality atoms from the right-hand side of the rules in Σ_1, we observe first that there is only one atom on the right-hand side of the rules in Σ_1. Obtain a set of rules Σ_2 by replacing every $x = y$ on the right-hand side by $E(x, y)$, for a new predicate name E, and adding the following guarded tgds, for every predicate name R appearing in Σ_1, $\mathbf{x} = x_1, \ldots, x_{r_R}$, and every i, j, k with $1 \leq i < j < k \leq r_R$:

$$\forall \mathbf{x}\, (R(\mathbf{x}) \wedge E(x_i, x_j) \rightarrow E(x_j, x_i)),$$
$$\forall \mathbf{x}\, (R(\mathbf{x}) \wedge E(x_i, x_j) \wedge E(x_j, x_k) \rightarrow E(x_i, x_k)),$$
$$\forall \mathbf{x}\, (R(\mathbf{x}) \wedge E(x_i, x_j) \rightarrow R(\mathbf{x}[x_i/x_j]) \wedge R(\mathbf{x}[x_j/x_i])).$$

Intuitively, these guarded tgds enforce that E behaves like a congruence relation under each possible guard R. It is routine to verify that φ_0 is satisfiable iff Σ_2 is satisfiable, and that query answering relative to φ_0 is the same as query answering relative to Σ_2.[3] $\square$

Consequently, we have:

Theorem 16. *Query answering in hornGF is in* PTIME *data complexity.*

B. Ehrenfeucht-Fraïssé games

The proof of Theorem 9 is standard and omitted. It is useful to define an analogue of the formula $\lambda_{\mathfrak{A},\ell,a}$ (in $\mathcal{ELU}$) for $\mathrm{GF}^\exists$. For a finite vocabulary τ, $\ell < \omega$, and variables $\mathbf{x} = x_1 \dots x_m$ we fix a finite set $\mathrm{GF}^\exists_\ell(\tau, m)$ of formulas in $\mathrm{GF}^\exists[\tau]$ of depth $\leq \ell$ and free variables among $x_1, \dots, x_m$ such that for every $\mathrm{GF}^\exists[\tau]$ formula λ of depth $\leq \ell$ and free variables among $x_1, \dots, x_m$ there exists a formula λ' in $\mathrm{GF}^\exists_\ell(\tau, m)$ such that λ and λ' are equivalent. For a structure $\mathfrak{A}$ and tuple $\mathbf{a}$ of length m we denote by $\lambda_{\mathfrak{A},\ell,\mathbf{a}}$ the conjunction of all formulas λ in $\mathrm{GF}^\exists_\ell(\tau, m)$ such that $\mathfrak{A} \models \lambda(\mathbf{a})$. We then have for all pointed structures $\mathfrak{A}, \mathbf{a}$ and $\mathfrak{B}, \mathbf{b}$ and $\ell < \omega$:

$$\mathfrak{B} \models \lambda_{\mathfrak{A},\ell,\mathbf{a}}(\mathbf{b}) \quad \text{iff} \quad \mathfrak{A}, \mathbf{a} \preceq^\ell_{\mathit{gsim}} \mathfrak{B}, \mathbf{b}. \tag{3}$$

We now come to the proof of Theorem 10. As announced, we are going to prove a stronger version in which the tuple $\mathbf{a}$ is replaced by a set X of tuples.

Lemma 13. *For any finite vocabulary τ, pointed τ-structures $\mathfrak{A}, X$ and $\mathfrak{B}, \mathbf{b}$, and any $\ell < \omega$, we have*

$$\mathfrak{A}, X \leq^\ell_{\mathit{hornGF}} \mathfrak{B}, \mathbf{b} \quad \textit{iff} \quad \exists X_0 \subseteq X : \mathfrak{A}, X_0 \preceq^\ell_{\mathit{ghsim}} \mathfrak{B}, \mathbf{b}$$

If $\mathfrak{A}$ and $\mathfrak{B}$ are finite, then

$$\mathfrak{A}, X \leq_{\mathit{hornGF}} \mathfrak{B}, \mathbf{b} \quad \textit{iff} \quad \exists X_0 \subseteq X : \mathfrak{A}, X_0 \preceq_{\mathit{ghsim}} \mathfrak{B}, \mathbf{b}$$

Proof. We first introduce an analogue of the formula $\rho_{\mathfrak{A},\ell,X}$ (for *horn*$\mathcal{ALC}$) in *hornGF*. For a finite vocabulary τ, $\ell < \omega$, and variables $\mathbf{x} = x_1 \dots x_m$ we fix a finite set $\mathit{hornGF}_\ell(m)$ of formulas in *hornGF* of depth $\leq \ell$ and free variables among $x_1, \dots, x_m$ such that for every *hornGF* formula φ of depth $\leq \ell$ and free variables among $x_1, \dots, x_m$ there exists a formula φ' in $\mathit{hornGF}_\ell(m)$ such that φ and φ' are equivalent. For a structure $\mathfrak{A}$ and set X of tuples of the same length m we denote by $\rho_{\mathfrak{A},\ell,X}$ the conjunction of all formulas φ in $\mathit{hornGF}_\ell(m)$ such that $\mathfrak{A} \models \varphi(\mathbf{a})$ for all $\mathbf{a} \in X$. In what follows, when using the formulas $\lambda_{\mathfrak{A},\ell,\mathbf{a}}$ and $\rho_{\mathfrak{A},\ell,X}$, the number m of variables used will always be clear from the context and not be mentioned explicitly.

As the second claim follows directly from the first claim, we prove the first one only.

For the proof of the implication from left to right, let the pointed structures $\mathfrak{A}, X_0$ and $\mathfrak{B}$, $\mathbf{b}_0$ be given with X_0 and $\mathbf{b}_0$ guarded tuples of the same length. Assume that $\mathfrak{A}, X_0 \leq^\ell_{\mathit{hornGF}} \mathfrak{B}, \mathbf{b}_0$. We show that there exists $X_0' \subseteq X_0$ such that $\mathfrak{A}, X_0' \preceq^\ell_{\mathit{ghsim}} \mathfrak{B}, \mathbf{b}_0$.

[3]The latter equivalence only holds in the case without the unique name assumption (UNA). With UNA, the equivalence holds under the additional condition that $\Sigma_2 \cup D \not\models E(a, b)$ for any distinct $a, b \in \mathsf{dom}(D)$.

