Interpolant Existence in the Guarded Fragment

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\textbf{Abstract.} The guarded fragment of FO fails to have the Craig interpolation property. Thus, not every valid implication between guarded formulas has a guarded interpolant. In this article we show that the existence of a guarded interpolant for such an implication is decidable in $3\text{ExpTime}$ in general, and in $2\text{ExpTime}$ if the arity of relation symbols is fixed.

\section{Introduction}

The guarded fragment of FO does not have the Craig interpolation property \cite{7, 6}. Thus, the existence of an interpolating formula for an implication cannot be deduced from its validity. In this article we show that, nevertheless, the existence of guarded interpolants between guarded formulas is decidable. The problem is in $3\text{ExpTime}$ in general and in $2\text{ExpTime}$ if the arity of relations is fixed. It follows, in particular, that the existence of explicit guarded definitions is also in $3\text{ExpTime}$ and, respectively, $2\text{ExpTime}$. We conjecture that these bounds are tight. We refer the reader to \cite{7, 6, 2, 3} for further information about interpolation in guarded logics.

\section{Preliminaries}

Let $\tau$ range over relational signatures not containing function or constant symbols. Denote by $\text{FO}(\tau)$ the set of first-order (FO) formulas constructed from atomic formulas $x = y$ and $R(x)$, $R \in \tau$, using conjunction, disjunction, negation, and existential and universal quantification. As usual, we write $\varphi(x)$ to indicate that the free variables in FO-formula $\varphi$ are all from $x$ and call a formula open if it has at least one free variable and a sentence otherwise. $\text{FO}(\tau)$ is interpreted in $\tau$-structures $\mathfrak{A} = (\text{dom}(\mathfrak{A}), (R^\mathfrak{A})_{R \in \tau})$, where $\text{dom}(\mathfrak{A})$ is the non-empty domain of $\mathfrak{A}$, and each $R^\mathfrak{A}$ is a relation over $\text{dom}(\mathfrak{A})$ whose arity matches that of $R$. We often drop $\tau$ and simply speak of structures $\mathfrak{A}$. In the guarded fragment (GF) of FO \cite{1, 4}, formulas are built from atomic formulas $R(x)$ and $x = y$ by applying the Boolean connectives and guarded quantifiers of the form

$$\forall y(\alpha(x, y) \rightarrow \varphi(x, y)) \quad \text{and} \quad \exists y(\alpha(x, y) \land \varphi(x, y))$$

where $\varphi(x, y)$ is a guarded formula and $\alpha(x, y)$ is an atomic formula that contains all variables in $x, y$. The formula $\alpha$ is called the guard of the quantifier.
GF(τ) denotes the set of all guarded formulas (also called GF-formulas) over the signature τ. The signature sig(ϕ) of a formula ϕ is the set of relation symbols used in it.

Let A be structure. It will be convenient to use the notation [a] = \{a_1, \ldots, a_n\} to denote the set of components of the tuple a = (a_1, \ldots, a_n) ∈ dom(A)^n. Similarly, for a tuple x = (x_1, \ldots, x_n) of variables we use [x] to denote the set \{x_1, \ldots, x_n\}.

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Let ϕ(x), ψ(x) be GF-formulas with the same free variables x. We call a GF-formula θ(x) a GF-interpolant for ϕ, ψ if sig(θ) ⊆ sig(ϕ) ∩ sig(ψ) and ϕ(x) ⊨ θ(x) and θ(x) ⊨ ψ(x). We are interested in GF-interpolant existence, the problem to decide for given ϕ(x), ψ(x) in GF whether a GF-interpolant for ϕ(x), ψ(x) exists.

In order to provide a model-theoretic characterization of when an interpolant exists, we introduce guarded τ-bisimulations [5]. A set G ⊆ dom(A) is guarded in A if G is a singleton or there exists R with A \models R(a) such that G = [a]. A tuple a ∈ dom(A)^n is guarded in A if [a] is a subset of some guarded set in A.

For tuples a = (a_1, \ldots, a_n) in A and b = (b_1, \ldots, b_n) in B we call a mapping p from [a] to [b] with p(a_i) = b_i for 1 ≤ i ≤ n (written p : a \mapsto b) a partial τ-isomorphism if p is an isomorphism from the τ-reduct of A|[a] onto B|[b].

A set I of partial τ-isomorphisms p : a \mapsto b from guarded tuples a in A to guarded tuples b in B is called a guarded τ-bisimulation if the following hold for all p : a \mapsto b ∈ I:

(i) for every guarded tuple a’ in A there exists a guarded tuple b’ in B and p’ : a’ \mapsto b’ ∈ I such that p’ and p coincide on [a] ∩ [a’].
(ii) for every guarded tuple b’ in B there exists a guarded tuple a’ in A and p’ : a’ \mapsto b’ ∈ I such that p’^{-1} and p^{-1} coincide on [b] ∩ [b’].

A pair A, a with a a tuple in A is called a pointed structure. Assume that a and b are (possibly not guarded) tuples in A and B. Then we say that the pointed structures A, a and B, b are guarded τ-bisimilar, in symbols A, a \sim_{GF,τ} B, b, if there exists a partial τ-isomorphism p : a \mapsto b and a guarded τ-bisimulation I such that Conditions (i) and (ii) hold for p. We write A, a \equiv_{GF,τ} B, b and call A, a and B, b GF(τ)-equivalent if A \models \varphi(a) iff B \models \varphi(b) holds for all formulas \varphi in GF(τ). The following equivalences are well known [5].

