Completeness for Game Logic

Sebastian Enqvist*, Helle Hvid Hansen†, Clemens Kupke‡, Johannes Marti§ and Yde Venema¶
*Stockholm University, SE-10691 Stockholm. Email: thesebastianenqvist@gmail.com
†Delft University of Technology, P.O. Box 5015, NL-2600 GA Delft. Email: h.h.hansen@tudelft.nl
‡University of Strathclyde, 16 Richmond St, Glasgow G1 1XQ, UK. Email: clemens.kupke@strath.ac.uk
§Universität Bremen, Postfach 330440, 28334 Bremen. Email: johannes.marti@gmail.com
¶University of Amsterdam, P.O. Box 94242, NL-1090 GE Amsterdam. Email: y.venema@uva.nl

***Abstract*—Game logic was introduced by Rohit Parikh in the 1980s as a generalisation of propositional dynamic logic (PDL) for reasoning about outcomes that players can force in determined 2-player games. Semantically, the generalisation from programs to games is mirrored by moving from Kripke models to monotone neighbourhood models. Parikh proposed a natural PDL-style Hilbert system which was easily proved to be sound, but its completeness has thus far remained an open problem.**

In this paper, we introduce a cut-free sequent calculus for game logic, and two cut-free sequent calculi that manipulate annotated formulas, one for game logic and one for the monotone μ-calculus, the variant of the polymodal μ-calculus where the semantics is given by monotone neighbourhood models instead of Kripke structures. We show these systems are sound and complete, and that completeness of Parikh's axiomatization follows. Our approach builds on recent ideas and results by Afshari & Leigh (LICS 2017) in that we obtain completeness via a sequence of proof transformations between the systems. A crucial ingredient is a validity-preserving translation from game logic to the monotone μ-calculus.

I. Introduction

A. Game logic, background and motivations

Game logic was introduced by Parikh in the 1980s [1] as a modal logic for reasoning about the outcomes that players can force in determined 2-player games. We refer to the two players as *Angel* and *Demon*, following [2]. A modal formula $\langle\gamma\rangle\varphi$ should be read as, *"Angel has a strategy in the game γ to ensure an outcome in which φ holds"*.

Syntactically, Parikh's game logic is an extension of propositional dynamic logic (PDL) [3] as games are composed from atomic games and constructors that denote sequential composition of games, as well as choice, iteration and test for Angel, and finally the dual operator which denotes swapping the roles of the two players. In Parikh's original language, the strategic ability of Demon is thus only implicitly expressed through the dual operator, and PDL programs can be viewed as 1-player games (played by Angel). Semantically, the generalisation from 1-player games to 2-player games is obtained by moving from Kripke structures to monotone neighbourhood structures. Game logic is thus a non-normal, monotone modal logic.

Just as PDL can be translated into the (normal) modal μ-calculus [4], game logic can be naturally translated into the monotone modal μ-calculus [5], and from there into the normal modal μ-calculus for the language that has two normal modalities for each monotone modality [6]. This was already sketched by Parikh in [1], and later improved in [2], [5] to show that the satisfiability of game logic is in EXPTIME. We refer to [1] and the survey [2] for applications of game logic and further results.

B. A landscape of logics for games

Parikh's game logic is probably the first of a family of logics designed to reason about different aspects of games. Since then, modal logics for multi-player games that can express strategic powers of groups of agents have appeared such as ATL [7] and Coalition Logic [8]. There are also logics that focus on 2-player games but go beyond game logic such as strategy logics [9], [10], which treat strategies as first-order objects, and dGL [11] which combines game operations and first-order quantification for hybrid games.

C. The challenge of completeness for game logic

It is a long-standing open question whether a complete proof system for game logic exists. The completeness result for dGL in [11] is of a rather different nature, since it concerns the completeness of a non-recursively enumerable logic relative to some oracle logic. Parikh proposed in [1] a natural-looking PDL-like Hilbert system Par, but a proof of its completeness has thus far remained an open problem. Only (relatively easy) partial results were known: completeness for the dual-free fragment [1], and for the iteration-free fragment [2], [5]. Giving a completeness proof similar to the one for PDL from [12] using canonical models seems impossible for the full language of game logic as such a proof essentially involves a filtration argument. It is not difficult to see, however, that game logic is not well-behaved with respect to filtrations.

The difficulty of showing completeness for the entire language of game logic can perhaps be explained by the fact that in the presence of both angelic iteration and dual, game logic (when interpreted over Kripke frames) spans all levels of the alternation hierarchy of the (normal) modal μ-calculus [13]. This is in stark contrast with PDL, LTL and CTL* which are all contained in low levels of the alternation hierarchy. Over Kripke models, game logic is thus a highly expressive fragment of the modal μ-calculus for which completeness is highly involved. The classical automata-based approach to the completeness of the μ-calculus from [14], [15] relies on the existence of "disjunctive" normal forms in the language of the μ-calculus. It is unlikely that a similar normal form can be defined for the more rigid game logic syntax, as occurrences of the $\times$-operator introduce greatest fixpoint operators that are invariably tied to conjunctions.

D. Main results and approach

In this paper, we introduce three cut-free sequent calculi, two for game logic and one for the monotone μ-calculus, that we show all to be sound and complete. The first of these is the system for game logic G which is a cut-free sequent calculus with deep inference rules. We show that G is complete, and that this implies completeness of Parikh's Hilbert system. One of the rules in G is a so-called *strengthened induction rule*, which is inspired by the strengthened induction rule in [16], and somewhat similar to Kozen's context rule [14, Proposition 5.7(vi)]. Our approach relies on game logic being able to express this rule. Just as it is convenient to work with μ-calculus formulas in negation normal form, the system G works on game logic formulas in *normal form*, where negation may only be applied to atomic propositions, and the dual game operator only to atomic games. Consequently, the system G is defined for the normal form language $\mathcal{L}_{\mathsf{NF}}$ which contains demonic game constructors as primitives. Given a game logic formula φ, $\mathsf{nf}(\varphi)$ is the formula obtained by bringing φ into dual and negation normal form.

The second system for game logic, called CloG, is a cut-free sequent calculus with a closure rule. In CloG, game logic formulas from $\mathcal{L}_{\mathsf{NF}}$ are annotated with *names* for formulas of the form $\langle\gamma^{\times}\rangle\varphi$. These names keep track of unfoldings of these greatest fixpoint formulas, and together with the closure rule they facilitate the detection of repeated unfolding of greatest fixpoints formulas in the same context (which closes the proof tree branch). Technically, this is achieved by imposing side conditions on the closure rule in CloG. These side conditions involve an order $\preccurlyeq$ on the set F consisting of game logic formulas of the form $\langle\gamma^{*}\rangle\varphi$ or $\langle\gamma^{\times}\rangle\varphi$. These game logic fixpoint formulas will be in 1-1 correspondence with fixpoint variables when we translate into the monotone μ-calculus.

The third system, CloM, is a cut-free sequent calculus for the monotone μ-calculus, also with a closure rule and name annotations. This system is a monotone variant of the system Clo for the normal modal μ-calculus introduced in [16]. In CloM, the side conditions are expressed with the usual (priority/subsumption) order $\leq$ on fixpoint variables where $x \leq y$ means that x is of higher priority than y.

Our approach to proving soundness and completeness builds on recent work by Afshari & Leigh. In [16] they presented a cut-free sequent calculus for the normal modal μ-calculus, and proved its completeness via a series of transformations through other proof systems, including the system Clo, and ending at the complete tableaux systems with names developed in [17] and [18]. We prove completeness of the systems G, CloG and CloM by showing that we can transform derivations as follows:

$$\mathsf{Par} \xleftarrow{1)} \mathsf{G} \xleftarrow{2)} \mathsf{CloG} \xleftarrow{3)} \mathsf{CloM} \xleftarrow{4)} \mathsf{Clo}$$

1) First, the transformation of G-derivations to Par-derivations goes via an intermediate Hilbert system $\mathsf{Par}_{\mathsf{Full}}$, which is an extension of Par to the full language which has angelic as well as explicit demonic operations and freely-placed duals and negations. These transformations are relatively straightforward using that Par essentially has cut via modus ponens.

2) The transformation of CloG-derivations to G-derivations requires non-trivial adaptations of the analogous result in [16, Theorem VI.1]. It uses a translation $(-)^{\bullet}$ that replaces annotations on game logic formulas with certain "deep insertions of demonic tests", which are the game logic analogues of the "deep disjunctions" of [16].

3) The transformation of CloM-derivations into CloG-derivations relies on a novel translation $(-)^{\sharp}$ from game logic into the monotone μ-calculus. This translation is truth- and validity-preserving, it commutes with fixpoint unfolding, and crucially, it reflects the order on fixpoint variables in $\varphi^{\sharp}$ into the order on fixpoint formulas in F. Note that the translation of game logic from [2] goes into the two-variable fragment of modal μ-calculus, and it is therefore not useful for the proof transformations in this paper. Indeed, we see the translation $(-)^{\sharp}$ as one of our main technical contributions.

4) Finally, we obtain completeness of CloM from the completeness of Clo [16] by transforming Clo-derivations into CloM-derivations via a validity-preserving translation $(-)^{t}$ which is the fixpoint extension of a well-known translation of monotone modal logic into normal modal logic [6].

To summarise, completeness of Parikh's system Par is obtained by the following argument. Assume that φ is a game logic formula that is valid over monotone neighbourhood models. As the above mentioned translations are validity-preserving, the normal modal μ-calculus formula $((\mathsf{nf}(\varphi))^{\sharp})^{t}$ is valid over Kripke models. By completeness of Clo, there is a Clo-derivation of $((\mathsf{nf}(\varphi))^{\sharp})^{t}$. By the above sequence of transformations, we obtain a Par-derivation of φ.

E. Outline

The paper is organised as follows. In Section II we recall the basic definitions of game logic, we introduce Parikh's Hilbert-style axiomatisation Par, we present the cut-free Gentzen style system G and show that G-derivations can be transformed into Par-derivations (Thm 11). In Section III, we introduce the annotated proof system CloG for game logic and show how CloG-derivations can be translated into G-derivations (Thm. 15). In Section IV we define the annotated system CloM for the monotone μ-calculus and prove its soundness and completeness by connecting it to the Clo-system from [16] using the standard simulation of monotone modal logic with a binormal modal logic. In Section V, we show how CloM-derivations can be transformed into corresponding CloG-derivations using the translation $(-)^{\sharp}$ of game logic into the monotone μ-calculus (Thm. 26). In Section VI, we apply the transformation results to prove soundness and completeness of CloG, G and Par. Finally, in Section VII we conclude and discuss related and future work. Due to space limitations we only provide proofs of key results. All proofs that have been omitted from the main text can be found in the appendix.

II. Two derivation systems

A. Game logic: basics

Throughout, we assume fixed countable sets P_0 and G_0 of atomic propositions and atomic games, respectively. Over these sets we shall define three distinct languages of game logic. Parikh's original language $\mathcal{L}_{\mathsf{Par}}$ only allows the angelic version of game constructors, while dual and negation may occur freely, The *normal form language* $\mathcal{L}_{\mathsf{NF}}$ allows both angelic and demonic game constructors, while negation and duals may only occur in front of atoms. The *full* language $\mathcal{L}_{\mathsf{Full}}$ allows all connectives and game constructors from the other two languages, and freely placed duals and negations.

Definition 1. The languages $\mathcal{L}_{\mathsf{Par}}$ and $\mathcal{L}_{\mathsf{NF}}$ consist of the formulas and games generated by the following grammars:

$$\begin{array}{lcl}
\mathcal{L}_{\mathsf{Par}} \ni \varphi & ::= & p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle\gamma\rangle\varphi,\ \gamma \in \mathcal{G}_{\mathsf{Par}} \\
\mathcal{G}_{\mathsf{Par}} \ni \gamma & ::= & g \mid \gamma\,;\gamma \mid \gamma \sqcup \gamma \mid \gamma^* \mid \gamma^d \mid \varphi?,\ \varphi \in \mathcal{L}_{\mathsf{Par}} \\
\mathcal{L}_{\mathsf{NF}} \ni \varphi & ::= & p \mid \neg p \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \langle\gamma\rangle\varphi,\ \gamma \in \mathcal{G}_{\mathsf{NF}} \\
\mathcal{G}_{\mathsf{NF}} \ni \gamma & ::= & g \mid g^d \mid \gamma\,;\gamma \mid \gamma \sqcup \gamma \mid \gamma \sqcap \gamma \mid \gamma^* \mid \gamma^\times \\
 & & \mid \varphi? \mid \varphi!,\ \varphi \in \mathcal{L}_{\mathsf{NF}} \\
\mathcal{L}_{\mathsf{Full}} \ni \varphi & ::= & p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle\gamma\rangle\varphi,\ \gamma \in \mathcal{G}_{\mathsf{Full}} \\
\mathcal{G}_{\mathsf{Full}} \ni \gamma & ::= & g \mid \gamma\,;\gamma \mid \gamma \sqcup \gamma \mid \gamma \sqcap \gamma \mid \gamma^* \mid \gamma^\times \mid \gamma^d \\
 & & \mid \varphi?,\ \varphi \in \mathcal{L}_{\mathsf{Full}}
\end{array}$$

where $p \in \mathsf{P}_0$ and $g \in \mathsf{G}_0$. In $\mathcal{L}_{\mathsf{Par}}$ and $\mathcal{L}_{\mathsf{Full}}$ we admit the connectives $\rightarrow, \wedge, \leftrightarrow$ as the usual abbreviations.

The game operations should be read as follows. The *composition* $\gamma\,;\delta$ means first play γ, then play δ. The *angelic choice* $\gamma \sqcup \delta$ is the game where Angel decides whether to play γ or δ. The *angelic iteration* γ^* is the game in which γ is played a finite, possibly zero, number of times, with Angel at the start and after each round deciding whether to stop or play one more round of γ. The *angelic test* $\psi?$ is the game in which φ is evaluated, and Angel immediately "loses" if ψ is false, and otherwise play continues. The *dual game* γ^d is the game in which the roles of the two players are interchanged, i.e., the strategies of Angel in γ^d are exactly the strategies of Demon in γ, and vice versa. The definitions of the demonic operations are such that (cf. [2]):

$$\gamma \sqcap \delta = (\gamma^d \sqcup \delta^d)^d, \quad \gamma^\times = ((\gamma^d)^*)^d, \quad \psi! = ((\neg\psi)?)^d \quad (1)$$

The interpretation of the demonic operations is obtained by replacing "Angel" with "Demon" in the above. However, since a modal formula $\langle\gamma\rangle\varphi$ expresses the strategic ability of Angel in γ, $\langle\gamma \sqcap \delta\rangle\varphi$ means that Angel has a strategy to achieve φ in both γ and δ, and $\langle\gamma^\times\rangle\varphi$ means that Angel has a strategy for maintaining φ indefinitely when playing γ repeatedly, and not knowing when the iteration terminates. Finally, $\psi!$ is the game in which Angel immediately "wins" if ψ is true. Hence, $\langle\psi!\rangle\varphi$ is true if at least one of ψ and φ is true.

We will often refer to formulas and games jointly as *terms*. We denote the *subterm relation* by $\trianglelefteq$, using $\triangleleft$ for the strict version. For example, $g^\times \triangleleft \langle g^\times\,; h\rangle p$ and $h \triangleleft (\langle h\rangle p?)\,; g$.

Formulas of the form $\langle\gamma^*\rangle\varphi$ or $\langle\gamma^\times\rangle\varphi$ will play the role of fixpoint variables on the game logic side. In particular, we need to define an order $\prec$ on them, but it is not immediately clear how to do that. For example, a naive approach based on the subformula-relation will not work, since we need that, e.g., $\langle(g^\times \sqcup h)^\times\rangle p \prec \langle g^\times\rangle p$. Our solution is to use the converse subterm relation on the game terms that label the modalities.

Definition 2. We define the set of least, greatest, respectively all, *fixpoint formulas* in $\mathcal{L}_{\mathsf{NF}}$ as follows:

$$\begin{array}{lcl}
F^* & := & \{\langle\gamma^*\rangle\varphi \mid \gamma \in \mathcal{G}_{\mathsf{NF}}, \varphi \in \mathcal{L}_{\mathsf{NF}}\}, \\
F^\times & := & \{\langle\gamma^\times\rangle\varphi \mid \gamma \in \mathcal{G}_{\mathsf{NF}}, \varphi \in \mathcal{L}_{\mathsf{NF}}\}, \\
F & := & F^* \cup F^\times.
\end{array}$$

We define an order $\prec$ on F by setting $\langle\gamma^\circ\rangle\varphi \prec \langle\delta^\dagger\rangle\psi$ for $\circ, \dagger \in \{*, \times\}$ if $\delta^\dagger \triangleleft \gamma^\circ$. We write $\langle\gamma^\circ\rangle\varphi \preccurlyeq \langle\delta^\dagger\rangle\psi$ if $\delta^\dagger \prec \gamma^\circ$ or $\delta^\dagger = \gamma^\circ$.

It should be clear that $\prec$ is transitive and irreflexive.

We need the following notion of (Fischer-Ladner) *closure*.

Definition 3. The *closure* $Cl(\xi)$ of a formula $\xi \in \mathcal{L}_{\mathsf{NF}}$ is the smallest subset of $\mathcal{L}_{\mathsf{NF}}$ that contains ξ and is closed under subformulas as well as the following rules: If $\langle\gamma^*\rangle\varphi \in Cl(\xi)$ then $\varphi \vee \langle\gamma\rangle\langle\gamma^*\rangle\varphi \in Cl(\xi)$. If $\langle\gamma^\times\rangle\varphi \in Cl(\xi)$ then $\varphi \wedge \langle\gamma\rangle\langle\gamma^\times\rangle\varphi \in Cl(\xi)$. If $\langle\psi?\rangle\varphi \in Cl(\xi)$ then $\psi \in Cl(\xi)$. If $\langle\psi!\rangle\varphi \in Cl(\xi)$ then $\psi \in Cl(\xi)$. The sets $F(\xi), F^*(\xi), F^\times(\xi)$ of all/least/greatest *fixpoint formulas of a formula* $\xi \in \mathcal{L}_{\mathsf{NF}}$ are given as $F(\xi) := F \cap Cl(\xi)$, etc.

The simplest way to define the semantics of these languages is as follows [2]. We denote by $\mathcal{M}(S)$ the set of all monotone maps $f\colon \wp(S) \to \wp(S)$. An *effectivity function for a game* γ *on a set* S is then a $E_\gamma \in \mathcal{M}(S)$, and $s \in E_\gamma(Y)$ means that at position s, Angel is effective for Y in γ, i.e., Angel has a strategy in γ that ensures that the outcome of γ is in Y.

Definition 4. A *game model* is a triple $\mathbb{S} = (S, E, V)$ such that $V : \mathsf{P}_0 \to \wp(S)$ is a *valuation* and $E\colon \mathsf{G}_0 \to \mathcal{M}(S)$ assigns an effectivity function on S to every atomic $g \in \mathsf{G}_0$. By a mutual induction on formulas and games, we define the *meaning* $[\![\varphi]\!]^{\mathbb{S}}$ of a formula φ in a model $\mathbb{S}$, and the effectivity function E_γ in $\mathbb{S}$ for complex games γ as follows:

$$\begin{array}{lcl}
[\![p]\!]^{\mathbb{S}} & := & V(p) \\
[\![\neg p]\!]^{\mathbb{S}} & := & S \setminus p \\
[\![\varphi \vee \psi]\!]^{\mathbb{S}} & := & [\![\varphi]\!]^{\mathbb{S}} \cup [\![\psi]\!]^{\mathbb{S}} \\
[\![\varphi \wedge \psi]\!]^{\mathbb{S}} & := & [\![\varphi]\!]^{\mathbb{S}} \cap [\![\psi]\!]^{\mathbb{S}} \\
[\![\langle\gamma\rangle\varphi]\!]^{\mathbb{S}} & := & E_\gamma([\![\varphi]\!]^{\mathbb{S}}) \\
E_g(X) & := & E(g)(X) \\
E_{(\gamma^d)}(X) & := & S \setminus E_\gamma(S \setminus X) \\
E_{\gamma\,;\,\delta}(X) & := & E_\gamma(E_\delta(X)) \\
E_{\gamma\sqcup\delta}(X) & := & E_\gamma(X) \cup E_\delta(X) \\
E_{\gamma\sqcap\delta}(X) & := & E_\gamma(X) \cap E_\delta(X) \\
E_{(\gamma^*)}(X) & := & \mathsf{lfp}\, Y.\, X \cup E_\gamma(Y) \\
E_{(\gamma^\times)}(X) & := & \mathsf{gfp}\, Y.\, X \cap E_\gamma(Y) \\
E_{(\varphi?)}(X) & := & [\![\varphi]\!]^{\mathbb{S}} \cap X \\
E_{(\varphi!)}(X) & := & [\![\varphi]\!]^{\mathbb{S}} \cup X
\end{array}$$

Notions like satisfiability, equivalence, etc., are defined in the standard way. In particular, a game formula φ is *valid*, notation: $\models \varphi$, if $[\![\varphi]\!]^{\mathbb{S}} = S$, for every game model $\mathbb{S} = (S, E, V)$.

Proposition 5. There are recursively defined, truth-preserving translations

$$\begin{array}{ll}
\mathsf{nf}(-): & \mathcal{L}_{\mathsf{Full}} \to \mathcal{L}_{\mathsf{NF}} \\
\mathsf{pa}(-): & \mathcal{L}_{\mathsf{Full}} \to \mathcal{L}_{\mathsf{Par}}
\end{array}$$

As a corollary of this, negation is definable in $\mathcal{L}_{\mathsf{NF}}$. We shall need the following explicit definition in the sequel.

Definition 6. By a mutual induction we define the *complementation* $\overline{\varphi} := \mathsf{nf}(\neg\varphi)$ of an $\mathcal{L}_{\mathsf{NF}}$-formula φ, and the *dual game* $\widetilde{\gamma}$ of an $\mathcal{L}_{\mathsf{NF}}$-game γ:

$$\begin{array}{lcl}
\overline{p} & := & \neg p \\
\overline{\neg p} & := & p \\
\overline{\varphi \vee \psi} & := & \overline{\varphi} \wedge \overline{\psi} \\
\overline{\varphi \wedge \psi} & := & \overline{\varphi} \vee \overline{\psi} \\
\overline{\langle\gamma\rangle\varphi} & := & \langle\widetilde{\gamma}\rangle\overline{\varphi}
\end{array}
\qquad
\begin{array}{lcl}
\widetilde{g} & := & g^d \\
\widetilde{(g^d)} & := & g \\
\widetilde{\gamma\,;\,\delta} & := & \widetilde{\gamma}\,;\,\widetilde{\delta} \\
\widetilde{\gamma \sqcup \delta} & := & \widetilde{\gamma} \sqcap \widetilde{\delta} \\
\widetilde{\gamma \sqcap \delta} & := & \widetilde{\gamma} \sqcup \widetilde{\delta} \\
\widetilde{(\gamma^*)} & := & (\widetilde{\gamma})^\times \\
\widetilde{(\gamma^\times)} & := & (\widetilde{\gamma})^* \\
\widetilde{(\varphi?)} & := & \overline{\varphi}! \\
\widetilde{(\varphi!)} & := & \overline{\varphi}?
\end{array}$$

The following proposition is proved by a straightforward induction. We leave the details to the reader.

Proposition 7. In any game model $\mathbb{S} = (S, E, V)$ we have

$$[\![\overline{\varphi}]\!]^{\mathbb{S}} = S \setminus [\![\varphi]\!]^{\mathbb{S}} \text{ and } E_{\widetilde{\gamma}} = E_{\gamma^d}$$

for any formula $\varphi \in \mathcal{L}_{\mathsf{NF}}$ and game $\gamma \in \mathcal{G}_{\mathsf{NF}}$.

B. Parikh's Hilbert-style system

The first axiom system for game logic was proposed and conjectured to be complete by Parikh [1]. This is a Hilbert-style system for the language $\mathcal{L}_{\mathsf{Par}}$ that axiomatises the angelic iteration with what Parikh calls *Bar Induction*. We will refer to this system as Par, and it is shown in Figure 1 below. For $\varphi \in \mathcal{L}_{\mathsf{Par}}$, we write $\mathsf{Par} \vdash \varphi$ if there is a Par-derivation of φ.

Par **Axioms:**

1) All propositional tautologies.
2) $\langle\gamma\,;\,\delta\rangle\varphi \leftrightarrow \langle\gamma\rangle\langle\delta\rangle\varphi$
3) $\langle\gamma \sqcup \delta\rangle\varphi \leftrightarrow \langle\gamma\rangle\varphi \vee \langle\delta\rangle\varphi$
4) $\langle\gamma^*\rangle\varphi \leftrightarrow \varphi \vee \langle\gamma\rangle\langle\gamma^*\rangle\varphi$
5) $\langle\psi?\rangle\varphi \leftrightarrow \psi \wedge \varphi$
6) $\langle\gamma^d\rangle\varphi \leftrightarrow \neg\langle\gamma\rangle\neg\varphi$

Par **Rules:**

$$\frac{\varphi \qquad \varphi \to \psi}{\psi}\ \mathsf{MP} \qquad \frac{\varphi \to \psi}{\langle\gamma\rangle\varphi \to \langle\gamma\rangle\psi}\ \mathsf{Mon} \qquad \frac{\langle\gamma\rangle\varphi \to \varphi}{\langle\gamma^*\rangle\varphi \to \varphi}\ \mathsf{BarInd}$$

Fig. 1. Axioms and rules of Par.

