Granular Spatial Calculi of Relative Directions or Movements with Parallelism: Consistent Account

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Abstract

The $\mathcal{OPRA}^*$ calculus family, originally suggested by Frank Dylla, adds parallelism to the $\mathcal{OPRA}$ calculus family which is very popular in Qualitative Spatio-temporal Reasoning (QSTR). Adding parallelism enables the direct representation of parallel moving objects, which is relevant in many applications like traffic monitoring. However, it turned out that it is hard to derive a sound geometric analysis. So far no sound spatial reasoning was supported. Our new generic analysis based on combining condensed semantics lower bounds with upper bounds from algebraic mappings of related calculi already leads to a close and sound approximization. This approximization can be easily augmented with a manual analysis of few geometrically underconstrained cases and then yields a complete analysis of possible configurations in this oriented point framework. This for the first time enables sound standard QSTR constraint reasoning for the $\mathcal{OPRA}^*$ calculus family.

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1 Introduction

Qualitative spatial representations provide mechanisms which characterize essential properties of objects or configurations and only make relatively coarse distinctions between spatial relations and configurations, and typically rely on relative comparisons rather than measuring. The concept of *qualitative space* then can be characterized by the following quotation from Galton [7]: "The divisions of qualitative space correspond to salient discontinuities in our apprehension of quantitative space". Qualitative spatial and temporal calculi as formally defined and investigated in the research area of qualitative spatio-temporal reasoning (QSTR) aim at modeling this human commonsense reasoning about space and time using qualitative

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relations for different spatial aspects such as topology (e.g., "included in"), direction ("to the left of"), and position as a combination of direction and distance, holding between elementary spatial entities such as points or regions. Coarse spatial knowledge can be used to represent incomplete and underdetermined knowledge systematically. This is especially useful if the task is to describe features of classes of configurations rather than features of individual configurations. For example, the observation that the goal keeper usually stands in front of the goal is true for a variety of ball games. A more specific expression about their position typically would have to refer to the corresponding configuration of a specific sport. Similarly, descriptions of allowed or desired spatial behavior are abstractions mapping infinite sets of possible quantitative configurations or trajectories to a single qualitative description. If qualitative spatial divisions serve as knowledge representation in a reasoning system, deductive inferences can be realized as constraint-based reasoning. Qualitative spatial calculi of relative directions are important for applications such as human-robot interaction, volunteered geographic information, scene understanding, outdoor robotic navigation [10].

A qualitative calculus consists of a set of base relations and a composition table; the latter enables spatial reasoning. There is a wide range of qualitative spatial/temporal calculi, and they are understood to varying levels of detail, see, e.g., the recent survey [3].

The calculi from the $\mathcal{DRA}$ [12, 11] and $\mathcal{OPRA}$ [9, 14] families are prominent examples of calculi of relative directions. They are available at varying granularities: $\mathcal{DRA}$ admits three granularities (variants $\mathcal{DRA}_c$, $\mathcal{DRA}_f$, $\mathcal{DRA}_{fp}$, i.e., "coarse-grained", "fine-grained" and "with parallelism and anti-parallelism"). In particular, $\mathcal{DRA}_{fp}$ extends $\mathcal{DRA}_f$ with the ability to capture parallelism, anti-parallelism, and positive and negative alignment. The $\mathcal{OPRA}$ family admits arbitrarily fine granularities, indicated by a subscript n. Already for small n, $\mathcal{OPRA}_n$ has a large number of base relations (72 for $n = 2$), which prohibits a manual computation of the composition table. For this reason, Moratz and Mossakowski [14] performed a systematic geometric analysis of oriented points in the 2D plane, resulting in a generic algorithm for computing the composition table in $\mathcal{OPRA}_n$ for any n.