Lemma 1. Let A, a and B, b be pointed structures and τ a signature. Then

A, a \sim_{GF,τ} B, b implies A, a \equiv_{GF,τ} B, b

and, conversely, if A and B are ω-saturated, then

A, a \equiv_{GF,τ} B, b implies A, a \sim_{GF,τ} B, b
We are now in the position to characterize the existence of GF-interpolants. Call GF-formulas $\varphi(x), \psi(x)$ jointly GF($\tau$)-consistent if there exist pointed models $\mathfrak{A}, \mathfrak{a}$ and $\mathfrak{B}, \mathfrak{b}$ with $\mathfrak{A} \models \varphi(\mathfrak{a})$ and $\mathfrak{B} \models \psi(\mathfrak{b})$ such that $\mathfrak{A}, \mathfrak{a} \sim_{GF, \tau} \mathfrak{B}, \mathfrak{b}$.

**Lemma 2.** Let $\varphi(x), \psi(x)$ be GF-formulas and let $\tau = \text{sig}(\varphi) \cap \text{sig}(\psi)$. There does not exist a GF-interpolant for $\varphi(x), \psi(x)$ iff $\varphi(x), \neg \psi(x)$ are jointly GF($\tau$)-consistent.

**Proof.** ($\Leftarrow$) Assume there is an interpolant $\theta(x)$ and let $\mathfrak{A}, \mathfrak{B}$ be structures and $\mathfrak{a}, \mathfrak{b}$ be tuples such that $\mathfrak{A} \models \varphi(\mathfrak{a})$ and $\mathfrak{B} \models \neg \psi(\mathfrak{b})$. Suppose further that $\mathfrak{A}, \mathfrak{a} \sim_{GF, \tau} \mathfrak{B}, \mathfrak{b}$. Since $\varphi(x) \models \theta(x)$, we have $\mathfrak{A} \models \theta(\mathfrak{a})$. By Lemma 1, we obtain $\mathfrak{B} \models \theta(\mathfrak{b})$. Finally, as $\theta(x) \models \psi(x)$, we obtain $\mathfrak{B} \models \psi(\mathfrak{b})$, a contradiction.

($\Rightarrow$) Suppose that for all structures $\mathfrak{A}, \mathfrak{B}$ and tuples $\mathfrak{a}, \mathfrak{b}$ such that $\mathfrak{A} \models \varphi(\mathfrak{a})$ and $\mathfrak{B} \models \neg \psi(\mathfrak{b})$ we have $\mathfrak{A}, \mathfrak{a} \not\sim_{GF, \tau} \mathfrak{B}, \mathfrak{b}$. Let $\Phi$ be defined by taking

$$\Phi = \{\varphi'(x) \in \text{GF}(\tau) \mid \varphi(x) \models \varphi'(x)\}$$

Clearly, $\varphi(x) \models \Phi$. We claim that also $\Phi \models \psi(x)$. To see this, let $\mathfrak{B}, \mathfrak{b}$ such that $\mathfrak{B} \models \Phi(\mathfrak{b})$. Let $\mathfrak{B}'$ be an $\omega$-saturated extension of $\mathfrak{B}$ and let $\mathfrak{A}, \mathfrak{a}$ be an $\omega$-saturated pointed structure realizing $\{\chi(x) \in \text{GF}(\tau) \mid \mathfrak{B} \models \chi(\mathfrak{b})\} \cup \{\varphi\}$ in $\mathfrak{a}$. By definition of $\Phi$ and Lemma 1, we have $\mathfrak{A}, \mathfrak{a} \sim_{GF, \tau} \mathfrak{B}', \mathfrak{b}$. By the initial assumption, we cannot have $\mathfrak{B}' \models \neg \psi(\mathfrak{b})$ and thus $\mathfrak{B} \models \psi(\mathfrak{b})$. By compactness, there is a finite subset $\Phi'$ of $\Phi$ such that $\Phi' \models \psi(x)$. The conjunction of the formulas in $\Phi'$ is the required interpolant. $\square$

The goal of the paper is to show the following Theorem.

**Theorem 1.** GF-interpolant existence is decidable in $3\text{ExpTime}$ in general and in $2\text{ExpTime}$ if arity of relation symbols is bounded by a constant.

Since Lemma 2 provides a reduction of GF-interpolant existence to joint GF($\tau$)-consistency, that is, the problem of deciding whether given $\varphi(x), \psi(x)$ are jointly GF($\tau$)-consistent, we will proceed by considering the latter.

### 4 Deciding Joint GF($\tau$)-consistency

To decide joint GF($\tau$)-consistency we pursue a mosaic approach based on types. Throughout the section, let $\varphi(x_0), \psi(x_0)$ be the input to joint GF($\tau$)-consistency, for some signature $\tau$. Let $\Xi = \{\varphi(x_0), \psi(x_0)\}$. Let width($\Xi$) denote the maximal arity of any relation symbol used in $\Xi$ and let $\text{fv}(\Xi)$ be the number of free variables in $\Xi$. Let $x_1, \ldots, x_{2n}$ be fresh variables, where $n := \max \{\text{width}(\Xi), \text{fv}(\Xi)\}$. We use $\text{cl}(\Xi)$ to denote the smallest set of GF-formulas that is closed under taking subformulas and single negation, and contains:

- $\Xi$,
- all formulae $x = y$ for distinct variables $x, y$;
- all formulae $\exists x R(xy)$, where $R$ is a relation symbol that occurs in $\Xi$ and $xy$ is a tuple of variables.
Let \( \mathfrak{A} \) be a structure, \( a \) a tuple of distinct elements from the domain of \( \mathfrak{A} \), and \( x \) a tuple of distinct variables in \( \{ x_1, \ldots, x_{2n} \} \) of the same length as \( a \). Consider the bijection \( \sigma : x \mapsto a \). Then the \( \Xi \)-type of \( a \) in \( \mathfrak{A} \) defined through \( \sigma \) is

\[
\text{tp}(\mathfrak{A}, \sigma : x \mapsto a) = \{ \theta \mid \mathfrak{A} \models_\sigma \theta, \theta \in \text{cl}(\Xi)[x] \},
\]

where \( \text{cl}(\Xi)[x] \) is obtained from \( \text{cl}(\Xi) \) by substituting in any formula \( \theta \in \text{cl}(\Xi) \) the free variables of \( \theta \) by variables in \( [x] \) in all possible ways. Note that the assumption that \( \sigma \) is bijective entails that \( \neg(x = y) \in \text{tp}(\mathfrak{A}, \sigma : x \mapsto a) \) for any two distinct \( x, y \in [x] \). We drop \( \sigma \) (and both \( \sigma \) and \( x \)) and write \( \text{tp}(\mathfrak{A}, x \mapsto a) \) (and \( \text{tp}(\mathfrak{A}, a) \), respectively), whenever they are obvious from the context. Any \( \Xi \)-type of some \( a \) through some \( \sigma : x \mapsto a \) is called a \( \Xi \)-type and simply denoted \( t(x) \). The set of all \( \Xi \)-types is denoted \( T(\Xi) \).

To decide joint GF(\( \tau \))-consistency of \( \varphi(x_0), \psi(x_0) \) we determine all sets \( \Phi \subseteq T(\Xi) \) using at most \( n \) variables from \( \{ x_1, \ldots, x_{2n} \} \) that can be satisfied in guarded \( \tau \)-bisimilar models in the following sense: there are models \( \mathfrak{A}_i, t \in \Phi \), realizing \( t \) in tuples \( a_i \) through assignments \( \sigma_i \) that are mutually guarded \( \tau \)-bisimilar on the images of shared variables between types. Such sets \( \Phi \) will be called \( \tau \)-mosaics and are the main ingredient of our approach. We can check whether \( \varphi(x_0), \psi(x_0) \) are jointly GF(\( \tau \)) consistent by simply checking whether there are types \( t_1(x), t_2(x) \) in a single \( \Phi \) such that we can replace the variables \( x_0 \) by variables in \( [x] \) and \( \varphi' \in t_1(x_1), \psi' \in t_2(x_2) \) for the resulting formulas \( \varphi', \psi' \). Thus, in what follows we aim to determine the characteristic properties of \( \tau \)-mosaics and show that they can be enumerated in triple exponential time in general. If \( \text{width}(\Xi) \) is fixed, we perform a closer analysis of the set of mosaics and show that double exponential time is sufficient.

To formulate the characteristic properties of \( \tau \)-mosaics, we require some notation. The restriction \( t(x)|_X \) of a \( \Xi \)-type \( t(x) \) to a set \( X \) of variables is the set of \( t \in t(x) \) with free variables among \( X \). The restriction \( \Phi|_X \) of a set \( \Phi \) of \( \Xi \)-types to \( X \) is defined as \( \{ t(x)|_X \mid t(x) \in \Phi \} \). Types \( t(x) \) and \( t'(x') \) coincide on \( X \) if \( t(x)|_X = t'(x')|_X \) and sets \( \Phi, \Phi' \) of \( \Xi \)-types coincide on \( X \) if \( \Phi|_X = \Phi'|_X \). A variable \( x \) is free in a \( \Phi \) if \( \Phi \) contains a type in which \( x \) is free.

A formula \( Q(x) \) of the form \( x = x \) or \( \exists y R(xy) \) with \( R \in \tau \) is called a \( \tau \)-guard (for \( x \)). It is called a strict \( \tau \)-guard if it is of the form \( x = x \) or \( y \) is empty, respectively. We call a set \( \Phi \subseteq T(\Xi) \) a \( \tau \)-mosaic if it satisfies the following conditions:

- \( \Phi \) is \( \tau \)-uniform: for all \( \tau \)-guards \( Q(z) \) and all \( t(x), s(y) \in \Phi \) with \( [z] \subseteq [x] \cap [y] \), \( Q(z) \in t(x) \) iff \( Q(z) \in s(y) \);
- closed under restrictions: if \( t(x) \in \Phi \) and \( X \subseteq [x] \), then \( t(x)|_X \in \Phi \);
- \( \tau \)-bisimulation saturated: for all \( t(x) \in \Phi \), all strict \( \tau \)-guards \( Q(y) \in t(x) \), and all \( t'(z) \in \Phi \) with \( [z] \subseteq [y] \), there is an \( s(y') \in \Phi \) such that \( t'(z) \subseteq s(y') \) and \( [y'] = [y] \).

Intuitively, \( \tau \)-uniformity reflects that guarded \( \tau \)-bisimulations preserve all \( \tau \)-guards and \( \tau \)-bisimulation saturatedness reflects the first condition for guarded
\(\tau\)-bisimulations. In addition to the properties above, we have to ensure that \(\tau\)-mosaics can be linked together. The next two conditions state when this is the case. We say that \(\tau\)-mosaics \(\Phi_1, \Phi_2\) are compatible if for \(\{i, j\} = \{1, 2\}\):

1. for every \(t(x) \in \Phi_i\) there is an \(s(y) \in \Phi_j\) such that \(t(x)\) and \(s(y)\) coincide on \([x] \cap [y]\);
2. if there are \(t(x) \in \Phi_i\) and \(s(y) \in \Phi_j\) and a \(\tau\)-guard \(Q(x) \in t(x)\) with \([z] \subseteq [x] \cap [y]\), then \(\Phi_i\) and \(\Phi_j\) coincide on \([z]\).