The system Par is easily seen to be sound. A main contribution of our paper is that we confirm Parikh's completeness conjecture. We prove Theorem 8 in section VI below.

Theorem 8 (Soundness and Completeness of Par)**.** For every formula $\varphi \in \mathcal{L}_{\mathsf{Par}}$, we have: $\mathsf{Par} \vdash \varphi$ iff $\models \varphi$.

C. The cut-free sequent system G *for game logic*

We now introduce a cut-free (Tait-style) sequent system G for game logic formulas in normal form. A *sequent* is thus defined as a finite set of $\mathcal{L}_{\mathsf{NF}}$-formulas (to be read disjunctively). For a finite set $\Phi \subseteq \mathcal{L}_{\mathsf{NF}}$, we define $\overline{\Phi} \in \mathcal{L}_{\mathsf{NF}}$ as the normal form $\overline{\bigvee \Phi}$ of $\neg(\bigvee \Phi)$.

The system G consists of several parts. Its core is the sequent calculus version of monotone modal logic as shown in Figure 2. In order to reason about game operators, in Fig. 3 we list some rules, each of which directly mirrors the semantic meaning of one game constructor.

$$\frac{}{\Phi, \overline{\Phi}}\ \mathsf{Ax} \qquad \frac{\Phi}{\Phi, \varphi}\ \mathsf{weak} \qquad \frac{\varphi, \psi}{\langle g\rangle\varphi, \langle g^d\rangle\psi}\ \mathsf{mod}_m$$

$$\frac{\Phi, \varphi, \psi}{\Phi, \varphi \vee \psi}\ \vee \qquad \frac{\Phi, \varphi \qquad \Phi, \psi}{\Phi, \varphi \wedge \psi}\ \wedge$$

Fig. 2. The basic rules of the sequent calculus mon-ML for Game Logic.

$$\frac{\Phi, \varphi \vee \langle\gamma\rangle\langle\gamma^*\rangle\varphi}{\Phi, \langle\gamma^*\rangle\varphi}\ * \qquad \frac{\Phi, \langle\gamma\rangle\varphi \vee \langle\delta\rangle\varphi}{\Phi, \langle\gamma \sqcup \delta\rangle\varphi}\ \sqcup \qquad \frac{\Phi, \psi \wedge \varphi}{\Phi, \langle\psi?\rangle\varphi}\ ?$$

$$\frac{\Phi, \varphi \wedge \langle\gamma\rangle\langle\gamma^\times\rangle\varphi}{\Phi, \langle\gamma^\times\rangle\varphi}\ \times \qquad \frac{\Phi, \langle\gamma\rangle\varphi \wedge \langle\delta\rangle\varphi}{\Phi, \langle\gamma \sqcap \delta\rangle\varphi}\ \sqcap \qquad \frac{\Phi, \psi \vee \varphi}{\Phi, \langle\psi!\rangle\varphi}\ !$$

Fig. 3. The sequent calculus rules GameOp for game operations.

In the third part of the G proof system we have the three "deep" derivation rules given in Figure 4. These rules are needed for technical reasons, as will become clear in some of the proofs further on.

The final ingredient of G is the strengthened induction rule ind_s in Figure 5. This rule, just like the homonymous rule in [16] on which it is based, detects unfoldings of greatest fixpoints in the same context. This may become clearer when

$$\frac{\Phi, \psi(\gamma)}{\Phi, \psi(\chi! \,;\gamma)}\ \mathsf{Mon}^{\mathsf{g}}_{\mathsf{d}} \qquad \frac{\Phi, \psi(\varphi)}{\Phi, \psi(\langle \chi! \rangle \varphi)}\ \mathsf{Mon}^{\mathsf{f}}_{\mathsf{d}} \qquad \frac{\Phi, \psi(\langle \gamma \rangle \langle \delta \rangle \varphi)}{\Phi, \psi(\langle \gamma \,; \delta \rangle \varphi)}\ ;_d$$

Fig. 4. Deep rules for Game Logic: DeepG. The notation $\psi(\varphi)$ should be read as follows: ψ is a context, i.e., a formula with a unique occurrence of a proposition letter p, and $\psi(\varphi)$ is the formula obtained by substituting p for φ in ψ.

we show in Theorem 15 how ind_s is used to translate the closure rule of the system CloG. In this sense, ind_s plays a role similar to the *context rule* [14, Proposition 5.7(vi)] in Kozen's completeness proof for the aconjunctive fragment of the modal μ-calculus. Only, Kozen's proof is based on satisfiability and the context rule therefore deals with least fixpoint unfoldings. Our approach is based on validity, and ind_s therefore detects greatest fixpoint unfoldings.

$$\frac{\Phi, \varphi \wedge \langle \gamma \rangle \langle (\overline{\Phi}! \,; \gamma)^{\times} \rangle \langle \overline{\Phi}! \rangle \varphi}{\Phi, \langle \gamma^{\times} \rangle \varphi}\ \mathsf{ind}_s$$

Fig. 5. Strengthened induction rule for Game Logic.

To obtain a more concrete understanding of the ind_s rule, think of the formula $\langle \gamma^{\times} \rangle \varphi$ as a greatest fixpoint formula $\nu x.\varphi \wedge \langle \gamma \rangle x$. The "standard" fixpoint rule for $\gamma^{\times}$ would read as follows: "from $\psi \to \varphi \wedge \langle \gamma \rangle \psi$ infer $\psi \to \langle \gamma^{\times} \rangle \varphi$", or, formulated as a Tait-style sequent rule:

$$\frac{\Phi, \varphi \wedge \langle \gamma \rangle \overline{\Phi}}{\Phi, \langle \gamma^{\times} \rangle \varphi}\ \mathsf{ind}$$

Now, observing that $\langle \overline{\Phi}! \rangle \varphi \equiv \overline{\Phi} \vee \varphi$, one may see that ind_s is indeed a variation of ind.

Some further understanding of the rule ind_s may be gained by establishing its *soundness*. For this purpose we may reason by contraposition, showing that the refutability of the conclusion of ind_s implies the refutability of its premise. It is not hard to see that this boils down to proving the following statement, which is formulated using the dual formulas and games.

Proposition 9. If $\chi \wedge \langle \gamma^{*} \rangle \varphi$ is satisfiable, then so is either $\chi \wedge \varphi$ or $\chi \wedge \langle \gamma \rangle \langle (\overline{\chi}? \,; \gamma)^{*} \rangle \langle \overline{\chi}? \rangle \varphi$.

In words, this Proposition states the following. Suppose that there is a situation where χ holds and where Angel has a strategy in the game γ^{*} ensuring the outcome φ. Suppose furthermore that χ and φ cannot be true simultaneously. Then there is a situation where χ holds, and where Angel has a strategy in γ^{*} which not only ensures that φ holds afterwards, but also guarantees that while playing γ^{*}, after each round of playing γ, the formula $\overline{\chi}$ holds.

The completeness of G will follow from the completeness of the system CloG, which we introduce in the next section. The proof of Theorem 10 will be outlined in Section VI.

Theorem 10 (Soundness and Completeness of G)**.** For all $\xi \in \mathcal{L}_{\mathsf{NF}}$, we have: $\mathsf{G} \vdash \xi$ iff $\models \xi$.

The following theorem states the transformation results between G and Par that are needed for transferring soundness from Par to G, and completeness from G to Par.

Theorem 11. We have:

1) For all $\varphi \in \mathcal{L}_{\mathsf{Par}}$, if $\mathsf{G} \vdash \mathsf{nf}(\varphi)$ then $\mathsf{Par} \vdash \varphi$.
2) For all $\xi \in \mathcal{L}_{\mathsf{NF}}$, if $\mathsf{G} \vdash \xi$ then $\mathsf{Par} \vdash \mathsf{pa}(\xi)$.

III. An annotated proof system

The completeness of G will follow from the completeness of the annotated tableau system CloG which we introduce now.

A. The CloG system for Game Logic

In CloG, formulas are annotated with names that are used to detect repeated unfoldings of greatest fixpoint formulas in the same context. With each greatest fixpoint formula $\varphi \in F^{\times}$ we associate a countable set N_{φ} of *names for* φ. We assume that $N_{\varphi} \cap N_{\psi} = \emptyset$ if $\varphi \neq \psi$. The set of all names is $N = \bigcup_{\varphi \in F^{\times}} N_{\varphi}$. Names will typically be denoted by $\mathsf{x}, \mathsf{y}, \ldots$ or with subscripts $\mathsf{x}_0, \mathsf{x}_1, \ldots$. Names inherit the order $\preccurlyeq$ on the set F of fixpoint formulas: For all $\mathsf{x} \in N_{\varphi}$, $\mathsf{y} \in N_{\psi}$, we define $\mathsf{x} \preccurlyeq \mathsf{y}$ iff $\varphi \preccurlyeq \psi$. For a sequence of names $\mathsf{a} = \mathsf{x}_0, \mathsf{x}_1, \ldots, \mathsf{x}_{n-1} \in N^{*}$ and a fixpoint formula $\varphi \in F$, we will write $\mathsf{a} \preccurlyeq \varphi$ if for all x_i occurring in a, $\mathsf{x}_i \in N_{\psi}$ such that $\psi \preccurlyeq \varphi$. The empty sequence is denoted by ε. An *annotation* is a sequence $\mathsf{a} = \mathsf{x}_0, \mathsf{x}_1, \ldots, \mathsf{x}_{n-1} \in N^{*}$ that is non-repeating and monotone w.r.t. $\preccurlyeq$, i.e., for all $i < n-1$, $\mathsf{x}_i \preccurlyeq \mathsf{x}_{i+1}$. An *annotated game logic formula* φ^{a} consists of a formula $\varphi \in \mathcal{L}_{\mathsf{NF}}$ and an annotation $\mathsf{a} \in N^{*}$. *Annotated* CloG*-sequents* are finite sets of annotated game logic formulas, and will be denoted by Φ, Ψ, etc.

The system CloG derives CloG-sequents using the axiom and rules in Figure 6. The *closure rule* $\mathsf{clo}_{\mathsf{x}}$ discharges *all* occurrences of the sequent $\Phi, \langle \gamma^{\times} \rangle \varphi^{\mathsf{ax}}$ appearing as an assumption above the proof node where the rule is applied. The side conditions ensure that no fixpoint formula of higher priority than $\langle \gamma^{\times} \rangle \varphi$ is unfolded between the application of $\mathsf{clo}_{\mathsf{x}}$ and its discharged assumption.

A CloG-proof is a finite tree of CloG-inferences in which each leaf is labelled by an axiom or a discharged assumption. Intuitively, a CloG-proof can be understood as a finitary representation of a non-wellfounded/circular proof. The discharged assumptions are the nodes where the circularity is detected. For a formula $\xi \in \mathcal{L}_{\mathsf{NF}}$, we write $\mathsf{CloG} \vdash \xi$ to mean that there is a CloG-proof of ξ^{ε}. Note that CloG is *analytic* in the sense that any CloG-proof of ξ^{ε} will contain only formulas from $Cl(\xi)$, and names for fixpoint formulas in $F^{\times}(\xi)$.

Completeness of CloG will follow from the completeness of the system CloM, which we introduce in Section IV-B. We prove Theorem 12 in Section VI.

Theorem 12 (Soundness and Completeness of CloG)**.** For all $\xi \in \mathcal{L}_{\mathsf{NF}}$, we have $\mathsf{CloG} \vdash \xi$ iff $\models \xi$.

$$\frac{}{p^{\varepsilon}, (\neg p)^{\varepsilon}}\ \mathsf{Ax1} \qquad \frac{\varphi^{\mathsf{a}}, \psi^{\mathsf{b}}}{(\langle g\rangle\varphi)^{\mathsf{a}}, (\langle g^d\rangle\psi)^{\mathsf{b}}}\ \mathsf{mod}_m$$

$$\frac{\Phi, \varphi^{\mathsf{a}}, \psi^{\mathsf{a}}}{\Phi, (\varphi \vee \psi)^{\mathsf{a}}}\ \vee \qquad \frac{\Phi, \varphi^{\mathsf{a}} \qquad \Phi, \psi^{\mathsf{a}}}{\Phi, (\varphi \wedge \psi)^{\mathsf{a}}}\ \wedge$$

$$\frac{\Phi, (\langle\gamma\rangle\varphi \vee \langle\delta\rangle\varphi)^{\mathsf{a}}}{\Phi, (\langle\gamma \sqcup \delta\rangle\varphi)^{\mathsf{a}}}\ \sqcup \qquad \frac{\Phi, (\langle\gamma\rangle\varphi \wedge \langle\delta\rangle\varphi)^{\mathsf{a}}}{\Phi, (\langle\gamma \sqcap \delta\rangle\varphi)^{\mathsf{a}}}\ \sqcap$$

$$\frac{\Phi}{\Phi, \varphi^{\mathsf{a}}}\ \mathsf{weak} \qquad \frac{\Phi, \varphi^{\mathsf{ab}},}{\Phi, \varphi^{\mathsf{axb}}}\ \mathsf{exp} \qquad \frac{\Phi, (\langle\gamma\rangle\langle\delta\rangle\varphi)^{\mathsf{a}}}{\Phi, (\langle\gamma\,;\delta\rangle\varphi)^{\mathsf{a}}}\ ;$$

$$(\mathsf{a} \preccurlyeq \langle\gamma^*\rangle\varphi)\ \frac{\Phi, (\varphi \vee \langle\gamma\rangle\langle\gamma^*\rangle\varphi)^{\mathsf{a}}}{\Phi, (\langle\gamma^*\rangle\varphi)^{\mathsf{a}}}\ * \qquad \frac{\Phi, (\psi \wedge \varphi)^{\mathsf{a}}}{\Phi, (\langle\psi?\rangle\varphi)^{\mathsf{a}}}\ ?$$

$$(\mathsf{a} \preccurlyeq \langle\gamma^\times\rangle\varphi)\ \frac{\Phi, (\varphi \wedge \langle\gamma\rangle\langle\gamma^\times\rangle\varphi)^{\mathsf{a}}}{\Phi, (\langle\gamma^\times\rangle\varphi)^{\mathsf{a}}}\ \times \qquad \frac{\Phi, (\psi \vee \varphi)^{\mathsf{a}}}{\Phi, (\langle\psi!\rangle\varphi)^{\mathsf{a}}}\ !$$

$$\begin{array}{c}[\Phi, \langle\gamma^\times\rangle\varphi^{\mathsf{ax}}]^{\mathsf{x}} \\ \vdots \end{array}$$

$$(\mathsf{a} \preccurlyeq \mathsf{x} \in N_{\langle\gamma^\times\rangle\varphi}, \mathsf{x} \notin \Phi, \mathsf{a})\ \frac{\Phi, (\varphi \wedge \langle\gamma\rangle\langle\gamma^\times\rangle\varphi)^{\mathsf{ax}}}{\Phi, (\langle\gamma^\times\rangle\varphi)^{\mathsf{a}}}\ \mathsf{clo}_{\mathsf{x}}$$

Fig. 6. The axiom and rules of the system CloG. In the side condition of clo_x, "$\mathsf{x} \notin \Phi, \mathsf{a}$" means that x does not occur in Φ or a.

B. Removing annotations with the bullet translation

In order to translate CloG-proofs into G-proofs, we must remove annotations. First, we introduce some notation. We let

$$\langle\!\langle\gamma\rangle\!\rangle\varphi := \begin{cases} \langle\!\langle\gamma_1\rangle\!\rangle\langle\!\langle\gamma_2\rangle\!\rangle\varphi & \text{if } \gamma = \gamma_1\,;\gamma_2 \\ \langle\gamma\rangle\varphi & \text{otherwise} \end{cases}$$

That is, if $\gamma = \gamma_1\,; \cdots ; \gamma_k$, and none of the game terms γ_i is itself a composition, then $\langle\!\langle\gamma\rangle\!\rangle\varphi = \langle\gamma_1\rangle \cdots \langle\gamma_k\rangle\varphi$.

Given a sequence $\vec{\varphi} = \varphi_1, \dots, \varphi_n$ of formulas and a game term γ, we define

$$\begin{aligned} \underleftarrow{\varphi!} &:= \varphi_n!\,;(\dots(\varphi_1!)\dots) \\ \underleftarrow{\varphi! \cdot \gamma} &:= \varphi_n!\,;(\dots(\varphi_1!\,;\gamma)\dots). \end{aligned}$$

We can now define the translation $(-)^{\bullet}$ which removes annotations. Intuitively, what this translation does is to weaken fixpoint formulas by adding dual tests corresponding to formulas associated with names in the annotation of a fixpoint formula. This will be used to "remember" contexts in which greatest fixpoint formulas have been unfolded. The translation needs to be set up carefully, so that it can be used to transform CloG-proofs to G-proofs. In particular, it is tailored to fit with the strengthened induction rule in G.

Definition 13. Assume that we have an assignment $\{\chi_{\mathsf{x}} \mid \mathsf{x} \in N\}$ of a game logic formula $\chi_{\mathsf{x}} \in \mathcal{L}_{\mathsf{NF}}$ to each name x. We define the *bullet translation* $(-)^{\bullet}$ from annotated game logic formulas to $\mathcal{L}_{\mathsf{NF}}$ by $\varphi^{\varepsilon\bullet} = \varphi$, and for non-empty annotations a as follows:

$$\begin{aligned} p^{\mathsf{a}\bullet} &:= p, & \beta(g, \mathsf{a}, \varphi) &:= g, \\ (\neg p)^{\mathsf{a}\bullet} &:= \neg p, & \beta(g^d, \mathsf{a}, \varphi) &:= g^d, \\ (\varphi \vee \psi)^{\mathsf{a}\bullet} &:= \varphi^{\mathsf{a}\bullet} \vee \psi^{\mathsf{a}\bullet}, & \beta(\psi?, \mathsf{a}, \varphi) &:= (\psi^{\mathsf{a}\bullet})?, \\ (\varphi \wedge \psi)^{\mathsf{a}\bullet} &:= \varphi^{\mathsf{a}\bullet} \wedge \psi^{\mathsf{a}\bullet}, & \beta(\psi!, \mathsf{a}, \varphi) &:= (\psi^{\mathsf{a}\bullet})!, \\ (\langle\gamma\rangle\varphi)^{\mathsf{a}\bullet} &:= \langle\!\langle\beta(\gamma, \mathsf{a}, \varphi)\rangle\!\rangle\varphi^{\mathsf{a}\bullet}, & \beta(\gamma^*, \mathsf{a}, \varphi) &:= \gamma^*, \end{aligned}$$

$$\begin{aligned} \beta(\gamma\,;\delta, \mathsf{a}, \varphi) &:= \beta(\gamma, \mathsf{a}, \langle\delta\rangle\varphi)\,; \beta(\delta, \mathsf{a}, \varphi), \\ \beta(\gamma \sqcup \delta, \mathsf{a}, \varphi) &:= \beta(\gamma, \mathsf{a}, \varphi) \sqcup \beta(\delta, \mathsf{a}, \varphi), \\ \beta(\gamma \sqcap \delta, \mathsf{a}, \varphi) &:= \beta(\gamma, \mathsf{a}, \varphi) \sqcap \beta(\delta, \mathsf{a}, \varphi). \end{aligned}$$

The crucial clause of the translation is the case for the demonic iteration. If $\mathsf{a} = \mathsf{b}\mathsf{x}_1 \cdots \mathsf{x}_n\mathsf{c}$, where $\mathsf{x}_1, \dots, \mathsf{x}_n$ are all the names for $\langle\gamma^\times\rangle\varphi$ in a, then we define

$$\beta(\gamma^\times, \mathsf{b}\mathsf{x}_1 \cdots \mathsf{x}_n\mathsf{c}, \varphi) := (\underleftarrow{\chi! \cdot \gamma})^\times ; \underleftarrow{\chi!}$$

where $\vec{\chi} := \chi_{\mathsf{x}_1}, \dots, \chi_{\mathsf{x}_n}$. Note that as a special case we have $\beta(\gamma^\times, \mathsf{a}, \varphi) = \gamma^\times$ if there are no names for $\langle\gamma^\times\rangle\varphi$ in a.

The bullet translation only affects the outermost fixpoint operators of a game term. This does, however, not mean that there is only ever one fixpoint affected in a formula. For instance when following the trace of the formula $\langle(g\,;(h^\times))^\times\rangle p$ in some CloG-proof the fixpoint might unravel such that we obtain the formula $\langle h^\times\rangle\langle(g\,;(h^\times))^\times\rangle p$. Applying the bullet translation to this formula might affect the outermost fixpoints of both modalities.

The following lemma shows how the bullet translation applies to annotated fixpoint formulas. It is needed in the proof of Theorem 15 below.

Lemma 14. Let $\mathsf{a} = \mathsf{b}\mathsf{x}_1 \dots \mathsf{x}_n$ where $\mathsf{x}_1, \dots, \mathsf{x}_n$ are all the names in a for $\langle\gamma^\times\rangle\varphi \in F^\times$. Then we have:

$$(\langle\gamma^\times\rangle\varphi)^{\mathsf{a}\bullet} = \langle(\underleftarrow{\chi! \cdot \gamma})^\times\rangle\langle\!\langle\underleftarrow{\chi!}\rangle\!\rangle\varphi^{\mathsf{b}\bullet} \text{ and} \tag{2}$$

$$(\varphi \wedge \langle\gamma\rangle\langle\gamma^\times\rangle\varphi)^{\mathsf{a}\bullet} = \varphi^{\mathsf{b}\bullet} \wedge \langle\!\langle\gamma\rangle\!\rangle\langle(\underleftarrow{\chi! \cdot \gamma})^\times\rangle\langle\!\langle\underleftarrow{\chi!}\rangle\!\rangle\varphi^{\mathsf{b}\bullet} \tag{3}$$

C. Embedding CloG *into* G

We are now ready to show how CloG-derivations can be transformed to G-derivations. This will be used in Section VI to transfer completeness from CloG to G and soundness from G to CloG.

Theorem 15. For all $\xi \in \mathcal{L}_{\mathsf{NF}}$, if $\mathsf{CloG} \vdash \xi$ then $\mathsf{G} \vdash \xi$.

Proof. Consider a game logic formula ξ and assume that π is a proof of ξ^{ε} in CloG. We assume that each application of the clo-rule in π introduces a distinct name, i.e., for any distinct pair of rule applications $\mathsf{clo}_{\mathsf{x}_1}$ and $\mathsf{clo}_{\mathsf{x}_2}$ in π we have $\mathsf{x}_1 \neq \mathsf{x}_2$. This assumption is w.l.o.g. as we can rename the variable names occurring in π appropriately if needed. The shape of the rules of CloG also imply that for each variable name x occurring in π, there is a corresponding occurrence of the $\mathsf{clo}_{\mathsf{x}}$-rule.

We now assign a formula χ_{x} to each variable name x occurring in π. This assignment is defined by induction on the distance of the (unique) $\mathsf{clo_x}$ instance in π from the root of π. Concretely, for a variable name x we consider the sequent Φ consisting of the side formulas of the application of $\mathsf{clo_x}$ in π and set $\chi_{\mathsf{x}} := \overline{\Phi^\bullet}$. Here the bullet translation of $\overline{\Phi}$ is well-defined as any variable name y occurring in Φ must have been introduced by an instance of $\mathsf{clo_y}$ that is closer to the root of the proof tree than $\mathsf{clo_x}$, so that the formula χ_{y} is already defined by the induction hypothesis.

We now show how to transform the CloG-proof π of ξ^{ε} into a G-proof of ξ by demonstrating that (i) for all (discharged) assumptions Φ of π there is a G-derivation of $\Phi^\bullet$, and (ii) for all CloG-rule applications Φ_1/Φ_2 in π there is a corresponding G derivation of ${\Phi_2}^\bullet$ from assumptions in ${\Phi_1}^\bullet$.