Define sets Z_k of pairs $(X, \mathbf{b})$ by setting $(X, \mathbf{b}) \in Z_k$ if $X \neq \emptyset$ is a set of guarded tuples in $\mathfrak{A}$ and $\mathbf{b}$ is a guarded tuple in $\mathfrak{B}$ of the same length as the tuples in X such that

(a) $\mathfrak{B} \models \rho_{\mathfrak{A},k,X}(\mathbf{b})$;
(b) $\mathfrak{A} \models \lambda_{\mathfrak{B},k,\mathbf{b}}(\mathbf{a})$ for all $\mathbf{a} \in X$.

Observe that if $(X, \mathbf{b}) \in Z_0$, then $(P, \mathbf{b})$ with

$$P = \{p_\mathbf{a} : \mathbf{b} \mapsto \mathbf{a} \mid \mathbf{a} \in X\}$$

is a well-defined link (as $\mathfrak{A} \models \lambda(\mathbf{a})$ implies $\mathfrak{B} \models \lambda(\mathbf{b})$ for all $\mathbf{a} \in X$ and $\lambda(\mathbf{x})$ in $\mathrm{GF}^\exists_0$). In what follows we do not distinguish between $(X, \mathbf{b})$ and the corresponding link. We show the following claim by induction over $k < \omega$.

Claim 1. For all $k < \omega$, non-empty sets X of tuples in $\mathfrak{A}$, and tuples $\mathbf{b}$ in $\mathfrak{B}$: if $(X, \mathbf{b}) \in Z_k$, then $\mathfrak{A}, X \preceq^k_{\mathit{ghsim}} \mathfrak{B}, \mathbf{b}$.

For $k = 0$, Claim 1 is proved as follows. Assume $(X, \mathbf{b}) \in Z_0$ and let $(P, \mathbf{b})$ be the corresponding link. For Condition (atom^g_h), observe that if $\mathfrak{A} \models p(b_i) = p(b_j)$ for all $p \in P$, then $\mathfrak{B} \models b_i = b_j$ since $\mathfrak{B} \models \rho_{\mathfrak{A},0,P[\mathbf{b}]}(\mathbf{b})$ and $(x = y)$ are in hornGF_0. Similarly, if $R \in \tau$ and $\mathfrak{A} \models R(p(\mathbf{b}))$ for all $p \in P$, then $\mathfrak{B} \models R(\mathbf{b})$ since $\mathfrak{B} \models \rho_{\mathfrak{A},0,P[\mathbf{b}]}(\mathbf{b})$ and $R(\mathbf{x})$ is in hornGF_0. Condition (sim^g_h) follows from Theorem 9.

Assume Claim 1 has been proved for k and let $(X, \mathbf{b}) \in Z_{k+1}$. Denote by $(P, \mathbf{b})$ the corresponding link. Condition (atom^g_h) can be proved in the same way as for $k = 0$. For Condition (forth^g_h), assume Y is an $R(\mathbf{b}_0, \mathbf{y})$-successor of $(P, \mathbf{b})$. We have to construct a tuple $\mathbf{b}'$ and set of tuples $Y' \subseteq Y$ with $(Y', \mathbf{b}_0\mathbf{b}') \in Z_k$. By definition,

$$\mathfrak{A} \models \exists \mathbf{y} \big(R(p(\mathbf{b}_0)\mathbf{y}) \wedge \rho_{\mathfrak{A},k,Y}(p(\mathbf{b}_0)\mathbf{y})\big),$$

for all $p \in P$. By Condition (a) and since $\exists \mathbf{y}(R(\mathbf{x}_0\mathbf{y}) \wedge \rho_{\mathfrak{A},k,Y}(\mathbf{x}_0\mathbf{y}))$ is of depth $\leq k + 1$,

$$\mathfrak{B} \models \exists \mathbf{y} \big(R(\mathbf{b}_0\mathbf{y}) \wedge \rho_{\mathfrak{A},k,Y}(\mathbf{b}_0\mathbf{y})\big).$$

Thus, there exists a tuple $\mathbf{b}'$ such that

$$\mathfrak{B} \models R(\mathbf{b}_0\mathbf{b}') \wedge \rho_{\mathfrak{A},k,Y}(\mathbf{b}_0\mathbf{b}').$$

Let

$$Y' = \{\mathbf{a} \in Y \mid \mathfrak{A} \models \lambda_{\mathfrak{B},k,\mathbf{b}_0\mathbf{b}'}(\mathbf{a})\}$$

To show that $(Y', \mathbf{b}_0\mathbf{b}') \in Z_k$ it suffices to show that Y' is non-empty and satisfies Condition (a) (Condition (b) holds by definition).

- Assume $Y' = \emptyset$. Then $\mathfrak{A} \not\models \lambda_{\mathfrak{B},k,\mathbf{b}_0\mathbf{b}'}(\mathbf{a})$ for any $\mathbf{a} \in Y$. Then $(\lambda_{\mathfrak{B},k,\mathbf{b}_0\mathbf{b}'} \rightarrow \bot)$ is equivalent to a conjunct of $\rho_{\mathfrak{A},k,Y}$. Then

  $$\mathfrak{B} \models (\lambda_{\mathfrak{B},k,\mathbf{b}_0\mathbf{b}'}(\mathbf{b}_0\mathbf{b}') \rightarrow \bot)$$

  by construction of $\mathbf{b}_0\mathbf{b}'$ and we have derived a contradiction.
- For Condition (a), assume that ψ is in *hornGF* and of depth $\leq k$ and $\mathfrak{A} \models \psi(\mathbf{a})$ for all $\mathbf{a} \in Y'$. We have to show that $\mathfrak{B} \models \psi(\mathbf{b}_0\mathbf{b}')$. We have

  $$\mathfrak{A} \models (\lambda_{\mathfrak{B},k,\mathbf{b}_0\mathbf{b}'} \rightarrow \psi)(\mathbf{a}),$$

for all $\mathbf{a} \in Y$. Then $(\lambda_{\mathfrak{B},k,\mathbf{b}_0\mathbf{b}'} \to \psi)$ is a conjunct of $\rho_{\mathfrak{A},k,Y}$ and so

$$\mathfrak{B} \models (\lambda_{\mathfrak{B},k,\mathbf{b}_0\mathbf{b}'} \to \psi)(\mathbf{b}_0\mathbf{b}').$$

From

$$\mathfrak{B} \models \lambda_{\mathfrak{B},k,\mathbf{b}_0\mathbf{b}'}(\mathbf{b}_0\mathbf{b}')$$

we obtain

$$\mathfrak{B} \models \psi(\mathbf{b}_0\mathbf{b}'),$$

as required.