Dylla and Lee [1, 2] extended $\mathcal{OPRA}$ in a way that is analogous to the way how $\mathcal{DRA}_{fp}$ extends $\mathcal{DRA}_f$. The resulting $\mathcal{OPRA}^*$ family refines $\mathcal{OPRA}$ with the ability to capture (anti-)parallelism and positive/negative alignment. It later turned out that the original algorithm for computing the composition table does not provide a sound geometric analysis, nor has an alternative algorithm been found yet. It is also far from obvious how to extend Moratz and Mossakowski's analysis to incorporate parallelism. For this reason, we develop an approach to compute the composition table of $\mathcal{OPRA}_n^*$ that relies on homomorphic embeddings into other calculi, geometric constraints on realizable triples of oriented points, and an enumeration of canonical configurations of triples of oriented points.

2 Qualitative Spatial and Temporal Reasoning

Objects and locations can be represented as simple, featureless points. In contrast, the $\mathcal{OPRA}_n$ calculus uses more complex basic entities: It is based on objects which are represented as oriented points. It is related to a calculus which is based on straight line segments (dipoles) [12]. Conceptually, the oriented points can be viewed as a transition from oriented line segments with concrete length to line segments with infinitely small length [11]. In this conceptualization the length of the objects no longer has any importance. Thus, only the orientation of the objects is modeled. *Opoints*, our term for oriented points, are specified as pair of a point and a orientation on the 2D-plane.

In a coarse representation, a single opoint induces the sectors depicted in Figure 1a. "Front", "Back", "Left", and "Right" are linear sectors. "Left-Front", "Right-Front", "Left-Back", and "Right-Back" are quadrants. The position of the point itself is denoted as "Same". This qualitative granularity corresponds to Freksa's double cross calculus [5].

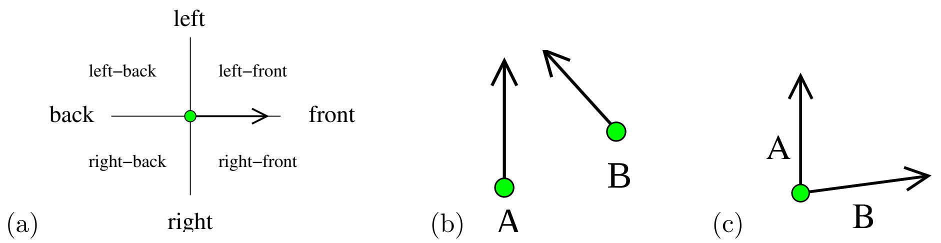

Figure 1 (a) An opoint and its qualitative spatial relative orientations. (b) and (c) Qualitative spatial relation between two opoints at (b) different positions, here $A \genfrac{}{}{0pt}{}{\text{leFr}}{\text{riFr}} B$, and (c) the same position, here $A \genfrac{}{}{0pt}{}{\text{riFr}}{\text{same}} B$.

A qualitative spatial relative direction relation between two opoints is represented by the sector in which the second opoint lies with respect to the first one and by the sector in which the first one lies with respect to the second one. For the general case of the two points having different positions we use the following relation symbols:

$$\genfrac{}{}{0pt}{}{\text{front}}{\text{front}}, \genfrac{}{}{0pt}{}{\text{leFr}}{\text{front}}, \genfrac{}{}{0pt}{}{\text{left}}{\text{front}}, \genfrac{}{}{0pt}{}{\text{leBa}}{\text{front}}, \genfrac{}{}{0pt}{}{\text{back}}{\text{front}}, \genfrac{}{}{0pt}{}{\text{riBa}}{\text{front}}, \genfrac{}{}{0pt}{}{\text{right}}{\text{front}}, \genfrac{}{}{0pt}{}{\text{riFr}}{\text{front}}, \genfrac{}{}{0pt}{}{\text{front}}{\text{leFr}}, \genfrac{}{}{0pt}{}{\text{leFr}}{\text{leFr}}, \ldots, \genfrac{}{}{0pt}{}{\text{riFr}}{\text{riFr}}.$$