Consider first the bullet translation of an arbitrary discharged assumption of an application of $\mathsf{clo_x}$ in π. Such a translation is of the form $\Phi^\bullet, (\langle\gamma^\times\rangle\varphi)^{\mathsf{ax}\bullet}$ for some annotated sequent Φ and a game logic formula $(\langle\gamma^\times\rangle\varphi)$. Furthermore, by definition we have $\chi_{\mathsf{x}} = \overline{\Phi^\bullet}$. Now consider the following G proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Phi^\bullet, \chi_{\mathsf{x}}}{\Phi^\bullet, \chi_{\mathsf{x}}, \theta}\,\mathsf{weak}}{\Phi^\bullet, \chi_{\mathsf{x}} \vee \theta}\,\vee}{\Phi^\bullet, \langle\chi_{\mathsf{x}}!\rangle\theta}\,! \qquad \dfrac{\dfrac{\dfrac{\dfrac{\Phi^\bullet, \chi_{\mathsf{x}}}{\Phi^\bullet, \chi_{\mathsf{x}}, \langle\underleftarrow{\chi! \cdot \gamma}\rangle\langle(\chi_{\mathsf{x}}!\,;\underleftarrow{\chi! \cdot \gamma})^\times\rangle\langle\chi_{\mathsf{x}}!\rangle\theta}\,\mathsf{weak}}{\Phi^\bullet, \chi_{\mathsf{x}} \vee \langle\underleftarrow{\chi! \cdot \gamma}\rangle\langle(\chi_{\mathsf{x}}!\,;\underleftarrow{\chi! \cdot \gamma})^\times\rangle\langle\chi_{\mathsf{x}}!\rangle\theta}\,\vee}{\Phi^\bullet, \langle\chi_{\mathsf{x}}!\rangle\langle\underleftarrow{\chi! \cdot \gamma}\rangle\langle(\chi_{\mathsf{x}}!\,;\underleftarrow{\chi! \cdot \gamma})^\times\rangle\langle\chi_{\mathsf{x}}!\rangle\theta}\,!}{\Phi^\bullet, \langle\chi_{\mathsf{x}}!\,;\underleftarrow{\chi! \cdot \gamma}\rangle\langle(\chi_{\mathsf{x}}!\,;\underleftarrow{\chi! \cdot \gamma})^\times\rangle\langle\chi_{\mathsf{x}}!\rangle\theta}\,;_d}{\dfrac{\Phi^\bullet, \langle\chi_{\mathsf{x}}!\rangle\theta \wedge \langle\chi_{\mathsf{x}}!\,;\underleftarrow{\chi! \cdot \gamma}\rangle\langle(\chi_{\mathsf{x}}!\,;\underleftarrow{\chi! \cdot \gamma})^\times\rangle\langle\chi_{\mathsf{x}}!\rangle\theta}{\Phi^\bullet, \langle(\chi_{\mathsf{x}}!\,;\underleftarrow{\chi! \cdot \gamma})^\times\rangle\langle\chi_{\mathsf{x}}!\rangle\theta}\,\times}\,\wedge$$

where $\theta := \langle\!\langle\underleftarrow{\chi!}\rangle\!\rangle\varphi^{\mathsf{a}\bullet}$ and $\vec{\chi} = \chi_{\mathsf{x}_1}, \dots, \chi_{\mathsf{x}_n}$ with $\mathsf{x}_1, \dots, \mathsf{x}_n$ being all names of $\langle\gamma^\times\rangle\varphi$ in a. The remaining assumption in this G proof is the sequent $\Phi^\bullet, \chi_{\mathsf{x}} = \Phi^\bullet, \overline{\Phi^\bullet}$. But in fact for *any* finite set $\Psi = \{\psi_1, \dots, \psi_n\}$ we can easily derive the sequent $\Psi, \overline{\Psi} = \Psi, \overline{\psi_1} \wedge \cdots \wedge \overline{\psi_n}$ in G using n instances of Ax and weak followed by an application of $\wedge$. Using Lemma 14 one can verify that

$$\Phi^\bullet, \langle(\chi_{\mathsf{x}}!\,;\underleftarrow{\chi! \cdot \gamma})^\times\rangle\langle\chi_{\mathsf{x}}!\rangle\theta = \Phi^\bullet, (\langle\gamma^\times\rangle\varphi)^{\mathsf{ax}\bullet}$$

which shows that we have constructed the required G derivation of the translated assumption.

We show claim (ii) above, ie., that for each rule application in π there is a corresponding G derivation. We only consider the rules exp and clo. For the other rules the reasoning is either trivial or it follows from reasoning that is similar but simpler as the one for clo.

Suppose that an instance of the exp-rule is applied in π to obtain $\Phi, \varphi^{\mathsf{axb}}$ from $\Phi, \varphi^{\mathsf{ab}}$. Let $\theta = \langle\gamma^\times\rangle\varphi'$ be the fixpoint formula corresponding to x and suppose w.l.o.g. that $\theta \in F^\times(\varphi)$ and that the bullet translation $\varphi^{\mathsf{ab}\bullet}$ is of the form $\psi(\langle\!\langle(\underleftarrow{\chi! \cdot \gamma})^\times; \underleftarrow{\chi!}\rangle\!\rangle\psi')$ where $\vec{\chi} = \chi_{\mathsf{x}_1} \dots \chi_{\mathsf{x}_n}$ are the context formulas corresponding to the names $\mathsf{x}_1, \dots, \mathsf{x}_n$ of θ that occur in ab. Let $\vec{\chi'} = \mathsf{x}_1 \dots \mathsf{x} \dots \mathsf{x}_n$ be the list of names of θ in axb. Then $\varphi^{\mathsf{axb}\bullet} = \psi(\langle\!\langle(\underleftarrow{\chi'! \cdot \gamma})^\times; \underleftarrow{\chi'!}\rangle\!\rangle\psi')$ and it is now easy to see that this formula is derivable from $\varphi^{\mathsf{ab}\bullet}$ in G by applying the $\mathsf{Mon^g_d}$-rule twice for each occurrence of θ that got expanded by the bullet translation.

Lastly, consider an application of the clo-rule in π that derives from $\Phi, (\varphi \wedge \langle\gamma\rangle\langle\gamma^\times\rangle\varphi)^{\mathsf{ax}}$ the conclusion $\Phi, (\langle\gamma^\times\rangle\varphi)^{\mathsf{a}}$. We need to construct a corresponding G derivation. First observe that by Lemma 14 we have

$$(\langle\gamma^\times\rangle\varphi)^{\mathsf{a}\bullet} = \langle(\underleftarrow{\chi! \cdot \gamma})^\times\rangle\langle\!\langle\underleftarrow{\chi!}\rangle\!\rangle\varphi^{\mathsf{b}\bullet},$$

where b and $\vec{\chi}$ are chosen as in the previous case. Furthermore

$$\begin{aligned}(\varphi \wedge \langle\gamma\rangle\langle\gamma^\times\rangle\varphi)^{\mathsf{ax}\bullet} &= \varphi^{\mathsf{b}\bullet} \wedge \langle\!\langle\gamma\rangle\!\rangle\langle(\chi_{\mathsf{x}}!\,;\underleftarrow{\chi! \cdot \gamma})^\times\rangle\langle\chi_{\mathsf{x}}!\rangle\langle\!\langle\underleftarrow{\chi!}\rangle\!\rangle\varphi^{\mathsf{b}\bullet}, \\ &= \varphi^{\mathsf{b}\bullet} \wedge \langle\!\langle\gamma\rangle\!\rangle\langle(\overline{\Phi^\bullet}!\,;\underleftarrow{\chi! \cdot \gamma})^\times\rangle\langle\overline{\Phi^\bullet}!\rangle\langle\!\langle\underleftarrow{\chi!}\rangle\!\rangle\varphi^{\mathsf{b}\bullet}\end{aligned}$$

where we again used Lemma 14 and the fact that $\chi_{\mathsf{x}} = \overline{\Phi^\bullet}$. Now we build the following G derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Phi^\bullet, \varphi^{\mathsf{b}\bullet} \wedge \langle\!\langle\gamma\rangle\!\rangle\langle(\overline{\Phi^\bullet}!;\underleftarrow{\chi! \cdot \gamma})^\times\rangle\langle\overline{\Phi^\bullet}!\rangle\langle\!\langle\underleftarrow{\chi!}\rangle\!\rangle\varphi^{\mathsf{b}\bullet}}{\Phi^\bullet, \varphi^{\mathsf{b}\bullet} \wedge \langle\gamma\rangle\langle(\overline{\Phi^\bullet}!;\underleftarrow{\chi! \cdot \gamma})^\times\rangle\langle\overline{\Phi^\bullet}!\rangle\langle\!\langle\underleftarrow{\chi!}\rangle\!\rangle\varphi^{\mathsf{b}\bullet}}\,;_d}{\Phi^\bullet, \varphi^{\mathsf{b}\bullet} \wedge \langle\underleftarrow{\chi! \cdot \gamma}\rangle\langle(\overline{\Phi^\bullet}!;\underleftarrow{\chi! \cdot \gamma})^\times\rangle\langle\overline{\Phi^\bullet}!\rangle\langle\!\langle\underleftarrow{\chi!}\rangle\!\rangle\varphi^{\mathsf{b}\bullet}}\,\mathsf{Mon^g_d}}{\Phi^\bullet, \langle\!\langle\underleftarrow{\chi!}\rangle\!\rangle\varphi^{\mathsf{b}\bullet} \wedge \langle\underleftarrow{\chi! \cdot \gamma}\rangle\langle(\overline{\Phi^\bullet}!;\underleftarrow{\chi! \cdot \gamma})^\times\rangle\langle\overline{\Phi^\bullet}!\rangle\langle\!\langle\underleftarrow{\chi!}\rangle\!\rangle\varphi^{\mathsf{b}\bullet}}\,\mathsf{Mon^f_d}}{\Phi^\bullet, \langle(\underleftarrow{\chi! \cdot \gamma})^\times\rangle\langle\!\langle\underleftarrow{\chi!}\rangle\!\rangle\varphi^{\mathsf{b}\bullet},}\,\mathsf{ind}_s$$

Here, the double lines indicate that multiple applications of the specified rule could be required to reach the next sequent. Using the equations given above the proof tree, we have given a G-derivation of $\Phi^\bullet, (\langle\gamma^\times\rangle\varphi)^{\mathsf{a}\bullet}$ from assumption $\Phi^\bullet, (\varphi \wedge \langle\gamma\rangle\langle\gamma^\times\rangle\varphi)^{\mathsf{ax}\bullet}$. This shows that for each instance of clo there is a corresponding G-derivation as required. ☐

IV. THE MONOTONE μ-CALCULUS

In this section we give the basic definitions of the monotone μ-calculus, and we introduce an annotated proof system for it.

A. The monotone μ-calculus: syntax and semantics

Additionally to the sets P_0 and G_0 from Section II-A we now also fix a countable set *Var* of *fixpoint variables*. We shall only consider μ-calculus formulas in negation normal form.

Definition 16. The language $\mathcal{L}^\mu_{\mathsf{NF}}$ of the *monotone μ-calculus* consists of the formulas:

$$\begin{aligned}\mathcal{L}^\mu_{\mathsf{NF}} \ni A, B \quad ::= \quad & p \mid \neg p \mid x \mid A \vee B \mid A \wedge B \\ & \mid \langle g\rangle A \mid \langle g^d\rangle A \mid \mu x.A \mid \nu x.A\end{aligned}$$

where $p \in \mathsf{P}_0$, $g \in \mathsf{G}_0$ and $x \in Var$.

We apply the usual notions concerning variable binding, writing $Var(A)/FVar(A)$ for the sets of all/all free variables in A. A formula A is a *sentence* if $FVar(A) = \emptyset$.

This is essentially the language of a multi-modal μ-calculus, except we write $\langle g^d\rangle\varphi$ instead of $[g]\varphi$ in order to stay closer to game logic syntax.

Given a $\mathcal{L}^\mu_{\mathsf{NF}}$-formula A, we define its (Fischer-Ladner) closure $Cl(A)$ in the usual way (via subformulas and unfoldings).

Our definition of the system CloM below crucially involves the following priority order $\leq_A$ on $Var(A)$, and a notion of well-namedness from [16].

Definition 17. Let A be a μ-calculus formula and $x, y \in Var(A)$. We write $x <^-_A y$ if for some subformula of A of the form $\sigma y.B$ where $\sigma \in \{\mu, \nu\}$, the variable x occurs freely in $\sigma y.B$. We denote by $<_A$ the transitive closure of $<^-_A$ on $Var(A)$. We denote by $\leq_A$ the reflexive, transitive closure of $<^-_A$ on $Var(A)$. We say that A is *locally well-named* if $<_A$ is irreflexive.

We show examples of $<^-_A$ in Examples 22 and 23 below.

The semantics of the monotone μ-calculus over the game models from Section II-A is standard.

Definition 18. We define the *meaning* $[\![A]\!]^{\mathbb{S}}_h$ of a formula $A \in \mathcal{L}^{\mu}_{\mathsf{NF}}$ in the game model $\mathbb{S} = (S, E, V)$, relative to an assignment $h : Var \to \wp(S)$, by a standard induction, where, e.g., $[\![\langle g^d \rangle A]\!]^{\mathbb{S}}_h := S \setminus E_g(S \setminus [\![A]\!]^{\mathbb{S}}_h)$, and

$$\begin{array}{rcl} [\![\mu x.A]\!]^{\mathbb{S}}_h & := & \mathsf{lfp}\, X.\, [\![A]\!]^{\mathbb{S}}_{h[x\mapsto X]} \\ [\![\nu x.A]\!]^{\mathbb{S}}_h & := & \mathsf{gfp}\, X.\, [\![A]\!]^{\mathbb{S}}_{h[x\mapsto X]}. \end{array}$$

Here $h[x \mapsto X]$ is the assignment h' given by $h'(x) := X$ and $h'(y) := h(y)$ for $y \neq x$.

The meaning of a sentence A in $\mathbb{S}$ does not depend on the assignment, and so we denote this set as $[\![A]\!]^{\mathbb{S}}$. Notions like satisfiability, validity, etc., are all defined in the standard way.

B. The CloM*-system for the monotone* μ*-calculus*

The proof system CloM is the monotone analogue of the annotated sequent system Clo for the μ-calculus from [16], with one further difference. In order to prove our key Proposition 24 below, we need more control over the order on fixpoint variables than allowed by the global order of [16]. Instead, we will use the fixed order $\leq_C$, for some ambient formula C, and define CloM to be parametric in such a C.

So fix a well-named μ-calculus formula C. In the derivation system CloM_C, formulas are annotated with sequences of names, as in the system CloG. To each variable $x \in Var(C)$ we link a set N_x of *names for* x, in such a way that $N_x \cap N_y = \emptyset$ if $x \neq y$. As in Section III-A, we let the names inherit the order $\leq_C$ over variables, introduce annotations, and extend the relation $\leq_C$ to hold between annotations and variables.

The proof rules of CloM_C are in Figure 7. A CloM_C-proof is a finite tree of inferences in which each leaf is labelled by an axiom or a discharged assumption. For a formula $A \in \mathcal{L}^{\mu}_{\mathsf{NF}}$, we write $\mathsf{CloM} \vdash A$ if there is a CloM_A-proof of A^{ε}.

Analogous to CloG, the system CloM is analytic in the sense that any CloM_C-proof of C^{ε} contains only formulas from $Cl(C)$ and names for variables in $Var(C)$. Hence, the order $\leq_C$ is defined for these names and variables.

We show that CloM is sound and complete for the semantics of the monotone μ-calculus given in Definition 18 by reduction to the soundness and completeness of the system Clo with respect to Kripke models, which has been proven in [16].

$$\dfrac{}{p^{\varepsilon}, (\neg p)^{\varepsilon}}\ \mathsf{Ax1} \qquad \dfrac{A^{\mathsf{a}}, B^{\mathsf{b}}}{\langle g \rangle A^{\mathsf{a}}, \langle g^d \rangle B^{\mathsf{b}}}\ \mathsf{mod}_m$$

$$\dfrac{\Gamma, A^{\mathsf{a}}, B^{\mathsf{a}}}{\Gamma, (A \vee B)^{\mathsf{a}}}\ \vee \qquad \dfrac{\Gamma, A^{\mathsf{a}} \qquad \Gamma, B^{\mathsf{a}}}{\Gamma, (A \wedge B)^{\mathsf{a}}}\ \wedge$$

$$\dfrac{\Gamma}{\Gamma, A^{\mathsf{a}}}\ \mathsf{weak} \qquad (\mathsf{a} \leq_C x)\ \dfrac{\Gamma, A(\mu x.A(x))^{\mathsf{a}}}{\Gamma, (\mu x.A(x))^{\mathsf{a}}}\ \mu$$

$$\dfrac{\Gamma, A^{\mathsf{ab}},}{\Gamma, A^{\mathsf{axb}}}\ \mathsf{exp} \qquad (\mathsf{a} \leq_C x)\ \dfrac{\Gamma, A(\nu x.A(x))^{\mathsf{a}}}{\Gamma, (\nu x.A(x))^{\mathsf{a}}}\ \nu$$

$$\begin{array}{c} [\Gamma, \nu x.A(x)^{\mathsf{ax}}]^{\mathsf{x}} \\ \vdots \\ (\mathsf{a} \leq_C \mathsf{x} \in N_x, \mathsf{x} \notin \Gamma, \mathsf{a})\ \dfrac{\Gamma, A(\nu x.A(x))^{\mathsf{ax}}}{\Gamma, (\nu x.A(x))^{\mathsf{a}}}\ \nu\text{-}\mathsf{clo}_{\mathsf{x}} \end{array}$$

Fig. 7. The axiom and rules of the system CloM_C.

The reduction uses a translation $(-)^t \colon \mathcal{L}^{\mu}_{\mathsf{NF}} \to \mathcal{L}^{2\mu}_{\mathsf{NF}}$ from monotone μ-calculus into normal μ-calculus that is based on well-known ideas, going back to [6], though our approach is closer to the one from [19, Ch.10], for simulating monotone modal logics with normal modal logics. The basic idea is that an effectivity function $E \colon \wp(S) \to \wp(S)$ corresponds to a monotone neighbourhood function $N \colon S \to \wp(\wp(S))$ which can be encoded with two Kripke relations on state space $S \cup \wp(S)$: A relation $R_N \subseteq S \times \wp(S)$ that relates states to their neighbourhoods, and a relation $R_{\ni} \subseteq \wp(S) \times S$ that relates neighbourhoods to their elements. Conversely, from two Kripke relations on a state space S one can define a monotone neighbourhood function $N \colon S \to \wp(\wp(S))$ using the same idea. The language $\mathcal{L}^{2\mu}_{\mathsf{NF}}$ is the modal μ-calculus language that has two (normal) modalities $\langle g_N \rangle$ and $\langle g_{\ni} \rangle$ for each $g \in \mathsf{G}_0$. The translation of atomic formulas, Boolean connectives and fixpoints is defined by straightforward recursion. For modalities, we take $(\langle g \rangle A)^t = \langle g_N \rangle [g_{\ni}](A^t)$, and $(\langle g^d \rangle A)^t = [g_N] \langle g_{\ni} \rangle (A^t)$. Using the model translatons described above, we can show that $(-)^t$ preserves satisfiability and validity.

Lemma 19. For all formulas $C \in \mathcal{L}^{\mu}_{\mathsf{NF}}$,

1) if C is satisfiable in a game model, then C^t is satisfiable in a Kripke model for $\mathcal{L}^{2\mu}_{\mathsf{NF}}$.
2) if C is valid over game models, then C^t is valid over Kripke models for $\mathcal{L}^{2\mu}_{\mathsf{NF}}$.

Lemma 19 gives the semantic part needed in the proof of the following theorem.

Theorem 20 (Soundness and Completeness of CloM)**.** For all $C \in \mathcal{L}^{\mu}_{\mathsf{NF}}$, C is valid (on game models) iff $\mathsf{CloM} \vdash C$.

A detailed proof of Theorem 20 is found in the appendix. Here we only give a sketch. What remains is to show that we can translate between proofs in CloM and proofs in the similar annotated proof system Clo [16] for the normal μ-calculus.

First, we note that the translation $(-)^t$ extends to annotated formulas and sequents in the obvious manner. For both directions, we transform proof trees starting from the root going up. The most interesting case is in the construction of a CloM-proof for $A \in \mathcal{L}^{\mu}_{\mathsf{NF}}$ from a Clo-proof of $A^t \in \mathcal{L}^{2\mu}_{\mathsf{NF}}$ when the modal rule from Clo is applied. So suppose some node v in a Clo-proof π is obtained from an application of the (normal) modal rule, and v is labelled with a sequent Γ^t where Γ is a sequent of annotated $\mathcal{L}^{\mu}_{\mathsf{NF}}$-formulas. Then Γ^t must have the form $\langle g_N\rangle[g_\ni]A_1, ..., \langle g_N\rangle[g_\ni]A_n, [g_N]\langle g_\ni\rangle B$, and hence Γ must have the form $\langle g\rangle C_1, ..., \langle g\rangle C_n, \langle g^d\rangle D$ where $C_1^t = A_1, ..., C_n^t = A_n$, $D^t = B$. By inspection of the rules of Clo, and assuming that $n \geq 2$ (since the other case is easier), we see that the subtree of the Clo-proof π rooted at v must have the following shape:

$$\dfrac{\dfrac{\dfrac{\vdots \atop A_i, B}{[g_\ni]A_i, \langle g_\ni\rangle B}\ \mathsf{mod}}{[g_\ni]A_1, ..., [g_\ni]A_n, \langle g_\ni\rangle B}\ \mathsf{weak}}{\langle g_N\rangle[g_\ni]A_1, ..., \langle g_N\rangle[g_\ni]A_n, [g_N]\langle g_\ni\rangle B}\ \mathsf{mod}$$

We mimic this by the following CloM-derivation steps:

$$\dfrac{\dfrac{C_i, D}{\langle g\rangle C_i, \langle g^d\rangle D}\ \mathsf{mod}}{\langle g\rangle C_1, ..., \langle g\rangle C_n, \langle g^d\rangle D}\ \mathsf{weak}$$

The label of the top node of this derivation translates to A_i, B, and so we can inductively continue the construction using the corresponding smaller subtree of π.

V. Game Logic and the monotone μ-calculus

In this section we define a novel translation from formulas in game logic to formulas in the monotone μ-calculus, and prove that if the translation of a formula is provable in CloM then the formula is provable in CloG.

A. Translating Game Logic to the monotone μ-calculus

It is shown in [5, sec. 6.4.2] that game logic can be translated into the two-variable fragment of the monotone μ-calculus. However, we use more than two variables because we need to keep track of the nesting of fixpoints. Before we give the formal definition of our translation, we first explain informally how we achieve this. Consider the translation of a game logic formula $\xi \in \mathcal{L}_{\mathsf{NF}}$. Formulas $\langle\gamma^\circ\rangle\varphi \in F(\xi)$ translate to fixpoint formulas of the form $\sigma x.A(x)$ on the μ-calculus side. In order to synchronise the translation across unfolding of fixpoint formulas, we syntactically encode $\langle\gamma^\circ\rangle\varphi$ into the fixpoint variable that it gives rise to in the translation of ξ.

Definition 21. We define the translation $(-)^\sharp\colon \mathcal{L}_{\mathsf{NF}} \to \mathcal{L}^{\mu}_{\mathsf{NF}}$ by a mutual induction on formulas and games as follows:

$$\begin{array}{rcl}
p^\sharp & := & p \\
(\neg p)^\sharp & := & \neg p \\
(\varphi \wedge \psi)^\sharp & := & \varphi^\sharp \wedge \psi^\sharp \\
(\varphi \vee \psi)^\sharp & := & \varphi^\sharp \vee \psi^\sharp \\
(\langle\gamma\rangle\varphi)^\sharp & := & \tau^{\varphi}_{\gamma}(\varphi^\sharp) \\
\tau^{\varphi}_{g}(A) & := & \langle g\rangle A \\
\tau^{\varphi}_{g^d}(A) & := & \langle g^d\rangle A \\
\tau^{\varphi}_{\gamma\sqcap\delta}(A) & := & \tau^{\varphi}_{\gamma}(A) \wedge \tau^{\varphi}_{\delta}(A) \\
\tau^{\varphi}_{\gamma\sqcup\delta}(A) & := & \tau^{\varphi}_{\gamma}(A) \vee \tau^{\varphi}_{\delta}(A) \\
\tau^{\varphi}_{\gamma^*}(A) & := & \mu x^{\langle\gamma^*\rangle\varphi}.A \vee \tau^{\langle\gamma^*\rangle\varphi}_{\gamma}(x^{\langle\gamma^*\rangle\varphi}) \\
\tau^{\varphi}_{\gamma^\times}(A) & := & \nu x^{\langle\gamma^\times\rangle\varphi}.A \wedge \tau^{\langle\gamma^\times\rangle\varphi}_{\gamma}(x^{\langle\gamma^\times\rangle\varphi}) \\
\tau^{\varphi}_{\gamma\,;\,\delta}(A) & := & \tau^{\langle\delta\rangle\varphi}_{\gamma}(\tau^{\varphi}_{\delta}(A)) \\
\tau^{\varphi}_{\psi?}(A) & := & \psi^\sharp \wedge A \\
\tau^{\varphi}_{\psi!}(A) & := & \psi^\sharp \vee A
\end{array}$$

Example 22. For $\varphi = \langle(a^*\,;(b^\times \sqcup c))^\times\rangle p$, the translation is

$$\begin{array}{rcl}
\varphi^\sharp & = & \nu x^{\varphi}.p \wedge \\
& & \quad \mu x^{\psi}.((\nu x^{\theta}.x^{\varphi} \wedge \langle b\rangle x^{\theta}) \vee \langle c\rangle x^{\varphi}) \vee \langle a\rangle x^{\psi} \\
\text{with } \psi & = & \langle a^*\rangle\langle b^\times \sqcup c\rangle\varphi \\
\theta & = & \langle b^\times\rangle\varphi
\end{array}$$

Applying the definitions of order on game logic fixpoint formulas (Def. 2) and μ-calculus fixpoint variables (Def. 17), we find that:

$$\varphi \prec \psi,\ \varphi \prec \theta \quad \text{and} \quad x^{\varphi} <^{-}_{\varphi^\sharp} x^{\psi},\ x^{\varphi} <^{-}_{\varphi^\sharp} x^{\theta}$$

Example 23. For $\varphi = \langle(a^*;(\langle b^\times\rangle p)?)^\times\rangle\langle c^*\rangle q$,

$$\begin{array}{rcl}
\varphi^\sharp & = & \nu x^{\varphi}.\,(\mu x^{\psi}.q \vee \langle c\rangle x^{\psi}) \wedge \\
& & \quad (\mu x^{\zeta}.((\nu x^{\theta}.p \wedge \langle b\rangle x^{\theta}) \wedge x^{\varphi}) \vee \langle a\rangle x^{\zeta}) \\
\text{with } \psi & = & \langle c^*\rangle q \\
\zeta & = & \langle a^*\rangle\langle(\langle b^\times\rangle p)?\rangle\varphi, \quad \text{and} \\
\theta & = & \langle b^\times\rangle p
\end{array}$$

Applying the definitions of order on game logic fixpoint formulas (Def. 2) and μ-calculus fixpoint variables (Def. 17), we find that:

$$\varphi \prec \zeta,\ \varphi \prec \theta \quad \text{and} \quad x^{\varphi} <^{-}_{\varphi^\sharp} x^{\zeta}$$

The above examples illustrate how the order on fixpoint variables in μ-calculus is reflected in game logic fixpoints along the translation, and that translations are always locally well-named. These are the syntactic properties of $(-)^\sharp$ that are crucial to our proofs.