To show Condition (*back*^g_h), assume that a guarded tuple $\mathbf{b}'$ is given. Consider w.l.o.g. an atomic formula $R(\mathbf{x}')$ with $\mathfrak{B} \models R(\mathbf{b}')$. Let $\mathbf{b}' = \mathbf{b}_0\mathbf{b}_1$ where $\mathbf{b}_0$ and $\mathbf{b}_1$ are such that $[\mathbf{b}_0] \subseteq [\mathbf{b}]$ and $[\mathbf{b}_1] \cap [\mathbf{b}] = \emptyset$. Take for every $\psi(\mathbf{x}_0\mathbf{x}_1)$ in *hornGF*$_k$ with $\mathfrak{B} \models \neg\psi(\mathbf{b}_0\mathbf{b}_1)$ a $p_\psi \in P$ and tuple $\mathbf{a}_\psi$ such that $\mathfrak{A} \models (\lambda_{\mathfrak{A},k,\mathbf{b}'} \wedge \neg\psi)(p_\psi(\mathbf{b}_0), \mathbf{a}_\psi)$. They exist because otherwise

$$\mathfrak{A} \models \forall \mathbf{x}_1(R(\mathbf{x}_0\mathbf{x}_1) \to (\lambda_{\mathfrak{A},k,\mathbf{b}'} \to \psi))(p(\mathbf{b}_0), \mathbf{x}_1)$$

for all $p \in P$ and so, by definition of Z_{k+1},:

$$\mathfrak{B} \models \forall \mathbf{x}_1(R(\mathbf{x}_0\mathbf{x}_1) \to (\lambda_{\mathfrak{A},k,\mathbf{b}'} \to \psi))(\mathbf{b}_0, \mathbf{x}_1))$$

and we have derived a contradiction. Now let $(P', \mathbf{b}')$ consist of all $p' : \mathbf{b}' \mapsto p_\psi(\mathbf{b}_0)\mathbf{a}_\psi$ with $p'|_{[\mathbf{b}_0]} = p|_{[\mathbf{b}_0]}$. Then $(P', \mathbf{b}')$ is as required for Condition (*back*^g_h).

Condition (*sim*^g_h) follows from Theorem 9 and so Claim 1 is proved.

It remains to prove that there exists $X'_0 \subseteq X_0$ with $(X'_0, \mathbf{b}_0) \in Z_\ell$. Set

$$X'_0 = \{\mathbf{a} \in X_0 \mid \mathfrak{A} \models \lambda_{\mathfrak{B},\ell,\mathbf{b}_0}(\mathbf{a})\}$$

Then $(X'_0, \mathbf{b}_0)$ satisfies Conditions (a) and (b) by definition. $X'_0 \neq \emptyset$ can be proved in the same way as the proof of $Y' \neq \emptyset$ given above.

For the proof of the implication from right to left, we show the following claim by induction over $k < \omega$.

Claim 2. For all $k < \omega$, if $(P, \mathbf{b}) \in Z_k$, then $\mathfrak{A} \models \varphi(p(\mathbf{b}))$ for all $p \in P$ implies $\mathfrak{B} \models \varphi(\mathbf{b})$, for all formulas φ in *hornGF* of depth $\leq k$.

For $k = 0$, Claim 2 follows from Condition (*atom*^g_h). Assume Claim 2 has been proved for k. We prove Claim 2 for $k+1$ by induction over the construction of φ. Thus, assume that Claim 2 has been proved for $\varphi', \varphi_1, \varphi_2 \in$ *hornGF*$_k$, and that $\varphi \in$ *hornGF*$_{k+1}$ is of the form $\varphi = \forall \mathbf{y}(G(\mathbf{xy}) \to \varphi'(\mathbf{xy}))$, $\varphi = \exists \mathbf{y}(G(\mathbf{xy}) \wedge \varphi'(\mathbf{xy}))$, $\varphi = \varphi_1 \wedge \varphi_2$, or $\varphi = \lambda \to \varphi'$, where λ is in $\mathrm{GF}^{\exists}_{k+1}$. Then we prove Claim 2 for φ. Assume $(P, \mathbf{b}) \in Z_{k+1}$ and $\mathfrak{A} \models \varphi(p(\mathbf{b}))$ for all $p \in P$.

- Assume $\varphi = \forall \mathbf{y}(G(\mathbf{xy}) \to \varphi'(\mathbf{xy}))$ and for a proof by contradiction that $\mathfrak{B} \not\models \varphi(\mathbf{b})$. Choose $\mathbf{b}'$ with $\mathfrak{B} \models G(\mathbf{bb}') \wedge \neg\varphi'(\mathbf{bb}')$. By Condition (*back*$^g_h$), there exists a link $(P', \mathbf{bb}')$ in Z_k such that for all $p' \in P'$ there exists $p \in P$ with $p|_{[\mathbf{b}]} = p'|_{[\mathbf{b}]}$. φ' has depth $\leq k$ and thus, as we assume that Claim 2 has been proved for k, there exists $p' \in P'$ such that $\mathfrak{A} \not\models \varphi'(p'(\mathbf{bb}'))$. There exists $p \in P$ with $p|_{[\mathbf{b}]} = p'|_{[\mathbf{b}]}$ and so $\mathfrak{A} \not\models \varphi(p(\mathbf{b}))$, and we have derived a contradiction.
- Assume $\varphi = \exists \mathbf{y}(G(\mathbf{xy}) \wedge \varphi'(\mathbf{xy}))$ and for a proof by contradiction that $\mathfrak{B} \not\models \exists \mathbf{y}(G(\mathbf{by}) \wedge \varphi'(\mathbf{by}))$. Let