Note that compatibility is a reflexive and symmetric relation. Let \(\mathcal{M}\) be a set of \(\tau\)-mosaics. We call \(\Phi \in \mathcal{M}\) existentially saturated in \(\mathcal{M}\) if for every \(t(x) \in \Phi\) and every formula \(\exists y (R(x', y) \land \lambda(x', y)) \in t(x)\) there is a some \(\Phi' \in \mathcal{M}\) such that \(\Phi, \Phi'\) are compatible and \(R(x', y') \land \lambda(x', y') \in t'(z)\) for some \(t'(z) \in \Phi'\) which coincides with \(t(x)\) on \([x] \cap [z]\). \(\mathcal{M}\) is called good if every \(\Phi \in \mathcal{M}\) is existentially saturated in \(\mathcal{M}\).

**Lemma 3.** Assume \(\mathcal{M}\) is good and let \(t_1(x_1), t_2(x_2) \in \Psi \in \mathcal{M}\). Then there are pointed models \(\mathfrak{A}_1, \mathfrak{a}_1\) and \(\mathfrak{A}_2, \mathfrak{a}_2\) and \(\sigma_i : x_i \mapsto \mathfrak{a}_i\) such that \(\mathfrak{A}_i \models t_i(\mathfrak{a}_i), i = 1, 2, \) and for \(Y = [x_1] \cap [x_2]\), \(\mathfrak{A}_i, \sigma_1(x_1|Y) \sim_{GF, \tau} \mathfrak{A}_2, \sigma_2(x_2|Y)\).

**Proof.** Let \(\Psi \in \mathcal{M}\). We assume w.l.o.g. that \(\mathcal{M}\) is closed under restrictions in the sense that for any \(\Phi \in \mathcal{M}\) and subset \(X\) of the free variables of \(\Phi, \Phi|_X \in \mathcal{M}\). (If it is not closed under restrictions simply add all \(\Phi|X\) with \(\Phi \in \mathcal{M}\) to \(\mathcal{M}\).

The resulting set is still good.) Define \(\tilde{\Psi} := \Psi_{|\mathcal{E}}\), that is, \(\tilde{\Psi}\) contains all \(\Xi\)-types in \(\Psi\) without free variables. By closure under restrictions of \(\mathcal{M}\), we have \(\Psi \in \mathcal{M}\). Assume \(\tilde{\Psi} = \{\tilde{t}_1, \ldots , \tilde{t}_m\}\). We construct structures \(\tilde{\mathfrak{A}}_i, i = 1, \ldots , m\), with \(\tilde{\mathfrak{A}}_i\) satisfying \(\tilde{t}_i\). For the construction, it is useful to employ notation for tree decompositions. A tree decomposition of an structure \(\mathfrak{A}\) is a triple \((T, E, bag)\) with \((T, E)\) a tree and bag a function that assigns to every \(t \in T\) a set \(bag(t) \subseteq \text{dom}(\mathfrak{A})\) such that

1. \(\mathfrak{A} = \bigcup_{t \in T} \mathfrak{A}_{bag(t)}\);
2. \(\{t \in T \mid a \in bag(t)\}\) is connected in \((T, E)\), for every \(a \in \text{dom}(\mathfrak{A})\).

We construct the structures \(\mathfrak{A}_i, i = 1, \ldots , m\) by giving a tree decomposition \((T_i, E_i, bag_i)\), starting with defining \((T_i, E_i)\). \(T_i\) is the set of all sequences

\[\sigma_n = (t_0(y_0), \Phi_0), \ldots , (t_n(y_n), \Phi_n)\]

such that \(t_0 = \tilde{t}_i, \Phi_0 = \tilde{\Psi}, t_j(y_j) \in \Phi_j\) for all \(j \leq n\), and for all \(j < n\):

- \(\Phi_j, \Phi_{j+1}\) are compatible, and
- \(t_i(y_j)\) and \(t_{j+1}(y_{j+1})\) coincide on \([y_j] \cap [y_{j+1}]\).

Then, \(E_i\) is the induced prefix-order on \(T_i\). We call \((t_n(y_n), \Phi_n)\) the tail of \(\sigma_n\).

Because the \(T_i\) are pairwise disjoint, we can define a single function bag. Below, we give an inductive definition with the aim to achieve that, for \(\sigma_n \in T_i\) of the form above with tail \((t_n(y_n), \Phi_n(x_n))\), \(bag(\sigma_n)\) satisfies \(t_n(y_n)\) in \(\mathfrak{A}_i\) under a canonical assignment \(v_{\sigma_n}\). For the construction, it is important to note
that we have \(-(x = y) \in t\) for any two distinct free variables \(x, y\) in a type \(t\), so that we can essentially use (copies of) the variables \(y, y\) as the domain of the bag(\(\sigma_n\)).

We inductively define bag starting with setting bag(\(\sigma_0\)) = \(\emptyset\) and \(v_{\sigma_0} = \emptyset\) for \(\sigma_0 = (t, \Phi_0)\). In the inductive step, assume that bag and \(v_{\sigma_{n-1}}\) have been defined on
\[
\sigma_{n-1} = (t_0(y_0), \Phi_0), \ldots, (t_{n-1}(y_{n-1}), \Phi_{n-1})
\]
The domain of bag(\(\sigma_n\)) contains
- fresh copies \(y'\) of the variables \(y \in [y_n] \setminus [y_{n-1}]\) and
- \(v_{\sigma_{n-1}}(y)\) for every \(y \in [y_n] \cap [y_{n-1}]\).