Proposition 24. For all $\xi \in \mathcal{L}_{\mathsf{NF}}$ the translation $\xi^\sharp$ is locally well-named, and for all $\varphi, \psi \in F(\xi)$ we have $x^{\varphi}, x^{\psi} \in Var(\xi^\sharp)$, and that $x^{\varphi} \leq_{\xi^\sharp} x^{\psi}$ implies $\varphi \preccurlyeq \psi$.

On the semantic side, our translation is adequate in the sense that it is truth- and validity preserving. Recall that $\mathcal{L}_{\mathsf{NF}}$ and $\mathcal{L}^{\mu}_{\mathsf{NF}}$ are both interpreted over game models, i.e., monotone neighbourhood models.

Proposition 25. For every $\xi \in \mathcal{L}_{\mathsf{NF}}$ and every game model $\mathbb{S}$ it holds that $[\![\xi]\!]^{\mathbb{S}} = [\![\xi^\sharp]\!]^{\mathbb{S}}$.

Proof. By a straightforward induction. □

B. From CloM *to* CloG

We now show how to construct a CloG-derivation of a game logic formula ξ from a CloM-derivation of $\xi^\sharp$.

For this purpose, we identify the set N_φ of names for $\varphi \in F(\xi)$ with the set N_{x^φ} of names for the variable $x^\varphi \in \mathit{Var}(\xi^\sharp)$. This is possible since both sets are defined to be arbitrary countable sets. We then extend the translation $(-)^\sharp$ to annotated formulas and sequents by taking

$$(\varphi^{\mathsf{a}})^\sharp := (\varphi^\sharp)^{\mathsf{a}} \quad \text{and} \quad \Phi^\sharp := \{(\varphi^{\mathsf{a}})^\sharp \mid \varphi^{\mathsf{a}} \in \Phi\}.$$

That is, the translation leaves annotations unchanged.

Theorem 26. For all $\xi \in \mathcal{L}_{\mathsf{NF}}$, if $\mathsf{CloM} \vdash \xi^\sharp$ then $\mathsf{CloG} \vdash \xi$.

Proof. We will prove the theorem by induction on the complexity of proof trees, and for a proper development of the induction we need to take care of derivations with open branches because the clo-rule allows to discharge assumptions. We shall write $\pi : \mathcal{A} \vdash_{\mathsf{CloM}_C} \Gamma$ to say that π is a CloM_C-derivation of Γ from assumptions in $\mathcal{A}$, and similarly for CloG-derivations with open assumptions.

More precisely, we shall prove, by induction on the complexity of CloM-derivations, that every $\mathsf{CloM}_{\xi^\sharp}$-proof π satisfies the following property:

for every game logic sequent Φ: if $\pi : \mathcal{A} \vdash_{\mathsf{CloM}_{\xi^\sharp}} \Phi^\sharp$ then there is a CloG-proof $\pi' : \mathcal{G} \vdash_{\mathsf{CloG}} \Phi$ where $\mathcal{G}^\sharp = \mathcal{A}$.	$(*)$

Two preliminary remarks are in order before we dive into the proof details. First, in the sequel we will often omit the annotation of formulas, for the sake of readability. And second, without loss of generality we may adopt the *injectivity assumption* stating that for each formula A in $\Phi^\sharp$ there is precisely one formula φ in Φ with $\varphi^\sharp = A$.

In the base case of our proof, the derivation π is either an application of the axiom Ax1 or a one-node derivation of a sequent $\Phi^\sharp$, where the set of assumptions of π is the singleton set $\{\Phi^\sharp\}$. In both cases it is straightforward to see that the derivation π', consisting of a single node labelled Φ, meets the requirements stated in $(*)$.

For the inductive step, first observe that we may assume that none of the formulas in Φ is of the form $\langle \gamma \,;\, \delta \rangle \psi$. Should $\varphi \in \Phi$ be of this form then we could apply the rule ; and subsequently work with the formula $\langle\gamma\rangle\langle\delta\rangle\psi$, for which it holds that $(\langle\gamma\rangle\langle\delta\rangle\psi)^\sharp = (\langle\gamma\,;\,\delta\rangle\psi)^\sharp$. This can be repeated until the resulting formula is of the required shape.

For the proof of the inductive step, we make a case distinction as to the last applied rule in the CloM-derivation π.

In case the last applied rule is the rule $\wedge$, then $\Phi^\sharp$ must be of the form $\Phi^\sharp = \Gamma, A_0 \wedge A_1$ and the rule $\wedge$ is applied to the premises Γ, A_0 and Γ, A_1. By our injectivity assumption there is precisely one formula φ in Φ such that $\Phi = \Psi, \varphi$, $\Psi^\sharp = \Gamma$ and $\varphi^\sharp = A_0 \wedge A_1$. But then it follows by the definition of the translation $(-)^\sharp$ and our assumption on the shape of the formulas in Φ that there are three possibilities: either (i) $\varphi = \varphi_0 \wedge \varphi_1$ such that $\varphi_0^\sharp = A_0$ and $\varphi_1^\sharp = A_1$, or (ii) $\varphi = \langle \gamma_0 \sqcap \gamma_1 \rangle \psi$ such that $(\langle\gamma_0\rangle\psi)^\sharp = A_0$ and $(\langle\gamma_1\rangle\psi)^\sharp = A_1$, or (iii) $\varphi = \langle \psi ? \rangle \chi$ such that $\psi^\sharp = A_0$ and $\chi^\sharp = A_1$.

The other cases being similar, we only consider case (ii). Here we have $\mathsf{CloM}_{\xi^\sharp}$-proofs π_0, π_1 of the sequents $\Psi^\sharp, \langle\gamma_0\rangle\psi^\sharp$ and $\Psi^\sharp, \langle\gamma_1\rangle\psi^\sharp$, from two respective sets of assumptions $\mathcal{A}_0$ and $\mathcal{A}_1$ such that $\mathcal{A}_0 \cup \mathcal{A}_1 = \mathcal{A}$. Use the induction hypothesis to obtain, for $i = 0, 1$, a set $\mathcal{G}_i$ of game logic sequents such that $\mathcal{G}_i^\sharp = \mathcal{A}_i$, as well as a CloG-proof $\pi_i' : \mathcal{A}_i \vdash \Psi, \langle\gamma_i\rangle\psi$. We then apply the rule $\wedge$ to get a proof of the sequent $\Psi, \langle\gamma_0\rangle\psi \wedge \langle\gamma_1\rangle\psi$, followed by the rule $\sqcap$ to derive the sequent $\Phi = \Psi, \langle\gamma_0 \sqcap \gamma_1\rangle\psi$. Finally, the set of assumptions of the resulting derivation π' is the set $\mathcal{G}_0 \cup \mathcal{G}_1$, which clearly satisfies the condition that $(\mathcal{G}_0 \cup \mathcal{G}_1)^\sharp = \mathcal{A}$.

The cases where the last rule applied in π is one of $\vee$, mod_m, or weak, are similarly easy to deal with; we omit the details.

Now consider the case where π ends with an application of the rule μ for a least fixpoint. We then have that $\Phi^\sharp = \Gamma, \mu x.A(x)^{\mathsf{a}}$, the premise of this application of μ is the sequent $\Gamma, A(\mu x.A(x))^{\mathsf{a}}$, and the side condition $\mathsf{a} \leq_{\xi^\sharp} x$ is fulfilled. As explained above we can assume that there is a single formula φ in Φ such that $\Phi = \{\Psi, \varphi\}$, $\varphi^\sharp\Psi = \Gamma$ and $\varphi^\sharp = \mu x.A(x)$. As we have already excluded the possibility that φ is a modality whose main operator is the composition it follows from the definition of the translation $(-)^\sharp$ that $\varphi = \langle\gamma^*\rangle\psi$ such that $A(x) = \psi^\sharp \vee \tau_\gamma^\varphi(x)$. Note that $x = x^{\langle\gamma^*\rangle\psi}$ by definition of the translation $(-)^\sharp$. Some further calculations show that

$$A(\mu x.A(x)) = \psi^\sharp \vee \tau_\gamma^\varphi(\mu x.A(x)) = (\psi \vee \langle\gamma\rangle\langle\gamma^*\rangle\psi)^\sharp.$$

We can thus apply the induction hypothesis to obtain a CloG-proof of the sequent $\Psi, \psi \vee \langle\gamma\rangle\langle\gamma^*\rangle\psi$, from a set of assumptions $\mathcal{G}$ satisfying $\mathcal{G}^\sharp = \mathcal{A}$. We then want to use the rule $*$ to obtain a proof of $\Phi = \Psi, \varphi = \Psi, \langle\gamma^*\rangle\psi$ from the same set $\mathcal{G}$ of assumptions. To do so we need to ensure that the side condition $\mathsf{a} \preccurlyeq \langle\gamma^*\rangle\psi$ is satisfied. Hence consider any name y that occurs in a and let χ be the fixpoint formula such that $\mathsf{y} \in N_\chi$. From the side condition $\mathsf{a} \leq_{\xi^\sharp} x$ it follows that $\mathsf{y} \leq_{\xi^\sharp} x$, and then from Proposition 24 that $\chi \preccurlyeq \langle\gamma^*\rangle\psi$, and hence we obtain the required $\mathsf{y} \preccurlyeq \langle\gamma^*\rangle\psi$.

If the last rule applied in π is the fixpoint rule ν for the greatest fixpoint then we can use a similar argument as in the paragraph using $\times$ instead of $*$.

Finally, consider the case where the last rule applied in π is ν-$\mathsf{clo}_{\mathsf{x}}$ for some name x, discharging the assumption $\Omega = \Gamma, \nu x.A(x)^{\mathsf{ax}}$. We then may observe that $\Phi^\sharp = \Gamma, \nu x.A(x)^{\mathsf{a}}$, that the premise of this application of ν-$\mathsf{clo}_{\mathsf{x}}$ is the sequent $\Gamma, A(\nu x.A(x))^{\mathsf{ax}}$, and that the side conditions $\mathsf{a} \leq_{\xi^\sharp} \mathsf{x}$ and $\mathsf{x} \notin \Gamma, \mathsf{a}$ are fulfilled. As explained above we can assume that there is a single formula φ in Φ such that $\Phi = \{\Psi, \varphi\}$,

$\Psi^\sharp = \Gamma$ and $\varphi^\sharp = \nu x.A(x)$. And, similar to the case of the rule μ discussed above, we may assume that $\varphi = \langle\gamma^\times\rangle\psi$ for γ and ψ such that $A(x) = \psi^\sharp \wedge \tau^\varphi_\gamma(x)$, and that

$$A(\nu x.A(x)) = \psi^\sharp \wedge \tau^\varphi_\gamma(\nu x.A(x)) = (\psi \wedge \langle\gamma\rangle\langle\gamma^\times\rangle\psi)^\sharp.$$

We then apply the induction hypothesis and obtain a CloG-derivation of the premise of the ν-$\mathsf{clo_x}$-rule from assumptions $\mathcal{G} \cup \mathcal{G}'$, where each sequent in $\mathcal{G}'$ translates to Ω (the assumption discharged by the application of the ν-$\mathsf{clo_x}$-rule with conclusion $\Phi^\sharp$). It follows that every sequent in $\mathcal{G}'$ must be of the form $\Theta, \varphi_0^{\mathsf{ax}}$ with $\Theta^\sharp = \Gamma$ and $\varphi_0^\sharp = \nu x.A(x)$. From $\varphi_0^\sharp = \nu x.A(x) = \varphi^\sharp$ it follows that $\varphi_0 = \varphi$ (syntactically), since we encode the formula $\varphi = \langle\gamma^\times\rangle\psi$ into the fixpoint variable x of its translation. That is, we may take $\mathcal{G}' = \{\Theta, \langle\gamma^\times\rangle\psi^{\mathsf{ax}} \mid \Theta \in \mathcal{L}\}$ for some set $\mathcal{L}$ with $\mathcal{L}^\sharp_\Sigma = \{\Gamma\}$. Note, however, that the sequents in $\mathcal{L}$ will generally not be identical to Ψ, which means that we cannot simply finish our proof with an application of the ν-$\mathsf{clo_x}$-rule of the CloG-system here. We need a more elaborate construction.

In fact we need to generalise the statement about Ψ and a to the observation below, where we let $\mathcal{S}$ be the (finite!) set of game logic sequents Σ such that $\Sigma^\sharp = \Gamma$.

CLAIM 1. For every $\Sigma \in \mathcal{S}$ there are game logic sequents $\mathcal{G}_\Sigma$ and $\mathcal{L}_\Sigma$ such that (†1) $\mathcal{G}^\sharp_\Sigma = \mathcal{A}$, $\mathcal{L}^\sharp_\Sigma = \{\Gamma\}$ and (†2) for every $\mathsf{b} = \mathsf{a}\mathsf{x}_1 \cdots \mathsf{x}_k$, with $\mathsf{x}_1, \ldots, \mathsf{x}_k$ names for $\langle\gamma^\times\rangle\psi$, there is a CloG-proof

$$\rho^{\mathsf{b}}_\Sigma : \mathcal{G}_\Sigma \cup \{\Theta, \langle\gamma^\times\rangle\psi^{\mathsf{b}} \mid \Theta \in \mathcal{L}_\Sigma\} \vdash \Sigma, (\psi \wedge \langle\gamma\rangle\langle\gamma^\times\rangle\psi)^{\mathsf{b}}$$

PROOF OF CLAIM Fix a sequent $\Sigma \in \mathcal{S}$. Repeating the argument that we just gave and that is directly based on the inductive hypothesis, we obtain sets of game logic sequents $\mathcal{G}_\Sigma$ and $\mathcal{L}_\Sigma$ satisfying condition (†1), together with a CloG-derivation $\rho^{\mathsf{ax}}_\Sigma$ of the sequent $\Sigma, (\psi \wedge \langle\gamma\rangle\langle\gamma^\times\rangle\psi)^{\mathsf{ax}}$ from the assumptions $\mathcal{G}_\Sigma \cup \{\Theta, \langle\gamma^\times\rangle\psi^{\mathsf{ax}} \mid \Theta \in \mathcal{L}_\Sigma\}$.

Now consider an annotation $\mathsf{b} = \mathsf{a}\overline{\mathsf{x}}$, where $\overline{\mathsf{x}} = \mathsf{x}_1 \cdots \mathsf{x}_k$, with $k \geq 1$. We will transform the derivation $\rho^{\mathsf{ax}}_\Sigma$ into the desired derivation ρ^{b}_Σ in two stages. First, we simply replace every occurrence of x as (part of) an annotation in $\rho^{\mathsf{ax}}_\Sigma$ with $\overline{\mathsf{x}}$. This transforms $\rho^{\mathsf{ax}}_\Sigma$ into a structure ρ' which is *almost* a proper CloG-proof. The only problem concerns applications in $\rho^{\mathsf{ax}}_\Sigma$ of the *expansion* rule exp of the form $\Delta, \varphi^{\mathsf{cd}}/\Delta, \varphi^{\mathsf{cxd}}$. In ρ' we may not be allowed to derive $\Delta, \varphi^{\mathsf{c}\overline{\mathsf{x}}\mathsf{d}}$ from $\Delta, \varphi^{\mathsf{cd}}$ by one application of the expansion rule, but we can easily take care of this problem in the second state of the construction, namely by deriving $\Delta, \varphi^{\mathsf{c}\overline{\mathsf{x}}\mathsf{d}}$ from $\Delta, \varphi^{\mathsf{cd}}$ by a *series* of applications of the expansion rule. This finishes the proof of the claim. ◀

We will use derivations of the form ρ^{b}_Σ as *building blocks* for our CloG-derivation of the sequent $\Psi, \langle\gamma^\times\rangle\psi^{\mathsf{a}}$. The idea is to first build up, step by step, a *pseudo-derivation* of $\Psi, \langle\gamma^\times\rangle\psi^{\mathsf{a}}$ which differs from a proper CloG-proof in that not all assumptions of prospective applications of the clo-rule are discharged. Once we have completed the construction of this pseudo-derivation, we transform it into a proper CloG-proof by taking care of these undischarged assumptions. To do this in a proper way we need to be precise about the annotations, and we need to introduce some auxiliary definitions.

Most importantly, we define a *pseudo-derivation* to be a proof in the derivation system CloG extended with the derivation rule D:

$$\mathsf{a} \preccurlyeq \mathsf{x} \in N_{\langle\gamma^\times\rangle\varphi}, \mathsf{x} \notin \Phi, \mathsf{a}\ \frac{\Phi, (\varphi \wedge \langle\gamma\rangle\langle\gamma^\times\rangle\varphi)^{\mathsf{ax}}}{\Phi, (\langle\gamma^\times\rangle\varphi)^{\mathsf{a}}}\ \mathsf{D_x}$$

Clearly, D is identical to the rule clo, apart from the fact that it does not require that the assumptions of the form $\Psi, \langle\gamma^\times\rangle\varphi^{\mathsf{ax}}$ in the proof tree leading up to the premise of D are discharged. We shall call a node t in a proof tree *dangling* if the rule applied at t is D. Observe that a pseudo-derivation is a proper CloG-derivation just in case it has no dangling nodes.

We now construct a pseudo-derivation for the sequent $\Psi, \langle\gamma^\times\rangle\psi^{\mathsf{a}}$. We shall make use of a set $\{\mathsf{x}_\Sigma \mid \Sigma \in \mathcal{S}\}$ of special, fresh names, all associated with the fixpoint formula $\langle\gamma^\times\rangle\psi$. Our starting point of the construction is the one-node derivation consisting of the sequent $\Psi, \langle\gamma^\times\rangle\psi^{\mathsf{a}}$.

Now suppose that the current approximation σ of the pseudo-derivation contains an assumption of the form $\Sigma, \langle\gamma^\times\rangle\psi^{\mathsf{b}}$, where $\Sigma \in \mathcal{S}$ and the annotation b is of the form $\mathsf{b} = \mathsf{a}\overline{\mathsf{x}}$ with x_Σ *not* occurring in the sequence $\overline{\mathsf{x}} = \mathsf{x}_{\Sigma_1} \cdots \mathsf{x}_{\Sigma_k}$. By our Claim 1, we may take a CloG-proof $\rho^{\mathsf{bx}_\Sigma}_\Sigma$ of the sequent $\Sigma, (\psi \wedge \langle\gamma\rangle\langle\gamma^\times\rangle\psi)^{\mathsf{bx}_\Sigma}$ from the assumptions $\mathcal{G}_\Sigma \cup \{\Theta, \langle\gamma^\times\rangle\psi^{\mathsf{bx}_\Sigma} \mid \Theta \in \mathcal{L}_\Sigma\}$. We adjoin copies of the derivation $\rho^{\mathsf{bx}_\Sigma}_\Sigma$ to the derivation tree, linking each leaf in the current approximation σ which is labelled as indicated, to the root of a copy of $\rho^{\mathsf{bx}_\Sigma}_\Sigma$ through an application of the rule $\mathsf{D}_{\mathsf{x}_\Sigma}$.

The above construction must terminate after finitely many steps, basically as a consequence of the fact that the set $\mathcal{S}$ is finite. Let ρ denote the pseudo-derivation that we arrive at in this way, and let $\mathcal{G}$ be the set of assumptions of ρ that belong to the set $\bigcup\{\mathcal{G}_\Sigma \mid \Sigma \in \mathcal{S}\}$; clearly then we have that $\mathcal{G}^\sharp = \mathcal{A}$.

It is not difficult to verify that the pseudo-derivation ρ satisfies the following conditions:

1) All leaves of ρ are labelled with an axiom, a sequent from $\mathcal{G}$, or else a sequent of the form $\Sigma, \langle\gamma^\times\rangle\psi^{\mathsf{b}}$, where $\Sigma \in \mathcal{S}$ and the annotation b is of the form $\mathsf{b} = \mathsf{a}\mathsf{x}_{\Sigma_1} \cdots \mathsf{x}_{\Sigma_k}$, with $\Sigma_1 = \Psi$, $\Sigma \in \{\Sigma_1, \ldots, \Sigma_k\}$, and the Σ_i are all distinct.
2) If a leaf l is labelled $\Sigma, \langle\gamma^\times\rangle\psi^{\mathsf{b}}$, where $\mathsf{b} = \mathsf{a}\mathsf{x}_{\Sigma_1} \cdots \mathsf{x}_{\Sigma_k}$, then the path from the root r of ρ to l passes through nodes $r = t_1, \ldots, t_k$, in that order, such that (a) every t_j is either dangling or the conclusion of an application of the clo-rule, and (b) the name x_{Σ_i} was introduced at the successor of t_i.
3) If t is a dangling node of ρ, labelled, say, with the sequent $\Sigma, \langle\gamma^\times\rangle\psi^{\mathsf{b}}$, and l is a leaf above t labelled with $\Sigma, \langle\gamma^\times\rangle\psi^{\mathsf{c}}$, then bx_Σ is an initial segment of c.

Step by step we will now transform this pseudo-derivation into a proper CloG-derivation. Clearly it suffices to prove that we can turn any pseudo-derivation satisfying the conditions 1) – 3) into a pseudo-derivation that still satisfies mentioned conditions, but has a smaller number of dangling nodes.

So let σ be such a pseudo-derivation, and pick a dangling node, say, t, that has maximal distance to the root; this means in particular that there are no dangling nodes above t. Let t and its successor be labelled with, respectively, the sequents $\Sigma, \langle\gamma^\times\rangle\psi^{\mathsf{a}\overline{\mathsf{x}}}$ and $\Sigma, (\psi \wedge \langle\gamma\rangle\langle\gamma^\times\rangle\psi)^{\mathsf{a}\overline{\mathsf{x}}\mathsf{x}_\Sigma}$, and let L_t be the set of leaves above t that are labelled with a sequent of the form $\Sigma, \langle\gamma^\times\rangle\psi^{\mathsf{b}}$. Now make a case distinction.

If L_t is empty, the pseudo-derivation does not record a proper circular dependency at t, so to speak. This is in fact the simplest case: we obtain a pseudo-derivation σ' from σ by (a) replacing D_x with $\times$ as the rule applied at t, and (b) simply erasing all occurrences of the name x_Σ in the pseudo-derivation above t.

If L_t is non-empty, consider an arbitrary leaf l in L_t, and let $\Sigma, \langle\gamma^\times\rangle\psi^{\mathsf{b}_l}$ be the sequent labelling l. It follows from condition 3 that b_l is of the form $\mathsf{a}\overline{\mathsf{x}}\mathsf{x}_\Sigma\mathsf{c}_l$ for some sequence c_l. Now, extend σ to σ' by attaching a successor l' to each $l \in L_t$ (so l' is a leaf in σ', but l is not), and label each such l' with the sequent $\Sigma, \langle\gamma^\times\rangle\psi^{\mathsf{a}\overline{\mathsf{x}}\mathsf{x}_\Sigma}$, so that we may obtain the sequent of l from that of l' by applications of the expansion rule. We then obtain the desired pseudo-derivation from σ' by discharging the assumption $\Sigma, \langle\gamma^\times\rangle\psi^{\mathsf{a}\overline{\mathsf{x}}\mathsf{x}_\Sigma}$ at every leaf l' with $l \in L_t$, and simultaneously changing the proof rule applied at node t into a (now legitimate) application of the $\mathsf{clo}_{\mathsf{x}_\Sigma}$-rule.

In both cases it is not hard to verify that the structure σ' is in fact a (pseudo-)derivation satisfying the clauses 1) – 3), that the node t is not a dangling node of σ', and that the transformation of σ into σ' has not created any new dangling node.

Finally, as a result of these transformations we obtain, as required, a CloG-derivation of the sequent $\Phi, \langle\gamma^\times\rangle\psi$ from the collection of assumptions $\mathcal{G}$ for which we already saw that $\mathcal{G}^\sharp = \mathcal{A}$.