  $$Y = \{\mathbf{a} \mid \mathfrak{A} \models (G \wedge \varphi')(\mathbf{a})\}$$

  Then Y is a $G(\mathbf{b}, \mathbf{y})$-successor of $(P, \mathbf{b})$ and so by Condition (*forth*^g_h) there exists $(P', \mathbf{bb}') \in Z_k$ such that $P'[\mathbf{bb}'] \subseteq Y$. Thus, by induction hypothesis, $\mathfrak{B} \models (G(\mathbf{bb}') \wedge \varphi'(\mathbf{bb}'))$. But then $\mathfrak{B} \models \varphi(\mathbf{b})$ and we have derived a contradiction.
- Assume $\varphi = \varphi_1 \wedge \varphi_2$ and for a proof by contradiction that $\mathfrak{B} \not\models (\varphi_1 \wedge \varphi_2)(\mathbf{b})$. We may assume w.l.o.g. that $\mathfrak{B} \not\models \varphi_1(\mathbf{b})$. But $\mathfrak{A} \models \varphi(p(\mathbf{b}))$ for all $p \in P$, and so $\mathfrak{A} \models \varphi_1(p(\mathbf{b}))$, for all $p \in P$. By IH, $\mathfrak{B} \models \varphi_1(\mathbf{b})$, and we have derived a contradiction.
- Assume $\varphi = \lambda \to \varphi'$ and for a proof by contradiction $\mathfrak{B} \not\models (\lambda \to \varphi')(\mathbf{b})$. Then $\mathfrak{B} \models \lambda(\mathbf{b})$ and $\mathfrak{B} \not\models \varphi'(\mathbf{b})$. By Theorem 9, $\mathfrak{A} \models \lambda(p(\mathbf{b}))$ for all $p \in P$, and, by IH, there exists $p \in P$ such that $\mathfrak{A} \not\models \varphi'(p(\mathbf{b}))$. Then there exists $p \in P$ with $\mathfrak{A} \not\models \varphi(p(\mathbf{b}))$ and we have derived a contradiction.

This finishes the proof. □

C. Model indistinguishability in hornGF

We introduce decision problems entailment, equivalence, CBE, and GuardedHornSim similar to Section IV. For example, entailment is the problem of deciding $\mathfrak{A}, \mathbf{a} \leq_{\mathit{hornGF}} \mathfrak{B}, \mathbf{b}$ on input $\mathfrak{A}, \mathfrak{B}, \mathbf{a}, \mathbf{b}$, and GuardedHornSim is the following problem:

- **Input**: structures $\mathfrak{A}$, $\mathfrak{B}$ and a set X of guarded tuples in $\mathfrak{A}$, and $\mathbf{b}$ a guarded tuple in $\mathfrak{B}$
- **Question:** Is $\mathfrak{A}, X \preceq_{\mathit{ghsim}} \mathfrak{B}, \mathbf{b}$?

The main theorem here is the following:

Theorem 11. *In hornGF, entailment, equivalence, and CBE are in* ExpTime. *Moreover, ℓ-entailment, ℓ-equivalence, and ℓ-CBE are in* ExpTime *for binary encoding of ℓ and in* PSpace *for unary encoding.*

Proof. Since entailment, equivalence, and CBE can be reduced to GuardedHornSim, it suffices to show that GuardedHornSim can be decided in ExpTime.

To this end, we start with noting that, for a given guarded tuple $\mathbf{b}$, the number of mappings $p : \mathbf{b} \mapsto p(\mathbf{b})$ is bounded by $|\mathfrak{A}|$: since $\mathbf{b}$ is a guarded tuple, there is some $R \in \tau$ and a tuple $\mathbf{a}$ such that $\mathbf{a} \in R^{\mathfrak{B}|_{[\mathbf{b}]}}$ and $[\mathbf{a}] = [\mathbf{b}]$. Every possible mapping p maps this atom to a different atom $R(\mathbf{a}')$ in $\mathfrak{A}$, thus the number of mappings is bounded by $|\mathfrak{A}|$. Hence, the size of a witnessing P with $X = P[\mathbf{b}]$ is bounded by $|\mathfrak{A}|$ as well, and we can try (in exponential time) all possible P.

It thus remains to check on input $\mathfrak{A}, \mathfrak{B}, (P, \mathbf{b})$ whether there is a guarded Horn simulation Z between $\mathfrak{A}$ and $\mathfrak{B}$ with $(P, \mathbf{b}) \in Z$. This can be realized using an alternating algorithm which implements the guarded Horn simulation game. We need an auxiliary notion and claim. Fix structures $\mathfrak{A}$, $\mathfrak{B}$ over some vocabulary τ and a link $(P_0, \mathbf{b}_0)$ between $\mathfrak{A}$ and $\mathfrak{B}$. Denote with $\mathsf{ar}(\tau)$ the maximal arity of symbols in τ. We call a guarded Horn simulation Z *normal for* $(P_0, \mathbf{b}_0)$ if, for every $(P, \mathbf{b}) \in Z$ with $(P, \mathbf{b}) \neq (P_0, \mathbf{b}_0)$, we have $|\mathbf{b}| \leq \mathsf{ar}(\tau)$. The following is routine to show:

Claim. There is a guarded Horn simulation Z with $(P_0, \mathbf{b}_0) \in Z$ iff there is a guarded Horn simulation Z with $(P_0, \mathbf{b}_0) \in Z$ which is normal for $(P_0, \mathbf{b}_0)$.

Based on this claim, we can devise the alternating algorithm. Let $\mathfrak{A}, \mathfrak{B}, (P_0, \mathbf{b}_0)$ be the input. The algorithm maintains links and proceeds in rounds. At a link $(P, \mathbf{b})$ it does the following:

1) Reject if $(P, \mathbf{b})$ does not satisfy Condition ($atom^g_h$).
2) Reject if $(P, \mathbf{b})$ does not satisfy Condition (sim^g_h).
3) For every set of guarded tuples Y and every atomic formula R such that Y is an $R(\mathbf{b}_0, \mathbf{y})$ successor of $(P, \mathbf{b})$:
   a) guess a link $(P', \mathbf{b}_0\mathbf{b})$ with $|\mathbf{b}_0\mathbf{b}| \leq \mathsf{ar}(\tau)$,
   b) reject if $P'[\mathbf{b}_0\mathbf{b}'] \not\subseteq Y$,
   c) replace $(P, \mathbf{b})$ with $(P', \mathbf{b}_0\mathbf{b})$.
4) for every guarded tuple $\mathbf{b}'$ with $|\mathbf{b}'| \leq \mathsf{ar}(\tau)$:
   a) guess a link $(P', \mathbf{b}')$ with $\mathbf{b}' \leq \mathsf{ar}(\tau)$,
   b) reject if for some $p' \in P'$, there is no $p \in P$ with $p|_{[\mathbf{b}]\cap[\mathbf{b}']} = p'|_{[\mathbf{b}\cap[\mathbf{b}']]}$;
   c) replace $(P, \mathbf{b})$ with $(P', \mathbf{b}')$.