We define \(v_{\sigma_n}(y)\) as the copy \(y'\) of \(y\) for \(y \in [y_n] \setminus [y_{n-1}]\) and set \(v_{\sigma_n}(y) := v_{\sigma_{n-1}}(y)\) for \(y \in [y_n] \cap [y_{n-1}]\). We then interpret the relations \(R\) in such a way that the atomic formulas in \(t_n(y_n)\) are satisfied under \(v_{\sigma_n}\), that is,
\[
\text{bag}(\sigma_n) = \{R(v_{\sigma_n}(y)) \mid R(y) \in t_n(y_n)\}
\]
Note that since, by definition, \(t_n(y_n)\) and \(t_{n-1}(y_{n-1})\) coincide on \([y_n] \cap [y_{n-1}]\), also bag(\(\sigma_n\)) and bag(\(\sigma_{n-1}\)) coincide on their shared variables.

To finish the construction of the \(A_i, i = 1, \ldots, m\), let \(A_i\) be the union of all bag(\(\sigma\)), \(\sigma \in T_i\). It is easy to see that \((T_i, E_i, \text{bag})\) is a tree decomposition of \(A_i\). We proceed to show that the guarded \(\tau\)-bisimulation mentioned in Lemma 3 indeed exists. For doing so, we prove the following auxiliary lemma. We call a tuple \(a\) a \(\tau\)-guarded in \(A\) if there exists a \(\tau\)-guard \(Q(x)\) such that \(A \models Q(a)\).

**Lemma 4.** For all \(i, j\) with \(1 \leq i, j \leq m\), we have:

1. For every \(\sigma \in T_i\) with \(\text{tail}(\sigma) = (t(y), \Phi)\), we have \(A_i \models t(v_{\sigma}(y))\);
2. Let \(H_{i,j}\) be the set of all mappings \(p_{\sigma, \sigma', \mathbf{z}}\) where
   - \(\sigma, \sigma' \in T_j\) with \(\text{tail}(\sigma) = (t(y), \Phi)\) and \(\text{tail}(\sigma') = (t'(y'), \Phi)\);
   - \(\mathbf{z}\) is a tuple with \([\mathbf{z}] \subseteq [y] \cap [y']\) and \(v_{\sigma}(\mathbf{z})\) is \(\tau\)-guarded in \(A_i\) (or, equivalently, \(v_{\sigma'}(\mathbf{z})\) is \(\tau\)-guarded in \(A_j\));
   - \(p_{\sigma, \sigma', \mathbf{z}} : v_{\sigma}(\mathbf{z}) \mapsto v_{\sigma'}(\mathbf{z})\).
   
   Then \(H_{i,j}\) is a guarded \(\tau\)-bisimulation between \(A_i\) and \(A_j\).

**Proof.** For Point 1, we prove by induction that, for all \(\sigma \in T_i\) with \(\text{tail}(\sigma) = (t(y), \Phi)\) and all formulas \(\varphi(\mathbf{z})\) with \([\mathbf{z}] \subseteq [y]\), we have:
\[
\varphi(\mathbf{z}) \in t(y) \text{ iff } A_i \models \varphi(v_{\sigma}(\mathbf{z}))
\]
The induction base is given by the definition of \(\text{bag}(\sigma)\). If \(\varphi\) is of the shape \(\neg \varphi', \varphi' \land \varphi'',\) or \(\varphi' \lor \varphi''\), the statement is immediate from the hypothesis. Consider now \(\varphi(\mathbf{z}) = \exists x(R(x, z) \land \lambda(z, x))\).

\((\Rightarrow)\) Since \(M\) is existentially saturated, there is a \(\Phi' \in M\) such that \(\Phi, \Phi'\) are compatible and \(R(x, z') \land \lambda(z', x') \in t'(y')\) for some \(t'(y') \in \Phi'\) such that \(t(y)\) and \(t'(y')\) coincide on \([y] \cap [y']\). By definition of \(T_i\) and compatibility of \(\Phi, \Phi'\), we have \(\sigma' = \sigma \cdot t'(y'), \Phi' \in T_j\). Moreover, by induction, we obtain that
\(T\)z satisfies \(R(z, x') \land \lambda(z, x')\) under \(v_{\sigma'}\). By definition of \(\text{bag}(\sigma)\) and \(\text{bag}(\sigma')\), we get \(T_i \models \varphi(v_{\sigma}(z))\).

(\(\Leftarrow\)) Conversely, assume \(T_i \models \varphi(v_{\sigma}(z))\). By construction, there is some \(\sigma' \in T_i\) such that \(v_{\sigma'}(z) = v_{\sigma}(z)\) and \(T_i\) satisfies \(R(z, x') \land \lambda(z, x')\) under \(v_{\sigma'}\), for some \(x'\). By induction hypothesis, \(R(z, x') \land \lambda(z, x') \in t'(y')\), where \(\text{tail}(\sigma') = (t'(y'), \Phi')\). Thus, \(\exists x(R(z, x) \land \lambda(z, x)) = \varphi(z) \in t'(y')\). As \(v_{\sigma'}(z) = v_{\sigma}(z)\), the construction of \(T_i\) implies that \(t'(y')\) and \(t(y)\) coincide on all subformulas over \(z\), hence \(\varphi(z) \in t(y)\).

For Point 2, observe first that the \(p_{\sigma, \sigma', z}\) are partial \(\tau\)-isomorphisms between \(\tau\)-guarded tuples since all \(\Phi \in \mathcal{M}\) are \(\tau\)-uniform. (In addition, the observation that \(v_{\sigma'}(z)\) is \(\tau\)-guarded in \(T_1\) if \(v_{\sigma'}(z)\) is \(\tau\)-guarded in \(T_2\) follows from the condition that \(\Phi\) is \(\tau\)-uniform.) By symmetry, it suffices to prove the first condition for guarded \(\tau\)-bisimulations.