This finishes the proof of Theorem 26, since the clo-rule was the last rule to be considered in the induction step. □

VI. Soundness and Completeness via Transformations

We now prove the soundness and completeness of the proof systems G (Theorem 10) and CloG (Theorem 12) as well as the completeness of Par (Theorem 8). We do this using the translations and transformations we introduced earlier. An overview is given by the following diagram. Here, Clo is the system from [16], and $(\mathcal{L}_{\mathsf{NF}})^{\mathsf{Ann}}$, $(\mathcal{L}^{\mu}_{\mathsf{NF}})^{\mathsf{Ann}}$ and $(\mathcal{L}^{2\mu}_{\mathsf{NF}})^{\mathsf{Ann}}$ denote, respectively, the sets of annotated formulas of $\mathcal{L}_{\mathsf{NF}}$, $\mathcal{L}^{\mu}_{\mathsf{NF}}$ and $\mathcal{L}^{2\mu}_{\mathsf{NF}}$.

$$\begin{array}{ccccccccc}
\mathsf{Par} & \xleftarrow{\text{Thm 11}} & \mathsf{G} & \xleftarrow{\text{Thm. 15}} & \mathsf{CloG} & \xleftarrow{\text{Thm. 26}} & \mathsf{CloM} & \longleftarrow & \mathsf{Clo} \\
\mathcal{L}_{\mathsf{Par}} & \underset{\mathsf{pa}(-)}{\overset{\mathsf{nf}(-)}{\rightleftarrows}} & \mathcal{L}_{\mathsf{NF}} & \xleftarrow{(-)^\bullet} & (\mathcal{L}_{\mathsf{NF}})^{\mathsf{Ann}} & \xrightarrow{(-)^\sharp} & (\mathcal{L}^{\mu}_{\mathsf{NF}})^{\mathsf{Ann}} & \xrightarrow{(-)^t} & (\mathcal{L}^{2\mu}_{\mathsf{NF}})^{\mathsf{Ann}}
\end{array}$$

The completeness of CloG and G is obtained from the completeness of CloM, and the fact that Proposition 25 implies that the translation $(-)^\sharp$ preserves validity over game models. Hence for all $\xi \in \mathcal{L}_{\mathsf{NF}}$ we find (†)

$$\models \xi \overset{\text{Prop. 25}}{\Rightarrow} \models \xi^\sharp \overset{\text{Thm. 20}}{\Rightarrow} \mathsf{CloM} \vdash \xi^\sharp \overset{\text{Thm. 26}}{\Rightarrow} \mathsf{CloG} \vdash \xi \overset{\text{Thm. 15}}{\Rightarrow} \mathsf{G} \vdash \xi$$

From the completeness of G, we obtain the completeness of Par as follows. For all $\varphi \in \mathcal{L}_{\mathsf{Par}}$, we have

$$\models \varphi \quad \overset{\text{Prop. 5}}{\Leftrightarrow} \quad \models \mathsf{nf}(\varphi) \quad \overset{(\dagger)}{\Rightarrow} \quad \mathsf{G} \vdash \mathsf{nf}(\varphi) \overset{\text{Thm 11(1)}}{\Rightarrow} \mathsf{Par} \vdash \varphi$$

To prove the soundness of G and CloG, let $\xi \in \mathcal{L}_{\mathsf{NF}}$. We then have,

$$\mathsf{CloG} \vdash \xi \overset{\text{Thm 15}}{\Rightarrow} \mathsf{G} \vdash \xi \overset{\text{Thm 11(2)}}{\Rightarrow} \mathsf{Par} \vdash \mathsf{pa}(\xi)$$

By the soundness of Par, it follows that $\mathsf{pa}(\xi)$ is valid over game models, and since $\mathsf{pa}(\xi)$ is equivalent with ξ by Proposition 5, also ξ is valid over game models.

VII. Conclusion

In this paper we introduced two cut-free sequent calculi for Parikh's game logic and established their soundness and completeness. From this result, we also obtained completeness of the original Hilbert-style proof system for game logic. This confirms a conjecture made by Parikh in [1]. The completeness of these two systems was obtained by translating game logic into the monotone μ-calculus, for which we also gave a cut-free sequent calculus that we showed to be sound and complete.

A. Discussion

Our proof makes essential use of ideas and results from Afshari and Leigh's paper [16]. In particular, the idea of using the proof systems CloG and CloM to obtain cut-free completeness is central here. An important reason that our approach is possible is that these annotated proof systems allow good control over the structure of proofs. In particular, formal proofs in CloG and CloM only contain formulas that are in the Fischer-Ladner closure of the formula at the root of the proof. This means that if the root formula of an annotated proof is the translation of a game logic formula, then indeed the entire proof can in a sense be carried out within game logic, modulo the translation. Also, the annotations provide a powerful machinery for keeping track of unfoldings of fixpoint formulas along traces in a proof tree. This is crucial in order to decide where to apply the strengthened induction rule when we construct cut-free sequent proofs from annotated ones.

B. Future research

Completeness for fixpoint logics is generally considered to be difficult as witnessed by the long wait for a completeness proof for the modal μ-calculus [14], [15] and game logic. Our work demonstrates that the techniques from Afshari & Leigh [16] can be transferred to other fixpoint logics, and we expect that it is the beginning of a fruitful line of research into cut-free complete proof systems for fixpoint logics.

More generally, we believe this approach can be used to provide cut-free complete proof systems for coalgebraic μ-calculi [20], [21], and for coalgebraic dynamic logics [22]. Also, there are many fragments of the modal μ-calculus that could be studied by similar techniques. As one example, it would be interesting to develop annotated proof systems for

CTL*, and see if this could help to simplify Reynold's axiomatization of CTL* [23]. It should also be checked whether our proof can be adapted to provide a cut-free complete proof system for PDL. An indication that this is possible is that the deep rules in our system G are reminiscent of display calculi. The latter have been successfully applied to obtain a complete proof system for PDL [24].

Going the opposite direction, similar techniques could potentially be applied to extensions of the μ-calculus, such as the two-way μ-calculus [25], hybrid μ-calculus [26], and alternating μ-calculus [7].

Finally, we would like to investigate applications of our cut-free proof systems for game logic to prove interpolation.

Acknowledgements

We would like to thank the anonymous LICS 2019 reviewers for their helpful comments and suggestions. We acknowledge financial support from Swedish Research Council grant 2015-01774, EPSRC grant EP/N015843/1, and ERC consolidator grant 647289 CODA.

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Appendix

A. Omitted proofs of Section II

Proposition 5 There are recursively defined, truth-preserving translations

$$\begin{aligned} \mathsf{nf}(-)\colon &\quad \mathcal{L}_{\mathsf{Full}} \to \mathcal{L}_{\mathsf{NF}} \\ \mathsf{pa}(-)\colon &\quad \mathcal{L}_{\mathsf{Full}} \to \mathcal{L}_{\mathsf{Par}} \end{aligned}$$

Below we give an explicit definition of the translations $\mathsf{nf}(-)$ and $\mathsf{pa}(-)$, leaving it for the reader to check that the translations land in the proper fragments and are truth-preserving.

Translating from $\mathcal{L}_{\mathsf{Par}}$ to $\mathcal{L}_{\mathsf{NF}}$ is simply taking dual and negation normal form.

Definition 27. We define the translation $\mathsf{nf}(-)\colon \mathcal{L}_{\mathsf{Full}} \to \mathcal{L}_{\mathsf{NF}}$ as follows:

$$\begin{array}{lcl}
\mathsf{nf}(p) & = & p \\
\mathsf{nf}(\neg p) & = & \neg p \\
\mathsf{nf}(\neg\neg\varphi) & = & \mathsf{nf}(\varphi) \\
\mathsf{nf}(\neg(\varphi \vee \psi)) & = & \mathsf{nf}(\neg\varphi) \wedge \mathsf{nf}(\neg\psi) \\
\mathsf{nf}(\neg(\varphi \wedge \psi)) & = & \mathsf{nf}(\neg\varphi) \vee \mathsf{nf}(\neg\psi) \\
\mathsf{nf}(\neg\langle\gamma\rangle\varphi) & = & \langle \mathsf{nf}(\gamma^d)\rangle \mathsf{nf}(\neg\varphi) \\
 & & \\
\mathsf{nf}(g) & = & g \\
\mathsf{nf}(g^d) & = & g^d \\
\mathsf{nf}((\gamma^d)^d) & = & \mathsf{nf}(\gamma) \\
\mathsf{nf}((\gamma \sqcup \delta)^d) & = & \mathsf{nf}(\gamma^d) \sqcap \mathsf{nf}(\delta^d) \\
\mathsf{nf}((\gamma \sqcap \delta)^d) & = & \mathsf{nf}(\gamma^d) \sqcup \mathsf{nf}(\delta^d) \\
\mathsf{nf}((\gamma; \delta)^d) & = & \mathsf{nf}(\gamma^d); \mathsf{nf}(\delta^d) \\
\mathsf{nf}((\gamma^*)^d) & = & \mathsf{nf}(\gamma^d)^\times \\
\mathsf{nf}((\gamma^\times)^d) & = & \mathsf{nf}(\gamma^d)^* \\
\mathsf{nf}((\varphi?)^d) & = & \mathsf{nf}(\neg\varphi)! \\
\mathsf{nf}((\varphi!)^d) & = & \mathsf{nf}(\neg\varphi)?
\end{array}$$

We translate from $\mathcal{L}_{\mathsf{NF}}$ to $\mathcal{L}_{\mathsf{Par}}$ by expanding demonic operations as dual angelic ones.

Definition 28. We define the translation $\mathsf{pa}(-)\colon \mathcal{L}_{\mathsf{Full}} \to \mathcal{L}_{\mathsf{Par}}$ as follows:

$$\begin{array}{lcl}
\mathsf{pa}(p) & = & p \\
\mathsf{pa}(\neg p) & = & \neg p \\
\mathsf{pa}(\varphi \wedge \psi) & = & \neg(\neg\mathsf{pa}(\varphi) \vee \neg\mathsf{pa}(\psi)), \\
\mathsf{pa}(\varphi \vee \psi) & = & \mathsf{pa}(\varphi) \vee \mathsf{pa}(\psi) \\
\mathsf{pa}(\langle\gamma\rangle\varphi) & = & \langle \mathsf{pa}(\gamma)\rangle \mathsf{pa}(\varphi) \\
 & & \\
\mathsf{pa}(g) & = & g \\
\mathsf{pa}(\gamma^d) & = & \gamma^d \\
\mathsf{pa}(\gamma \sqcup \delta) & = & \mathsf{pa}(\gamma) \sqcup \mathsf{pa}(\delta) \\
\mathsf{pa}(\gamma \sqcap \delta) & = & (\mathsf{pa}(\gamma)^d \sqcup \mathsf{pa}(\delta)^d)^d, \\
\mathsf{pa}(\gamma; \delta) & = & \mathsf{pa}(\gamma); \mathsf{pa}(\delta) \\
\mathsf{pa}(\gamma^*) & = & \mathsf{pa}(\gamma)^* \\
\mathsf{pa}(\gamma^\times) & = & ((\mathsf{pa}(\gamma)^d)^*)^d \\
\mathsf{pa}(\varphi?) & = & \mathsf{pa}(\varphi)? \\
\mathsf{pa}(\varphi!) & = & ((\neg\mathsf{pa}(\varphi))?)^d
\end{array}$$

B. Omitted proofs of Section III

1) Lemmas 14 and 34: We will work towards proving Lemmas 14 and 34. To this end, we introduce some auxiliary notions. For $\mathsf{a} \in N^*$ and $\Gamma \subseteq F^\times$, let $\mathsf{a}\restriction_\Gamma$ denote the subsequence of a of all names x in a such that $\mathsf{x} \in N_\varphi$ for $\varphi \in \Gamma$. Similarly, for $X \subseteq N$ we write $\mathsf{a}\restriction_X$ to denote the subsequence of a consisting of names from X. By minor abuse of notation, we may write $\mathsf{a} \subseteq X$ to indicate that all names occurring in a are in X.

We define the set $S^\times(\varphi) \subseteq F^\times(\varphi)$ of *surface level greatest fixpoints* of $\varphi \in \mathcal{L}_{\mathsf{NF}}$ as follows.

$$\begin{array}{rclrcl}
S^\times(p) & = & \emptyset & S^\times(g, \varphi) & = & \emptyset \\
S^\times(\neg p) & = & \emptyset & S^\times(g^d, \varphi) & = & \emptyset \\
S^\times(\varphi \wedge \psi) & = & S^\times(\varphi) \cup S^\times(\psi) & S^\times(\gamma^*, \varphi) & = & \emptyset \\
S^\times(\varphi \vee \psi) & = & S^\times(\varphi) \cup S^\times(\psi) & S^\times(\gamma^\times, \varphi) & = & \{\langle\gamma^\times\rangle\varphi\} \\
S^\times(\langle\gamma\rangle\varphi) & = & S^\times(\gamma, \varphi) \cup S^\times(\varphi) & S^\times(\psi?, \varphi) & = & S^\times(\psi) \\
 & & & S^\times(\psi!, \varphi) & = & S^\times(\psi)
\end{array}$$

$$\begin{aligned}
S^\times(\gamma \sqcup \delta, \varphi) &= S^\times(\gamma, \varphi) \cup S^\times(\delta, \varphi) \\
S^\times(\gamma \sqcap \delta, \varphi) &= S^\times(\gamma, \varphi) \cup S^\times(\delta, \varphi) \\
S^\times(\gamma\,; \delta, \varphi) &= S^\times(\gamma, \langle\delta\rangle\varphi) \cup S^\times(\delta, \varphi)
\end{aligned}$$

In other words, $S^\times(\varphi)$ are those $\langle\gamma^\times\rangle\psi$ in $F^\times(\varphi)$ such that $\gamma^\times$ is not a direct subterms of another (angelic or demonic) iteration operator. For example, $S^\times(\langle g^\times ; h^\times\rangle p) = \{\langle g^\times\rangle\langle h^\times\rangle p, \langle h^\times\rangle p\}$, $S^\times(\langle g^\times \sqcup h^\times\rangle p) = \{\langle g^\times\rangle p, \langle h^\times\rangle p\}$, $S^\times(\langle(g^\times)\rangle^* p) = \emptyset$, and $S^\times(\langle(\langle g^\times\rangle p?)^\times\rangle q) = \{\langle(\langle g^\times\rangle p?)^\times\rangle q\}$. Note that $\langle\gamma^\times\rangle\psi \in S^\times(\varphi)$ does not imply that $\langle\gamma^\times\rangle\psi \trianglelefteq \varphi$.

Lemma 29. For all annotations $\mathsf{a} \in N^*$, formulas $\varphi \in \mathcal{L}_{\mathsf{NF}}$, game terms $\gamma \in \mathcal{G}_{\mathsf{NF}}$ and $X \subseteq N$:

1) If $\mathsf{a}\restriction_{S^\times(\varphi)} \subseteq X$, then $\varphi^{\mathsf{a}\bullet} = \varphi^{\mathsf{a}\restriction_X\bullet}$.
2) If $\mathsf{a}\restriction_{S^\times(\gamma,\varphi)} \subseteq X$, then $\beta(\gamma, \mathsf{a}, \varphi) = \beta(\gamma, \mathsf{a}\restriction_X, \varphi)$.

Proof. We prove the two claims by a mutual induction on the subterm-relation, i.e., for any $t \in \mathcal{L}_{\mathsf{NF}} \cup \mathcal{G}_{\mathsf{NF}}$ the I.H. stipulates that

- claim (1) holds for all formulas ψ such that $\psi \triangleleft t$ and
- claim (2) holds for all game terms δ such that $\delta \triangleleft t$.

For the first claim, the only interesting induction step is for modal formulas $\langle\gamma\rangle\varphi \in \mathcal{L}_{\mathsf{NF}}$. Suppose $\mathsf{a}\restriction_{S^\times(\langle\gamma\rangle\varphi)} \subseteq X$. We have $S^\times(\gamma, \varphi) \subseteq S^\times(\langle\gamma\rangle\varphi)$ and thus $\mathsf{a}\restriction_{S^\times(\gamma,\varphi)} \subseteq \mathsf{a}\restriction_{S^\times(\langle\gamma\rangle\varphi)} \subseteq X$. By the induction hypothesis of (2) on γ we get:

$$\beta(\gamma, \mathsf{a}, \varphi) = \beta(\gamma, \mathsf{a}\restriction_X, \varphi)$$

Furthermore $S^\times(\varphi) \subseteq S^\times(\langle\gamma\rangle\varphi)$, so we have $\mathsf{a}\restriction_{S^\times(\varphi)} \subseteq X$. So by the induction hypothesis of (1) on φ we get:

$$\varphi^{\mathsf{a}\bullet} = \varphi^{\mathsf{a}\restriction_X\bullet}$$

Putting these observations together we get:

$$\begin{aligned}
(\langle\gamma\rangle\varphi)^{\mathsf{a}\bullet} &= \langle\!\langle \beta(\gamma, \mathsf{a}, \varphi)\rangle\!\rangle \varphi^{\mathsf{a}\bullet} \\
&= \langle\!\langle \beta(\gamma, \mathsf{a}\restriction_X, \varphi)\rangle\!\rangle \varphi^{\mathsf{a}\restriction_X\bullet} \\
&= \langle\gamma\rangle\varphi^{\mathsf{a}\restriction_X\bullet}
\end{aligned}$$

as required.

We now turn to the induction on game terms, for item (2): the atomic cases for $\gamma = g$ or $\gamma = g^d$ are trivial, and the induction steps for $\sqcup, \sqcap$ are straightforward. For composition suppose that $S^\times(\gamma\,;\delta,\varphi) \subseteq X$. Since $S^\times(\gamma\,;\delta,\varphi) = S^\times(\gamma, \langle\delta\rangle\varphi) \cup S^\times(\delta,\varphi)$ we have $\mathsf{a}\restriction_{S^\times(\gamma,\langle\delta\rangle\varphi)} \subseteq X$ and $\mathsf{a}\restriction_{S^\times(\delta,\varphi)} \subseteq X$. So the induction hypothesis on γ and δ gives:

$$\begin{aligned}\beta(\gamma\,;\delta,\mathsf{a},\varphi) &= \beta(\gamma,\mathsf{a},\langle\delta\rangle\varphi)\,;\beta(\delta,\mathsf{a},\varphi)\\ &= \beta(\gamma,\mathsf{a}\restriction_X,\langle\delta\rangle\varphi)\,;\beta(\delta,\mathsf{a}\restriction_X,\varphi)\\ &= \beta(\gamma\,;\delta,\mathsf{a}\restriction_X,\varphi)\end{aligned}$$

For angelic tests, suppose that $\mathsf{a}\restriction_{S^\times(\psi?,\varphi)} \subseteq X$. Since $S^\times(\psi?,\varphi) = S^\times(\psi) \cup S^\times(\varphi)$ we get $\mathsf{a}\restriction_{S^\times(\psi)} \subseteq X$. Using the induction hypothesis of (1) on the formula ψ, we now get:

$$\begin{aligned}\beta(\psi?,\mathsf{a},\varphi) &= \psi^{\mathsf{a}\bullet}?\\ &= \psi^{\mathsf{a}\restriction_X\bullet}?\\ &= \beta(\psi?,\mathsf{a}\restriction_X,\varphi)\end{aligned}$$

The reasoning for ! is the same.

The induction step for $*$ is trivial since $\beta(\gamma^*,\mathsf{a},\varphi) = \gamma^*$ for any annotation a. Finally, the induction step for $\times$ is handled as follows: suppose that $\mathsf{a}\restriction_{S^\times(\gamma^\times,\varphi)} \subseteq X$. Let $\mathsf{x}_1,\dots,\mathsf{x}_n$ be all the names in a for the fixpoint $\langle\gamma^\times\rangle\varphi$. Since $S^\times(\gamma^\times,\varphi) = \{\langle\gamma^\times\rangle\varphi\}$, this means that X contains $\mathsf{x}_1,\dots,\mathsf{x}_n$, and so these are also all the names for $\langle\gamma^\times\rangle\varphi$ in $\mathsf{a}\restriction_X$. From this it is immediate from the definition of $\beta(-,-,-)$ that:

$$\beta(\gamma^\times,\mathsf{a},\varphi) = \beta(\gamma^\times,\mathsf{a}\restriction_X,\varphi)$$

as required. □

Lemma 30. For all $\delta,\gamma \in \mathcal{G}_{\mathsf{NF}}$ and $\varphi,\psi \in \mathcal{L}_{\mathsf{NF}}$:

1) If $\langle\delta^\times\rangle\psi \in S^\times(\varphi)$ then $\delta^\times \lhd \varphi$.
2) If $\langle\delta^\times\rangle\psi \in S^\times(\gamma,\varphi)$ then $\delta^\times \unlhd \gamma$.

Proof. By a mutual induction on φ and γ The easy argument is left to the reader. □

Lemma 31. For all annotations $\mathsf{a} \in N^*$, $\gamma \in \mathcal{G}_{\mathsf{NF}}$, $\varphi \in \mathcal{L}_{\mathsf{NF}}$ and $\circ \in \{\times,*\}$, if $\mathsf{a} \preccurlyeq \langle\gamma^\circ\rangle\varphi$ then $\beta(\gamma,\mathsf{a},\langle\gamma^\circ\rangle\varphi) = \gamma$.

Proof. By Lemma 29 (taking $X = S^\times(\gamma,\langle\gamma^\circ\rangle\varphi)$), we have that $\beta(\gamma,\mathsf{a},\langle\gamma^\circ\rangle\varphi) = \beta(\gamma,\mathsf{a}\restriction_{S^\times(\gamma,\langle\gamma^\circ\rangle\varphi)},\langle\gamma^\circ\rangle\varphi)$. Hence, it suffices to show that $\mathsf{a}\restriction_{S^\times(\gamma,\langle\gamma^\circ\rangle\varphi)} = \varepsilon$, because $\beta(\gamma,\varepsilon,\langle\gamma^\circ\rangle\varphi) = \gamma$.

So let $\mathsf{x} \in N_{\langle\delta^\times\rangle\psi}$ be a name occurring in a. By the assumption that $\mathsf{a} \preccurlyeq \langle\gamma^\circ\rangle\varphi$, it follows that $\langle\delta^\times\rangle\psi \preccurlyeq \langle\gamma^\circ\rangle\varphi$, i.e., $\gamma^\circ \unlhd \delta^\times$ and hence $\gamma \lhd \delta^\times$, so it is *not* the case that $\delta^\times \unlhd \gamma$. By Lemma 30(2) this entails that $\langle\delta^\times\rangle\psi \notin S^\times(\gamma,\langle\gamma^\circ\rangle\varphi)$. We have therefore shown that $\mathsf{a}\restriction_{S^\times(\gamma,\langle\gamma^\circ\rangle\varphi)} = \varepsilon$, which concludes the proof. □

We denote by $C(\varphi)$, the number of occurrences of the demonic iteration symbol $^\times$ in the formula $\varphi \in \mathcal{L}_{\mathsf{NF}}$. A precise, inductive definition is left to the reader.

Lemma 32. For all $\varphi,\psi \in \mathcal{L}_{\mathsf{NF}}$ and $\gamma \in \mathcal{G}_{\mathsf{NF}}$:

1) If $\psi \in S^\times(\varphi)$ then $C(\psi) \le C(\varphi)$.
2) If $\psi \in S^\times(\gamma,\varphi)$ then $C(\psi) \le C(\gamma) + C(\varphi)$

Proof. The two items can be proved by a straightforward mutual induction on the complexity of game terms γ and formulas φ. □

Lemma 33. Consider a sequence $\mathsf{a} = \mathsf{b}\mathsf{x}_1\dots\mathsf{x}_n$ where $\mathsf{x}_1,\dots,\mathsf{x}_n$ are all the names of a fixpoint $\langle\gamma^\times\rangle\varphi$. Then we have that $\varphi^{b\bullet} = \varphi^{\mathsf{a}\bullet}$

Proof. First, we observe that $\varphi^{\mathsf{a}\bullet} = \varphi^{\mathsf{a}\restriction_{S^\times(\varphi)}\bullet}$. This is an instance of Lemma 29, if we take X to be the set of all names associated with a fixpoint in $S^\times(\varphi)$. It therefore suffices to prove that $\mathsf{x}_1,\dots,\mathsf{x}_n$ do not appear in $\mathsf{a}\restriction_{S^\times(\varphi)}$. Since $\mathsf{x}_1,\dots,\mathsf{x}_n$ are all the names for $\langle\gamma^\times\rangle\varphi$, it suffices to prove $\langle\gamma^\times\rangle\varphi \notin S^\times(\varphi)$. By Lemma 32(1), $\langle\gamma^\times\rangle\varphi \in S^\times(\varphi)$ would imply that $C(\langle\gamma^\times\rangle\varphi) \le C(\varphi)$, which is clearly impossible. □

We are now ready to prove Lemma 14.

Proof of Lemma 14 For item (2), we apply the definition of the bullet translation to obtain:

$$\begin{aligned}(\langle\gamma^\times\rangle\varphi)^{\mathsf{a}\bullet} &= \langle\!\langle\beta(\gamma^\times,\mathsf{a},\varphi)\rangle\!\rangle\varphi^{\mathsf{a}\bullet}\\ &= \langle(\underleftarrow{\chi!\cdot\gamma})^\times\rangle\langle\!\langle\underleftarrow{\chi!}\rangle\!\rangle\varphi^{\mathsf{a}\bullet}\end{aligned}$$

The result now follows since $\varphi^{\mathsf{a}\bullet} = \varphi^{b\bullet}$, by Lemma 33.