The algorithm accepts after $N := 2^{|\mathfrak{A}|} \cdot (|\mathrm{dom}(\mathfrak{B})|^{\mathsf{ar}(\tau)} + 1) + 1$ where $|\mathfrak{A}|$ is length of the representation of $\mathfrak{A}$, that is, roughly the number of ground atoms. Note that steps 1) and 2) correspond to Conditions ($atom^g_h$) and (sim^g_h), respectively, and steps 3) and 4) correspond to ($forth^g_h$) and ($back^g_h$), respectively. Thus, to establish correctness it suffices to prove that after N steps, we know that there is a normal guarded Horn simulation Z with $(P_0, \mathbf{b}_0)$. This is a consequence of the fact that there are at most $N - 1$ links:

- the number of possible tuples $\mathbf{b}$ is bounded by $\mathrm{dom}(\mathfrak{B})^{\mathsf{ar}(\tau)} + 1$, by normality;
- as argued in the beginning of the proof, for a fixed guarded tuple $\mathbf{b}$ the number of possible sets P is bounded by $2^{|\mathfrak{A}|}$.

Thus, after N steps, the algorithm has visited a link twice and can stop. This argument also shows that the size of each link is bounded by some polynomial in the size of the representations of $\mathfrak{A}$ and $\mathfrak{B}$. Moreover, we can count (in binary) up to N in polynomial space. Overall, the algorithm is an alternating PSPACE algorithm which suffices to show an EXPTIME upper bound.

The upper bounds for the restricted problems are obtained in the same way from the alternating algorithm as discussed in Lemma 7 for *horn*$\mathcal{ALC}$. □

D. Expressive Completeness

We remind the reader of the definition of ω-saturated structures. Let τ be a finite vocabulary and assume that $\mathfrak{A}$ is a τ-structure. Then $\mathfrak{A}$ is *ω-saturated* if for all tuples $\mathbf{a}$ in $\mathrm{dom}(\mathfrak{A})$ and all sets $\Gamma(\mathbf{xy})$ of FO[τ]-formulas with $\mathbf{y}$ and $\mathbf{a}$ of the same length, if $\mathfrak{A} \models \exists \mathbf{x}(\bigwedge_{\varphi\in\Gamma'} \varphi(\mathbf{xa}))$ for all finite subsets Γ' of Γ, then there exists a tuple $\mathbf{b}$ in $\mathrm{dom}(\mathfrak{A})$ such that $\mathfrak{A} \models \Gamma(\mathbf{ba})$. Every satisfiable set of FO[τ]-formulas is satisfiable in an ω-saturated structure [52].

Theorem 12 A FO-formula is equivalent to a *hornGF*-formula just in case it is preserved under FO-restricted generalized guarded Horn simulations.

Proof. The direction from left to right is straightforward. Conversely, suppose $\varphi(\mathbf{x}_0)$ is preserved under generalized guarded Horn simulations. Let $\mathrm{cons}(\varphi)$ be the set of all $\psi(\mathbf{x}_0)$ in *hornGF* entailed by $\varphi(\mathbf{x}_0)$. By compactness, it suffices to show $\mathrm{cons}(\varphi) \models \varphi$. Let $\mathfrak{B}$ be an ω-saturated model satisfying $\mathrm{cons}(\varphi)(\mathbf{b}_0)$ for some tuple $\mathbf{b}_0$ in $\mathrm{dom}(\mathfrak{B})$. We show $\mathfrak{B} \models \varphi(\mathbf{b}_0)$. For any tuple $\mathbf{b}$ and tuple $\mathbf{x}$ of variables of the same length as $\mathbf{b}$, we denote by $\lambda_{\mathfrak{B},\mathbf{b}}(\mathbf{x})$ the set of guarded existential positive $\lambda(\mathbf{x})$ with $\mathfrak{B} \models \lambda(\mathbf{b})$. Let $\mathcal{C}$ be the set of all sets $\Gamma(\mathbf{x}_0)$ of FO-formulas with $\mathfrak{B} \models \Gamma(\mathbf{b}_0)$ and such that $\Gamma(\mathbf{x}_0) \cup \{\varphi(\mathbf{x}_0)\}$ is satisfiable and take, for any $\Gamma(\mathbf{x}_0) \in \mathcal{C}$, an ω-saturated structure $\mathfrak{A}_\Gamma$ and tuple $\mathbf{a}_\Gamma$ with $\mathfrak{A}_\Gamma \models (\Gamma \cup \{\varphi\})(\mathbf{a}_\Gamma)$. Let $\mathfrak{A}$ be the disjoint union of $(\mathfrak{A}_\Gamma \mid \Gamma \in \mathcal{C})$ and let Z be the set of pairs $(X, \mathbf{b})$ such that

(a) for any $\psi(\mathbf{x}) \in$ *hornGF*, if $(\mathfrak{A}_\Gamma \mid \Gamma \in \mathcal{C}) \models \psi(\mathbf{a})$ for all $\mathbf{a} \in X$, then $\mathfrak{B} \models \psi(\mathbf{b})$;
(b) there exists a set $\Phi(\mathbf{x}) \supseteq \lambda_{\mathfrak{B},\mathbf{b}}$ of FO-formulas such that X is the set of all tuples $\mathbf{a}$ with $(\mathfrak{A}_\Gamma \mid \Gamma \in \mathcal{C}) \models \Phi(\mathbf{a})$.

Each $(X, \mathbf{b})$ in Z can be regarded as a link of the form $(P, \mathbf{b})$ with $X = P[\mathbf{b}]$. We show that Z is an FO-restricted guarded Horn simulation between $(\mathfrak{A}_\Gamma \mid \Gamma \in \mathcal{C})$ and $\mathfrak{B}$.

Assume $(X, \mathbf{b})$ is given. Let Γ_X be a set of FO-formulas that defines X in $(\mathfrak{A}_\Gamma \mid \Gamma \in \mathcal{C})$. Let $(P, \mathbf{b})$ be the link corresponding to $(X, \mathbf{b})$. We check the conditions.

The Condition ($atom^g_h$) follows from Condition (a).