Let \(p \in H_{i,j}\). Then we have \(\sigma \in T_i, \sigma' \in T_j\) with \(\text{tail}(\sigma) = (t(y), \Phi)\) and \(\text{tail}(\sigma') = (t'(y'), \Phi')\) and we have a tuple \(z\) such that \([z] \subseteq [y] \cap [y']\) and \(v_{\sigma'}(z)\) is \(\tau\)-guarded in \(T_i\) and \(p = p_{\sigma, \sigma', z}\). Thus, there is a \(\tau\)-guard \(Q(z)\) with \(T_i \models Q(v_{\sigma'}(z))\) and \(T_j \models Q(v_{\sigma'}(z))\). Consider any tuple \(b\) with \(T_i \models R(b)\) for some \(R \in \tau\). We have to show that there exists a mapping \(p_{\sigma, \sigma', z} \in H_{i,j}\) with domain \([b]\) which coincides with \(p_{\sigma, \sigma', z}\) on \([v_{\sigma}(z)] \cap [b]\). We distinguish on whether or not that intersection is empty.

**Intersection is empty.** The existence of such a mapping follows from \(\tau\)-bisimulation saturatedness: to see this observe that, as we have a tree decomposition, there exists \(\rho_0 \in T_i\) such that \([b] \subseteq \text{dom}(\text{bag}(\rho_0))\). Let \(\text{tail}(\rho_0) = (s(x_0), \Omega)\). Then there exists a tuple \(y_0\) with \([y_0] \subseteq [x_0]\) such that \(v_{\rho_0}(y_0) = b\). We have \(R(y_0) \in s(x_0)\). As \(t_j \in \Omega\), by \(\tau\)-bisimulation saturatedness of \(\Omega\), there exists \(s'(y_0) \in \Omega\) such that \(t_j \subseteq s'(y_0)\) and \([y_0] = [y_0]\). But then \(R(y_0) \in s'(y_0)\). Also \(\rho = (t_j, \Phi') \cdot (s'(y_0), \Omega) \in T_j\). Thus \(p_{\rho_0, \rho, y_0}\) is as required.

**Intersection is not empty.** As we have a tree decomposition, there exists \(\rho_0 \in T_i\) such that \([b] \subseteq \text{dom}(\text{bag}(\rho_0))\). Let \(\text{tail}(\rho_0) = (s(x_0), \Omega)\). Then there exists a tuple \(z'\) with \([z'] \subseteq [x_0]\) such that \(v_{\rho_0}(z') = b\). We distinguish the following cases:

(a) \(\rho_0 = \sigma\);
(b) \(\rho_0 \neq \sigma\).

Assume first that (a) holds. Then \((s(x_0), \Omega) = (t(y), \Phi)\) and \(b = v_{\sigma}(z')\). We use \(\tau\)-bisimulation saturatedness of \(\Phi\). Consider the restriction \(z''\) of \(z'\) to \([z] \cap [z']\) and the restriction \(t'(y')\) of \(t'(y')\) to \([z'']\). Then there exists \(s'(z''_0) \in \Phi\) such that \(t'(y')|_{[z'']} \subseteq s'(z''_0) \in \Phi\) and \([z''] = [z']\). Let \(\sigma'' = \sigma' \cdot (s'(z''_0), \Phi) \in T_j\). Then \(p_{\sigma, \sigma'', z''_0}\) is as required, as \(\Phi\) is \(\tau\)-uniform.

Assume now that Point (b) holds. Consider the restriction \(z''\) of \(z'\) to \([z] \cap [z']\) and the restriction \(t'(y')\) of \(t'(y')\) to \([z'']\). Consider the restriction \(\Phi|_{[z'']}\) of \(\Phi\) to \([z'']\). By closure under restrictions, \(\Phi|_{[z'']} \in \mathcal{M}\). Observe that \(\Phi, \Phi|_{[z'']}\) and \(\Phi|_{[z'']}\), \(\Omega\) are compatible: indeed, in the tree decomposition all bags on the path from \(\sigma\) to \(\rho_0\) have a tail \((\cdot, \Omega')\) satisfying \(\Phi|_{[z'']} \subseteq \Omega'\). Thus \(t'(y')|_{[z'']} \in \Omega\). Using
the fact that $\Omega$ is $\tau$-bisimulation saturated, one can now show that there exists $s'(z'_0) \in \Omega$ such that $t'(y')|_{[z'_0]} \subseteq s'(z'_0)$ and $[z'_0] = [z']$. We then have

$$\rho = \sigma \cdot (t'(y')|_{[z']} \cdot \Phi|_{[z']} \cdot (s'(z'_0), \Omega) \in T_j$$

and $p_{\rho, \rho, z'_0}$ is as required.

To complete the proof of Lemma 3, assume w.l.o.g. that $i = 1, 2$. Take $\rho_i = (t_i, \Psi_i) \cdot (t_i, y_i, \Psi_i) \in T_i$, for $i = 1, 2$. Consider the tuples $a_i := v_{\rho_i}(x_i)$. By Lemma 4, $A_i = t_i(a_i)$. Also by Lemma 4, for any tuple $z$ with $|z| \subseteq [x_1] \cap [x_2]$ and such that $v_{\rho_i}(z)$ is $\tau$-guarded in $A_i$ or $A_2$, we have $p_{\rho_1, \rho_2, z} : v_{\rho_1}(z) \rightarrow v_{\rho_2}(z) \in H_{1.2}$. But then, as any two $p_{\rho_1, \rho_2, z}$ coincide on the intersection of their domains, we have for $Y = [x_1] \cap [x_2]$, $A_1$, $v_{\rho_1}(x_1|_Y) \sim_{GF, \tau} A_2, v_{\rho_2}(x_2|_Y)$, as required.