The abbreviated sector name for the sector where the second opoint position is located from the perspective of the first opoint is the lower part of the relation symbol. Conversely, the sector name for the relative position of the first opoint location using the second opoint as a reference is put atop the other abbreviated sector name. We thus obtain 8×8 base relations for two opoints having different positions. The configuration in Figure 1b is expressed via the relation $A \genfrac{}{}{0pt}{}{\text{leFr}}{\text{riFr}} B$. If both opoints share the same position, the lower relation symbol part is the word "same" and the upper part denotes the orientation of the second opoint w.r.t. the first; see Figure 1c. Altogether we obtain 72 different atomic relations (8×8 general relations plus 8 with the opoints at the same position). These relations are jointly exhaustive and pairwise disjoint (JEPD). The relation $\genfrac{}{}{0pt}{}{\text{front}}{\text{same}}$ is the identity relation. The granularity of the $\mathcal{OPRA}$ version we just described is $n = 2$, so this calculus version is called $\mathcal{OPRA}_2$. The general schema for arbitrary m is described below.

The $\mathcal{OPRA}_2^*$ calculus [1, 2] is similar to $\mathcal{OPRA}_2$. The important extension is a refinement of the relations by marking them with letters '+' or '−', 'P' or 'A', according to whether the two orientations of the oriented points are positive (e.g. turning the first opoint in the direction of the second opoint would need a mathematically positive turn), negative, parallel or anti-parallel.

A comprehensive simulation using the $\mathcal{OPRA}$ calculus for an important subtask was built by Dylla et. al. [16]. Their system SailAway simulates the behaviour of different vessels following declarative (written) navigation rules for collision avoidance. This system can be used to verify whether a given set of rules leads to stable avoidance between potentially colliding vessels. The different vessel categories that determine their right-of-way priorities are represented in an ontology. The vessel's movement is described by a method called conceptual neighborhood-based reasoning (CNH reasoning). CNH reasoning describes whether two spatial configurations of objects can be transformed into each other by small changes [6]. A CNH transformation can be an object movement in a short period of time.

Instead of using $\mathcal{OPRA}_4$, like in the original SailAway system, we use this domain to show how the $\mathcal{OPRA}_2^*$ calculus can model parallel movement like in a typical overtake (e.g. catch up with and pass while travelling in the same direction) event. Fig. 2 shows a CNH transition diagram which represents relative trajectories of two vessels during such an overtake event (for an earlier version of qualitative navigation simulation, see [4]). The depicted sequence between two vessels A and B is: $A \genfrac{}{}{0pt}{}{\text{leBa}}{\text{riFr}}\mathsf{P}\, B \rightarrow A \genfrac{}{}{0pt}{}{\text{left}}{\text{right}}\mathsf{P}\, B \rightarrow A \genfrac{}{}{0pt}{}{\text{leFr}}{\text{riBa}}\mathsf{P}\, B$.

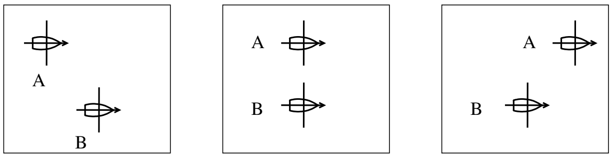

Figure 2 Representation of vessel navigation with conceptual neighbourhood in $\mathcal{OPRA}_2^*$.

Preliminaries. A *qualitative calculus* $\mathcal{A} = (U_\mathcal{A}, R_\mathcal{A})$ consists of a set $U_\mathcal{A}$ called the *universe* of $\mathcal{A}$ and a set $R_\mathcal{A}$ of binary relations on $U_\mathcal{A}$ called *base relations* that are *JEPD* (jointly exhaustive and pairwise disjoint), i.e. $r \cap s = \emptyset$ for $r, s \in R_\mathcal{A}$ with $r \neq s$ and $\bigcup_{r \in R_\mathcal{A}} r = U_\mathcal{A} \times U_\mathcal{A}$. Furthermore, if r is a base relation, then the *converse* $r^\smile = \{(a, b) \mid (b, a) \in r\}$ must be a base relation as well. A *general relation* is a union of base relations.