For item (3), we apply the bullet translation again to get:

$$\begin{aligned}&(\varphi\wedge\langle\gamma\rangle\langle\gamma^\times\rangle\varphi)^{\mathsf{a}\bullet}\\ =\;&\varphi^{\mathsf{a}\bullet}\wedge\langle\!\langle\beta(\gamma,\mathsf{a},\langle\gamma^\times\rangle\varphi)\rangle\!\rangle\langle\!\langle\beta(\gamma^\times,\mathsf{a},\varphi)\rangle\!\rangle\varphi^{\mathsf{a}\bullet}\\ =\;&\varphi^{\mathsf{a}\bullet}\wedge\langle\!\langle\beta(\gamma,\mathsf{a},\langle\gamma^\times\rangle\varphi)\rangle\!\rangle\langle(\underleftarrow{\chi!\cdot\gamma})^\times\rangle\langle\!\langle\underleftarrow{\chi!}\rangle\!\rangle\varphi^{\mathsf{a}\bullet}\\ =\;&\varphi^{b\bullet}\wedge\langle\!\langle\beta(\gamma,\mathsf{a},\langle\gamma^\times\rangle\varphi)\rangle\!\rangle\langle(\underleftarrow{\chi!\cdot\gamma})^\times\rangle\langle\!\langle\underleftarrow{\chi!}\rangle\!\rangle\varphi^{b\bullet}\end{aligned}$$

where for the last step we used Lemma 33 again. By Lemma 31, we get $\beta(\gamma,\mathsf{a},\langle\gamma^\times\rangle\varphi) = \gamma$, and so we are done.

We also need the following simpler analogue of Lemma 14 for least fixpoints.

Lemma 34. For a least fixpoint formula $\langle\gamma^*\rangle\varphi \in F^*$ and an annotation $\mathsf{a} \preccurlyeq \langle\gamma^*\rangle\varphi$, we have

$$(\varphi\vee\langle\gamma\rangle\langle\gamma^*\rangle\varphi)^{\mathsf{a}\bullet} = \varphi^{\mathsf{a}\bullet}\vee\langle\gamma\rangle\langle\gamma^*\rangle\varphi^{\mathsf{a}\bullet}. \tag{4}$$

Proof. We use the definition of the bullet translation to compute as follows:

$$\begin{aligned}(\varphi\vee\langle\gamma\rangle\langle\gamma^*\rangle\varphi)^{\mathsf{a}\bullet} &= \varphi^{\mathsf{a}\bullet}\vee\langle\beta(\gamma,\mathsf{a},\langle\gamma^*\rangle\varphi)\rangle\langle\beta(\gamma^*,\mathsf{a},\varphi)\rangle\varphi^{\mathsf{a}\bullet}\\ &= \varphi^{\mathsf{a}\bullet}\vee\langle\beta(\gamma,\mathsf{a},\langle\gamma^*\rangle\varphi)\rangle\langle\gamma^*\rangle\varphi^{\mathsf{a}\bullet}\end{aligned}$$

By Lemma 31 we get $\beta(\gamma,\mathsf{a},\langle\gamma^*\rangle\varphi) = \gamma$, and so we are done. □

2) Missing cases in Theorem 15: We now give more details for the translation of some of the rules from CloG to G in the proof of Theorem 15.

For the weakening rule this is trivial. The case of Ax1: $\Phi = p^{\varepsilon}, \overline{p}^{\varepsilon}$ is also immediate as $\Phi^{\bullet} = p, \overline{p}$ which is an instance of Ax. Equally straightforward to translate are the CloG rules dealing with the Boolean and the (basic) modal operators. Here one simply has to observe that these connectives commute with the bullet translation. Concerning the rules for tests and for angelic and demonic choice we consider only the demonic choice operator in detail and leave the other similar cases to the reader. Suppose we derive $\Phi, (\langle \gamma \sqcap \delta \rangle \varphi)^{\mathsf{a}}$ within the CloG proof π via an application of the $\sqcap$-rule. The translation of the assumption of the rule is $\Phi^{\bullet}, (\langle \gamma \rangle \varphi)^{\mathsf{a}\bullet} \wedge (\langle \delta \rangle \varphi)^{\mathsf{a}\bullet}$. Spelling out the details of the bullet translation, this can be rewritten as $\Phi^{\bullet}, \langle\!\langle \beta(\gamma, \mathsf{a}, \varphi) \rangle\!\rangle \varphi^{\mathsf{a}\bullet} \wedge \langle\!\langle \beta(\delta, \mathsf{a}, \varphi) \rangle\!\rangle \varphi^{\mathsf{a}\bullet}$. From here we obtain the following G derivation steps:

$$\dfrac{\dfrac{\Phi^{\bullet}, \langle\!\langle \beta(\gamma, \mathsf{a}, \varphi) \rangle\!\rangle \varphi^{\mathsf{a}\bullet} \wedge \langle\!\langle \beta(\delta, \mathsf{a}, \varphi) \rangle\!\rangle \varphi^{\mathsf{a}\bullet}}{\Phi^{\bullet}, \langle \beta(\gamma, \mathsf{a}, \varphi) \rangle \varphi^{\mathsf{a}\bullet} \wedge \langle \beta(\delta, \mathsf{a}, \varphi) \rangle \varphi^{\mathsf{a}\bullet}}\,(*)}{\Phi^{\bullet}, \langle \beta(\gamma \sqcap \delta, \mathsf{a}, \varphi) \rangle \varphi^{\mathsf{a}\bullet}}\,\sqcap$$

where (*) possibly involves applying the $;_d$-rule multiple times. It is easy to see that the conclusion of the G derivation is equal to $\Phi^{\bullet}, (\langle \gamma \sqcap \delta \rangle \varphi)^{\mathsf{a}\bullet}$ as required.

To see how to deal with the $*$-rule consider an application of the rule in π of the form

$$\dfrac{\Phi, (\varphi \vee \langle \gamma \rangle \langle \gamma^{*} \rangle \varphi)^{\mathsf{a}}}{\Phi, (\langle \gamma^{*} \rangle \varphi)^{\mathsf{a}}}\,*$$

The premise of this rule translates to $\Phi^{\bullet}, \varphi^{\mathsf{a}\bullet} \vee \langle \gamma \rangle \langle \gamma^{*} \rangle \varphi^{\mathsf{a}\bullet}$ - this can be seen using Lemma 34 and the side condition of the $*$-rule. An application of the $*$-rule in G yields $\Phi^{\bullet}, \langle \gamma^{*} \rangle \varphi^{\mathsf{a}\bullet}$ which in turn equals $\Phi^{\bullet}, (\langle \gamma^{*} \rangle \varphi)^{\mathsf{a}\bullet}$ as required.

For the $\times$ rule consider a rule application

$$\dfrac{\Phi, (\varphi \wedge \langle \gamma \rangle \langle \gamma^{\times} \rangle \varphi)^{\mathsf{a}}}{\Phi, (\langle \gamma^{\times} \rangle \varphi)^{\mathsf{a}}}\,\times$$

By the side condition of the $\times$-rule we can assume that a is of the form $\mathsf{b}\mathsf{x}_1 \dots \mathsf{x}_n$ where $\mathsf{x}_1 \dots \mathsf{x}_n$ are all the names of $\langle \gamma^{\times} \rangle \varphi$ occurring in a. Applying Lemma 14 we get that the premise of the rule translates to $\Phi^{\bullet}, \varphi^{\mathsf{b}\bullet} \wedge \langle\!\langle \gamma \rangle\!\rangle \langle (\underleftarrow{\chi! \cdot \gamma})^{\times} \rangle \langle\!\langle \underleftarrow{\chi!} \rangle\!\rangle \varphi^{\mathsf{b}\bullet}$ where $\vec{\chi}$ consists of the context sequents as in previous cases. Consider now the following G derivation steps:

$$\dfrac{\dfrac{\dfrac{\Phi^{\bullet}, \varphi^{\mathsf{b}\bullet} \wedge \langle\!\langle \gamma \rangle\!\rangle \langle (\underleftarrow{\chi! \cdot \gamma})^{\times} \rangle \langle\!\langle \underleftarrow{\chi!} \rangle\!\rangle \varphi^{\mathsf{b}\bullet}}{\Phi^{\bullet}, \varphi^{\mathsf{b}\bullet} \wedge \langle \gamma \rangle \langle (\underleftarrow{\chi! \cdot \gamma})^{\times} \rangle \langle\!\langle \underleftarrow{\chi!} \rangle\!\rangle \varphi^{\mathsf{b}\bullet}}\,;_d}{\Phi^{\bullet}, \langle\!\langle \underleftarrow{\chi!} \rangle\!\rangle \varphi^{\mathsf{b}\bullet} \wedge \langle \underleftarrow{\chi! \cdot \gamma} \rangle \langle (\underleftarrow{\chi! \cdot \gamma})^{\times} \rangle \langle\!\langle \underleftarrow{\chi!} \rangle\!\rangle \varphi^{\mathsf{b}\bullet}}\,\mathsf{Mon}^{\mathsf{g}}_{\mathsf{d}}, \mathsf{Mon}^{\mathsf{f}}_{\mathsf{d}}}{\Phi^{\bullet}, \langle (\underleftarrow{\chi! \cdot \gamma})^{\times} \rangle \langle\!\langle \underleftarrow{\chi!} \rangle\!\rangle \varphi^{\mathsf{b}\bullet},}\,\times$$

where the double line indicates multiple applications of the deep monotonicity rules. Observe now that the conclusion $\Phi^{\bullet}, \langle (\underleftarrow{\chi! \cdot \gamma})^{\times} \rangle \langle\!\langle \underleftarrow{\chi!} \rangle\!\rangle \varphi^{\mathsf{b}\bullet}$, is by Lemma 14 equal to $\Phi^{\bullet}, (\langle \gamma^{\times} \rangle \varphi)^{\mathsf{a}\bullet}$ as required.

C. Omitted proofs of Section IV

This part of the appendix contains definitions and lemmas that lead up to a detailed proof of Theorem 20.

First, we translate monotone μ-calculus into normal μ-calculus by extending the translation from [6], [19] of monotonic modal logic into normal bimodal logic. More precisely, we define the language $\mathcal{L}^{2\mu}_{\mathsf{NF}}$ to be the set of modal μ-calculus formulas over the set of labels $L = \{g_N \mid g \in \mathsf{G}_0\} \cup \{g_{\ni} \mid g \in \mathsf{G}_0\}$ defined in the usual way, and interpret $\mathcal{L}^{2\mu}_{\mathsf{NF}}$ over Kripke models that have an accessibility relation for each label in L. We define the translation $(-)^t \colon \mathcal{L}^{\mu}_{\mathsf{NF}} \to \mathcal{L}^{2\mu}_{\mathsf{NF}}$ follows:

- $p^t = p^t$ and $(\neg p)^t = \neg p$.
- For Boolean connectives: compositionally.
- $(\mu x.A)^t = \mu x.(A^t)$ and $(\nu x.A)^t = \nu x.(A^t)$,
- $(\langle g \rangle A)^t = \langle g_N \rangle [g_{\ni}](A^t)$,
- $(\langle g^d \rangle A)^t = [g_N] \langle g_{\ni} \rangle (A^t)$.

Proof of Lemma 19 Item 1) Given a game model $\mathbb{S} = (S, E, V)$ we construct a Kripke model $\mathbb{S}^t$ for $\mathcal{L}^{2\mu}_{\mathsf{NF}}$ with state space $S \cup \wp(S)$ and valuation $V'(p) = V(p)$, by taking

$R_{g_N} = \{(s, U) \in S \times \wp(S) \mid U \in E_g(s)\}$,
$R_{g_{\ni}} = \{(U, s) \in \wp(S) \times S \mid s \in U\}$.

It is straightforward to show by induction that for all $s \in S$ and all $C \in \mathcal{L}^{\mu}_{\mathsf{NF}}$: $s \in [\![C]\!]^{\mathbb{S}}$ iff $s \in [\![C^t]\!]^{\mathbb{S}}$. It follows that $(-)^t$ preserves satisfiability.

Item 2) Given a Kripke $\mathcal{L}^{2\mu}_{\mathsf{NF}}$- model $\mathcal{K}$ with state space W and valuation V, we can construct a game model $\mathcal{K}_m$ with the same state space and valuation, and by defining for $g \in \mathsf{G}_0$ an effectivity function E_g by

$$E_g(Z) = \{w \in W \mid \exists v \in W (w R_{g_N} v \text{ and } R_{g_{\ni}}[v] \subseteq Z)\}.$$

It is again routine to show that for all Kripke models $\mathcal{K}$ and for all $C \in \mathcal{L}^{\mu}_{\mathsf{NF}}$, we have: $[\![C]\!]^{\mathcal{K}_m} = [\![C^t]\!]^{\mathcal{K}}$.

From this it follows that if C is valid over game models (for $\mathcal{L}^{\mu}_{\mathsf{NF}}$) then C^t is valid over Kripke models (for $\mathcal{L}^{2\mu}_{\mathsf{NF}}$). ☐

We will now show how to transform derivations from the system Clo [16] into CloM using the translation $(-)^t$.

Lemma 35. For all $C \in \mathcal{L}^{\mu}_{\mathsf{NF}}$, if $\mathsf{Clo} \vdash C^t$ then $\mathsf{CloM}_C \vdash C$.

Proof. Given a Clo-proof π for C^t, we shall construct step by step a tree rooted at a node labelled C in which every edge is labelled by a proof rule of CloM_C, every node is labelled by a sequent or an expression $[\Gamma]^{\mathsf{x}}$ marking the sequent Γ as a discharged assumption. We also construct a map h sending each node in the tree to some node in the proof tree π, such that the label of $h(u)$ is Γ^t if the label of u is Γ. Here, we are extending the translation $(-)^t$ to annotated sequents in the obvious way. Rather than performing an induction on the size of proofs, the construction will simply proceed from the root up, and will be carried out in such a way that the end result will clearly be a CloM_C-proof.

We start by creating a root node labelled with C^{ε} and letting h map this node to the root of the proof tree π. Whenever we create a node in the construction that is mapped via h to a discharged assumption $[\Gamma^t]^{\mathsf{x}}$, we make sure to label that node by the discharged assumption $[\Gamma]^{\mathsf{x}}$. Note that we can always assume that for each A in the label of $h(l)$, there is at most one B in the label of l with $B^t = A$. (Otherwise, we just apply the weakening rule.)

Now, given that we have constructed the tree up to some point, but we do not yet have a proper CloM_C-proof, we pick a leaf l of the tree that is not a discharged assumption and not an axiom of CloM_C, and continue the construction by a case distinction. Most of the cases are handled in a trivial manner, and we only give two examples of the easy cases: if the leaf l is mapped to a node $h(l)$ labelled $\Gamma, \nu x.A^{\mathsf{a}}$, and is the conclusion to an instance of the ν-$\mathsf{clo}_{\mathsf{x}}$-rule with premise $\Gamma, A(\nu x.A)^{\mathsf{a}}$, then there must be some Ψ, B such that $\Psi^t = \Gamma$, $B = A$ and l is labelled $\Psi, \nu x.B$. We add a new child of l which we label $\Psi, B(\nu x.B)^{\mathsf{ax}}$. We let h map this child to the premise of $h(l)$, and label the edge by the rule ν-$\mathsf{clo}_{\mathsf{x}}$. For another example, if the leaf l is mapped to $h(l)$ labelled $\Gamma, A \wedge B$, and is the conclusion to an instance of the $\wedge$-rule with premises Γ, A and Γ, B, then l must be labelled $\Psi, C \wedge D$ where $\Psi^t = \Gamma$, $C^t = A$ and $D^t = B$. We add two new children of l which we label Ψ, C and Ψ, D respectively. We extend the map h by sending each child to its corresponding premise, and labelling the edges by the $\wedge$-rule.

The only non-trivial case is when the node $h(l)$ is a conclusion to an instance of the modal rule. This is only possible if $h(l)$ has a label of the form $\langle g_N\rangle[g_\ni]A_1, ..., \langle g_N\rangle[g_\ni]A_n, [g_N]\langle g_\ni\rangle B$, where l is labelled with the sequent $\langle g\rangle C_1, ..., \langle g\rangle C_n, \langle g^d\rangle D$ and $C_1^t = A_1, ..., C_n^t = A_n$, $D^t = B$. By inspection of the rules of Clo, and assuming that $n \geq 2$ (since the other case is easier), we can see that the subtree of the proof π rooted at the node $h(l)$ must have the following shape, since these are the only rules that are applicable:

$$\dfrac{\dfrac{\dfrac{\vdots}{A_i, B}}{[g_\ni]A_i, \langle g_\ni\rangle B}\,\mathsf{mod}}{\dfrac{[g_\ni]A_1, ..., [g_\ni]A_n, \langle g_\ni\rangle B}{\langle g_N\rangle[g_\ni]A_1, ..., \langle g_N\rangle[g_\ni]A_n, [g_N]\langle g_\ni\rangle B}\,\mathsf{mod}}\,\mathsf{weak}$$

for some $i \in \{1, ..., n\}$. We thus continue the construction as follows:

$$\dfrac{\dfrac{C_i, D}{\langle g\rangle C_i, \langle g^d\rangle D}\,\mathsf{mod}}{\langle g\rangle C_1, ..., \langle g\rangle C_n, \langle g^d\rangle D}\,\mathsf{weak}$$

Finally, we extend the map h by sending the new leaf labelled C_i, D to the node in the previous figure labelled A_i, B. ☐

Lemma 36. For all $C \in \mathcal{L}^{\mu}_{\mathsf{NF}}$, if $\mathsf{CloM} \vdash C$ then $\mathsf{Clo} \vdash C^t$.

Proof. The proof is very similar to the proof of Lemma 35 above. In this case –again starting from the root– we turn a CloM_C-proof π of a given formula $C \in \mathcal{L}^{\mu}_{\mathsf{NF}}$ into a corresponding Clo-proof of C^t. Unlike in the proof of Lemma 35 we only sketch the construction. First we note that all cases of non-modal rules are even easier than in the proof of Lemma 35 as the non-modal CloM_C-rules are instances of Clo-rules.

For the modal case suppose that we have constructed a partial Clo-proof of C^t with a leaf labelled with $(\langle g\rangle D_1)^t, (\langle g^d\rangle D_2)^t$ and that the corresponding node in the CloM_C-proof π labelled with $\langle g\rangle D_1, \langle g^d\rangle D_2$ has been derived via an application of the modal rule:

$$\dfrac{\dfrac{\vdots}{D_1, D_2}}{\langle g\rangle D_1, \langle g^d\rangle D_2}\,\mathsf{mod}$$

By definition we have $(\langle g\rangle D_1)^t = \langle g_N\rangle[g_\ni]D_1^t$ and $(\langle g^d\rangle D_2)^t = [g_N]\langle g_\ni\rangle D_2^t$. Therefore we can extend the Clo-proof of C^t as follows:

$$\dfrac{\dfrac{D_1^t, D_2^t}{[g_\ni]D_1^t, \langle g_\ni\rangle D_2^t}\,\mathsf{mod}}{\langle g_N\rangle[g_\ni]D_1^t, [g_N]\langle g_\ni\rangle D_2^t}\,\mathsf{mod}$$

This finishes the proof sketch. ☐

Proof of Theorem 20 *Soundness:* Let $C \in \mathcal{L}^{\mu}_{\mathsf{NF}}$, such that $\mathsf{CloM} \vdash C$. Then by Lemma 36, $\mathsf{Clo} \vdash C^t$ and hence by the soundness of Clo it follows that C^t is valid on all Kripke models for $\mathcal{L}^{2\mu}_{\mathsf{NF}}$. Suppose now that C would not be valid in all game models, i.e., there is a game model $\mathcal{M}$ and a state w in $\mathcal{M}$ such that $\mathcal{M}, w \vDash \overline{C}$. Since $(-)^t$ preserves satisfiability (Lemma 19(1)), it follows that $\overline{C}^t$ is satisfiable in a Kripke model for $\mathcal{L}^{2\mu}_{\mathsf{NF}}$. Finally, since $\overline{C}^t$ is equivalent with $\overline{C^t}$ this would imply that C^t is not valid, a contradiction.

Completeness: Assume that C is valid over game models. Then by Lemma 19(2), C^t is valid over Kripke models for $\mathcal{L}^{2\mu}_{\mathsf{NF}}$. From the completeness of Clo [16], it follows that $\mathsf{Clo} \vdash C^t$, and hence by Lemma 35 that $\mathsf{CloM} \vdash C$. ☐

D. Omitted proofs of Section V

1) Proof of Proposition 24: This part of the appendix contains a detailed proof for Proposition 24, which summarises the most important properties of our translation from game logic into the monotone μ-calculus.

We address the claims in the proposition with separate lemmas. The first of this lemmas entails that $x^\varphi \in Var(\xi^\sharp)$ for all $\varphi \in F(\xi)$:

Lemma 37.

1) For all game logic formulas φ, ψ: If $\psi \in F(\varphi)$ then $x^\psi \in Var(\varphi^\sharp)$.
2) For all game logic formulas χ, ψ and games γ: If $\psi \in F(\gamma, \chi)$ then $x^\psi \in Var(\tau^\chi_\gamma(\chi^\sharp))$.

Proof. Both items are are proven with a mutual induction over the complexity of the formula φ and of the game term γ.

Base case (1): If φ is of the form p or $\neg p$, then $F(\varphi) = \emptyset$, hence (1) holds trivially. *Base case (2):* Similarly, if γ is is of the form g or g^d, then $F(\gamma, \varphi) = \emptyset$, hence (2) holds trivially.

Induction hypotheses: Assume (1) holds for all formulas that are proper subterms of φ or γ. Assume (2) holds for all games that are proper subterms of φ or γ.

Induction step (1): Suppose $\varphi = \varphi_1 \vee \varphi_2$. We then have that $F(\varphi) = F(\varphi_1) \cup F(\varphi_2)$. Let $i \in \{1, 2\}$ and assume $\psi \in F(\varphi_i)$ then by IH for (1), we have $x^\psi \in \mathit{Var}(\varphi_i^\sharp)$ and hence $x^\psi \in \mathit{Var}(\varphi^\sharp) = \mathit{Var}(\varphi_1^\sharp) \cup \mathit{Var}(\varphi_2^\sharp)$.

The argument for $\varphi = \varphi_1 \wedge \varphi_2$ is similar.

Suppose now $\varphi = \langle\delta\rangle\varphi_0$. We then have $F(\varphi) = F(\delta, \varphi_0) \cup F(\varphi_0)$. For all $\psi \in F(\varphi_0)$, we have by the IH for (1) that $x^\psi \in \mathit{Var}(\varphi_0^\sharp)$, and since $\varphi_0^\sharp$ is a subterm of $(\langle\delta\rangle\varphi_0)^\sharp = \tau_\delta^{\varphi_0}(\varphi_0^\sharp)$, it follows that $\mathit{Var}(\varphi_0^\sharp) \subseteq \mathit{Var}(\varphi^\sharp)$. For all $\psi \in F(\delta, \varphi_0)$, we have by the IH for (2) that $x^\psi \in \mathit{Var}(\tau_\delta^{\varphi_0}(\varphi_0^\sharp)) = \mathit{Var}(\varphi^\sharp)$.

Induction step (2): Let χ be an arbitrary game logic formula. Suppose that $\gamma = \gamma_1 \sqcup \gamma_2$. We then have $F(\gamma, \chi) = F(\gamma_1, \chi) \cup F(\gamma_2, \chi)$. Let $i \in \{1, 2\}$ and assume that $\psi \in F(\gamma_i, \chi)$. Then by IH for (2), we have that $x^\psi \in \mathit{Var}(\tau_{\gamma_i}^\chi(\chi^\sharp)) \subseteq \mathit{Var}(\tau_{\gamma_1}^\chi(\chi^\sharp) \vee \tau_{\gamma_2}^\chi(\chi^\sharp)) = \mathit{Var}(\tau_\gamma^\chi(\chi^\sharp))$. The case for $\gamma = \gamma_1 \sqcap \gamma_2$ is similar.

Suppose $\gamma = \gamma_1 ; \gamma_2$. We then have $F(\gamma, \chi) = F(\gamma_1, \langle\gamma_2\rangle\chi) \cup F(\gamma_2, \chi)$. If $\psi \in F(\gamma_1, \langle\gamma_2\rangle\chi)$, then by IH for (2), we have that $x^\psi \in \mathit{Var}(\tau_{\gamma_1}^{\langle\gamma_2\rangle\chi}((\langle\gamma_2\rangle\chi)^\sharp)) = \mathit{Var}(\tau_{\gamma_1;\gamma_2}^\chi(\chi^\sharp))$. If $\psi \in F(\gamma_2, \chi)$, then by IH for (2), we have that $x^\psi \in \mathit{Var}(\tau_{\gamma_2}^\chi(\chi^\sharp)) \subseteq \mathit{Var}(\tau_{\gamma_1}^{\langle\gamma_2\rangle\chi}(\tau_{\gamma_2}^\chi(\chi^\sharp))) = \mathit{Var}(\tau_{\gamma_1;\gamma_2}^\chi(\chi^\sharp))$.