**Lemma 5.** Let $\mathcal{A}_1, a_1$ and $\mathcal{A}_2, a_2$ be pointed structures with $a_1$ and $a_2$ tuples with pairwise distinct elements of length $m \leq \text{fr}(\mathcal{Z})$ and let $\tau$ be a signature. Consider assignments $x_0 \mapsto a_i$ with $[x_0] \subseteq \{x_0, \ldots, x_{2n}\}$. If $\mathcal{A}_1, a_1 \sim_{GF, \tau} \mathcal{A}_2, a_2$, then there exists a good set $M$ and some $\Psi \in \mathcal{M}$ such that

- all $\Phi \in \mathcal{M}$ with $\Phi \neq \Psi$ use at most width$(\mathcal{Z})$ many free variables;
- there exist types $t_1(x_0), t_2(x_0) \in \Psi$ such that $t_i(x_0) = tp(\mathcal{A}_i, x_0 \mapsto a_i)$ for $i = 1, 2$ and all types $t(y) \in \Psi \setminus \{t_1(x_0), t_2(x_0)\}$ use at most width$(\mathcal{Z})$ free variables among $[x_0]$.

**Proof.** Assume w.l.o.g. that $\mathcal{A}_1$ and $\mathcal{A}_2$ are disjoint. For any tuples $b_i$ in $\mathcal{A}_i$ and $b_2$ in $\mathcal{A}_j$ with $i, j \in \{1, 2\}$, we use $tp(b_1 \mapsto b_2)$ to denote $tp(\mathcal{A}_i, x_1 \mapsto a_1)$ and we write $b_1 \sim_{GF, \tau} b_2$ if $\mathcal{A}_1, b_1 \sim_{GF, \tau} \mathcal{A}_j, b_2$. Define $M$ as follows. Take any tuple $a_i$ of distinct elements in $\mathcal{A}_i, i \in \{1, 2\}$. Take a tuple $x$ from $\{x_1, \ldots, x_{2n}\}$ such that $\sigma : x \mapsto a$ is a bijection. The let $\Phi_{a,x}$ containing all types $tp(\sigma' : x_{1Y} \mapsto b)$ with $Y \subseteq [x] \mapsto b$ and $b$ in either $\mathcal{A}_1$ or $\mathcal{A}_2$ such that $\sigma(x_{1Y}) \sim_{GF, \tau} \sigma'(x_{1Y})$

Let $M$ contain all such $\Phi_{a,x}$ with $a$ of length at most $\text{width}(\mathcal{Z})$ and $x$ from $\{x_1, \ldots, x_{2n}\}$. Moreover, if $m > \text{width}(\mathcal{Z})$, then add $\widehat{\Phi}_{a_1, x_0}$ to $M$, where $\widehat{\Phi}_{a_1, x_0}$ is obtained from $\Phi_{a_1, x_0}$ by removing all $t$ distinct from $t_1(x_0)$ and $t_2(x_0)$ using more than $\text{width}(\mathcal{Z})$ many free variables.

We show that $M$ is as required. By definition, $tp(\mathcal{A}_1, x_0 \mapsto a_1)$, $tp(\mathcal{A}_2, x_0 \mapsto a_2) \in \Phi_{a_1, x_0} \in M$.

For the next steps we first assume that instead of $\widehat{\Phi}_{a_1, x_0}$ we have $\Phi_{a_1, x_0}$ in $M$. Then observe that if we have any $\Phi \in M$ and $t(x'), s(x'') \in \Phi$, then we can assume that $\Phi = \Phi_{a,x}$, we have a bijection $\sigma$ from $x$ to $x'$, $x' = x_{1Y}$ and $x'' = x_{1Y'}$ for appropriate sets of variables $Y', Y'' \subseteq [x]$, and there are $\sigma' : x_{1Y'} \mapsto A_i$ and $\sigma'' : x_{1Y''} \mapsto A_j$ such that $\sigma'(x_{1Y'}) \sim_{GF, \tau} \sigma(x_{1Y'})$ and $\sigma''(x_{1Y''}) \sim_{GF, \tau} \sigma(x_{1Y''})$. Then $\sigma'(x_{1Y \cap Y'}) \sim_{GF, \tau} \sigma''(x_{1Y \cap Y''})$. We now show that each $\Phi_{a,x}$ is $\tau$-uniform, closed under restrictions, and $\tau$-bisimulation saturated.

1. Every $\Phi_{a,x} \in M$ is $\tau$-uniform: let $t(x')$, $s(x'') \in \Phi_{a,x}$ be as above and assume that $Q(z)$ is a $\tau$-guard with $[z] \subseteq [x'] \cap [x'']$. Then $[z] \subseteq Y' \cap Y''$ and so $Q(z)$ in $t(x')$ iff $Q(z) \in s(x'')$ since $\sigma'(x_{1Y \cap Y'}) \sim_{GF, \tau} \sigma''(x_{1Y \cap Y''})$, as required.
2. To show τ-bisimulation saturatedness let \( \Phi_{a,x} \in M \) and \( t(x'), s(x'') \in \Phi_{a,x} \) be as above and let \( R(y) \in t(x') \) with \( [x''] \subseteq [y] \) be a strict τ-guard. We have \( Y'' \subseteq [y] \subseteq Y' \) and \( \sigma'(x_{Y''}) \sim_{GF,\tau} \sigma''(x_{Y''}) \). Let \( H \) be the guarded τ-bisimulation witnessing this. By the definition of guarded \( \tau \)-bisimulations, there exists \( p \in H \) with domain \( \sigma'(x_{[y]}) \) such that \( p \circ \sigma'_{[y]} = \sigma'' \). Now we expand \( \sigma'' \) to the domain \([y]\) by setting \( \hat{\sigma} := p \circ \sigma'_{[x_{[y]}]} \). Let \( b' \) be the image of \( x_{[y]} \) under \( \hat{\sigma} \). Then the type \( \text{tp}(\hat{\sigma} : x_{[y]} \mapsto b') \) is as required.