Every qualitative calculus $\mathcal{A} = (U_\mathcal{A}, R_\mathcal{A})$ gives rise to an algebraic structure via *weak composition* of relations from $R_\mathcal{A}$ in the following way. If $r, s \in 2^{R_\mathcal{A}}$ are general relations, then $r \diamond s = \{t \in R_\mathcal{A} \mid r \circ s \cap t \neq \emptyset\}$, where $r \circ s$ is the usual set-theoretic composition.

We define the $\mathcal{OPRA}_n$ and $\mathcal{OPRA}_n^*$ families of calculi as introduced in [9, 2]. Their universe is the set $\mathcal{O} = \mathbb{R} \times \mathbb{R} \times [0, 2\pi)$ of *opoints* in the 2D-plane. In the $\mathcal{OPRA}_n$, every opoint $p = (x, y, \phi)$ is associated with n lines, all intersecting at (x, y) and pointing to the directions $\{\phi + i \cdot \frac{\pi}{n} \mid i = 0, \ldots, n-1\}$. These n lines partition $\mathbb{R} \times \mathbb{R} \setminus \{(x, y)\}$ into $2n$ sections which are numbered 0 to $2n-1$: The ray which p points towards ϕ has number 0; the other sections are numbered counter-clockwise, so 1-dimensional (2-dimensional) rays are assigned even (odd) numbers. If $(u, v) \in \mathbb{R}^2$, we write $\mathsf{pos}(u, v, p) = i$ if (u, v) is in the i-th section of p, and $\mathsf{pos}(u, v, p) = s$ if $(u, v) = (x, y)$ (s = *same*).

The base relation between two opoints $p_1 = (x_1, y_1, \phi_1)$ and $p_2 = (x_2, y_2, \phi_2)$ is described by the location of p_2 relative to p_1 and the location of p_1 relative to p_2. For $i, j \in \{0, \ldots, 2n-1\}$ let $\angle_i^j$ be the set of all pairs (p_1, p_2) of opoints $p_1 = (x_1, y_1, \phi_1)$ and $p_2 = (x_2, y_2, \phi_2)$ such that $i = \mathsf{pos}(x_2, y_2, p_1)$ and $j = \mathsf{pos}(x_1, y_1, p_2)$. For $i \in \{0, \ldots, 2n-1\}$ let $\angle_s^i$ be the set of all pairs (p_1, p_2) of opoints $p_1 = (x_1, y_1, \phi_1)$ and $p_2 = (x_2, y_2, \phi_2)$ such that $x_1 = x_2$ and $y_1 = y_2$ and ϕ_2 points into section i of p_1. Now $\mathcal{OPRA}_n$ is the qualitative calculus with universe $\mathcal{O}$ and base relations $\{\angle_i^j \mid 0 \leq i, j \leq 2n-1\} \cup \{\angle_s^i \mid 0 \leq i \leq 2n-1\}$.

The calculus $\mathcal{OPRA}_n^*$ refines $\mathcal{OPRA}_n$ by adding information about parallelism. Let $\alpha(p_1, p_2) = \phi_2 - \phi_1$ if $\phi_2 - \phi_1 \geq 0$ and $\alpha(p_1, p_2) = \phi_2 - \phi_1 + 2\pi$ otherwise. Every $\mathcal{OPRA}_n$ base relation $\angle_i^j$ can be partitioned into four relations, some of which will be empty:

$$\angle_i^j\mathsf{P} = \angle_i^j \cap \{(p_1, p_2) \mid \alpha(p_1, p_2) = 0\} \qquad \angle_i^j+ = \angle_i^j \cap \{(p_1, p_2) \mid 0 < \alpha(p_1, p_2) < \pi\}$$
$$\angle_i^j\mathsf{A} = \angle_i^j \cap \{(p_1, p_2) \mid \alpha(p_1, p_2) = \pi\} \qquad \angle_i^j- = \angle_i^j \cap \{(p_1, p_2) \mid \pi < \alpha(p_1, p_2) < 2\pi\}$$

The base relations of $\mathcal{OPRA}_n^*$ are all non-empty relations of the form $\angle_i^j*$, where $0 \leq i, j \leq 2n-1$ and $* \in \{\mathsf{P}, +, \mathsf{A}, -\}$ as well as all relations of the form $\angle_s^i$, where $0 \leq i \leq 2n-1$.