Suppose $\gamma = \delta^*$. We then have $F(\gamma, \chi) = \{\langle\delta^*\rangle\chi\} \cup F(\delta, \langle\delta^*\rangle\chi)$. If $\psi = \langle\delta^*\rangle\chi$, we use that $\tau_{\delta^*}^\chi(\chi^\sharp) = \mu x^{\langle\delta^*\rangle\chi}.\chi^\sharp \vee \tau_\delta^{\langle\delta^*\rangle\chi}(x^{\langle\delta^*\rangle\chi})$ and hence $x^\psi = x^{\langle\delta^*\rangle\chi} \in \mathit{Var}(\tau_{\delta^*}^\chi(\chi^\sharp))$. If $\psi \in F(\delta, \langle\delta^*\rangle\chi)$, then by IH for (2), we have that $x^\psi \in \mathit{Var}(\tau_\delta^{\langle\delta^*\rangle\chi}((\langle\delta^*\rangle\chi)^\sharp))$. It therefore suffices to show that $\mathit{Var}(\tau_\delta^{\langle\delta^*\rangle\chi}((\langle\delta^*\rangle\chi)^\sharp)) \subseteq \mathit{Var}(\tau_{\delta^*}^\chi(\chi^\sharp)) = \mathit{Var}(\mu x^{\langle\delta^*\rangle\chi}.\chi^\sharp \vee \tau_\delta^{\langle\delta^*\rangle\chi}(x^{\langle\delta^*\rangle\chi}))$. Since the context $\tau_\delta^{\langle\delta^*\rangle\chi}(-)$ occurs on both sides of the inclusion, it suffices to show that $\mathit{Var}((\langle\delta^*\rangle\chi)^\sharp) \subseteq \mathit{Var}(\tau_{\delta^*}^\chi(\chi^\sharp))$, and this holds since $(\langle\delta^*\rangle\chi)^\sharp = \tau_{\delta^*}^\chi(\chi^\sharp)$.

Suppose $\gamma = \varphi?$. By IH for (1) it follows that $x^\psi \in \mathit{Var}(\varphi^\sharp)$. Since $\tau_{\varphi?}^\chi(\chi^\sharp) = \varphi^\sharp \vee \chi^\sharp$, we have that $x^\psi \in \mathit{Var}(\tau_{\varphi?}^\chi(\chi^\sharp))$.

The case for $\gamma = \varphi!$ is similar. □

To show that $\xi^\sharp$ is locally well-named and that $x^\varphi \leq_{\xi^\sharp} x^\psi$ implies $\varphi \preccurlyeq \psi$ we prove a series of lemmas leading to the central property from Lemma 43 below.

Lemma 38. For all game logic formulas φ, all games γ and all μ-calculus formulas A, we have that $\mathit{FVar}(\tau_\gamma^\varphi(A)) \subseteq \mathit{FVar}(A)$, i.e., all free variables in $\tau_\gamma^\varphi(A)$, occur free in A.

Proof. The proof is a simple induction on the complexity of γ in the recursive clauses defining $\tau_\gamma^\varphi(A)$. □

Definition 39. A *call triple* is a triple (γ, φ, A) where γ is a game term, φ is a game logic formula and A is a formula of the monotone μ-calculus. The set of call triples is denoted by CTr.

Definition 40. Given a game logic formula ξ, the set of *recursive calls of* τ *on* ξ, denoted C_ξ, is the least fixpoint of the monotone map $c_\xi : \mathsf{P}(\mathsf{CTr}) \to \mathsf{P}(\mathsf{CTr})$ defined by setting $t \in c_\xi(X)$, for $X \subseteq \mathsf{CTr}$, iff:

- $t = (\gamma, \varphi, \varphi^\sharp)$ for some subformula $\langle\gamma\rangle\varphi$ of ξ, or
- t is of the form (γ, φ, A) or (γ', φ, A) where $(\gamma \sqcup \gamma', \varphi, A) \in X$, or
- t is of the form (γ, φ, A) or (γ', φ, A) where $(\gamma \sqcap \gamma', \varphi, A) \in X$, or
- t is of the form $(\gamma, \langle\gamma^\times\rangle\varphi, x^{\langle\gamma^\times\rangle\varphi})$ and there exists some A for which $(\gamma^\times, \varphi, A) \in X$,
- t is of the form $(\gamma, \langle\gamma^*\rangle\varphi, x^{\langle\gamma^*\rangle\varphi})$ and there exists some A for which $(\gamma^*, \varphi, A) \in X$,
- t is of the form (γ', φ, A) or $(\gamma, \langle\gamma'\rangle\varphi, \tau_\varphi^{\gamma'}(A))$ for some γ, γ', φ and A such that $((\gamma ; \gamma'), \varphi, A) \in X$.

Lemma 41. Let ξ be a game logic formula and let $\eta x^{\langle\delta^\circ\rangle\varphi}.B$ be a subformula of $\xi^\sharp$. Then there exists some A such that

$$\eta x^{\langle\delta^\circ\rangle\varphi}.B = \tau_{\delta^\circ}^\varphi(A)$$

and, furthermore, $(\delta^\circ, \varphi, A) \in C_\xi$.

Proof. We show with mutual induction on the complexity on subformula ψ of ξ and game terms γ occurring in ξ that

1) If $\eta x^{\langle\delta^\circ\rangle\varphi}.B$ is a subformula of $\psi^\sharp$ then there exists some A such that $\eta x^{\langle\delta^\circ\rangle\varphi}.B = \tau_\varphi^{\delta^\circ}(A)$ and, furthermore, $(\delta^\circ, \varphi, A) \in C_\xi$.
2) If $\eta x^{\langle\delta^\circ\rangle\varphi}.B$ is a subformula of $\tau_\gamma^\chi(D)$ and $(\gamma, \chi, D) \in C_\xi$ then it is either a subformula of D or there exists some A such that $\eta x^{\langle\delta^\circ\rangle\varphi}.B = \tau_\varphi^{\delta^\circ}(A)$ and, furthermore, $(\delta^\circ, \varphi, A) \in C_\xi$.

In the inductive argument we distinguish the following cases:

- In the base case we have that either $\psi = p$, $\psi = \neg p$, $\gamma = g$ or $\gamma = g^d$. The inductive claims are trivially satisfied for all of these cases because $\eta x^{\langle\delta^\circ\rangle\psi}.B$ is not a subformula of either $\psi^\sharp$ or of τ_γ^χ.
- If $\psi = \chi_1 \wedge \chi_2$ or $\psi = \chi_1 \vee \chi_2$ and $\eta x^{\langle\delta^\circ\rangle\varphi}.B$ is a subformula of $\psi^\sharp$ then $\eta x^{\langle\delta^\circ\rangle\varphi}.B$ is already a subformula of $\chi_i^\sharp$ for some $i \in \{1, 2\}$. It follows from (1) in the inductive assumption for χ_i that there is some A such that $\eta x^{\langle\delta^\circ\rangle\varphi}.B = \tau_\varphi^{\delta^\circ}(A)$ and that $(\delta^\circ, \varphi, A) \in C_\xi$. This is already the inductive claim we needed to show.
- Consider the case where $\psi = \langle\gamma\rangle\chi$ and assume that $\eta x^{\langle\delta^\circ\rangle\varphi}.B$ is a subformula of $\psi^\sharp = \tau_\gamma^\chi(\chi^\sharp)$. Because $\langle\gamma\rangle\chi$ is a subformula of ξ it follows from the definition of C_ξ that $(\gamma, \chi, \chi^\sharp) \in C_\xi$. We can thus apply (2) from inductive hypotheses to γ to obtain that either $\eta x^{\langle\delta^\circ\rangle\varphi}.B$ is a subformula of $\chi^\sharp$ or there exists some A such that $\eta x^{\langle\delta^\circ\rangle\varphi}.B = \tau_\varphi^{\delta^\circ}(A)$ and $(\delta^\circ, \varphi, A) \in C_\xi$. In the latter case we are done and in the former we can apply (1) from the inductive hypothesis to the subformula χ of ψ and obtain the required properties as well.
- For the cases where $\gamma = \gamma_1 \sqcap \gamma_2$ or $\gamma = \gamma_1 \sqcup \gamma_2$ assume that $\eta x^{\langle\delta^\circ\rangle\varphi}.B$ is a subformula of $\tau_\gamma^\chi(D)$ and

that $(\gamma, \chi, D) \in C_\xi$. From the former it follows then that $\eta x^{\langle\delta^\circ\rangle\varphi}.B$ is already a subformula of $\tau^\chi_{\gamma_i}(D)$ for some $i \in \{1,2\}$. From the latter it follows with the closure conditions of C_ξ that $(\gamma_i, \chi, D) \in C_\xi$. We can thus immediately apply (2) of the inductive hypothesis for the respective γ_i to obtain the required property.
- The interesting case is where $\gamma = \rho^\circ$ with $\circ \in \{*, \times\}$. We only consider the case where $\circ = *$. Assume that $(\rho^*, \chi, D) \in C_\xi$ and that $\eta x^{\langle\delta^\circ\rangle\varphi}.B$ is a subformula of
$$\tau^\chi_{\rho^*}(D) = \mu x^{\langle\rho^*\rangle\chi}.D \vee \tau^{\langle\rho^*\rangle\varphi}_\rho(x^{\langle\rho^*\rangle\chi}).$$
There are three possibilities how the latter might be the case.
First, $\eta x^{\langle\delta^\circ\rangle\varphi}.B$ might be equal to $\tau^\chi_{\rho^*}(D)$. In that case $\eta x^{\langle\delta^\circ\rangle\varphi}.B$ is of the required shape and $(\rho^*, \chi, D) \in C_\xi$ holds by assumption.
Second, it might be that $\eta x^{\langle\delta^\circ\rangle\varphi}.B$ is a subformula of D, in which case we are done immediately.
Third and last, $\eta x^{\langle\delta^\circ\rangle\varphi}.B$ might be a subformula of $\tau^{\langle\rho^*\rangle\chi}_\rho(x^{\langle\rho^*\rangle\chi})$. In that case we can apply (2) of the inductive hypothesis to ρ because $(\rho, \langle\rho^*\rangle\chi, x^{\langle\rho^*\rangle\chi}) \in C_\xi$ follows from the assumption that $(\rho^*, \chi, D) \in C_\xi$ together with the closure conditions from the definition of C_ξ. Hence, it follows that either $\eta x^{\langle\delta^\circ\rangle\varphi}.B$ is a subformula of $x^{\langle\rho^*\rangle\chi}$, which is impossible, or that there is some A such that $\eta x^{\langle\delta^\circ\rangle\varphi}.B = \tau^{\delta^\circ}_\varphi(A)$ and, furthermore, $(\delta^\circ, \varphi, A) \in C_\xi$. . The latter is exactly what we have to show.
- For the case where $\gamma = \gamma_1 ; \gamma_2$ assume that $\eta x^{\langle\delta^\circ\rangle\varphi}.B$ is a subformula of $\tau^\chi_{\gamma_1;\gamma_2}(D) = \tau^{\langle\gamma_2\rangle\chi}_{\gamma_1}(\tau^\chi_{\gamma_2}(D))$ and that $(\gamma_1 ; \gamma_2, \chi, D) \in C_\xi$. From the latter it follows with the definition of C_ξ that $(\gamma_1, \langle\gamma_2\rangle\chi, \tau^\chi_{\gamma_2}(D)) \in C_\xi$. Hence, we can apply (2) of the induction hypothesis to γ_1 and obtain that either $\eta x^{\langle\delta^\circ\rangle\varphi}.B$ is already of the required shape or it is a subterm of $\tau^\chi_{\gamma_2}(D)$. In the latter case we can use the fact that $(\gamma_2, \chi, D) \in C_\xi$, which also follows from $(\gamma_1 ; \gamma_2, \chi, D) \in C_\xi$, to apply the induction hypothesis again, this time to γ_2, and obtain that $\eta x^{\langle\delta^\circ\rangle\varphi}.B$ is either of the required shape or that it is a subterm of D. This is precisely what we need to show.
- Of the cases where $\gamma = \alpha?$ or $\gamma = \alpha!$ we just consider the former because the latter goes analogously. Assume that $\eta x^{\langle\delta^\circ\rangle\varphi}.B$ is a subformula of $\tau^\chi_{\alpha?}(D) = \alpha^\sharp \vee D$ and that $(\alpha?, \chi, D) \in C_\xi$. The former means that $\eta x^{\langle\delta^\circ\rangle\varphi}.B$ is either a subformula of D, in which case we are done, or it is a subformula of $\alpha^\sharp$. But if $\eta x^{\langle\delta^\circ\rangle\varphi}.B$ is a subformula of $\alpha^\sharp$ we can apply (1) to the subformula α of ξ to conclude that $\eta x^{\langle\delta^\circ\rangle\varphi}.B$ must be of the required shape.

□

Lemma 42. Let ξ be a game logic formula, let (γ, φ, A) be a recursive call of τ on ξ and let $x^{\langle\delta^\circ\rangle\psi}$ be a bound variable of $\xi^\sharp$. If $x^{\langle\delta^\circ\rangle\psi}$ is free in A, then γ is a proper subterm of δ°.

Proof. We prove this by least fixpoint induction: let X be the set of all call triples (γ, φ, A) such that, for every bound variable $x^{\langle\delta^\circ\rangle\psi}$ of $\xi^\sharp$ such that $x^{\langle\delta^\circ\rangle\psi}$ is free in A, we have that γ is a strict subterm of δ. We show that $c_\xi(X) \subseteq X$, hence $C_\xi \subseteq X$ as required.

So suppose that the triple t is in $c_\xi(X)$. We make a case distinction:

- $t = (\gamma, \varphi, \varphi^\sharp)$ for some subformula $\langle\gamma\rangle\varphi$ of ξ. Then $t \in X$ since no bound variable of ξ appears free in $\varphi^\sharp$, which means that the defining condition of the set X holds trivially.
- t is of the form (γ, φ, A) or (γ', φ, A) where $(\gamma \sqcup \gamma', \varphi, A) \in X$. Given a bound variable $x^{\langle\delta^\circ\rangle\psi}$ of $\xi^\sharp$ such that $x^{\langle\delta^\circ\rangle\psi}$ is free in A, since $(\gamma \sqcup \gamma', \varphi, A) \in X$ it follows that $\gamma \sqcup \gamma'$ is a proper subterm of δ°. So clearly this also holds for both γ and γ'. Hence $(\gamma, \varphi, A) \in X$ and $(\gamma', \varphi, A) \in X$ as required.
- The case where t is of the form (γ, φ, A) or (γ', φ, A) for $(\gamma \sqcap \gamma', \varphi, A) \in X$ is similar.
- Suppose t is of the form $(\gamma, \langle\gamma^\times\rangle\varphi, x^{\langle\gamma^\times\rangle\varphi})$ and there exists some A for which $(\gamma^\times, \varphi, A) \in X$. Given a bound variable $x^{\langle\delta^\circ\rangle\psi}$ of $\xi^\sharp$, the only possible way that $x^{\langle\delta^\circ\rangle\psi}$ can appear free in $x^{\langle\gamma^\times\rangle\varphi}$ is if $x^{\langle\delta^\circ\rangle\psi} = x^{\langle\gamma^\times\rangle\varphi}$, so $\langle\delta^\circ\rangle\psi = \langle\gamma^\times\rangle\varphi$ and therefore $\delta^\circ = \gamma^\times$. Since γ is a proper subterm of $\gamma^\times$, this means that the required conclusion holds.
- The case where t is of the form $(\gamma, \langle\gamma^*\rangle\varphi, x^{\langle\gamma^*\rangle\varphi})$ is similar.
- Suppose t is of the form (γ', φ, A) for some γ, γ', φ and A such that $((\gamma ; \gamma'), \varphi, A) \in X$. If $x^{\langle\delta^\circ\rangle\psi}$ is a bound variable of $\xi^\sharp$ that appears free in A, then since $((\gamma ; \gamma'), \varphi, A) \in X$ it follows that $\gamma ; \gamma'$ is a proper subterm of δ°. Hence, so is γ'.
- Finally, suppose t is of the form $(\gamma, \langle\gamma'\rangle\varphi, \tau^{\gamma'}_\varphi(A))$ for some γ, γ', φ and A such that $((\gamma ; \gamma'), \varphi, A) \in X$. If $x^{\langle\delta^\circ\rangle\psi}$ is a bound variable of $\xi^\sharp$ that appears free in $\tau^{\gamma'}_\varphi(A)$, then by Lemma 38 $x^{\langle\delta^\circ\rangle\psi}$ appears free in A as well. Since $((\gamma ; \gamma'), \varphi, A) \in X$ it follows that $\gamma ; \gamma'$ is a proper subterm of δ°, hence so is γ.

We have shown that $t \in c_\xi(X)$ implies $t \in X$, so the proof is finished. □

Lemma 43. Let ξ be a game logic formula and $\langle\gamma^\circ\rangle\varphi, \langle\delta^\dagger\rangle\psi \in F(\xi)$, where $\circ, \dagger \in \{*, \times\}$. If $x^{\langle\delta^\dagger\rangle\psi} <^-_{\xi^\sharp} x^{\langle\gamma^\circ\rangle\varphi}$ then $\langle\delta^\dagger\rangle\psi \prec \langle\gamma^\circ\rangle\varphi$.

Proof. We consider the case that $\gamma^\circ = \gamma^\times$. The case where $\circ = *$ is similar.

Assume that the variable $x^{\langle\delta^\dagger\rangle\psi}$ is free in some subformula $\nu x^{\langle\gamma^\times\rangle\varphi}.B$ of $\xi^\sharp$. By Lemma 41, there exists some A such that
$$\nu x^{\langle\gamma^\times\rangle\varphi}.B = \tau^{\gamma^\times}_\varphi(A)$$
and furthermore, $(\gamma^\times, \varphi, A) \in C_\xi$. But we have:
$$\tau^\varphi_{\gamma^\times}(A) = \nu x^{\langle\gamma^\times\rangle\varphi}.A \wedge \tau^{\langle\gamma^\times\rangle\varphi}_\gamma(x^{\langle\gamma^\times\rangle\varphi}),$$
so
$$\nu x^{\langle\gamma^\times\rangle\varphi}.B = \nu x^{\langle\gamma^\times\rangle\varphi}.A \wedge \tau^{\langle\gamma^\times\rangle\varphi}_\gamma(x^{\langle\gamma^\times\rangle\varphi}),$$

so the variable $x^{\langle\delta^\dagger\rangle\psi}$ must occur freely in either A or in $\tau_\gamma^{\langle\gamma^\times\rangle\varphi}(x^{\langle\gamma^\times\rangle\varphi})$. If the variable $x^{\langle\delta^\dagger\rangle\psi}$ occurs freely in $\tau_\gamma^{\langle\gamma^\times\rangle\varphi}(x^{\langle\gamma^\times\rangle\varphi})$ then by Lemma 38, $\langle\gamma^\times\rangle\varphi = \langle\delta^\dagger\rangle\psi$, hence $x^{\langle\gamma^\times\rangle\varphi} = x^{\langle\delta^\dagger\rangle\psi}$. But this contradicts our assumption that $x^{\langle\delta^\dagger\rangle\psi} <^-_{\xi^\sharp} x^{\langle\gamma^\times\rangle\varphi}$, since $<^-_{\xi^\sharp}$ is an irreflexive relation.

So the only possibility is that $x^{\langle\delta^\dagger\rangle\psi}$ occurs freely in A. Since $(\gamma^\times, \varphi, A) \in C_\xi$ was a recursive call of τ on ξ, it follows from Lemma 42 that $\gamma^\times$ is a proper subterm of $\delta^\dagger$. Hence $\langle\delta^\dagger\rangle\psi \prec \langle\gamma^\times\rangle\varphi$ as required. $\square$

Proof of Proposition 24 From Lemma 43 it follows that $\xi^\sharp$ is locally well-named: If $\xi^\sharp$ was not locally well-named then the transitive closure $<_{\xi^\sharp}$ of $<^-_{\xi^\sharp}$ would be reflexive and hence by Lemma 43 the relation $\prec$ would be reflexive as well. But $\prec$ is irreflexive, since the strict subterm relation is irreflexive.

That $x^\varphi \leq_{\xi^\sharp} x^\psi$ implies $\varphi \preccurlyeq \psi$ follows also from Lemma 43 because $\leq_{\xi^\sharp}$ is the reflexive and transitive closure of $<^-_{\xi^\sharp}$ and $\preccurlyeq$ is transitive and reflexive, as it is defined via the subterm relation. $\square$

E. Omitted proofs of Section VI

We prove completeness for Par from completeness of G by going via an intermediate Hilbert system $\mathsf{Par}_{\mathsf{Full}}$ for the language $\mathcal{L}_{\mathsf{Full}}$. Note that $\mathcal{L}_{\mathsf{Par}} \subseteq \mathcal{L}_{\mathsf{Full}}$. The system $\mathsf{Par}_{\mathsf{Full}}$ is defined as the extension of Par with the axioms and rules listed in Figure 8 below.

Additional axioms:

- $\langle\gamma \sqcap \delta\rangle\varphi \leftrightarrow \langle\gamma\rangle\varphi \wedge \langle\delta\rangle\varphi$
- $\langle\gamma^\times\rangle\varphi \leftrightarrow \varphi \wedge \langle\gamma\rangle\langle\gamma^\times\rangle\varphi$
- $\langle\psi!\rangle\varphi \leftrightarrow \psi \vee \varphi$

Additional rule:

$$\frac{\varphi \to \langle\gamma\rangle\varphi}{\varphi \to \langle\gamma^\times\rangle\varphi}\ \mathsf{BarInd}^\times$$

Fig. 8. Additional axioms and rules of $\mathsf{Par}_{\mathsf{Full}}$.

The next lemma shows that $\mathsf{Par}_{\mathsf{Full}}$ is a conservative extension of Par.

Lemma 44. For all $\varphi \in \mathcal{L}_{\mathsf{Full}}$, if $\mathsf{Par}_{\mathsf{Full}} \vdash \varphi$ then $\mathsf{Par} \vdash \mathsf{pa}(\varphi)$.

Proof. We need to check that the $\mathsf{pa}()$-translation of every axiom of $\mathsf{Par}_{\mathsf{Full}}$ is derivable in Par, and that the $\mathsf{pa}()$-translation of every instance of a rule of $\mathsf{Par}_{\mathsf{Full}}$ is admissible in Par. We shall allow defined propositional connectives like $\to, \leftrightarrow, \wedge$ as abbreviations here.