To show that Condition ($forth^{gg}_h$) holds, assume that Y is a set of guarded tuples such that there is a set Γ_Y of FO-formulas with free variables among $\mathbf{x}_0\mathbf{y}$ defining Y in $(\mathfrak{A}_\Gamma \mid \Gamma \in \mathcal{C})$. Assume first that Y is an $R(\mathbf{b}_0\mathbf{y})$-successor of $(P, \mathbf{b})$ and $\mathbf{b}_0$ is not empty. We have to show that there exists $(P', \mathbf{b}_0\mathbf{b}') \in Z$ such that $P'[\mathbf{b}_0\mathbf{b}'] \subseteq Y$. Let $\rho_{(\mathfrak{A}_\Gamma|\Gamma\in\mathcal{C}),Y}$ be the set of all formulas ρ in *hornGF* with free variables among $\mathbf{x}_0\mathbf{y}$ such that $(\mathfrak{A}_\Gamma \mid \Gamma \in \mathcal{C}) \models \rho(\mathbf{a}_0\mathbf{a})$ for all $\mathbf{a}_0\mathbf{a} \in Y$. For every $\rho(\mathbf{x}_0\mathbf{y})$ in $\rho_{(\mathfrak{A}_\Gamma|\Gamma\in\mathcal{C}),Y}$, and $p \in P$:

$$(\mathfrak{A}_\Gamma \mid \Gamma \in \mathcal{C}) \models \exists \mathbf{y}(R(p(\mathbf{b}_0), \mathbf{y}) \wedge \rho(p(\mathbf{b}_0), \mathbf{y})).$$

Thus

$$\mathfrak{B} \models \exists \mathbf{y}(R(\mathbf{b}_0\mathbf{y}) \wedge \rho(\mathbf{b}_0\mathbf{y})),$$

for every $\rho(\mathbf{x}_0\mathbf{y})$ in $\rho_{(\mathfrak{A}_\Gamma|\Gamma\in\mathcal{C}),Y}$. By ω-saturatedness of $\mathfrak{B}$, there exists a tuple $\mathbf{b}'$ such that

$$\mathfrak{B} \models R(\mathbf{b}_0\mathbf{b}'), \quad \mathfrak{B} \models \rho_{(\mathfrak{A}_\Gamma|\Gamma\in\mathcal{C}),Y}(\mathbf{b}_0\mathbf{b}').$$

Now we set

$$Y' = \{\mathbf{a} \in Y \mid (\mathfrak{A}_\Gamma \mid \Gamma \in \mathcal{C}) \models \lambda_{\mathfrak{B},\mathbf{b}_0\mathbf{b}'}(\mathbf{a})\}$$

One can show that $(Y', \mathbf{b}_0\mathbf{b}')$ is as required. To show that $(Y', \mathbf{b}_0\mathbf{b}') \in Z$ it suffices to show that Y' is not empty and satisfies Conditions (a) and (b).

- Assume $Y' = \emptyset$. Then
$$(\mathfrak{A}_\Gamma \mid \Gamma \in \mathcal{C}) \not\models \lambda_{\mathfrak{B},\mathbf{b}_0\mathbf{b}'}(\mathbf{a})$$
for any $\mathbf{a} \in Y$. Then $(\Gamma_Y \cup \lambda_{\mathfrak{B},\mathbf{b}_0\mathbf{b}'})(\mathbf{x}_0\mathbf{y})$ is not satisfied in any $\mathfrak{A}_\Gamma$. By compactness and ω-saturatedness of every $\mathfrak{A}_\Gamma$, there exist finite subsets Γ_0 of Γ_Y and $\lambda'_{\mathfrak{B},\mathbf{b}_0\mathbf{b}'}$ of $\lambda_{\mathfrak{B},\mathbf{b}_0\mathbf{b}'}$ such that $\Gamma_0 \cup \lambda'_{\mathfrak{B},\mathbf{b}_0\mathbf{b}'}$ is not satisfied in any $\mathfrak{A}_\Gamma$. But then
$$(\mathfrak{A}_\Gamma \mid \Gamma \in \mathcal{C}) \models (\lambda'_{\mathfrak{B},\mathbf{b}_0\mathbf{b}'} \rightarrow \bot)(\mathbf{a}),$$
for all $\mathbf{a} \in Y$. $(\lambda'_{\mathfrak{B},\mathbf{b}_0\mathbf{b}'} \rightarrow \bot)$ is then in *hornGF* and so
$$\mathfrak{B} \models (\lambda'_{\mathfrak{B},\mathbf{b}_0\mathbf{b}'} \rightarrow \bot)(\mathbf{b}_0\mathbf{b}'),$$
by construction of $\mathbf{b}_0\mathbf{b}'$, and we have derived a contradiction.
- For Condition (a), assume that ρ is in *hornGF* and $\mathfrak{A} \models \rho(\mathbf{a})$ for all $\mathbf{a} \in Y'$. We have to show that $\mathfrak{B} \models \rho(\mathbf{b}_0\mathbf{b}')$. But this can be shown similarly to the non-emptiness proof above.
- For Condition (b), observe that
$$\Phi = (\Gamma_Y \cup \lambda_{\mathfrak{B},\mathbf{b}_0\mathbf{b}'})(\mathbf{x}_0\mathbf{y})$$
is as required.

Assume next that Y is an $R(\mathbf{y})$-successor of $(P, \mathbf{b})$ and Y intersects with all $\mathfrak{A}_\Gamma$. We have to show that there exists $(P', \mathbf{b}') \in Z$ such that $P'[\mathbf{b}'] \subseteq Y$. As Y intersects with every $\mathfrak{A}_\Gamma$, it follows that
$$(\mathfrak{A}_\Gamma \mid \Gamma \in \mathcal{C}) \models \exists \mathbf{y}(R(\mathbf{y}) \wedge \rho(\mathbf{y}))$$
for every $\rho(\mathbf{y})$ in $\rho_{(\mathfrak{A}_\Gamma \mid \Gamma \in \mathcal{C}),Y}$. Thus,
$$\mathfrak{B} \models \exists \mathbf{y}(R(\mathbf{y}) \wedge \rho(\mathbf{y})),$$
for every $\rho(\mathbf{y})$ in $\rho_{(\mathfrak{A}_\Gamma \mid \Gamma \in \mathcal{C}),Y}$. By ω-saturatedness of $\mathfrak{B}$, there exists a tuple $\mathbf{b}'$ such that
$$\mathfrak{B} \models R(\mathbf{b}'), \quad \mathfrak{B} \models \rho_{(\mathfrak{A}_\Gamma \mid \Gamma \in \mathcal{C}),Y}(\mathbf{b}')$$
Consider $\lambda_{\mathfrak{B},\mathbf{b}'}(\mathbf{y})$. We let
$$Y' = \{\mathbf{a} \in Y \mid (\mathfrak{A}_\Gamma \mid \Gamma \in \mathcal{C}) \models \lambda_{\mathfrak{B},\mathbf{b}'}(\mathbf{a})\}$$
One can now show similarly to the previous case that $(Y', \mathbf{b}')$ is as required.