Let r, s, t be base relations. We say that the triple (r, s, t) is *realizable*, if $r \diamond s \ni t$ and that the triple is *impossible* otherwise. For a realizable triple (r, s, t), we say that $(p_1, p_2, p_3) \in \mathcal{O}^3$ realizes (r, s, t), if $r(p_1, p_2)$, $s(p_2, p_3)$ and $t(p_1, p_3)$. Computing the composition table of a calculus is the same as computing the set of realizable triples.

3 Composition table of $\mathcal{OPRA}_2^*$

We compute the $\mathcal{OPRA}_2^*$ composition table twice, using two different algorithms which are performed independently of each other: (1) We enumerate realizable triples, using a condensed semantics approach in the spirit of [9, 11]. Since every realizable triple (r, s, t)

certifies that $r \diamond s \ni t$, the enumeration yields a *lower bound* for the composition table, that is, a subset of the set of all realizable triples. In contrast to [8], the condensed semantics approach does not generate realizable triples randomly but via a systematic enumeration that exploits the geometric properties of the underlying calculus. (2) Starting from the set of all triples, we eliminate impossible triples by computing homomorphisms $\mathcal{OPRA}_2^* \to \mathcal{OPRA}_2$ and $\mathcal{OPRA}_2^* \to \mathcal{OPRA}_1^*$, and by observing angular, location, and permutation constraints. This way we obtain an *upper bound* for the composition table.

The lower bound is obviously a subset of the upper bound. After computing both using the algorithms described, it will turn out that, for $\mathcal{OPRA}_2^*$, the upper and lower bound coincide. This implies that either of them computes the $\mathcal{OPRA}_2^*$ composition table.

3.1 Lower bound

Our aim in this section is to compute a lower bound for the composition table, i.e., a list (set) of configurations of opoint triples that are guaranteed to be contained in the table. Using the condensed semantics approach analogously to previous work on the Dipole calculus [11] and the $\mathcal{OPRA}$ calculus [9], we found a qualitative abstraction in a discrete geometry that has a mapping to the equivalence classes in the $\mathbb{R}^2$ plane of the original model domain.

We used a set of different qualitative triangles relevant for positions of three opoints. In the first triangle all three opoints are on the same location. In the second location triple two points are on the same position and the third point is at a different location. In a grid we constructed specific configurations of opoints as vertices of the following list of triangles. The first vertex is fixed at position $(0,0)$ the second vertex is fixed at position $(8,0)$. With a third vertex at the positions $(4,0),(4,2),(4,3),(4,4),(4,8)$ we constructed five triangles. At each vertex there are only limited qualitatively different options for opoints in our $\mathcal{OPRA}_2^*$ domain. We used 32 orientations for opoints at each vertex. Then the exhaustive enumeration of all opoint options (e.g. including permutations of the three arguments) for all three vertices for all seven three location configurations generates a lower bound for the composition table. With this approach there is no guarantee that every possible opoint triple w.r.t. the $\mathcal{OPRA}_2^*$ domain is constructed. So our condensed semantics method provides only a lower bound without the guarantee that all entries in the composition table are complete. Therefore we augmented our approach with an upper bound using a method based on abstract algebra [15] presented next.

3.2 Upper bound

We describe the upper bound algorithm. We first introduce a homomorphism technique to derive information about $\mathcal{OPRA}_2^*$ from $\mathcal{OPRA}_2$ and $\mathcal{OPRA}_1^*$, making use of the fact that the composition tables for the latter calculi are known [14, 2]. Then we improve the upper bound using two methods which we call angular constraints and location constraints. The last two methods are then refined by considering permutations of relations in a triple.