Case: $\langle\gamma \sqcap \delta\rangle\varphi \leftrightarrow \langle\gamma\rangle\varphi \wedge \langle\delta\rangle\varphi$. The translation of this axiom becomes:

$$\langle(\mathsf{pa}(\gamma)^d \sqcup \mathsf{pa}(\delta)^d)^d\rangle\mathsf{pa}(\varphi) \leftrightarrow \langle\mathsf{pa}(\gamma)\rangle\mathsf{pa}(\varphi) \wedge \langle\mathsf{pa}(\delta)\rangle\mathsf{pa}(\varphi)$$

We derive this through the following chain of provable equivalences in Par:

$$\begin{aligned} & \langle(\mathsf{pa}(\gamma)^d \sqcup \mathsf{pa}(\delta)^d)^d\rangle\mathsf{pa}(\varphi) \\ \Leftrightarrow\ & \neg\langle\mathsf{pa}(\gamma)^d \sqcup \mathsf{pa}(\delta)^d\rangle\neg\mathsf{pa}(\varphi) \\ \Leftrightarrow\ & \neg(\langle\mathsf{pa}(\gamma)^d\rangle\neg\mathsf{pa}(\varphi) \vee \langle\mathsf{pa}(\delta)^d\rangle\neg\mathsf{pa}(\varphi)) \\ \Leftrightarrow\ & \neg\langle\mathsf{pa}(\gamma)^d\rangle\neg\mathsf{pa}(\varphi) \wedge \neg\langle\mathsf{pa}(\delta)^d\rangle\neg\mathsf{pa}(\varphi) \\ \Leftrightarrow\ & \neg\neg\langle\mathsf{pa}(\gamma)\rangle\neg\neg\mathsf{pa}(\varphi) \wedge \neg\neg\langle\mathsf{pa}(\delta)\rangle\neg\neg\mathsf{pa}(\varphi)) \\ \Leftrightarrow\ & \langle\mathsf{pa}(\gamma)\rangle\mathsf{pa}(\varphi) \wedge \langle\mathsf{pa}(\delta)\rangle\mathsf{pa}(\varphi)) \end{aligned}$$

Case: $\langle\gamma^\times\rangle\varphi \leftrightarrow \varphi \wedge \langle\gamma\rangle\langle\gamma^\times\rangle\varphi$. The translation of this axiom becomes:

$$\langle((\mathsf{pa}(\gamma)^d)^*)^d\rangle\mathsf{pa}(\varphi) \leftrightarrow \mathsf{pa}(\varphi) \wedge \langle\mathsf{pa}(\gamma)\rangle\langle((\mathsf{pa}(\gamma)^d)^*)^d\rangle\mathsf{pa}(\varphi)$$

We derive this through the following chain of provable equivalences in Par:

$$\begin{aligned} & \langle((\mathsf{pa}(\gamma)^d)^*)^d\rangle\mathsf{pa}(\varphi) \\ \Leftrightarrow\ & \neg\langle(\mathsf{pa}(\gamma)^d)^*\rangle\neg\mathsf{pa}(\varphi) \\ \Leftrightarrow\ & \neg(\neg\mathsf{pa}(\varphi) \vee \langle\mathsf{pa}(\gamma)^d\rangle\langle(\mathsf{pa}(\gamma)^d)^*\rangle\neg\mathsf{pa}(\varphi)) \\ \Leftrightarrow\ & \neg(\neg\mathsf{pa}(\varphi) \vee \neg\langle\mathsf{pa}(\gamma)\rangle\neg\langle(\mathsf{pa}(\gamma)^d)^*\rangle\neg\mathsf{pa}(\varphi)) \\ \Leftrightarrow\ & \mathsf{pa}(\varphi) \wedge \langle\mathsf{pa}(\gamma)\rangle\neg\langle(\mathsf{pa}(\gamma)^d)^*\rangle\neg\mathsf{pa}(\varphi) \\ \Leftrightarrow\ & \mathsf{pa}(\varphi) \wedge \langle\mathsf{pa}(\gamma)\rangle\langle((\mathsf{pa}(\gamma)^d)^*)^d\rangle\mathsf{pa}(\varphi) \end{aligned}$$

Case: $\langle\psi!\rangle\varphi \leftrightarrow \psi \vee \varphi$ The translation of this axiom becomes:

$$\langle(\neg\mathsf{pa}(\psi)?)^d\rangle\mathsf{pa}(\varphi) \leftrightarrow \mathsf{pa}(\psi) \vee \mathsf{pa}(\varphi)$$

We derive this through the following chain of provable equivalences in Par:

$$\begin{aligned} & \langle(\neg\mathsf{pa}(\psi)?)^d\rangle\mathsf{pa}(\varphi) \\ \Leftrightarrow\ & \neg\langle\neg\mathsf{pa}(\psi)?\rangle\neg\mathsf{pa}(\varphi) \\ \Leftrightarrow\ & \neg(\neg\mathsf{pa}(\psi) \wedge \neg\mathsf{pa}(\varphi)) \\ {}'\Leftrightarrow\ & \mathsf{pa}(\psi) \vee \mathsf{pa}(\varphi) \end{aligned}$$

Case:

$$\frac{\varphi \to \langle\gamma\rangle\varphi}{\varphi \to \langle\gamma^\times\rangle\varphi}$$

We show that the translation of this rule is derivable in Par. It then follows that it is admissible. The translation of the premise of this rule is $\mathsf{pa}(\varphi) \to \langle\mathsf{pa}(\gamma)\rangle\mathsf{pa}(\varphi)$, and the conclusion becomes $\mathsf{pa}(\varphi) \to \langle((\mathsf{pa}(\gamma)^d)^*)^d\rangle\mathsf{pa}(\varphi)$. So suppose that $\mathsf{Par} \vdash \mathsf{pa}(\varphi) \to \langle\mathsf{pa}(\gamma)\rangle\mathsf{pa}(\varphi)$. Then $\mathsf{Par} \vdash \neg\langle\mathsf{pa}(\gamma)\rangle\mathsf{pa}(\varphi) \to \neg\mathsf{pa}(\varphi)$, which gives $\mathsf{Par} \vdash \langle\mathsf{pa}(\gamma)^d\rangle\neg\mathsf{pa}(\varphi) \to \neg\mathsf{pa}(\varphi)$. Bar Induction gives:

$$\mathsf{Par} \vdash \langle(\mathsf{pa}(\gamma)^d)^*\rangle\neg\mathsf{pa}(\varphi) \to \neg\mathsf{pa}(\varphi)$$

Contraposition now gives:

$$\mathsf{Par} \vdash \mathsf{pa}(\varphi) \to \neg\langle(\mathsf{pa}(\gamma)^d)^*\rangle\neg\mathsf{pa}(\varphi),$$

and the desired conclusion follows by the equivalence $\neg\langle(\mathsf{pa}(\gamma)^d)^*\rangle\neg\mathsf{pa}(\varphi) \Leftrightarrow \langle((\mathsf{pa}(\gamma)^d)^*)^d\rangle\mathsf{pa}(\varphi)$. This concludes the proof. $\square$

The convenience of working in a Hilbert system for the full language is that we can prove the following lemma.

Lemma 45. For all $\varphi \in \mathcal{L}_{\mathsf{Full}}$, $\mathsf{Par}_{\mathsf{Full}} \vdash \varphi \leftrightarrow \mathsf{nf}(\varphi)$.

Proof. Straightforward induction on the complexity of formulas. □

Since we eventually want to connect G-provability of $\mathsf{nf}(\varphi)$ with Par-provability of φ via $\mathsf{Par}_{\mathsf{Full}}$, the following proposition takes care of one of the steps.

Proposition 46. For all $\varphi \in \mathcal{L}_{\mathsf{Par}}$, if $\mathsf{Par}_{\mathsf{Full}} \vdash \mathsf{nf}(\varphi)$ then $\mathsf{Par} \vdash \varphi$.

Proof. Due to Lemma 45, it suffices to show that for all $\varphi \in \mathcal{L}_{\mathsf{Par}}$, if $\mathsf{Par}_{\mathsf{Full}} \vdash \varphi$ then $\mathsf{Par} \vdash \varphi$. This follows from Lemma 44. since, if $\varphi \in \mathcal{L}_{\mathsf{Par}}$ then $\mathsf{pa}(\varphi) = \varphi$. □

We now show that we can translate G-derivations into $\mathsf{Par}_{\mathsf{Full}}$-derivations.

Proposition 47. For all sequents $\Phi \subseteq \mathcal{L}_{\mathsf{NF}}$,

$$\mathsf{G} \vdash \Phi \quad \text{implies} \quad \mathsf{Par}_{\mathsf{Full}} \vdash \bigvee \Phi.$$

Consequently, for all $\xi \in \mathcal{L}_{\mathsf{NF}}$, if $\mathsf{G} \vdash \xi$ then $\mathsf{Par}_{\mathsf{Full}} \vdash \xi$.

Proof. To prove this proposition, we need to prove that the disjunction of any axiom of G is derivable in $\mathsf{Par}_{\mathsf{Full}}$, and that every rule R of G is admissible in $\mathsf{Par}_{\mathsf{Full}}$ in the following sense: if the disjunction of the premise of an instance of R is derivable in $\mathsf{Par}_{\mathsf{Full}}$, then so is the disjunction of the corresponding conclusion. Axioms of G are taken care of by Lemma 45, since they are of the form $\Phi, \overline{\Phi}$, and the disjunction of such a sequent is the $\mathsf{nf}()$-translation of a propositional tautology. For admissibility of the G-rules ind_s and $\mathsf{Mon}^{\mathsf{g}}_{\mathsf{d}}$ is shown in Lemmas 49 and 48 below. The other cases are straightforward, and we leave the details to the reader. □

Lemma 48. The rule ind_s is admissible in the system $\mathsf{Par}_{\mathsf{Full}}$: if $\mathsf{Par}_{\mathsf{Full}} \vdash \bigvee \Gamma \vee (\varphi \wedge \langle\gamma\rangle\langle(\overline{\Gamma}! \,;\gamma)^{\times}\rangle\langle\overline{\Gamma}!\rangle\varphi)$ then $\mathsf{Par}_{\mathsf{Full}} \vdash \bigvee \Gamma \vee \langle\gamma^{\times}\rangle\varphi$ as well.

Proof. Suppose that:

$$\mathsf{Par}_{\mathsf{Full}} \vdash \bigvee \Gamma \vee (\varphi \wedge \langle\gamma\rangle\langle(\overline{\Gamma}! \,;\gamma)^{\times}\rangle\langle\overline{\Gamma}!\rangle\varphi)$$

We show that $\mathsf{Par}_{\mathsf{Full}} \vdash \bigvee \Gamma \vee \langle\gamma^{\times}\rangle\varphi$. Note that we can rewrite the assumption as:

$$(*) \quad \mathsf{Par}_{\mathsf{Full}} \vdash \overline{\Gamma} \to (\varphi \wedge \langle\gamma\rangle\langle(\overline{\Gamma}! \,;\gamma)^{\times}\rangle\langle\overline{\Gamma}!\rangle\varphi)$$

and the desired conclusion as: $\mathsf{Par}_{\mathsf{Full}} \vdash \overline{\Gamma} \to \langle\gamma^{\times}\rangle\varphi$.

CLAIM 2. $\mathsf{Par}_{\mathsf{Full}} \vdash \overline{\Gamma} \to \varphi \wedge \langle\gamma\rangle\langle(\overline{\Gamma}! \,;\gamma)^{\times}\rangle\varphi$

PROOF OF CLAIM By our assumption $(*)$ we get $\mathsf{Par}_{\mathsf{Full}} \vdash \overline{\Gamma} \to \varphi$, and since $\mathsf{Par}_{\mathsf{Full}} \vdash \langle\overline{\Gamma}!\rangle\varphi \leftrightarrow \overline{\Gamma} \vee \varphi$ we get $\mathsf{Par}_{\mathsf{Full}} \vdash \langle\overline{\Gamma}!\rangle\varphi \to \varphi$. Together with Monotonicity applied to the consequent in $(*)$, we get

$$\mathsf{Par}_{\mathsf{Full}} \vdash \overline{\Gamma} \to \varphi \wedge \langle\gamma\rangle\langle(\overline{\Gamma}! \,;\gamma)^{\times}\rangle\varphi$$

as required. ◀

CLAIM 3. $\mathsf{Par}_{\mathsf{Full}} \vdash \langle(\overline{\Gamma}! \,;\gamma)^{\times}\rangle \to \langle\gamma^{\times}\rangle\langle(\overline{\Gamma}! \,;\gamma)^{\times}\rangle$

PROOF OF CLAIM First, by simply unfolding the fixpoint and applying propositional reasoning we get:

$$\mathsf{Par}_{\mathsf{Full}} \vdash \langle(\overline{\Gamma}! \,;\gamma)^{\times}\rangle \to \langle\overline{\Gamma}! \,;\gamma\rangle\langle(\overline{\Gamma}! \,;\gamma)^{\times}\rangle$$

But the consequent of this implication is equivalent to:

$$\overline{\Gamma} \vee \langle\gamma\rangle\langle(\overline{\Gamma}! \,;\gamma)^{\times}\rangle$$

By Claim 2 applied to the left disjunct of this formula we get:

$$\mathsf{Par}_{\mathsf{Full}} \vdash \langle(\overline{\Gamma}! \,;\gamma)^{\times}\rangle \to \langle\gamma\rangle\langle(\overline{\Gamma}! \,;\gamma)^{\times}\rangle$$

By the Bar Induction rule we get:

$$\mathsf{Par}_{\mathsf{Full}} \vdash \langle(\overline{\Gamma}! \,;\gamma)^{\times}\rangle \to \langle\gamma^{\times}\rangle\langle(\overline{\Gamma}! \,;\gamma)^{\times}\rangle$$

as required. ◀

CLAIM 4. $\mathsf{Par}_{\mathsf{Full}} \vdash \overline{\Gamma} \to \langle(\overline{\Gamma}! \,;\gamma)^{\times}\rangle\varphi$

PROOF OF CLAIM By Claim 2 we get $\vdash \overline{\Gamma} \to \varphi \wedge \langle\gamma\rangle\langle(\overline{\Gamma}! \,;\gamma)^{\times}\rangle\varphi$. By propositional reasoning we have:

$$\mathsf{Par}_{\mathsf{Full}} \vdash \langle\gamma\rangle\langle(\overline{\Gamma}! \,;\gamma)^{\times}\rangle\varphi \to \overline{\Gamma} \vee \langle\gamma\rangle\langle(\overline{\Gamma}! \,;\gamma)^{\times}\rangle\varphi$$

But the consequent of this implication is equivalent to $\langle\overline{\Gamma}! \,;\gamma\rangle\langle(\overline{\Gamma}! \,;\gamma)^{\times}\rangle\varphi$, so we get:

$$\vdash \overline{\Gamma} \to \varphi \wedge \langle\overline{\Gamma}! \,;\gamma\rangle\langle(\overline{\Gamma}! \,;\gamma)^{\times}\rangle\varphi$$

But the consequent of this implication is just the unfolding of the fixpoint $\langle(\overline{\Gamma}! \,;\gamma)^{\times}\rangle\varphi$, so we get:

$$\mathsf{Par}_{\mathsf{Full}} \vdash \overline{\Gamma} \to \langle(\overline{\Gamma}! \,;\gamma)^{\times}\rangle\varphi$$

as required. ◀

CLAIM 5. $\mathsf{Par}_{\mathsf{Full}} \vdash \langle(\overline{\Gamma}! \,;\gamma)^{\times}\rangle\varphi \to \varphi$

PROOF OF CLAIM Just unfold the fixpoint $\langle(\overline{\Gamma}! \,;\gamma)^{\times}\rangle\varphi$ to $\varphi \wedge \langle\overline{\Gamma} \,;\gamma\rangle\langle(\overline{\Gamma}! \,;\gamma)^{\times}\rangle\varphi$. ◀

We can now prove the lemma: by Claim 4 we have:

$$\text{(i)} \quad \mathsf{Par}_{\mathsf{Full}} \vdash \overline{\Gamma} \to \langle(\overline{\Gamma}! \,;\gamma)^{\times}\rangle\varphi$$

Combining (i) with Claim 3 we get:

$$\text{(ii)} \quad \mathsf{Par}_{\mathsf{Full}} \vdash \overline{\Gamma} \to \langle\gamma^{\times}\rangle\langle(\overline{\Gamma}! \,;\gamma)^{\times}\rangle\varphi$$

By Claim 5 and Monotonicity we get:

$$\text{(iii)} \quad \mathsf{Par}_{\mathsf{Full}} \vdash \langle\gamma^{\times}\rangle\langle(\overline{\Gamma}! \,;\gamma)^{\times}\rangle\varphi \to \langle\gamma^{\times}\rangle\varphi$$

Putting (ii) and (iii) together we get:

$$\mathsf{Par}_{\mathsf{Full}} \vdash \overline{\Gamma} \to \langle\gamma^{\times}\rangle\varphi$$

as required. □

Lemma 49. The rule $\mathsf{Mon}^{\mathsf{g}}_{\mathsf{d}}$ is derivable in the system $\mathsf{Par}_{\mathsf{Full}}$.

Proof. We shall prove this by induction on the complexity of formulas in dual normal form. To keep notation simple, in this proof we abbreviate $\mathsf{Par}_{\mathsf{Full}} \vdash \varphi$ by $\vdash \varphi$, and $\mathsf{Par}_{\mathsf{Full}} \vdash \varphi \to \psi$ by $\varphi \vdash \psi$. The induction hypothesis on a formula $\varphi(\delta)$ is that $\varphi(\delta) \vdash \varphi(\chi! \,;\delta)$.

CLAIM 6. For every game term $\gamma(\delta)$ in dual normal form and every formula φ, we can prove the following implication in $\mathsf{Par}_{\mathsf{Full}}$:

$$\langle\gamma(\delta)\rangle\varphi \to \langle\gamma(\chi!\,;\delta)\rangle\varphi$$

provided that the main induction hypothesis holds for every formula θ corresponding to a subterm $\theta!$ or $\theta?$ of $\gamma(\delta)$.

PROOF OF CLAIM We prove that the implication holds for all φ by induction on the complexity of the game term $\gamma(\delta)$, treating δ as an atomic case.

Atomic case, $\gamma(\delta) = \delta$. For all φ we have:

$$\langle\delta\rangle\varphi \vdash \chi \vee \langle\delta\rangle\varphi \vdash \langle\chi!\rangle\langle\delta\rangle\varphi \vdash \langle\chi!\,;\delta\rangle\varphi$$

as required.

Atomic case for game terms g or g^d: trivial.

Case for $\theta(\delta)?$ or $\theta(\delta)!$: follows immediately from the induction hypothesis on θ, since $\vdash \langle\theta(\delta)?\rangle\varphi \leftrightarrow \theta(\delta) \wedge \varphi$ and $\vdash \langle\theta(\delta)!\rangle\varphi \leftrightarrow \theta(\delta) \vee \varphi$.

Case for $\sqcup$: the induction hypothesis on the subterms $\gamma_1(\delta)$ and $\gamma_2(\delta)$ of $(\gamma_1 \sqcup \gamma_2)(\delta)$ gives $\vdash \langle\gamma_1(\delta)\rangle\varphi \to \langle\gamma_1(\chi!\,;\delta)\rangle\varphi$ and $\vdash \langle\gamma_2(\delta)\rangle\varphi \to \langle\gamma_2(\chi!\,;\delta)\rangle\varphi$. We get:

$$\begin{array}{rcl}\langle\gamma_1(\delta) \sqcup \gamma_2(\delta)\rangle\varphi & \vdash & \langle\gamma_1(\delta)\rangle\varphi \vee \langle\gamma_2(\delta)\rangle\varphi \\ & \vdash & \langle\gamma_1(\chi!\,;\delta)\rangle\varphi \vee \langle\chi!\,;\gamma_2(\delta)\rangle\varphi \\ & \vdash & \langle\gamma_1(\chi!\,;\delta) \sqcup \gamma_2(\chi!\,;\delta)\rangle\varphi\end{array}$$

as required.

Case for $\sqcap$: similar.

Case for ;: consider the formula $\langle\gamma_1(\delta)\,;\gamma_2(\delta)\rangle\varphi$. The induction hypothesis on $\gamma_2(\delta)$ instantiated for the formula φ gives

$$\vdash \langle\gamma_2(\delta)\rangle\varphi \to \langle\gamma_2(\chi!\,;\delta)\rangle\varphi$$

By monotonicity we get:

$$\vdash \langle\gamma_1(\delta)\rangle\langle\gamma_2(\delta)\rangle\varphi \to \langle\gamma_1(\delta)\rangle\langle\gamma_2(\chi!\,;\delta)\rangle\varphi$$

But the induction hypothesis on $\gamma_1(\delta)$ instantiated for the formula $\langle\gamma_2(\chi!\,;\delta)\rangle\varphi$ gives:

$$\langle\gamma_1(\delta)\rangle\langle\gamma_2(\chi!\,;\delta)\rangle\varphi \to \langle\gamma_1(\chi!\,;\delta)\rangle\langle\gamma_2(\chi!\,;\delta)\rangle\varphi$$

Putting these implications together we get:

$$\vdash \langle\gamma_1(\delta)\rangle\langle\gamma_2(\delta)\rangle\varphi \to \langle\gamma_1(\chi!\,;\delta)\rangle\langle\gamma_2(\chi!\,;\delta)\rangle\varphi$$

The required result now follows from the reduction axioms for ; applied to both the antecedent and the consequent in this implication.

Case for $*$: by the induction hypothesis we have, for every formula ψ, $\vdash \langle\gamma(\delta)\rangle\psi \to \langle\gamma(\chi!\,;\delta)\rangle\psi$. We wish to show that for all φ, we have $\vdash \langle\gamma(\delta)^*\rangle\varphi \to \langle\gamma(\chi!\,;\delta)^*\rangle\varphi$. By the induction hypothesis instantiated with the formula $\psi = \langle\gamma(\chi!\,;\delta)^*\rangle\varphi$ we have

$$\vdash \langle\gamma(\delta)\rangle\langle\gamma(\chi!\,;\delta)^*\rangle\varphi \to \langle\gamma(\chi!\,;\delta)\rangle\langle\gamma(\chi!\,;\delta)^*\rangle\varphi$$

But by the unfolding axiom for angelic iteration and propositional reasoning we get:

$$\vdash \langle\gamma(\chi!\,;\delta)\rangle\langle\gamma(\chi!\,;\delta)^*\rangle\varphi \to \langle\gamma(\chi!\,;\delta)^*\rangle\varphi$$

Hence, putting these two implications together, we get:

$$\vdash \langle\gamma(\delta)\rangle\langle\gamma(\chi!\,;\delta)^*\rangle\varphi \to \langle\gamma(\chi!\,;\delta)^*\rangle\varphi$$

By the Bar Induction rule for angelic iteration, we now get:

$$(\dagger) \quad \vdash \langle\gamma(\delta)^*\rangle\langle\gamma(\chi!\,;\delta)^*\rangle\varphi \to \langle\gamma(\chi!\,;\delta)^*\rangle\varphi$$

Applying unfolding and propositional reasoning again, we get:

$$\vdash \varphi \to \langle\gamma(\chi!\,;\delta)^*\rangle\varphi$$

By the monotonicity rule we get:

$$(\ddagger) \quad \vdash \langle\gamma(\delta)^*\rangle\varphi \to \langle\gamma(\delta)^*\rangle\langle\gamma(\chi!\,;\delta)^*\rangle\varphi$$

Putting together the implications $(\dagger)$ and $(\ddagger)$, we get:

$$\vdash \langle\gamma(\delta)^*\rangle\varphi \to \langle\gamma(\chi!\,;\delta)^*\rangle\varphi$$

as required.

Case for $\times$: by the induction hypothesis we have, for every formula ψ, $\vdash \langle\gamma(\delta)\rangle\psi \to \langle\gamma(\chi!\,;\delta)\rangle\psi$. We wish to show that for all φ, we have $\vdash \langle\gamma(\delta)^\times\rangle\varphi \to \langle\gamma(\chi!\,;\delta)^\times\rangle\varphi$. By the induction hypothesis instantiated with the formula $\psi = \langle\gamma(\delta)^\times\rangle\varphi$ we have

$$\vdash \langle\gamma(\delta)\rangle\langle\gamma(\delta)^\times\rangle\varphi \to \langle\gamma(\chi!\,;\delta)\rangle\langle\gamma(\delta)^\times\rangle\varphi$$

But by unfolding $\langle\gamma(\delta)^\times\rangle\varphi$ to $\varphi \wedge \langle\gamma(\delta)\rangle\langle\gamma(\delta)^\times\rangle\varphi$, we see that:

$$\vdash \langle\gamma(\delta)^\times\rangle\varphi \to \langle\gamma(\delta)\rangle\langle\gamma(\delta)^\times\rangle\varphi$$

Putting together the implications we have established so far, we get:

$$\vdash \langle\gamma(\delta)^\times\rangle\varphi \to \langle\gamma(\chi!\,;\delta)\rangle\langle\gamma(\delta)^\times\rangle\varphi$$

By the Bar Induction rule for $\times$ we now get:

$$(\dagger) \quad \vdash \langle\gamma(\delta)^\times\rangle\varphi \to \langle\gamma(\chi!\,;\delta)^\times\rangle\langle\gamma(\delta)^\times\rangle\varphi$$

But we have $\vdash \langle\gamma(\delta)^\times\rangle\varphi \to \varphi$, so by monotonicity we get:

$$(\ddagger) \quad \vdash \langle\gamma(\chi!\,;\delta)^\times\rangle\langle\gamma(\delta)^\times\rangle\varphi \to \langle\gamma(\chi!\,;\delta)^\times\rangle\varphi$$

Putting together $(\dagger)$ and $(\ddagger)$ we get:

$$\vdash \langle\gamma(\delta)^\times\rangle\varphi \to \langle\gamma(\chi!\,;\delta)^\times\rangle\varphi$$

as required.

◀

We can now complete the main induction: the atomic cases for literals and induction steps for $\vee, \wedge$ are easy. The only interesting step is for a formula of the form $(\langle\gamma\rangle\varphi)(\delta) =$

$\langle\gamma(\delta)\rangle\varphi(\delta)$. By the induction hypothesis on $\varphi(\delta)$ we get $\vdash \varphi(\delta) \to \varphi(\chi!\,;\delta)$, so by monotonicity we get

$$\vdash \langle\gamma(\chi!\,;\delta)\rangle\varphi(\delta) \to \langle\gamma(\chi!\,;\delta)\rangle\varphi(\chi!\,;\delta)$$

By the induction hypothesis on all subformulas θ occurring in subterms $\theta!$ or $\theta?$ of $\gamma(\delta)$, we can apply Claim 6 and get

$$\vdash \langle\gamma(\delta)\rangle\varphi(\delta) \to \langle\gamma(\chi!\,;\delta)\rangle\varphi(\delta)$$

Putting together these implications we get:

$$\vdash \langle\gamma(\delta)\rangle\varphi(\delta) \to \langle\gamma(\chi!\,;\delta)\rangle\varphi(\chi!\,;\delta)$$

as required. □

We are now ready to prove the transformations between G and Par.

Proof of Theorem 11 For item 1, let $\varphi \in \mathcal{L}_{\mathsf{Par}}$ such that $\mathsf{G} \vdash \mathsf{nf}(\varphi)$. By Proposition 47, $\mathsf{Par}_{\mathsf{Full}} \vdash \mathsf{nf}(\varphi)$, and by Proposition 46, we obtain $\mathsf{Par} \vdash \varphi$.

For item 2, let $\xi \in \mathcal{L}_{\mathsf{NF}}$ such that $\mathsf{G} \vdash \xi$. By Proposition 47, $\mathsf{Par}_{\mathsf{Full}} \vdash \xi$, and since $\mathcal{L}_{\mathsf{NF}} \subseteq \mathcal{L}_{\mathsf{Full}}$, we obtain $\mathsf{Par} \vdash \mathsf{pa}(\xi)$ from Lemma 44.