To show Condition $(back_h^g)$, assume that a guarded tuple $\mathbf{b}'$ is given. Consider w.l.o.g. an atomic formula $R(\mathbf{x}')$ with $\mathfrak{B} \models R(\mathbf{b}')$. Let $\mathbf{b}' = \mathbf{b}_0\mathbf{b}_1$, where $\mathbf{b}_0$ and $\mathbf{b}_1$ are such that $[\mathbf{b}_0] \subseteq [\mathbf{b}]$ and $[\mathbf{b}_1] \cap [\mathbf{b}] = \emptyset$. Let X' denote the set of all tuples $\mathbf{a}_0\mathbf{a}_1$ such that there exists $\Gamma \in \mathcal{C}$ with $\mathbf{a}_0\mathbf{a}_1$ in $\mathfrak{A}_\Gamma$ and such that

- $\mathfrak{A}_\Gamma \models \lambda_{\mathfrak{A},\mathbf{b}'}(\mathbf{a}_0\mathbf{a}_1)$ and
- there exists a tuple $\mathbf{a}''$ with $\mathfrak{A}_\Gamma \models \Gamma_X(\mathbf{a}_0\mathbf{a}'')$.

Define $\Gamma_{X'}$ as the set of all formulas
$$\exists \mathbf{x}_1(\lambda(\mathbf{x}_0\mathbf{x}_1) \wedge \psi(\mathbf{x}_0))$$
where $\lambda(\mathbf{x}_0\mathbf{x}_1) \in \lambda_{\mathfrak{A},\mathbf{b}'}(\mathbf{x}_0\mathbf{x}_1)$ and $\psi(\mathbf{x}_0) = \exists \mathbf{x}''\psi'(\mathbf{x}_0\mathbf{x}'')$ for some $\psi(\mathbf{x}_0\mathbf{x}'') \in \Gamma_X(\mathbf{x}_0\mathbf{x}'')$. Then, by ω-saturatedness of all $\mathfrak{A}_\Gamma$ with $\Gamma \in \mathcal{C}$, we have that $\Gamma_{X'}$ defines X':
$$X' = \{\mathbf{a}_0\mathbf{a}_1 \mid (\mathfrak{A}_\Gamma \mid \Gamma \in \mathcal{C}) \models \Gamma_{X'}(\mathbf{a}_0\mathbf{a}_1)\}$$
We show that $(X', \mathbf{b}')$ is as required. Condition (b) is satisfied by definition. For Condition (a), assume $\rho(\mathbf{x}_0\mathbf{x}_1)$ is in *hornGF* and $\mathfrak{B} \not\models \rho(\mathbf{b}_0\mathbf{b}_1)$. It suffices to show that there exists $\mathbf{a}_0\mathbf{a}_\rho \in X'$ such that $(\mathfrak{A}_\Gamma \mid \Gamma \in \mathcal{C}) \models \neg\rho(\mathbf{a}_0\mathbf{a}_\rho)$. But $\mathbf{a}_0\mathbf{a}_\rho$ exists because otherwise there exists $\lambda \in \lambda_{(\mathfrak{A}_\Gamma \mid \Gamma \in \mathcal{C}),\mathbf{b}'}$ such that
$$(\mathfrak{A}_\Gamma \mid \Gamma \in \mathcal{C}) \models \forall \mathbf{x}_1(R(\mathbf{x}_0\mathbf{x}_1) \rightarrow (\lambda \rightarrow \rho))(p(\mathbf{b}_0)\mathbf{x}_1) \quad (4)$$
for all $p \in P$ and so, by definition of Z,
$$\mathfrak{B} \models \forall \mathbf{x}_1(R(\mathbf{x}_0\mathbf{x}_1) \rightarrow (\lambda \rightarrow \rho))(\mathbf{b}_0, \mathbf{x}_1),$$
and we have derived a contradiction. To prove (4), assume (4) does not hold. Then, for every $\lambda' \in \lambda_{(\mathfrak{A}_\Gamma \mid \Gamma \in \mathcal{C}),\mathbf{b}'}$, there exists $p \in P$ such that
$$(\mathfrak{A}_\Gamma \mid \Gamma \in \mathcal{C}) \models \exists \mathbf{x}_1(\lambda' \wedge \neg\rho)(p(\mathbf{b}_0), \mathbf{x}_1).$$
By definition, it follows that for every $\lambda' \in \lambda_{(\mathfrak{A}_\Gamma \mid \Gamma \in \mathcal{C}),\mathbf{b}'}$ there exists $\Gamma \in \mathcal{C}$ such that $\mathfrak{A}_\Gamma$ realizes
$$\Gamma_X(\mathbf{x}_0\mathbf{x}'') \cup \{(\lambda' \wedge \neg\rho)(\mathbf{x}_0\mathbf{x}_1)\}$$
But then, by ω-saturatedness, compactness, and the definition of $(\mathfrak{A}_\Gamma \mid \Gamma \in \mathcal{C})$, there exists $\Gamma \in \mathcal{C}$ such that $\mathfrak{A}_\Gamma$ realizes
$$\Gamma_X(\mathbf{x}_0\mathbf{x}'') \cup \lambda_{(\mathfrak{A}_\Gamma \mid \Gamma \in \mathcal{C}),\mathbf{b}'}(\mathbf{x}_0\mathbf{x}_1) \cup \{\neg\rho(\mathbf{x}_0\mathbf{x}_1)\}$$
which implies that there exists $\mathbf{a}_0\mathbf{a}_\rho \in X'$ such that $(\mathfrak{A}_\Gamma \mid \Gamma \in \mathcal{C}) \models \neg\rho(\mathbf{a}_0\mathbf{a}_\rho)$.

Finally, Condition (sim_h^{gg}) can be proved using ω-saturatedness of $\mathfrak{B}$ and all $\mathfrak{A}_\Gamma$. □

EXTENDING *horn$\mathcal{ALC}$* WITH THE ∇-OPERATOR

Denote by $\mathcal{ELU}_\nabla$ the extension of $\mathcal{ELU}$ with the ∇-operator defined by setting
$$\nabla R.C \;=\; \exists R.\top \sqcap \forall R.C.$$
Thus, *$\mathcal{ELU}_\nabla$-concepts* are given by the grammar
$$C, D \;::=\; A \;\mid\; \top \;\mid\; C \sqcap D \;\mid\; C \sqcup D \;\mid\; \exists R.C \;\mid\; \nabla R.C.$$
We define *horn$\mathcal{ALC}_\nabla$* in the same way as *horn$\mathcal{ALC}$* (see Definition 1) with the exception that now L is an arbitrary $\mathcal{ELU}_\nabla$-concept. The following lemma shows that, modulo the standard translation, *horn$\mathcal{ALC}_\nabla$* is a fragment of Horn FO.