Homomorphisms to $\mathcal{OPRA}_2$ and $\mathcal{OPRA}_1^*$. Let $(U_\mathcal{A}, R_\mathcal{A})$ and $(U_\mathcal{B}, R_\mathcal{B})$ be qualitative calculi. We observe that every map $f : R_\mathcal{A} \to R_\mathcal{B}$ with the condition $(*)$ $f(r \diamond s) \subseteq f(r) \diamond f(s)$ yields an upper bound for the composition table of $\mathcal{A}$ by $r \diamond s \subseteq f^{-1}(f(r \diamond s)) \subseteq f^{-1}(f(r) \diamond f(s))$, so for every cell $r \diamond s$ in the table, $f^{-1}(f(r) \diamond f(s))$ is an upper bound that can be computed using the composition in $\mathcal{B}$. We give a sufficient condition for a map f to have condition $(*)$.

A function $f : U_\mathcal{A} \to U_\mathcal{B}$ is said to *induce a map on base relations* if for every base relation $r \in R_\mathcal{A}$ there exists a base relation $s \in R_\mathcal{B}$ s.t. $f(r) \subseteq s$. In this case, we denote the induced map $R_\mathcal{A} \to R_\mathcal{B}$ by f as well. The following lemma is proved in [13].

▶ **Lemma 1.** *If $f : U_{\mathcal{A}} \to U_{\mathcal{B}}$ induces a map on base relations, then $f(r \diamond s) \subseteq f(r) \diamond f(s)$.*

Now we establish the first upper bound for the composition table of $\mathcal{OPRA}_2^*$. Since $\mathcal{OPRA}_2^*$ combines features from two calculi, we obtain upper bounds from two natural homomorphisms, namely the quotient homomorphisms $f : \mathcal{OPRA}_2^* \to \mathcal{OPRA}_2$ and $g : \mathcal{OPRA}_2^* \to \mathcal{OPRA}_1^*$, which are both induced by the identity on $\mathcal{O}$. Both f and g induce a map on base relations, so by Lemma 1, they yield two upper bounds for the composition table of $\mathcal{OPRA}_2^*$, which can be calculated from the known composition tables of $\mathcal{OPRA}_2$ and $\mathcal{OPRA}_1^*$. The homomorphism f forgets the information about parallelism, whereas g maps the regions of $\mathcal{OPRA}_2^*$ to the coarser ones of $\mathcal{OPRA}_1^*$. Formally, $f(\angle_i^j *) = \angle_i^j$ with $* \in \{\mathsf{P}, \mathsf{A}, +, -\}$, and $g(\angle_i^j *) = \angle_{\rho(i)}^{\rho(j)} *$ with $* \in \{\mathsf{P}, \mathsf{A}, +, -\}$, where ρ maps the number of a section in $\mathcal{OPRA}_2$ to that of the corresponding section in $\mathcal{OPRA}_1$, so $\rho(0) = 0$, $\rho(1) = \rho(2) = \rho(3) = 1$, $\rho(4) = 2$ and $\rho(5) = \rho(6) = \rho(7) = 3$.

Angular constraints. We describe the method of *angular constraints*, which excludes triples that are impossible due to contradictory information about the angle of the third point relative to the first point. Consider two opoints $p_i = (x_i, y_i, \phi_i) \in \mathcal{O}$, where $i \in \{1, 2\}$. We first describe how to obtain a constraint on $\alpha(p_1, p_2)$. Let the relative angle $\mathsf{a}(p_1, p_2)$ be the number of the section in which ϕ_2 points relative to p_1. Precisely, let $\alpha = \alpha(p_1, p_2)$, then

$$\mathsf{a}(p_1, p_2) = \begin{cases} 0 & \text{if } \alpha = 0 \\ 1 & \text{if } 0 < \alpha < \frac{\pi}{2} \\ 2 & \text{if } \alpha = \frac{\pi}{2} \\ 3 & \text{if } \frac{\pi}{2} < \alpha < \pi \\ 4 & \text{if } \alpha = \pi \\ 5 & \text{if } \pi < \alpha < \frac{3\pi}{2} \\ 6 & \text{if } \alpha = \frac{3\pi}{2} \\ 7 & \text{if } \frac{3\pi}{2} < \alpha < 2\pi \end{cases}$$

and if r is a base relation, we define $\mathsf{a}(r) = \{\mathsf{a}(p_1, p_2) \mid p_1 r p_2\}$.