Finally we show that every \( \Phi \in M \) is existentially saturated in \( M \). Assume \( \Phi_{a,x} \) is given. Assume \( \exists y(R(x', y) \land \lambda(x', y)) \in t(x_{[y]}) = \text{tp}(\sigma' : x_{[y]} \mapsto b) \) with \( Y \subseteq [x] \) and \( b \) w.l.o.g. in \( \Phi_1 \). Then \( \Phi_1 \models \exists y(R(x', y) \land \lambda(x', y)) \). Then we find an assignment \( \sigma'' \) for the variables in \([x']\) which coincides with \( \sigma' \) on \([x']\) such that \( \Phi_1 \models \sigma'' \sim_{GF,\tau} R(x', y) \land \lambda(x', y) \). Take a tuple \( c \) of distinct elements with \([c] = [\sigma'''(x'y)]\) and a tuple \( y' \) of variables in \([x_1, \ldots, x_{2n}]\) such that \([x'] = [x] \cap [y']\) and we have a bijection \( \rho : y' \mapsto c \) which coincides with \( \sigma'' \) on \([x']\). Then \( \rho(y'_{[x']}) \sim_{GF,\tau} \sigma(x_{[x']}) \) and so \( \Phi_{a,x} \) and \( \Phi_{e,y'} \) are compatible and \( \Phi_{e,y'} \) is as required.

For the proof with \( \Phi_{a,1,x_0} \) instead of \( \Phi_{a,1,x_0} \) in \( M \) observe that \( \Phi_{a,1,x_0} \) is τ-uniform and τ-bisimulation saturated as \( \Phi_{a,1,x_0} \) behaves in exactly the same way as \( \Phi_{a,1,x_0} \) regarding τ-guarded \( Q(y) \). For the same reason all elements of \( M \) are still existentially saturated in \( M \).

As a consequence of Lemmas 3 and 5, the following Conditions are equivalent.

1. \( \varphi(x_0), \psi(x_0) \) are jointly \( GF(\tau) \)-consistent;
2. there is a good set \( M = \{\Phi_0\} \cup M' \) and \( \Xi \)-types \( t_1(x), t_2(x) \in \Phi_0 \) such that:
   (a) \( t_1(x), t_2(x) \) have \( \text{fv}(\Xi) \) free variables and one can replace the variables in \([x_0]\) by variables in \( x \) such that \( \varphi' \in t_1(x), \psi' \in t_2(x) \) for the resulting formulas \( \varphi', \psi' \);
   (b) all \( \Xi \)-types \( t(y) \in \Phi_0 \setminus \{t_1(x), t_2(x)\} \) use at most \( \text{width}(\Xi) \) free variables among \([x]\);
   (c) all mosaics in \( M' \) use at most \( \text{width}(\Xi) \) free variables.

Hence, it suffices to provide an algorithm for deciding Condition 2.

**Lemma 6.** On input \( \varphi(x_0), \psi(x_0) \), Condition 2 can be decided in time triple exponential in the size of \( \varphi(x_0), \psi(x_0) \) in general, and double exponential in the size of \( \varphi(x_0), \psi(x_0) \) if \( \text{width}(\Xi) \) is bounded by a constant.

**Proof.** We proceed as follows to identify a good set \( M \) that satisfies Condition 2. For every pair of \( \Xi \)-types \( t_1(x), t_2(x) \) that satisfy Condition 2(a) enumerate all τ-mosaics \( \Phi_0 \) which satisfy Condition 2(b), that is, \( t_1(x), t_2(x) \in \Phi_0 \) and all types in \( \Phi_0 \) except \( t_1(x), t_2(x) \) use at most \( \text{width}(\Xi) \) free variables among \([x]\). For each such \( \Phi_0 \) start with the set \( M_0 \) consisting of \( \Phi_0 \) and all mosaics \( \Phi \) with at most \( \text{width}(\Xi) \) free variables. Then exhaustively remove mosaics from \( M_0 \) that are not existentially saturated. It can be verified that the fixpoint is good.

Accept if the fixpoint still contains \( \Phi_0 \). Reject if for no choice of \( \Phi_0 \) this is the case.
Correctness of the algorithm is straightforward, so it remains to analyze its run time. For this purpose, let \( r \) be the number of subformulas (of formulas) in \( \Xi \) and \( \ell \geq 0 \). Observe that a subformula with \( \ell \) free variables has at most \((2n)^\ell\) instantiations with variables from \( x_1, \ldots, x_{2n} \). Since for every such instantiated formula either the formula itself or its negation is contained in any type, there are at most \( 2^{(2n)^\ell} \) many types with \( \ell \) free variables. Thus, there are only double exponentially many choices for \( t_1(x), t_2(x) \) and \( \Phi_0 \). Moreover, the initial sets \( M_0 \) are of size triple exponential in the size of \( \varphi(x_0), \psi(x_0) \) in general, and double exponential in the size of \( \varphi(x_0), \psi(x_0) \) if \( \text{width}(\Xi) \) is bounded by a constant.

Checking whether some \( \Phi \) is existentially saturated in some set of mosaics \( M \) can be done in time polynomial in the size of \( M \). Since in every round at least one mosaic is removed, the overall run time is as stated in the Lemma.

From the equivalence of Conditions 1 and 2, and Lemma 6 we finally obtain:

**Theorem 2.** Joint GF(\( \tau \))-consistency of \( \varphi(x_0), \psi(x_0) \) can be decided in time triple exponential in the size of \( \varphi(x_0), \psi(x_0) \) in general, and double exponential in the size of \( \varphi(x_0), \psi(x_0) \) if the arity of relation symbols is bounded by a constant.

**References**