Lemma 14. *Every horn$\mathcal{ALC}_\nabla$-concept is equivalent to a Horn formula.*

Proof. We proceed by induction on the construction of *horn$\mathcal{ALC}_\nabla$*-concepts. The only non-trivial case is $L \rightarrow C$, where L is an $\mathcal{ELU}_\nabla$-concept and $C^\dagger$ is equivalent to a Horn formula. This case follows from the claim below.

Claim. If $H(\mathbf{z})$ is any Horn formula and $L^\dagger(x)$ is the standard translation of an $\mathcal{ELU}_\nabla$-concept L, then $L^\dagger(x) \rightarrow H(\mathbf{z})$ is equivalent to a Horn formula.

Proof of claim is by induction on the construction of L. The basis of induction and the cases when L is A, $\top$ and $C \sqcup D$ are obvious.

Case $(C^\dagger \wedge D^\dagger)(x) \to H(\mathbf{z})$. This formula is equivalent to $C^\dagger(x) \to (D^\dagger(x) \to H(\mathbf{z}))$. By IH, $D^\dagger(x) \to H(\mathbf{z})$ is equivalent to some Horn formula $H'(x, \mathbf{z})$, and by the same reason $C^\dagger(x) \to H'(x, \mathbf{z})$ is also equivalent to a Horn formula.

Case $(\exists R.C)^\dagger(x) \to H(\mathbf{z})$. This formula is equivalent to $\forall y\,((R(x,y) \wedge C^\dagger(y)) \to H(\mathbf{z}))$ with a fresh y, which is clearly equivalent to a Horn formula, by IH.

Case $(\nabla R.C)^\dagger \to H(\mathbf{z})$. Its standard translation is equivalent to

$$\exists y\, R(x,y) \to [\forall y(R(x,y) \to C^\dagger(y)) \to H(\mathbf{z})].$$

One can check that this formula has the same models as

$$\exists y\, R(x,y) \to \exists y[R(x,y) \wedge (C^\dagger(y) \to H(\mathbf{z}))],$$

which is equivalent to a Horn formula, by IH. □

Definition 10 ($\mathcal{ELU}_\nabla$-simulation). An *$\mathcal{ELU}_\nabla$-simulation* between τ-structures $\mathfrak{A}$ and $\mathfrak{B}$ is a relation $Z \subseteq \mathrm{dom}(\mathfrak{A}) \times \mathrm{dom}(\mathfrak{B})$ if the following conditions hold:

- (*atom*) for any $A \in \tau$, if $(a,b) \in Z$ and $a \in A^{\mathfrak{A}}$, then $b \in A^{\mathfrak{B}}$;
- (*forth*) for any $R \in \tau$, if $(a,b) \in Z$ and $(a,a') \in R^{\mathfrak{A}}$, then there exists $b' \in \mathrm{dom}(\mathfrak{B})$ with $(b,b') \in R^{\mathfrak{B}}$ and $(a',b') \in Z$;
- (*back*) for any $R \in \tau$, if $(a,b) \in Z$, $a \in (\exists R.\top)^{\mathfrak{A}}$, and $(b,b') \in R^{\mathfrak{B}}$, then there is $a' \in \mathrm{dom}(\mathfrak{A})$ with $(a,a') \in R^{\mathfrak{A}}$ and $(a',b') \in Z$.

We write $\mathfrak{A}, a \preceq_\nabla \mathfrak{B}, b$ if there exists a $\mathcal{ELU}_\nabla$-simulation Z between $\mathfrak{A}$ and $\mathfrak{B}$ such that $(a,b) \in Z$.

Theorem 17 (Ehrenfeucht-Fraïssé game for $\mathcal{ELU}_\nabla$). *For any finite vocabulary τ, pointed τ-structures $\mathfrak{A}, a$ and $\mathfrak{B}, b$, and any $\ell < \omega$, we have*

$$\mathfrak{A}, a \le^\ell_{\mathcal{ELU}_\nabla} \mathfrak{B}, b \quad \textit{iff} \quad \mathfrak{A}, a \preceq^\ell_\nabla \mathfrak{B}, b.$$

If $\mathfrak{A}$ and $\mathfrak{B}$ are finite, then

$$\mathfrak{A}, a \le_{\mathcal{ELU}_\nabla} \mathfrak{B}, b \quad \textit{iff} \quad \mathfrak{A}, a \preceq_\nabla \mathfrak{B}, b.$$

Definition 11 (Horn$_\nabla$-simulation). A *Horn$_\nabla$ simulation* Z between τ-structures $\mathfrak{A}$ and $\mathfrak{B}$ is a Horn simulation such that, in addition,

- (*sim$_\nabla$*) if $(X,b) \in Z$, then $\mathfrak{B}, b \preceq_\nabla \mathfrak{A}, a$ for every $a \in X$.

We write $\mathfrak{A}, X \preceq_{horn_\nabla} \mathfrak{B}, b$ if there exists a Horn$_\nabla$ simulation Z between $\mathfrak{A}$ and $\mathfrak{B}$ such that $(X,b) \in Z$.

Theorem 18 (Ehrenfeucht-Fraïssé game for *horn*$\mathcal{ALC}_\nabla$). *For any finite vocabulary τ, pointed τ-structures $\mathfrak{A}, a$ and $\mathfrak{B}, b$, and any $\ell < \omega$, we have*

$$\mathfrak{A}, a \le^\ell_{horn\mathcal{ALC}_\nabla} \mathfrak{B}, b \quad \textit{iff} \quad \mathfrak{A}, a \preceq^\ell_{horn_\nabla} \mathfrak{B}, b.$$

If $\mathfrak{A}$ and $\mathfrak{B}$ are finite, then

$$\mathfrak{A}, a \le_{horn\mathcal{ALC}_\nabla} \mathfrak{B}, b \quad \textit{iff} \quad \mathfrak{A}, a \preceq_{horn_\nabla} \mathfrak{B}, b.$$