Now assume we have a triple (r, s, t) of base relations and want to know if it is realizable. If the triple is realized by three opoints p_1, p_2, p_3, then $\mathsf{a}(p_1, p_2) \in \mathsf{a}(r)$, $\mathsf{a}(p_2, p_3) \in \mathsf{a}(s)$ and $\mathsf{a}(p_1, p_3) \in \mathsf{a}(t)$. At the same time, $\mathsf{a}(r)$ and $\mathsf{a}(s)$ impose another constraint on $\mathsf{a}(t)$ by composing the possible angles. If these two constraints on $\mathsf{a}(t)$ have an empty intersection, then (r, s, t) is an impossible triple. Figure 3 shows an example.

The subroutine `isAngleCombinationPossible` gets as an input a triple (r, s, t) of base relations and returns a Boolean indicating whether the triple is impossible due to contradictory information about the relative angle. If at least one of $\mathsf{ang}(r, s)$ and $\mathsf{ang}(s, t)$ is even, then the resulting constraint for $\mathsf{ang}(r, t)$ is the singleton set $S = \{(\mathsf{ang}(r, s) + \mathsf{ang}(s, t)) \bmod 8\}$; otherwise $S = \{(u-1) \bmod 8, u \bmod 8, (u+1) \bmod 8 \mid u = \mathsf{ang}(r, s) + \mathsf{ang}(s, t)\}$. If $\mathsf{ang}(r, t) \notin S$, then `false` is returned, otherwise `true`.

Location constraints. The next improvement is obtained by *location constraints.* Here we exclude triples that are impossible due to contradictory information about the location of the third point relative to the first. Figure 4 shows how to identify such impossible triples. The algorithm `isLocationCombinationPossible` gets as input a triple (r, s, t) of base relations and returns a Boolean indicating whether the triple is ruled out for the said reason. To achieve this, we assume there is a triple (p_1, p_2, p_3) realizing (r, s, t). From r and

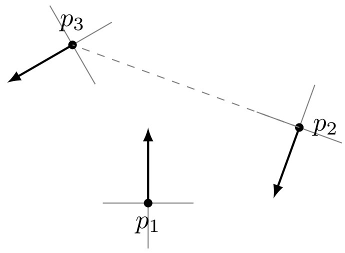

Figure 3 If $r = \angle_7^7+$, $s = \angle_6^3-$, and $t = \angle_7^3\mathsf{P}$, then $\mathsf{a}(r) = \{3\}$, $\mathsf{a}(s) = \{7\}$ and $\mathsf{a}(t) = \{0\}$. At the same time, $\mathsf{a}(r)$ and $\mathsf{a}(s)$ impose the constraint $\{1, 2, 3\}$ on $\mathsf{a}(t)$, and since $\{1, 2, 3\} \cap \{0\} = \emptyset$, the triple (r, s, t) is impossible. The image shows opoints p_1, p_2, p_3 such that $r(p_1, p_2)$ and $s(p_2, p_3)$. Under these circumstances, it is impossible that p_1 and p_3 are parallel, so $t(p_1, p_3)$ can be ruled out.

s, we compute a constraint on the location of p_3 relative to p_1, by systematically analyzing all possible cases and exploiting symmetry. See [13] for a complete list of all cases with visualizations.

Permutation constraints. Let r, s, t be base relations. It is easy to see that the triples (r, s, t), $(r^\smile, t, s)$, $(s, t^\smile, r^\smile)$, $(s^\smile, r^\smile, t^\smile)$, $(t, s^\smile, r)$ and $(t^\smile, r, s^\smile)$ are either all realizable or all impossible: if (x, y, z) realizes one of these triples, then its permutations $(x, z, y), \dots, (z, y, x)$ realize the other triples. Hence the final step of computing the upper bound traverses all triples $(r, s, t) \in B^3$; whenever one such triple has been excluded by some homomorphism, angular constraint or location constraint, then its other five permutations are excluded, too.

Upper bound algorithm. Algorithm 1 is the final algorithm for the upper bound. Recall that f (g) is the homomorphism from $\mathcal{OPRA}_2^*$ to $\mathcal{OPRA}_2$ (to $\mathcal{OPRA}_1^*$).

3.3 Discussion

The lower bound from Section 3.1 is correct since it generates only realizable triples. The upper bound from Section 3.2 is correct since it eliminates only impossible triples. Our implementation shows that both bounds coincide for $\mathcal{OPRA}_2^*$, so our method computes the correct composition table for this calculus.

In principle, our method can be applied to other members of the $\mathcal{OPRA}_n^*$ family. However, the approaches to computing both the lower and upper bound rely on heuristics, and it is not reasonable to expect that the lower and upper bounds will always coincide. If they do not, then the method will only yield a "range" of possible composition tables and, in order to compute the table precisely, it would be necessary to find an appropriate refinement of the

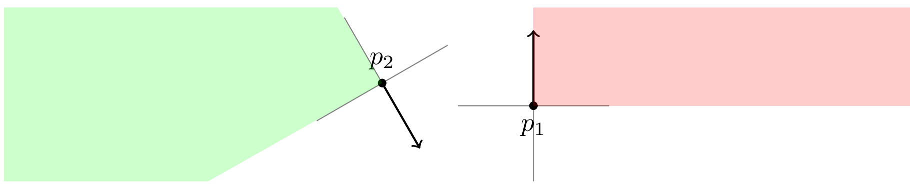

Figure 4 $(\angle_1^1-, \angle_5^1+, \angle_7^3-)$ is an impossible triple: the first 2 relations force the third opoint into the green area, which is contained in sections 1,2,3 of p_1, so p_3 cannot be in section 7 of p_1 (red).

Algorithm 1 Upper bound for the $\mathcal{OPRA}_2^*$ composition table.

Result: an upper bound U on the set of realizable triples
$U \leftarrow$ all triples (r, s, t) of base relations
foreach *triple* (r, s, t) **do**
  if $t \notin f^{-1}(f(r) \diamond f(s))$ **then** remove (r, s, t) from U
  if $t \notin g^{-1}(g(r) \diamond g(s))$ **then** remove (r, s, t) from U
  if *not* `isAngleCombinationPossible(`r,s,t`)` **then** remove (r, s, t) from U
  if *not* `isLocationCombinationPossible(`r,s,t`)` **then** remove (r, s, t) from U
foreach *triple* (r, s, t) **do**
  if *(r,s,t) is not in* U **then**
    remove (r^-, t, s), (s, t^-, r^-), (s^-, r^-, t^-), (t, s^-, r) and (t^-, r, s^-) from U
return U

upper bound (e.g., by observing further constraints) and/or the lower bound (by extending the enumeration). An obvious candidate is $\mathcal{OPRA}_6^*$, in whose definition the quadrants from $\mathcal{OPRA}_2^*$ are replaced by twelfth-planes enclosing an angle of 30°. Since that angle cannot be represented by integer ratios, our current enumeration, will no longer be complete, as it relies on integer arithmetics.

The success of our method on $\mathcal{OPRA}_2^*$ is largely due to two properties: (a) point-based calculi such as $\mathcal{OPRA}$ and $\mathcal{OPRA}^*$ exhibit a relatively simple and regular structure, which permits a complete geometric analysis such as to the one in [15]; (b) homomorphisms from $\mathcal{OPRA}_n^*$ to related calculi with established composition tables are easy to find. Whether our method yields useful results for calculi beyond the $\mathcal{OPRA}$ and $\mathcal{OPRA}^*$ families remains speculative and requires a thorough investigation of the previous two properties.

4 Conclusion

We presented our new generic analysis of the $\mathcal{OPRA}^*$ calculus family, which adds parallelism to the $\mathcal{OPRA}$ calculus family. Our analysis is based on combining condensed semantics lower bounds with upper bounds from algebraic mappings of related calculi. This for the first time enables sound standard QSTR constraint reasoning for $\mathcal{OPRA}^*$.

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